CORRIGENDUM TO THE ARTICLE "SHEETS AND ASSOCIATED VARIETIES OF AFFINE VERTEX ALGEBRAS", ADV. MATH, 320 (2017), 157-209.

TOMOYUKI ARAKAWA AND ANNE MOREAU

It is claimed in the main theorem of [2] (Theorem 1.1) that

$$\operatorname{gr} V_{-1}(\mathfrak{sl}_n) \cong \mathbb{C}[J_\infty \overline{\mathbb{S}_{\mathfrak{l}_1}}] \quad \text{and} \quad \operatorname{gr} V_{-m}(\mathfrak{sl}_{2m}) \cong \mathbb{C}[J_\infty \overline{\mathbb{S}_{\mathfrak{l}_0}}]$$

as Poisson vertex algebras, but this is not true in general (the rest of the theorem is correct).

Indeed, using `Macaulay2` one can compute the Hilbert series of $J_\infty \overline{\mathbb{S}_{\mathfrak{l}_1}}$, and we obtain for $n = 4$ that

$$H(J_\infty \overline{\mathbb{S}_{\mathfrak{l}_1}}, q) = 1 + 15q + 115q^2 + 620q^3 + 2785q^4 + o(q^4).$$

On the other hand, from the results of [1] it is possible to obtain an explicit character formula of $\operatorname{gr} V_{-1}(\mathfrak{sl}_4))$ as in [3], and we obtain for $n = 4$ that

$$\chi_{V_{-1}(\mathfrak{sl}_4)}(q) = 1 + 15q + 115q^2 + 620q^3 + 2765q^4 + o(q^4).$$

Hence $J_\infty \overline{\mathbb{S}_{\mathfrak{l}_1}}$ and $SS(V_{-1}(\mathfrak{sl}_4))$ are not isomorphic as schemes.

The error comes from the fact that the isomorphisms of Theorem 4.1 of [2] hold only as reduced schemes. However, in Corollary 4.2 of [2] the arc spaces $J_\infty \overline{\mathbb{O}}$ and $J_\infty \overline{\mathbb{S}}$ needs not to be reduced. In addition, the hypothesis that the singular support $SS(V)$ is $J_\infty \mathbb{C}^*$-invariant is necessary in Theorem 4.1 although we show below (see Lemma 3) that this holds for the vertex algebras $V_{-1}(\mathfrak{sl}_n)$ considered in [2, Th. 1.1].

To summarize, Theorem 1.1 of [2] has to be replaced by

Theorem 1 (replacement of [2, Th. 1.1]). (1) *For $n \geqslant 4$,*

$$\tilde{X}_{V_{-1}(\mathfrak{sl}_n)} \cong \overline{\mathbb{S}_{\mathfrak{l}_1}}$$

as schemes, where $\mathfrak{l}_1$ is the standard Levi subalgebra of $\mathfrak{sl}_n$ generated by all simple roots except α_1. Moreover $V_{-1}(\mathfrak{sl}_n)$ is a quantization of the infinite jet scheme $J_\infty \overline{\mathbb{S}_{\mathfrak{l}_1}}$ of $\overline{\mathbb{S}_{\mathfrak{l}_1}}$ in the sense that

$$SS(V_{-1}(\mathfrak{sl}_n)) \cong J_\infty \overline{\mathbb{S}_{\mathfrak{l}_1}}$$

as topological spaces, that is, $SS(V_{-1}(\mathfrak{sl}_n))_{\mathrm{red}} \cong (J_\infty \overline{\mathbb{S}_{\mathfrak{l}_1}})_{\mathrm{red}}$.

(2) *For $m \geqslant 2$,*

$$\tilde{X}_{V_{-m}(\mathfrak{sl}_{2m})} \cong \overline{\mathbb{S}_{\mathfrak{l}_0}}$$

as schemes, where $\mathfrak{l}_0$ is the standard Levi subalgebra of $\mathfrak{sl}_{2m}$ generated by all simple roots except α_m.

Theorem 4.1 of [2] has to be replaced by:

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Theorem 2 (replacement of [2, Th. 4.1])**.** *Let V be a quotient vertex algebra $V^k(\mathfrak{g})$. Suppose that $X_V = \overline{G.\mathbb{C}^* x}$ for some $x \in \mathfrak{g}$ and that $J_\infty \mathbb{C}^*.x$ is contained in the reduced singular support $SS(V)_{\rm red}$, where the action of $J_\infty \mathbb{C}^*$ on $J_\infty \mathfrak{g}$ is induced by the $\mathbb{C}^*$-action on $\mathfrak{g}$. Then*

$$SS(V) = J_\infty X_V = J_\infty \overline{G.\mathbb{C}^* x} = \overline{J_\infty G.\mathbb{C}^* x},$$

as topological spaces. In particular, $SS(V)_{\rm red} = (J_\infty X_V)_{\rm red}$.

The proof in unchanged (except that in the last sentence, "$J_\infty \mathbb{C}^*$-invariant" has to be replaced by "contains $J_\infty \mathbb{C}^*.x$", provided that we work on $\mathbb{C}$-points. The equality (1) in Corollary 4.2 of [2] still holds but as topological spaces. The second one has to be removed. Moreover, the lemma below (Lemma 3) has to added in order to apply Theorem 2 to the vertex algebra $V_{-1}(\mathfrak{sl}_n)$ we consider in Theorem 1.

Finally, the second sentence in the proof of Theorem 1.1 (1), page 183, has to be replaced by "The second statement follows from Theorem 4.1 (now Theorem 2) and Lemma 3", the second sentence in the proof of Theorem 1.1 (2), page 195, has to be removed.

Lemma 3. *Assume that $V = V_{-1}(\mathfrak{sl}_n(\mathbb{C}))$. Then the hypothesis of Theorem 2 are satisfied.*

Proof. Let $x \in \mathfrak{g}^* \cong \mathfrak{g}$ be such that $X_V = \overline{\mathbb{S}_{min}} = \overline{G.\mathbb{C}^* x}$.

We have a natural map $\varphi_x \colon \mathbb{C}^* \to X_V$, $t \mapsto t.x$. It gives a morphism from R_V to $\mathcal{O}(\mathbb{C}^*) = \mathbb{C}[t, t^{-1}]$ which induces a morphism from $J_\infty R_V$ to $\mathcal{O}(J_\infty \mathbb{C}^*) = \mathbb{C}[t_0, t_0^{-1}, t_1, t_2, \ldots]$. Note that $J_\infty \mathbb{C}^*$ is nothing but $(\pi^{\mathbb{C}}_{\infty,0})^{-1}(\mathbb{C}^*)$, where $\pi^{\mathbb{C}}_{\infty,0}$ is the canonical projection from $J_\infty \mathbb{C}$ to $\mathbb{C}$.

On the other hand, we have a surjective Poisson vertex algebra morphism

$$\rho \colon J_\infty R_V \twoheadrightarrow \operatorname{gr} V.$$

The question is to know whether the morphism $J_\infty \varphi_x^* \colon J_\infty R_V \to \mathcal{O}(J_\infty \mathbb{C}^*)$ factorizes through ρ. Indeed, if so, then it implies that $J_\infty \mathbb{C}^*.x$ is contained in $SS(V)_{\rm red}$. We will prove that it is true for some $x \in \mathfrak{g}$ such that $X_V = \overline{G.\mathbb{C}^* x}$.

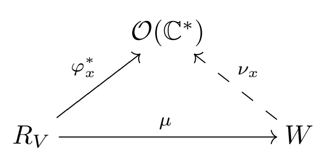
Let $\hat{\mu}$ be the morphism

$$\hat{\mu} \colon \operatorname{gr} V \longrightarrow \mathbb{C}[J_\infty T^* \mathbb{C}^n]$$

induced from the embedding from $V = V_{-1}(\mathfrak{sl}_n)$ to $\mathcal{D}^{ch}_{\mathbb{C}^n}$ (see [2, Th. 7.13]). Denote by μ the restriction to $R_V \subset \operatorname{gr} V$ of $\hat{\mu}$. Note that μ is the morphism from

$$R_V = \mathbb{C}[\mathfrak{g}]/I$$

to $\mathbb{C}[T^* \mathbb{C}^n]$ such that the image of $e_{i,j} + I \in R_V$ is $-\beta_j \gamma_i$ and the image of $h_i + I \in R_V$ is $-\beta_i \gamma_i + \beta_{i+1} \gamma_{i+1}$, with $\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n$ the natural coordinates on $T^* \mathbb{C}^n \cong \mathbb{C}^{2n}$. Let $W \subset \mathcal{O}(T^* \mathbb{C}^n)$ be the image of μ. Note that we have $J_\infty W \subset \mathcal{O}(J_\infty T^* \mathbb{C}^n)$. It suffices to show that for some x such that $X_V = \overline{G.\mathbb{C}^* x}$, there is a well-defined morphism ν_x from W to $\mathcal{O}(\mathbb{C}^*)$ such that $\nu_x \circ \mu = \varphi_x^*$, that is, such that the following diagram commutes:



Indeed, assume the statement proven, and pick x satisfying the above conditions. We observe that $\hat{\mu} \circ \rho = J_\infty \mu$ (its comes from the identification of $J_\infty \mathfrak{g}^*$ with $S(t^{-1}\mathfrak{g}[t^{-1}])$). Hence, we get the commutative diagram:

$$\begin{array}{ccc} J_\infty R_V & \xrightarrow{J_\infty \varphi_x^*} & \mathcal{O}(J_\infty \mathbb{C}^*) \\ {\scriptstyle\rho}\downarrow & \searrow^{J_\infty \mu} & \uparrow{\scriptstyle J_\infty \nu_x} \\ \operatorname{gr} V & \xrightarrow[\hat{\mu}]{} & J_\infty W \end{array}$$

Therefore, for such an x, $J_\infty \varphi_x^*$ factorizes through $\operatorname{gr} V$: we have $\rho \circ \hat{\mu} \circ J_\infty \nu_x = J_\infty \varphi_x^*$.

To complete the proof, it thus remains to find such an x. It is enough to show that for some x, $\ker \mu \subset \ker \varphi_x$. Note that

$$\ker \varphi_x^* = \{f \in \mathbb{C}[\mathfrak{g}]/I \mid f(tx) = 0 \text{ for all } t \in \mathbb{C}^*\} = \bigcap_{t \in \mathbb{C}^*} \mathfrak{m}_{tx},$$

where $\mathfrak{m}_x$ is the maximal ideal of R_V associated with x. We have $I \subset \ker \mu$ because μ is well-defined. Since $I \subset \ker \mu$, we have $Z_\mu := \operatorname{Specm}(\mathbb{C}[\mathfrak{g}]/\ker \mu) \subset \operatorname{Specm}(\mathbb{C}[\mathfrak{g}]/I) = \overline{\mathbb{S}_{min}}$. Recall now that $\overline{\mathbb{S}_{min}} = G.\mathbb{C}^* \varpi_1 \cup \overline{\mathbb{O}_{min}}$, and let us now prove that there is $x \in G.\mathbb{C}^* \varpi_1$ such that $\ker \mu \subset \mathfrak{m}_{tx}$ for all $t \in \mathbb{C}^*$. Since Z_μ is G-invariant and $\mathbb{C}^*$-invariant, it is enough to prove that there is $x \in G.\mathbb{C}^* \varpi_1$ such that $\ker \mu \subset \mathfrak{m}_x$. Assume the contrary. We expect a contradiction. We have $Z_\mu \cap G.\mathbb{C}^* \varpi_1 = \varnothing$ by our assumption, and so $Z_\mu \subset \overline{\mathbb{O}_{min}}$. So the defining ideal of $\overline{\mathbb{O}_{min}}$ would be contained in $\ker \mu$. But this is not true, since the Casimir element,

$$\Omega = \sum_{1 \leqslant i \neq j \leqslant n} e_{i,j} e_{j,i} + \sum_{i=1}^{n} h_i \varpi_i,$$

is not in $\ker \mu$. Indeed, the coefficient of $\beta_2 \gamma_1 \beta_1 \gamma_2$ in $\mu(\Omega) \in \mathbb{C}[\beta_i, \gamma_i \mid i = 1, \dots, n]$ is

$$\begin{aligned} & 2 - 2h_1^*(\varpi_1) + h_2^*(\varpi_1) + h_1^*(\varpi_2) \\ & = \begin{cases} 2 - \dfrac{2(n-1)}{n} + \dfrac{n-2}{n} + \dfrac{n-2}{n} = \dfrac{2(n-1)}{n} \neq 0 & \text{if } n > 4, \\ \dfrac{3}{2} \neq 0 & \text{if } n = 4. \end{cases} \end{aligned}$$

This proves the expected statement, and so completes the proof. □

References

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 JAPAN

Email address: arakawa@kurims.kyoto-u.ac.jp

Laboratoire Painlevé, CNRS U.M.R. 8524, 59655 Villeneuve d'Ascq Cedex , France

Email address: anne.moreau@univ-lille.fr