$(\Pi(\mathcal{D}_1), \preceq_{\mathcal{D}_1})$ *is a homomorphic image of* $\mathcal{P}(\mathcal{D}_2) = (\Pi(\mathcal{D}_2), \preceq_{\mathcal{D}_2})$ *under the natural mapping* (*i.e.,* $T_2 \in \Pi(\mathcal{D}_2)$ *is made to correspond to* T_1 *if* $T_2 \subseteq T_1$).

(Proof) Suppose $\mathcal{D}_1 \subseteq \mathcal{D}_2$. For each $i = 1, 2$, distinct elements e and e' of E belong to different components of $\Pi(\mathcal{D}_i)$ if and only if there exists an $X \in \mathcal{D}_i$ such that $|\{e, e'\} \cap X| = 1$. Therefore, since $\mathcal{D}_1 \subseteq \mathcal{D}_2$, $\Pi(\mathcal{D}_2)$ is a refinement of $\Pi(\mathcal{D}_1)$. Also, since we have $T_2 \preceq_{\mathcal{D}_2} T_2'$ if and only if $T_2' \subseteq X \in \mathcal{D}_2$ implies $T_2 \subseteq X$ and since $\mathcal{D}_1 \subseteq \mathcal{D}_2$, $T_2' \subseteq X \in \mathcal{D}_1$ implies $T_2 \subseteq X$ if $T_2 \preceq_{\mathcal{D}_2} T_2'$. Consequently, for T_1, $T_1' \in \Pi(\mathcal{D}_1)$ such that $T_2 \subseteq T_1$ and $T_2' \subseteq T_1'$, we have $T_1 \preceq_{\mathcal{D}_1} T_1'$.

The converse is easy. Q.E.D.

Theorem 3.32: *For* $\mathcal{D}_0 \in \mathbf{D}$ *we have*

$$\dim \mathrm{F}(\mathcal{D}_0) = |E| - |\Pi(\mathcal{D}_0)|, \tag{3.122}$$

where $\dim \mathrm{F}(\mathcal{D}_0)$ *is the dimension of the face* $\mathrm{F}(\mathcal{D}_0)$.

(Proof) The dimension of the face $\mathrm{F}(\mathcal{D}_0)$ is equal to that of the affine set

$$\mathrm{M}(\mathcal{D}_0) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{D}_0\colon\ x(X) = f(X)\}. \tag{3.123}$$

Since the rank of the coefficient matrix in the right-hand side of (3.123) is equal to $|\Pi(\mathcal{D}_0)|$, we have (3.122). Q.E.D.

It may be noted that the extreme point theorem (Theorem 3.22) and the extreme ray theorem (Theorem 3.26) easily follow from Theorems 3.30 and 3.32 and Lemma 3.31.

Lemma 3.33: *Suppose* $\mathcal{D}_0 \in \mathbf{D}$ *and let*

$$\mathcal{C}_0\colon\ \emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = E \tag{3.124}$$

be a maximal chain of $\mathcal{D}_0$. *Then,*

$$\mathrm{F}(\mathcal{C}_0) = \mathrm{F}(\mathcal{D}_0). \tag{3.125}$$

(Proof) Since $\Pi(\mathcal{C}_0) = \Pi(\mathcal{D}_0)$, we have $x \in \mathrm{F}(\mathcal{C}_0)$ if and only if $x \in \mathrm{F}(\mathcal{D}_0)$. Q.E.D.

Theorem 3.34: *Suppose* $\mathcal{D}_0 \in \mathbf{D}$. *Then a base* $x \in \mathrm{B}(f)$ *is an extreme point of the face* $\mathrm{F}(\mathcal{D}_0)$ *if and only if, for a maximal chain*

$$\mathcal{C}\colon\ \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E \tag{3.126}$$

where $\hat{\mathcal{F}}$ is defined by (3.213). Therefore, we can replace f and $\mathcal{F}$ in (3.219) and (3.221) by $\hat{f}$ and $\hat{\mathcal{F}}$, respectively.

For $Y \subset E$, if $\{E - X_i \mid i \in I\}$ is a partition of $E - Y$, we call $\{X_i \mid i \in I\}$ a *copartition of* $E - Y$ *augmented by* Y.

Theorem 3.55: *For each nonempty* $Y \subset E$,

$$\hat{f}_2(Y) = \min\Big\{\sum_{i \in I} \hat{f}(X_i) - (|I| - 1)\hat{f}(E) \ \Big| \ \{E - X_i \mid i \in I\}\colon \text{ a partition of } E - Y, \ \forall i \in I\colon X_i \in \hat{\mathcal{F}}\Big\}. \tag{3.226}$$

(Proof) If f and $\mathcal{F}$ in (3.219) and (3.221) are replaced by $\hat{f}$ and $\hat{\mathcal{F}}$, we can restrict admissible families $\mathcal{G}$ in (3.220)∼(3.224) to those which satisfy (3.220)∼ (3.224) (with $\mathcal{F}$ replaced by $\hat{\mathcal{F}}$ in (3.221)) and the following (i)∼(iv):

(i) $\mathcal{G}$ is a cross-free family, (3.227)

(ii) for any $X_i, X_j \in \mathcal{G}$ we have $X_i \cap X_j \neq \emptyset$, (3.228)

(iii) $\mathcal{G}$ does not contain a subfamily which forms a copartition of E, (3.229)

(iv) $E \notin \mathcal{G}$. (3.230)

(Here, (i)∼(iii) follow from Theorems 3.51, 3.53 and 3.54. (iv) follows from the form of (3.219).) From (i), the family $\mathcal{G} = (X_i \mid i \in I)$ can be represented by a pair of a tree $T = (V, A)$ and a family

$$\mathcal{P} = (P_v \mid v \in V), \tag{3.231}$$

where $A = \{a_i \mid i \in I\}$ and nonempty P_v's form a partition of E as in Lemma 3.49. From (ii), T is a directed tree. (For, if there were distinct arcs a_i and a_j in T such that $\partial^- a_i = \partial^- a_j$, we would have $X_i \cap X_j = \emptyset$.)

Let v_0 be the root of T. If $P_{v_0} = \emptyset$, then $\mathcal{G}$ contains a subfamily which form a copartition of E. Therefore, $P_{v_0} \neq \emptyset$ due to (iii). Since for each $e \in E$ the number of i's for which $e \in X_i$ should be taken from the fixed set of two distinct values of (3.222) and (3.223), for any leaf u of T every vertex $w \notin \{u, v_0\}$ lying on the unique path $Q(v_0, u)$ connecting v_0 with u in T gives

$$P_w = \emptyset, \tag{3.232}$$

with $E' = E - \{e\}$, *where*

$$\mathcal{D}_1 = \{X \mid e \notin X \in \mathcal{D}\}, \tag{3.253}$$

$$\mathcal{D}_2 = \{E - X \mid e \in X \in \mathcal{D}\}, \tag{3.254}$$

f' *is the restriction of* f *to* $\mathcal{D}_1$, *and* g' *is the restriction of* $f^\#$ *to* $\mathcal{D}_2$.

Conversely, every generalized polymatroid in $\mathbf{R}^{E'}$ *is obtained in this way. For each generalized polymatroid* $\mathrm{P}(f', g')$ *with* $\mathcal{D}_i \subseteq 2^{E'}$ $(i = 1, 2)$ *such a base polyhedron* $\mathrm{B}(f)$ *in* $\mathbf{R}^E$ *with* $E = E' \cup \{e\}$ *is unique up to translation along the new axis* e, *and the two polyhedra* $\mathrm{P}(f', g')$ *and* $\mathrm{B}(f)$ *are isomorphic with each other under the projection of the hyperplane* $x(E) = f(E)$ *onto the hyperplane* $x(e) = 0$ *along the axis* e.

(Proof) The base polyhedron $\mathrm{B}(f)$ is the solution set of

$$x(X) \le f(X) \quad (X \in \mathcal{D}), \tag{3.255}$$

$$x(E) = f(E). \tag{3.256}$$

Choose an element $e \in E$. From (3.256) we have

$$x(e) = f(E) - x(E - \{e\}). \tag{3.257}$$

Substituting (3.257) into (3.255), we have

$$\forall X \in \mathcal{D} \text{ with } e \notin X\colon\ x(X) \le f(X), \tag{3.258}$$

$$\forall X \in \mathcal{D} \text{ with } e \in X\colon\ x(E - X) \ge f(E) - f(X). \tag{3.259}$$

(3.258) and (3.259) are rewritten as

$$\forall X \in \mathcal{D}_1\colon\ x(X) \le f'(X), \tag{3.260}$$

$$\forall Y \in \mathcal{D}_2\colon\ x(Y) \ge g'(Y), \tag{3.261}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are, respectively, defined by (3.253) and (3.254), f' is the restriction of f to $\mathcal{D}_1$ and g' is the restriction of $f^\#$ to $\mathcal{D}_2$. It follows from (3.260) and (3.261) that the projection of $\mathrm{B}(f)$ along the axis e on the hyperplane $x(e) = 0$ is the generalized polymatroid $\mathrm{P}(f', g')$ in $\mathbf{R}^{E'}$ with $E' = E - \{e\}$. Note that (3.251) follows from the submodularity of f.

Now, we show the converse. For an arbitrary generalized polymatroid $\mathrm{P}(f', g')$ in $\mathbf{R}^{E'}$ with a submodular system $(\mathcal{D}_1, f')$ and a supermodular system $(\mathcal{D}_2, g')$ on E' let e be a new element not in E' and define

$$E = E' \cup \{e\}, \tag{3.262}$$

Define a polyhedron

$$\mathrm{P}_*(f) = \{x \mid x \in \mathbf{R}^E,\ \forall (X,Y) \in 3^E\colon\ x(X) - x(Y) \le f(X,Y)\}. \quad (3.276)$$

The polyhedron $\mathrm{P}_*(f)$ is called a *polypseudomatroid* ([Chandrasekaran+Kabadi88], [Kabadi + Chandrasekaran90]) and f its *rank function* (see Fig. 3.8). Polyhedral studies are also made in [Nakamura88b] and [Qi88,89]. A *pseudomatroid* is a set-theoretical version of a polypseudomatroid. [Since $f(X,Y)$ is submodular both in X with any fixed Y and in Y with any fixed X, f is called a *bisubmodular function* (like 'bilinear' in a bilinear form) and a polypseudomatroid a *bisubmodular polyhedron* later in [Bouchet+Cunningham95], [Ando+Fuji96], [Fuji+Patkar95], etc. (Note that $f(X,Y)$ is not submodular in (X,Y) in general as a bilinear form is not linear.) Also see [Borovik+Gelfand+White03] for related topics in Coxeter matroids. Bisubmodular polyhedra first appeared in [Dunstan+Welsh73].]

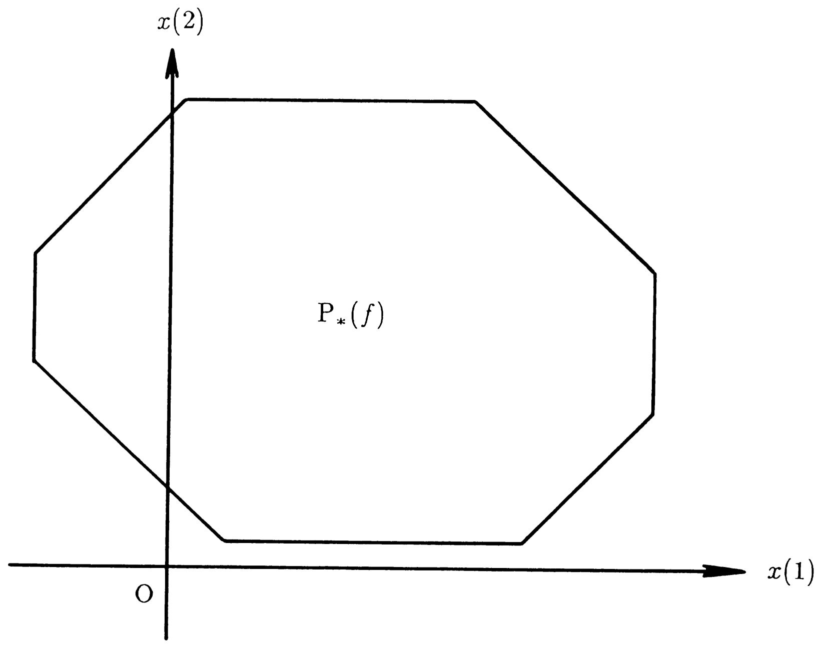

Figure 3.8: A polypseudomatroid.

We call a pair $(S,T) \in 3^E$ such that $S \cup T = E$ an *orthant* of $\mathbf{R}^E$. For each orthant (S,T) denote by $2^{(S,T)}$ the set of all the pairs (X,Y) such that $X \subseteq S$ and $Y \subseteq T$.

It should be noted that we have (4.19)∼(4.21) if (4.16) and (4.17) hold for an appropriate numbering of u_i's and v_i's.

Lemma 4.5 will play a very fundamental rôle in developing algorithms for solving the intersection problem and other related problems.

(b) An algorithm and the intersection theorem

We now consider Problem $P_1{}'$ described by (4.2). Given a feasible flow φ in network $\mathcal{N} = (G = (E, E'; A), c, \mathbf{S}_1, \mathbf{S}_2)$, define the *auxiliary network* $\mathcal{N}_\varphi = (G_\varphi = (V, A_\varphi), c_\varphi)$ associated with φ as follows. $G_\varphi = (V, A_\varphi)$ is a directed graph, called the *auxiliary graph* associated with φ, with vertex set V and arc set A_φ given by

$$\begin{aligned} V &= E \cup E' \cup \{s^+, s^-\}, && (4.22)\\ A_\varphi &= S_\varphi^+ \cup A_\varphi^+ \cup A^* \cup B^* \cup A_\varphi^- \cup S_\varphi^-, && (4.23) \end{aligned}$$

where

$$\begin{aligned} S_\varphi^+ &= \{(s^+, v) \mid v \in E - \mathrm{sat}^+(\partial^+\varphi)\}, && (4.24)\\ A_\varphi^+ &= \{(u, v) \mid v \in \mathrm{sat}^+(\partial^+\varphi),\ u \in \mathrm{dep}^+(\partial^+\varphi, v) - \{v\}\}, && (4.25)\\ A^* &= A, && (4.26)\\ B^* &= \{(e', e) \mid e \in E\}, && (4.27)\\ A_\varphi^- &= \{(u, v) \mid u \in \mathrm{sat}^-(\partial^-\varphi),\ v \in \mathrm{dep}^-(\partial^-\varphi, u) - \{u\}\}, && (4.28)\\ S_\varphi^- &= \{(v, s^-) \mid v \in E' - \mathrm{sat}^-(\partial^-\varphi)\}. && (4.29) \end{aligned}$$

Here, $\partial^+\varphi = (\partial\varphi)^E$, $\partial^-\varphi = -(\partial\varphi)^{E'}$, and sat^+ and dep^+ (sat^- and dep^-) are, respectively, the saturation function and the dependence function defined with respect to submodular system $(\mathcal{D}_1, f_1)$ on E ($(\mathcal{D}_2, f_2)$ on E'). Note that B^* is the set of the reorientations of arcs of A. We also define the capacity function c_φ: $A_\varphi \to \mathbf{R} \cup \{+\infty\}$ by

$$c_\varphi(a) = \begin{cases} \hat{c}^+(\partial^+\varphi, v) & (a = (s^+, v) \in S_\varphi^+),\\ \tilde{c}^+(\partial^+\varphi, v, u) & (a = (u, v) \in A_\varphi^+),\\ +\infty & (a \in A^* \cup B^*),\\ \tilde{c}^-(\partial^-\varphi, u, v) & (a = (u, v) \in A_\varphi^-),\\ \hat{c}^-(\partial^-\varphi, v) & (a = (v, s^-) \in S_\varphi^-), \end{cases} \qquad (4.30)$$

From (4.77) (and Theorem 4.9) we see that there exists no common base in $\mathrm{B}(f_1)$ and $\mathrm{B}(f_2)$.

When f_1 and f_2 are integer-valued and initial bases b_1 and b_2 are integral, bases b_1 and b_2 obtained during the execution of the above algorithm are integral and hence the algorithm terminates after repeating (a)∼(c) of Step 1 at most $b_1(T_\beta^+) - b_2(T_\beta^+)$ times.

For general rank functions f_1 and f_2 we adopt the lexicographic ordering technique described in Section 4.1.c ([Schönsleben80], [Lawler + Martel82a]). When finding a shortest path from T_β^+ to T_β^- by the breadth-first search, for each $u \in V$ search arc (u, v) in A_β^1 (or A_β^2) earlier than arc (u, v') in A_β^1 (or A_β^2) if $\pi(v) < \pi(v')$, for a fixed numbering $\pi\colon V \to \{1, 2, \cdots, |V|\}$ of V. By this modification the algorithm terminates after repeating Cycle (a)∼(c) of Step 1 $\mathrm{O}(|E|^3)$ times.

5. Neoflows

In this section we consider the submodular flow problem, the independent flow problem and the polymatroidal flow problem, which we call *neoflow problems*. We discuss the equivalence among these neoflow problems and give algorithms for solving them.

5.1. Neoflows

We first give the definitions of the submodular flow problem, the independent flow problem and the polymatroidal flow problem.

(a) Submodular flows

Let $G = (V, A)$ be a graph with a vertex set V and an arc set A. Also let $\overline{c}\colon A \to \mathbf{R} \cup \{+\infty\}$ be an *upper capacity function* and $\underline{c}\colon A \to \mathbf{R} \cup \{-\infty\}$ be a *lower capacity function*. A function $\gamma\colon A \to \mathbf{R}$ is a *cost function*. Let $\mathcal{F} \subseteq 2^V$ be a crossing family with $\emptyset,\ V \in \mathcal{F}$ and $f\colon \mathcal{F} \to \mathbf{R}$ be a crossing-submodular function on the crossing family $\mathcal{F}$ with $f(\emptyset) = f(V) = 0$. (See Section 2.3 for the definition of crossing-submodular function on a crossing family.) Denote this network by $\mathcal{N}_S = (G = (V, A), \underline{c}, \overline{c}, \gamma, (\mathcal{F}, f))$.

From Theorem 4.13 we have

Theorem 5.1 [Frank84]: *There exists a feasible flow for the submodular flow problem P_S satisfying* (5.1b) *and* (5.1c) *if and only if*

$$\forall X \in \mathcal{D}\colon\ (\kappa_{\underline{c},\overline{c}})^{\#}(X) \le f(X) \tag{5.34}$$

or

$$\forall X \in \mathcal{D}\colon\ \overline{c}(\Delta^- X) - \underline{c}(\Delta^+ X) + f(X) \ge 0, \tag{5.35}$$

where for each $X \subseteq V$ $\Delta^+ X = \{a \mid a \in A,\ \partial^+ a \in X,\ \partial^- a \in V - X\}$ and $\Delta^- X = \{a \mid a \in A,\ \partial^- a \in X,\ \partial^+ a \in V - X\}$.

Moreover, if $\overline{c}$, $\underline{c}$ and f are integer-valued and P_S is feasible, there exists an integral feasible flow.

(Proof) Immediate from Theorem 4.13. Q.E.D.

A feasible flow for the submodular flow problem can be obtained by the use of the algorithm shown in Section 4.3.

Frank [Frank84] showed feasibility theorems for the cases where f is an intersecting-submodular function and where f is a crossing-submodular function. We can give Frank's result by combining Theorems 5.1 and 2.6. That is,

Corollary 5.2 [Frank84]:

(i) *When f is an intersecting-submodular function on an intersecting family $\mathcal{F}$ such that $\emptyset, V \in \mathcal{F}$ and $f(\emptyset) = f(V) = 0$, the submodular flow problem P_S described by* (5.1) *has a feasible flow if and only if we have*

$$(\kappa_{\underline{c},\overline{c}})^{\#}(X) \le \sum_{i \in I} f(X_i) \tag{5.36}$$

for each $X \subseteq V$ and disjoint $X_i \in \mathcal{F}$ $(i \in I)$ such that $X = \bigcup_{i \in I} X_i$.

(ii) *When f is a crossing-submodular function on a crossing family $\mathcal{F}$ such that $\emptyset, V \in \mathcal{F}$ and $f(\emptyset) = f(V) = 0$, the submodular flow problem P_S has a feasible flow if and only if we have*

$$(\kappa_{\underline{c},\overline{c}})^{\#}(X) \le \sum_{i \in I} \sum_{j \in J_i} f(X_{ij}) \tag{5.37}$$

for each $X \subseteq V$, codisjoint $X_i \subseteq V$ $(i \in I)$ and disjoint $X_{ij} \in \mathcal{F}$ $(j \in J_i)$ (for each $i \in I$) such that $X = \bigcap_{i \in I} X_i$ and $X_i = \bigcup_{j \in J_i} X_{ij}$ $(i \in I)$.

$$\overline{\xi}(a) = 0 \quad (a \in A,\ \overline{c}(a) = +\infty), \tag{5.66d}$$

$$\underline{\xi},\ \overline{\xi},\ \eta \geq 0, \tag{5.66e}$$

where $\underline{\xi}$, $\overline{\xi}$: $A \to \mathbf{R}$, η: $\mathcal{D} \to \mathbf{R}$ and we should regard (5.66a) as the objective function with the terms $\underline{\xi}(a)\underline{c}(a)$ $(a \in A,\ \underline{c}(a) = -\infty)$ and $\overline{\xi}(a)\overline{c}(a)$ $(a \in A,\ \overline{c}(a) = +\infty)$ being suppressed.

Because of (5.62) there is a maximal chain

$$\mathcal{C}\colon\ \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = V \tag{5.67}$$

of $\mathcal{D}$ such that p is constant on each quotient $S_k - S_{k-1}$ $(k = 1, 2, \cdots, n)$. Using this chain $\mathcal{C}$ and the potential p, define

$$\eta(S_k) = p_k - p_{k+1} \quad (k = 1, 2, \cdots, n-1), \tag{5.68}$$

where p_k is the value of p taken in $S_k - S_{k-1}$ and note that $\eta(S_k) > 0$. Also define $\eta(X) = 0$ for other $X \in \mathcal{D}$. Moreover, define

$$\underline{\xi}(a) \;=\; \gamma(a) + p(\partial^+ a) - p(\partial^- a) \quad (a \in A,\ \overline{c}(a) = +\infty), \tag{5.69}$$

$$\overline{\xi}(a) \;=\; -\gamma(a) - p(\partial^+ a) + p(\partial^- a) \quad (a \in A,\ \underline{c}(a) = -\infty), \tag{5.70}$$

and for each arc $a \in A$ with $\overline{c}(a) < +\infty$ and $\underline{c}(a) > -\infty$ define $\underline{\xi}(a)$ and $\overline{\xi}(a)$ such that $\underline{\xi}(a)$, $\overline{\xi}(a) \geq 0$ and (5.66b) holds. We can easily see that thus defined $\underline{\xi}$, $\overline{\xi}$, η satisfy (5.66b)$\sim$(5.66e), where note that for each arc $a \in A$ with $\overline{c}(a) = +\infty$ and $\underline{c}(a) = -\infty$ we have $\gamma(a) + p(\partial^+ a) - p(\partial^- a) = 0$ due to (5.60) and (5.61).

Since the dual of Problem P_S has a feasible solution and the feasibility of the primal problem P_S is assumed, there exists an optimal solution of Problem P_S.

The "only if" part: Suppose that there is a negative cycle in $\hat{\mathcal{N}}$ relative to the length function $\hat{\gamma}$, and let Q be such a negative cycle in $\hat{\mathcal{N}}$. Then for any positive α, if we define φ': $A \to \mathbf{R}$ by (5.50), φ' is feasible for Problem P_S because of the definition of $\hat{\mathcal{N}}$ and we have (5.51). Since α (> 0) is arbitrary and $\gamma_\varphi(Q) < 0$, Problem P_S does not have a finite optimal solution. Q.E.D.

We also have

Theorem 5.6 [Edm+Giles77]: *The system of linear inequalities*

$$\underline{c}(a) \leq \varphi(a) \leq \overline{c}(a) \quad (a \in A), \tag{5.71}$$

Now, consider a supermodular system $(\mathcal{D}^+, g^+)$ on S^+ instead of submodular system $(\mathcal{D}^+, f^+)$ and also consider the following system of inequalities.

$$\underline{c}(a) \leq \varphi(a) \leq \overline{c}(a) \quad (a \in A), \tag{5.121}$$

$$\partial\varphi(v) = 0 \quad (v \in V - (S^+ \cup S^-)), \tag{5.122}$$

$$(\partial\varphi)^{S^+} \in \mathrm{P}(g^+), \tag{5.123}$$

$$-(\partial\varphi)^{S^-} \in \mathrm{P}(f^-), \tag{5.124}$$

where $\mathrm{P}(g^+)$ is the supermodular polyhedron associated with $(\mathcal{D}^+, g^+)$. Then we have the following theorem.

Theorem 5.10: *There exists a feasible flow φ satisfying* (5.121) − (5.124) *if and only if we have for each $U \subseteq V$ such that $S^+ \cap U \in \mathcal{D}^+$ and $S^- \cap U \in \mathcal{D}^-$*

$$g^+(S^+ \cap U) - f^-(S^- \cap U) \leq \overline{c}(\Delta^+ U) - \underline{c}(\Delta^- U) \tag{5.125}$$

and for each $U \subseteq V$ such that $S^+ \cup S^- \subseteq U$

$$0 \leq \overline{c}(\Delta^+ U) - \underline{c}(\Delta^- U). \tag{5.126}$$

Moreover, if there exists a feasible flow and $\overline{c}$, $\underline{c}$, g^+ and f^- are integer-valued, then there exists an integral feasible flow.

(Proof) Define $\mathcal{D} \subseteq 2^V$ and $g\colon \mathcal{D} \to \mathbf{R}$ by

$$\mathcal{D} = \{U \mid U \subseteq V,\ S^+ \cap U \in \mathcal{D}^+,\ S^- \cap U \in \mathcal{D}^-\}, \tag{5.127}$$

$$g(U) = \begin{cases} g^+(S^+ \cap U) - f^-(S^- \cap U) & (U \in \mathcal{D},\ (S^+ \cup S^-) - U \neq \emptyset) \\ 0 & (U \in \mathcal{D},\ S^+ \cup S^- \subseteq U). \end{cases} \tag{5.128}$$

If there is a feasible flow, we must have

$$g^+(S^+) - f^-(S^-) \leq 0. \tag{5.129}$$

Also, (5.125) with $U = V$ implies (5.129). Therefore, we assume (5.129). Due to (5.129), the function $g\colon \mathcal{D} \to \mathbf{R}$ defined by (5.128) is a supermodular function on the distributive lattice $\mathcal{D}$ with $\emptyset,\ V \in \mathcal{D}$ and $g(\emptyset) = g(V) = 0$. We have $x \in \mathrm{B}(g)$ if and only if

$$x^{S^+} \in \mathrm{P}(g^+),\ \ -x^{S^-} \in \mathrm{P}(f^-),\ \ x^{V-(S^+ \cup S^-)} = \mathbf{0}, \tag{5.130}$$

Putting $p' \leftarrow p$ and $\gamma' \leftarrow \hat{\gamma}$ and repeating the above argument for other vertices, we see that the total change of any potential difference is bounded by $\alpha|V|$. Q.E.D.

From this lemma we have

Theorem 5.13: *Let γ' be an α-approximation of γ and p' be an optimal potential in $\mathcal{N}' = (G = (V, A), \underline{c}, \overline{c}, \gamma', (\mathcal{D}, f))$. Then for any optimal submodular flow φ in $\mathcal{N} = (G = (V, A), \underline{c}, \overline{c}, \gamma, (\mathcal{D}, f))$,*

(1) $\forall a \in A$: $\gamma_{p'}(a) > \alpha|V| \Longrightarrow \varphi(a) = \underline{c}(a)$,

(2) $\forall a \in A$: $\gamma_{p'}(a) < -\alpha|V| \Longrightarrow \varphi(a) = \overline{c}(a)$,

(3) $\forall u, v \in V$: $p'(u) - p'(v) > \alpha|V| \Longrightarrow v \notin \mathrm{dep}(\partial\varphi, u)$.

(Proof) The present theorem follows from Lemma 5.12 and Theorem 5.3. Q.E.D.

Theorem 5.13 gives a basis for a strongly polynomial algorithm for submodular flows.

We show a strongly polynomial algorithm which consists of the repeated applications of a procedure called *Fundamental Cycle*.

Fundamental Cycle

Input: Lower and upper capacity functions $\underline{c}$ and $\overline{c}$; a partition $\mathcal{W} = \{W^i \mid i \in I\}$ of V; a representation of $\mathbf{S} = \oplus_{i \in I} \mathbf{S}^i$ as a direct sum of submodular systems $\mathbf{S}^i = (\mathcal{D}^i, f^i)$ on W^i $(i \in I)$; a graph $H = (V, D)$ with connected components $H^i = (W^i, E^i)$ $(i \in I)$ which are strongly connected; and a set $A^0 = \{a \mid a \in A,\ \underline{c}(a) = \overline{c}(a)\}$ of all tight arcs. (*Comment*: At the initial application of this procedure we put $I = \{1\}$, $W^1 = V$ and $H = (V, D)$ with $D = \{(u, v) \mid u, v \in V,\ u \neq v\}$.)
Output: A nonnegative real M, and if $M \neq 0$, modified $\underline{c}$, $\overline{c}$, $\mathcal{W}$, $\mathbf{S}$, H and A^0. (*Comment*: When $M = 0$, the set of all the submodular flows in the current network $\mathcal{N} = (G = (V, A), \underline{c}, \overline{c}, \gamma, \mathbf{S})$ is exactly the set of all the optimal submodular flows in the original network. When $M \neq 0$, $\underline{c}$, $\overline{c}$, $\mathcal{W}$, H and A^0 have been modified in such a way that all the input characteristics are maintained and that at least one of the following two properties holds:

(ii) if $(\gamma_p^s)_{p'}(a) > \frac{1}{2}|V|$, then put $\bar{c}(a) \leftarrow \underline{c}(a)$ and $A^0 \leftarrow A^0 \cup \{a\}$,

where $(\gamma_p^s)_{p'}(a) = \gamma_p^s(a) + p'(\partial^+ a) - p'(\partial^- a)$.

(4-2) For each arc $(u, v) \in D$ of the graph H, if $p'(v) - p'(u) > \frac{1}{2}|V|$, then delete (u, v) from D and from E_i to which (u, v) belongs.

(4-3) For each $i \in I$ do the following (4-3-1)∼(4-3-3):

(4-3-1) Find a maximal chain $\emptyset = U_0^i \subset U_1^i \subset \cdots \subset U_{k_i}^i = W^i$ of upper ideals of H^i. (*Comment*: An *upper ideal* of a graph is a vertex set which no arcs enter.)

(4-3-2) Define

$$\mathbf{S}_{i_s}(= (\mathcal{D}^{i_s}, f^{i_s})) = (\mathcal{D}^i, f^i) \cdot U_s^i / U_{s-1}^i \qquad (s = 1, 2, \cdots, k_i),$$

$$W^{i_s} = U_s^i - U_{s-1}^i \qquad (s = 1, 2, \cdots, k_i).$$

(*Comment*: W^{i_s} $(s = 1, 2, \cdots, k_i)$ are the vertex sets of the strongly connected components of H^i.)

(4-3-3) Delete from the graph H all the arcs connecting distinct subsets W^{i_s} $(s = 1, 2, \cdots, k_i)$.

(4-4) Put

$$\begin{aligned} I &\leftarrow \{i_s \mid s = 1, 2, \cdots, k_i,\ i \in I\}, \\ \mathbf{S} &\leftarrow \oplus_{i \in I} \mathbf{S}_i, \\ \mathcal{W} &\leftarrow \{W^i \mid i \in I\}. \end{aligned}$$

(End)

To find an optimal submodular flow in the original network we repeatedly apply the procedure, Fundamental Cycle. We show the validity and the strong polynomiality of this algorithm.

At any stage of the algorithm the input to Fundamental Cycle is referred to as the current network with the current capacity functions, the current submodular systems, etc.

Theorem 5.14: *At any stage of the algorithm the following statements are valid.*

Since no vector in $\mathrm{ER}(X)$ can be expressed as a nonnegative linear combination of the other vectors in $\mathrm{ER}(X)$, it suffices to prove that every vector in $\mathrm{C}_f(X)$ can be expressed as a nonnegative linear combination of vectors in $\mathrm{ER}(X)$.

Let v be an arbitrary vector in $\mathrm{C}_f(X)$. From (6.53),

$$v(X - Y) \geq 0 \quad (X \supseteq Y \in \mathcal{D}), \tag{6.59}$$

$$v(Y - X) \leq 0 \quad (X \subseteq Y \in \mathcal{D}). \tag{6.60}$$

Suppose that each arc of $B^*(\mathcal{P}) - \Delta^-(X)$ has the infinite upper capacity and the zero lower capacity and that each arc of $\Delta^-(X)$ has the zero upper and lower capacities. Then it easily follows from (6.59), (6.60) and the feasibility theorem for network flows (Theorem 1.3) [Hoffman60] ([Ford + Fulkerson62]) that there exist a nonnegative flow $\varphi\colon B^*(\mathcal{P}) \to \mathbf{R}_+$ in $G(\mathcal{P})$ with $\varphi(a) = 0$ $(a \in \Delta^-(X))$, a nonpositive vector $x \in \mathbf{R}^E_-$ with $x(e) = 0$ $(e \notin E^+ - X)$ and a nonnegative vector $y \in \mathbf{R}^E_+$ with $y(e) = 0$ $(e \notin E^- \cap X)$ such that

$$v = \partial\varphi + x + y, \tag{6.61}$$

where $\partial\varphi$ is the boundary of φ in $G(\mathcal{P})$. (6.61) gives an expression of v as a nonnegative linear combination of vectors in $\mathrm{ER}(X)$. Q.E.D.

We see from Lemmas 6.5 and 6.7 that $\mathrm{C}_f(X)$ is the direct product of the characteristic cones of the supermodular polyhedron $\partial f^X(X)$ and the submodular polyhedron $\partial f_X(\emptyset)$, when $f(\emptyset) = 0$. Hence, Theorem 6.12 may also follow from Theorem 3.26.

It should be noted that if v in $\mathrm{C}_f(X)$ satisfies $v(X) = 0$, then $y = \mathbf{0}$ in (6.61) and that if v satisfies $v(E - X) = 0$, then $x = \mathbf{0}$ in (6.61). Theorem 3.26 (the extreme ray theorem for base polyhedra) also follows from this theorem.

6.3. The Lovász Extensions of Submodular Functions

Consider a submodular function $f\colon \mathcal{D} \to \mathbf{R}$ on a simple distributive lattice $\mathcal{D} = \mathbf{2}^{\mathcal{P}}$ with $\mathcal{P} = (E, \preceq)$. We assume $f(\emptyset) = 0$.

Define the convex function $\hat{f}\colon \mathbf{R}^E \to \mathbf{R} \cup \{+\infty\}$ by

$$\hat{f}(c) = \max\{(c, x) \mid x \in \mathrm{P}(f)\} \tag{6.62}$$

(Proof) If f is a submodular function, then its extension $\hat{f}$ is given by (6.62) and hence is a convex function. Conversely, suppose that the extension $\hat{f}$ of f is a convex function. By definition, for any $X,\ Y \in \mathcal{D}$

$$\begin{aligned}\hat{f}(\chi_X + \chi_Y) &= \hat{f}(\chi_{X\cup Y} + \chi_{X\cap Y}) \\ &= f(X \cup Y) + f(X \cap Y).\end{aligned} \tag{6.68}$$

Since $\hat{f}$ is a positively homogeneous convex function, we also have

$$\hat{f}(\chi_X + \chi_Y) \le \hat{f}(\chi_X) + \hat{f}(\chi_Y) = f(X) + f(Y). \tag{6.69}$$

From (6.68) and (6.69) f is a submodular function on $\mathcal{D}$. Q.E.D.

Theorem 6.13 shows the close relationship between the submodularity and the convexity. The results in Sections 6.1 and 6.2 can be viewed from the theory of convex functions through this theorem. However, the integrality result in Theorem 6.3 does not follow directly from the ordinary convex analysis; it is truly a combinatorial deep result.

Define

$$\mathrm{P}(\mathcal{D}) = \text{ the convex hull of vectors } \chi_A \ (A \in \mathcal{D}). \tag{6.70}$$

Lemma 6.14 [Lovász83]: *For a submodular function $f\colon \mathcal{D} \to \mathbf{R}$ we have*

$$\min\{f(X) \mid X \in \mathcal{D}\} = \min\{\hat{f}(c) \mid c \in \mathrm{P}(\mathcal{D})\}. \tag{6.71}$$

(Proof) Immediate from Theorem 6.13 and (6.66), the positive homogeneity of $\hat{f}$. Q.E.D.

Lemma 6.15: *For any $c \in \mathrm{P}(\mathcal{D})$ there uniquely exists a nonempty chain*

$$B_1 \subset B_2 \subset \cdots \subset B_p \tag{6.72}$$

of $\mathcal{D}$ such that c is expressed as a convex combination

$$c = \sum_{i=1}^{p} \mu_i \chi_{B_i} \tag{6.73}$$

with $\mu_i > 0$ $(i = 1, 2, \cdots, p)$ and $\sum_{i=1}^{p} \mu_i = 1$.

where the last equality follows from Lemma 6.14 with f replaced by $f - x$. (6.79) is equivalent to $x \in \partial f(A)$. Q.E.D.

Theorem 6.17: *Let c be an arbitrary vector in $\mathrm{P}(\mathcal{D})$ and suppose that c is expressed as* (6.73) *with* (6.72). *Then, we have*

$$\partial \tilde{f}(c) = \bigcap\{\partial f(B_i) \mid i = 1, 2, \cdots, p\}. \tag{6.80}$$

(Proof) We have $x \in \partial \tilde{f}(c)$ if and only if

$$\forall\, b \in \mathrm{P}(\mathcal{D})\colon\ (b - c, x) \le \hat{f}(b) - \hat{f}(c). \tag{6.81}$$

From (6.72) and (6.73), (6.81) is rewritten as

$$\begin{aligned} \sum_{i=1}^{p} \mu_i(f(B_i) - x(B_i)) \quad &\le \quad \min\{\hat{f}(b) - (b, x) \mid b \in \mathrm{P}(\mathcal{D})\} \\ &= \quad \min\{f(X) - x(X) \mid X \in \mathcal{D}\} \end{aligned} \tag{6.82}$$

due to Lemma 6.14. Furthermore, since $\sum_{i=1}^{p} \mu_i = 1$ and $\mu_i > 0$ $(i = 1, 2, \cdots, p)$, (6.82) is equivalent to

$$\begin{aligned} f(B_i) - x(B_i) \quad = \quad &\min\{f(X) - x(X) \mid X \in \mathcal{D}\} \\ &(i = 1, 2, \cdots, p) \end{aligned} \tag{6.83}$$

or

$$x \in \bigcap\{\partial f(B_i) \mid i = 1, 2, \cdots, p\}. \tag{6.84}$$

Q.E.D.

For any maximal chain $\mathcal{C}\colon \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E$ of $\mathcal{D}$, denote by $\mathrm{P}(\mathcal{C})$ the n-simplex with vertices χ_{S_i} $(i = 0, 1, \cdots, n)$.

Lemma 6.18: *The collection of $\mathrm{P}(\mathcal{C})$'s for all the maximal chains $\mathcal{C}$ of $\mathcal{D}$ forms a simplicial subdivision of $\mathrm{P}(\mathcal{D})$. Moreover, for two maximal chains $\mathcal{C}^i$: $\emptyset = S_0^i \subset S_1^i \subset \cdots \subset S_n^i = E$ $(i = 1, 2)$ the n-simplices $\mathrm{P}(\mathcal{C}^i)$ $(i = 1, 2)$ have a common facet if and only if for some k with $1 \le k \le n - 1$ we have*

$$S_j^1 = S_j^2 \quad (0 \le j \le n,\ j \ne k). \tag{6.85}$$

(Proof) The first half of this lemma follows from the uniqueness property of Lemma 6.15. Moreover, any facet of the n-simplex $\mathrm{P}(\mathcal{C}^i)$ corresponds to where $\lambda_0 \equiv -\infty$, $\lambda_{p+1} \equiv +\infty$, $\mathcal{L}(\lambda)$'s are the same in each interval $(\lambda_i, \lambda_{i+1})$ $(i = 0, 1, \cdots, p)$, $|\mathcal{L}(\lambda_i)| \geq 2$ $(i = 1, 2, \cdots, p)$ and $|\mathcal{L}(\lambda)| = 1$ $(\lambda \in (\lambda_i, \lambda_{i+1})$, $i = 0, 1, \cdots, p)$. Moreover, for each $i = 1, 2, \cdots, p$, because of the finiteness character there exists a (sufficiently small) positive number ϵ such that

$$\mathcal{L}(\lambda_i - \epsilon) \subseteq \mathcal{L}(\lambda_i), \tag{7.112}$$

$$\mathcal{L}(\lambda_i + \epsilon) \subseteq \mathcal{L}(\lambda_i). \tag{7.113}$$

Since f_1 is monotone decreasing, we have from (7.112) and (7.113)

$$S^-(\lambda_i) \in \mathcal{L}(\lambda_i - \epsilon), \tag{7.114}$$

$$S^+(\lambda_i) \in \mathcal{L}(\lambda_i + \epsilon). \tag{7.115}$$

From (7.114) and (7.115) we have (7.103)∼(7.106). Also, since the set of the quotients $S^+(\lambda_i) - S^+(\lambda_{i-1})$ $(i = 1, 2, \cdots, p)$ with $S^+(\lambda_0) \equiv \emptyset$ is a partition of E into nonempty subsets of E due to Theorem 7.14 and (7.103)∼(7.106), we have $p \leq |E|$. Q.E.D.

The λ_i $(i = 1, 2, \cdots, p)$ in (7.101) are called *critical values* for the pair of submodular systems $(\mathcal{D}_0, f_0)$ and $(\mathcal{D}_1, f_1)$. Denote $\mathbf{S}_0 = (\mathcal{D}_0, f_0)$ and $\mathbf{S}_1 = (\mathcal{D}_1, f_1)$. Submodular systems $\mathbf{S}_i$ $(i = 0, 1)$ are decomposed according to the distributive lattice $\mathcal{L}^* = \bigcup_{\lambda \in \mathbf{R}} \mathcal{L}(\lambda)$ as follows. Choose any maximal chain

$$\mathcal{C}\colon\ \emptyset = A_0 \subset A_1 \subset \cdots \subset A_k = E \tag{7.116}$$

of $\mathcal{L}^*$ and then decompose $\mathbf{S}_i$ $(i = 0, 1)$ into their minors

$$\mathbf{S}_i \cdot A_j / A_{j-1} \quad (j = 1, 2, \cdots, k), \tag{7.117}$$

where $\mathbf{S}_i \cdot A_j / A_{j-1}$ is the set minor of $\mathbf{S}_i$ obtained by restricting $\mathbf{S}_i$ to A_j and contracting A_{j-1}. Such a set of decompositions of $\mathbf{S}_i$ $(i = 0, 1)$ is called the *principal partition* of the pair of $\mathbf{S}_i$ $(i = 0, 1)$. By the poset on the partition $\{A_j - A_{j-1} \mid j = 1, 2, \cdots, k\}$ of E which is uniquely determined by $\mathcal{L}^*$ (see Section 3.2.a), the corresponding poset structure is defined on the set of minors (7.117) for each $i = 0, 1$. We can show that the decompositions (7.117) do not depend on the choice of a maximal chain in $\mathcal{L}^*$ ([Nakamura + Iri81], [Tomi + Fuji82]), due to Theorem 7.17 shown below.

Lemma 7.16: *Let μ: $\mathcal{D}_0 \to \mathbf{R}$ be a modular function on a distributive lattice $\mathcal{D}_0 \subseteq 2^E$ with $\emptyset, E \in \mathcal{D}_0$. We have $\mu(X) = 0$ for all $X \in \mathcal{D}_0$*

Remark: An integral vector $z \in \mathbf{Z}^V$ and a maximal chain $\mathcal{C} : S_0(=\emptyset) \subset S_1 \subset \cdots \subset S_n(= V)$ of the Boolean lattice 2^V define an n-dimensional simplex given by the convex hull of $z + \chi_{S_i}$ $(i = 0, 1, \cdots, n)$. The collection of such simplices for all integral vectors z and for all maximal chains $\mathcal{C}$ forms a simplicial division of $\mathbf{R}^V$ due to Freudenthal (Fig. 16.3) (see, e.g., [Todd76] and [Yang99]). We call any face of such a simplex in Freudenthal's simplicial division *Freudenthal's simplex cell*.

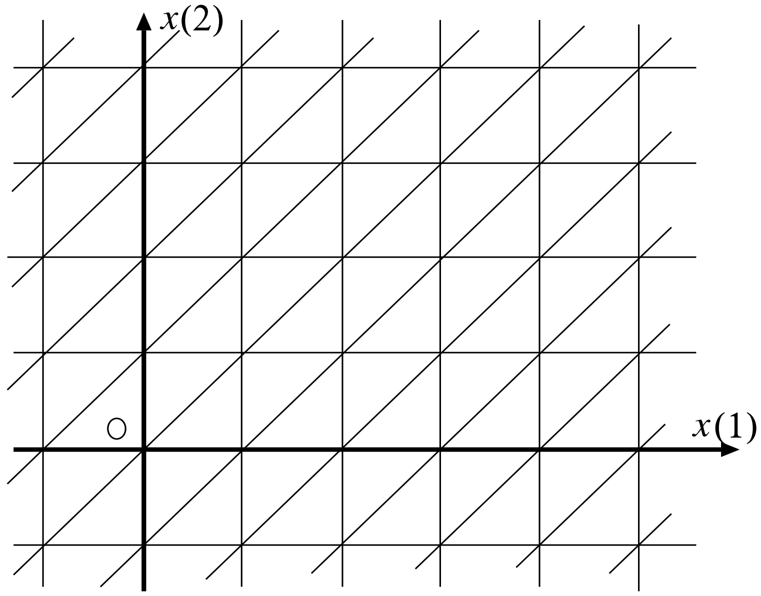

Figure 16.3: Freudenthal's simplicial division of $\mathbf{R}^{\{1,2\}}$.

Note that the truncated Lovász extensions of submodular (set) functions defined by (6.76) are exactly L$^\natural$-convex functions with their effective domains being contained in the unit hypercube $[0, 1]^V$. Hence, we have

Lemma 16.10: *A function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is a domain-integral L$^\natural$-convex function if and only if*

(a) *f is a locally polyhedral convex function and*

(b) *for each integral vector $z \in \mathbf{Z}^V$ and each set $W \subseteq V$ such that $z, z + \chi_W \in \mathrm{dom} f$, the restriction of $f(x) - f(z)$ in x on the interval*

$[z, z+\chi_W]$ is the truncated Lovász extension (on $\mathbf{R}^W$) of a submodular (set) function whose domain is imbedded in $\mathbf{R}^V$ and translated by z.

For a function $h : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with dom$h \neq \emptyset$, if a locally polyhedral (not necessarily convex) function $\hat{h} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is obtained by (L$^\natural$2$'$) with f being replaced by h, then we call $\hat{h}$ the *Lovász-Freudenthal extension* of h. From (L$^\natural$1$'$) and (L$^\natural$2$'$) we have

Theorem 16.11: *A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is an L$^\natural$-convex function on $\mathbf{Z}^V$ if and only if the Lovász-Freudenthal extension of f is a convex function.*

The concept of a domain-integral L$^\natural$-convex function with its domain being a box was considered by P. Favati and F. Tardella [Favati+Tardella90], who called it a *submodular integrally convex function*. It should be noted that L$^\natural$- and L-convex functions of [Favati+Tardella90] and [Murota98b] are originally defined on integral lattice points in $\mathbf{Z}^V$, while we are here considering locally polyhedral convex functions defined on $\mathbf{R}^V$ of real (or rational) vectors that are uniquely determined from the values on $\mathbf{Z}^V$ by the scheme of (L$^\natural$2$'$) given above (also see [Murota03a, Sections 6.11 and 7.8]).

The origin of the following characterization is found in [Favati+Tardella 90] (also see [Fuji+Murota00]).

Theorem 16.12: *A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with* dom$f \neq \emptyset$ *is an L$^\natural$-convex function on $\mathbf{Z}^V$ if and only if for each $p, q \in \mathbf{Z}^V$*

$$f(p) + f(q) \geq f(\lceil (p+q)/2 \rceil) + f(\lfloor (p+q)/2 \rfloor). \qquad (16.23)$$

(Proof) (The if part): We assume without loss of generality that dom$f$ is full-dimensional. Suppose that (16.23) holds for each $p, q \in \mathbf{Z}^V$. It suffices to prove the convexity of the Lovász-Freudenthal extension $\hat{f}$ of f on the union of two adjacent full-dimensional Freudenthal's simplex cells. We have adjacent simplex cells of the following two types. For an integral vector z in dom$f$ and a linear ordering $(v_1, v_2, \cdots, v_n)$, defining

$$S_i = \{v_1, v_2, \cdots, v_i\} \qquad (i = 1, 2, \cdots, n) \qquad (16.24)$$

and $S_0 = \emptyset$, consider

(I) two simplices formed by

$$z + \chi_{S_i} \quad (i = 0, 1, \cdots, n) \tag{16.25}$$

and by

$$z - \chi_{v_n}, \quad z + \chi_{S_i} \ (i = 0, 1, \cdots, n-1), \tag{16.26}$$

where the common face of the two simplices is formed by the n points $z + \chi_{S_i}$ $(i = 0, 1, \cdots, n-1)$, and $p = z + \mathbf{1}$ and $q = z - \chi_{v_n}$ are the points of the two simplices outside the common face, and

(II) two simplices formed by

$$z + \chi_{S_i} \quad (i = 0, 1, \cdots, n) \tag{16.27}$$

and by

$$z + \chi_{S_i} \ (i = 0, 1, \cdots, k, k+2, \cdots, n), \quad z + \chi_{S_k \cup \{v_{k+2}\}} \tag{16.28}$$

for some k with $0 \leq k \leq n-2$, where the common face of the two simplices is formed by the n points $z + \chi_{S_i}$ $(i = 0, 1, \cdots, k, k+2, \cdots, n)$, and $p = z + \chi_{S_{k+1}}$ and $q = z + \chi_{S_k \cup \{v_{k+2}\}}$ are the points of the two simplices outside the common face.

Since we have

$$p + q = \lceil (p+q)/2 \rceil + \lfloor (p+q)/2 \rfloor, \tag{16.29}$$

it follows from (16.23) and the definition of the Lovász-Freudenthal extension $\hat{f}$ that

$$\begin{aligned} \frac{1}{2}\{\hat{f}(p) + \hat{f}(q)\} &= \frac{1}{2}\{f(p) + f(q)\} \\ &\geq \frac{1}{2}\{f(\lceil (p+q)/2 \rceil) + f(\lfloor (p+q)/2 \rfloor)\} \\ &= \hat{f}((p+q)/2). \end{aligned} \tag{16.30}$$

Here note that for (I) we have $\lceil (p+q)/2 \rceil = z + \chi_{S_{n-1}}$ and $\lfloor (p+q)/2 \rfloor = z$ and for (II) $\lceil (p+q)/2 \rceil = z + \chi_{S_{k+2}}$ and $\lfloor (p+q)/2 \rfloor = z + \chi_{S_k}$. Since these two points $\lceil (p+q)/2 \rceil$ and $\lfloor (p+q)/2 \rfloor$ are vertices of the common face, $(p+q)/2$ belongs to the common face. Hence, it follows from (16.30) that the Lovász-Freudenthal extension $\hat{f}$ of f restricted to the union of the two adjacent simplex cells is convex.

(The only-if part): Suppose that f is an L$^\natural$-convex function on $\mathbf{Z}^V$. Consider the Lovász-Freudenthal extension $\hat{f}$ of f. Then, because of the convexity of $\hat{f}$ and the definition of the Lovász-Freudenthal extension, we have for each $p, q \in \mathbf{Z}^V$

$$\begin{aligned} \hat{f}(p) + \hat{f}(q) &\geq 2\hat{f}((p+q)/2) \\ &= 2\hat{f}((\lceil (p+q)/2 \rceil + \lfloor (p+q)/2 \rfloor)/2) \\ &= \hat{f}(\lceil (p+q)/2 \rceil) + \hat{f}(\lfloor (p+q)/2 \rfloor). \end{aligned} \tag{16.31}$$

Q.E.D.

We give characterizations of integral L$^\natural$-convex sets as follows. (Recall that by a polyhedron we mean a convex polyhedron.)

Theorem 16.13: *For a polyhedron P in $\mathbf{R}^V$ the following four statements are equivalent*:

(1) *P is an integral L$^\natural$-convex set.*

(2) *P is a polyhedron formed by the union of Freudenthal's simplex cells.*

(3) *P is an integral polyhedron and for any integral points $p, q \in P$ we have $\lfloor (p+q)/2 \rfloor$, $\lceil (p+q)/2 \rceil \in P$.*

(4) *P is an integral polyhedron and for each integral point $p \in P$,*

$$\mathcal{D}_p^+ \equiv \{X \mid X \subseteq V,\ p + \chi_X \in P\}, \quad \mathcal{D}_p^- \equiv \{X \mid X \subseteq V,\ p - \chi_X \in P\} \tag{16.32}$$

are distributive lattices with join $\cup$ and meet $\cap$, and $P \cap [p, p+\mathbf{1}]$ and $P \cap [p-\mathbf{1}, p]$ are, respectively, equal to the convex hull of χ_X $(X \in \mathcal{D}_p^+)$ and that of χ_X $(X \in \mathcal{D}_p^-)$.

(Proof) By (L$^\natural$2$'$) in the characterization of domain-integral L$^\natural$-convex functions and Theorem 16.12 we see that (1), (2) and (3) are equivalent. So, we show the equivalence between (4) and $\{(1), (2), (3)\}$. Suppose (3). Then for each integral point $p \in P$ and $X, Y \subseteq V$ such that $p + \chi_X, p + \chi_Y \in P$, we have $\lfloor (p + \chi_X + p + \chi_Y)/2 \rfloor = p + \chi_{X \cap Y}$ and $\lceil (p + \chi_X + p + \chi_Y)/2 \rceil = p + \chi_{X \cup Y}$. Similarly, for $X, Y \subseteq V$ such that $p - \chi_X, p - \chi_Y \in P$ we have $\lfloor (p - \chi_X + p - \chi_Y)/2 \rfloor = p - \chi_{X \cup Y}$ and $\lceil (p - \chi_X + p - \chi_Y)/2 \rceil = p - \chi_{X \cap Y}$. Hence (4) holds. Conversely, suppose (4). Then it follows that for each integral point $p \in P$ sets $P \cap [p, p+\mathbf{1}]$ and $P \cap [p-\mathbf{1}, p]$ are the unions of Freudenthal's simplex cells. Hence (2) holds. Q.E.D.

Minimizers of a domain-integral L$^\natural$-convex function are characterized by the following.

Theorem 20.1: *For a domain-integral L$^\natural$-convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, a vector $x \in \mathbf{Z}^V$ is a minimizer of f if and only if x is a minimizer of the restriction of f to $[x - \mathbf{1}, x] \cup [x, x + \mathbf{1}]$.*

(Proof) Let $n = |V|$. Any n-dimensional Freudenthal's simplex cell that includes $x \in \mathbf{Z}^V$ is given by $n+1$ integral points $x - \chi_W$, $x - \chi_W + \chi_{\{v_1,\cdots,v_k\}}$ $(k = 1, 2, \cdots, n)$ for some linear ordering $(v_1, v_2, \cdots, v_n)$ of V with $W = \{v_1, v_2, \cdots, v_l\}$ for some l $(0 \le l \le n)$. We can easily see that these points belong to $[x - \mathbf{1}, x] \cup [x, x + \mathbf{1}]$. Hence the present theorem holds. Q.E.D.

The following property for L-/L$^\natural$-convex functions is fundamental and useful for introducing scaling techniques. For any positive integer k we say that f is an *L$^\natural$-convex function on* $(k\mathbf{Z})^V$ if $f_k : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ defined by $f_k(z) = f(kz)$ for all $z \in \mathbf{Z}^V$ is an L$^\natural$-convex function on $\mathbf{Z}^V$.

Theorem 20.2: *An L-convex (L$^\natural$-convex) function f on $\mathbf{Z}^V$ is also an L-convex (L$^\natural$-convex) function on $(k\mathbf{Z})^V$ for any positive integer k.*

(Proof) Because of Lemma 16.6 it suffices to prove the present theorem for any L-convex function f on $\mathbf{Z}^V$. Since f is a submodular function on $(k\mathbf{Z})^V$ and satisfies $f(x + \alpha k\mathbf{1}) = f(x) + \alpha kr$ $(x \in (k\mathbf{Z})^V)$, it follows from Theorem 16.9 that f is an L-convex function on $(k\mathbf{Z})^V$. Q.E.D.

Iwata [Iwata99] pointed out that a polynomial algorithm for minimizing L-convex functions could be obtained by combining Theorem 20.2 and the proximity theorem shown in [Iwata+Shigeno02] (see Theorem 20.9 given below) by the use of any polynomial algorithm for submodular (set) function minimization. See [Murota03b] for a faster algorithm for L-convex function minimization.

20.2. M- and M$^\natural$-convex Functions

As a generalization of Theorem 3.16 we have the following characterization of minimizers of M- and M$^\natural$-convex functions.

Theorem 20.3: *For an M-convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ a vector $x \in \mathrm{dom} f$ is a minimizer of f if and only if for each $u, v \in V$ and each $\alpha > 0$ we have*

$$f(x + \alpha(\chi_u - \chi_v)) \ge f(x). \tag{20.1}$$

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