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A quantum double construction in Rel[†]

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We study bialgebras and Hopf algebras in the compact closed category **Rel** of sets and binary relations. Various monoidal categories with extra structure arise as the categories of (co)modules of bialgebras and Hopf algebras in **Rel**. In particular, for any group G, we derive a ribbon category of crossed G-sets as the category of modules of a Hopf algebra in **Rel** that is obtained by the quantum double construction. This category of crossed G-sets serves as a model of the braided variant of propositional linear logic.

1. Introduction

Many important examples of *traced monoidal categories* (Joyal *et al.* 1996) and *ribbon categories* (*tortile monoidal categories*) (Shum 1994; Turaev 1994) have emerged in mathematics and theoretical computer science during the last two decades. Ribbon categories of particular interest to mathematicians are those of linear representations of *quantum groups* (*quasi-triangular Hopf algebras*) (Drinfel'd 1987; Kassel 1995). In many of these examples, there are braidings (Joyal and Street 1993) that are not symmetries: in terms of the graphical presentation (Joyal and Street 1991; Selinger 2011), the braid $c =$ is distinguished from its inverse $c^{-1} =$, and this is the key property for providing non-trivial invariants (or denotational semantics) of knots, tangles and so on (Freyd and Yetter 1989; Kassel 1995; Turaev 1994; Yetter 2001) as well as solutions of the quantum Yang–Baxter equation (Drinfel'd 1987; Kassel 1995) and 3-dimensional topological quantum field theory (Bakalov and Kirilov 2001). In theoretical computer science, major examples include categories with fixed-point operators used in denotational and algebraic semantics (Bloom and Ésik 1993; Hasegawa 1999; Hasegawa 2009; Ştefănescu 2000), and the category of sets and binary relations and its variations used for models of linear logic (Girard 1987) and game semantics (Joyal 1977; Melliès 2004). Moreover, the Int-construction (Joyal *et al.* 1996) provides a rich class of models of the Geometry of Interaction (Girard 1989; Abramsky *et al.* 2002; Haghverdi and Scott 2011) and, more generally, bi-directional information flow (Hildebrandt *et al.* 2004; Katsumata 2008). In most of these cases the braiding is a symmetry, so is identified with .

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Although it is nice to know that all these examples share a common structure, it is also striking that the important examples from mathematics are almost disjoint from those from computer science[†]. Is this just a matter of taste, or is it the case that categories used in computer science cannot host structures interesting for mathematicians (non-symmetric braidings in particular)?

In this paper we demonstrate that there are mathematically interesting structures in a category preferred by computer scientists. Specifically, we focus on the category **Rel** of sets and binary relations. **Rel** is a compact closed category (Kelly and Laplaza 1980), that is, a ribbon category in which the braiding is a symmetry. We study bialgebras and Hopf algebras in **Rel**, and show that various monoidal categories with extra structure, like traces and autonomy, can be derived as the categories of (co)modules of bialgebras in **Rel**. As a most interesting example, for any group G, we consider the associated Hopf algebra in **Rel**, and apply the *quantum double construction* (Drinfel'd 1987) to it. The resulting Hopf algebra is equipped with a universal R-matrix as well as a universal twist. We show that the category of its modules is the category of *crossed G-sets* (Freyd and Yetter 1989; Whitehead 1949) and suitable binary relations, featuring non-symmetric braiding and non-trivial twist.

While the results mentioned above are interesting in their own right, we hope that this work serves as a useful introduction to the theory of quantum groups for researchers working on semantics of computation, and that it helps to connect these two research areas, which deserve to interact much more.

Related work

Hopf algebras have been extensively studied in connection with quantum groups (Drinfel'd 1987): standard references include Kassel (1995) and Majid (1995). The idea of using Hopf algebras for modelling various non-commutative linear logics goes back to Blute (1996), where the focus is on Hopf algebras in the $*$-autonomous category of topological vector spaces. As far as we know, there is no published result on Hopf algebras in **Rel**. Since Freyd and Yetter (1989), categories of crossed G-sets have appeared frequently as typical examples of braided monoidal categories. In the standard setting of finite-dimensional vector spaces, modules of the quantum double of a Hopf algebra A amount to the crossed A-bimodules (Kassel 1995; Kassel and Turaev 1995), and our result is largely an adaptation of such a standard result to **Rel**. However, we are not aware of any characterisation in the literature of crossed G-sets in terms of a quantum double construction.

Organisation of the paper

In Section 2, we recall basic notions and facts for monoidal categories and bialgebras. In Section 3, we examine some bialgebras in **Rel** that arise from monoids and

[†] Important exceptions are the dagger compact closed categories used in the study of quantum information protocols (Abramsky and Coecke 2004), though they do not feature non-symmetric braidings. Note that the category of crossed G-sets we introduce later in the paper is actually a dagger tortile category in the sense of Selinger (2011).

groups, and study the categories of (co)modules. Section 4 is devoted to a quantum double construction in **Rel**. In this development, we give a simplified description of the quantum double construction in terms of the Int-construction on traced symmetric monoidal categories. In Section 5, we observe that the ribbon Hopf algebra constructed in the previous section gives rise to a ribbon category of crossed G-sets, and look at some elements of this category. We discuss how this category can be used as a model of braided linear logic in Section 6. Finally, Section 7 presents the paper's conclusions.

2. Monoidal categories and bialgebras

2.1. *Monoidal categories*

A *monoidal* (*tensor*) *category* (Mac Lane 1971; Joyal and Street 1993) $\mathscr{C} = (\mathscr{C}, \otimes, I, a, l, r)$ consists of a category $\mathscr{C}$, a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, an object $I \in \mathscr{C}$ and natural isomorphisms

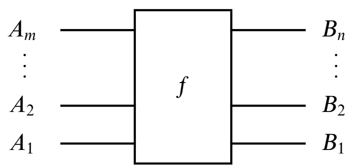
$$\begin{aligned} a_{A,B,C} &: (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C) \\ l_A &: I \otimes A \xrightarrow{\sim} A \\ r_A &: A \otimes I \xrightarrow{\sim} A \end{aligned}$$

subject to the standard coherence diagrams. A monoidal category $\mathscr{C} = (\mathscr{C}, \otimes, I, a, l, r)$ is said to be strict if a, l, r are identity morphisms. For simplicity, in most of this paper we will pretend that our monoidal categories are strict; Mac Lane's coherence theorem ensures that there is no loss of generality in doing so.

In the rest of this paper, we will make use of the graphical presentation of morphisms in monoidal categories (Joyal and Street 1991; Selinger 2011). A morphism

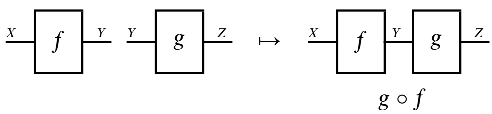
$$f : A_1 \otimes A_2 \otimes \cdots \otimes A_m \to B_1 \otimes B_2 \otimes \cdots \otimes B_n$$

in a monoidal category will be drawn as:

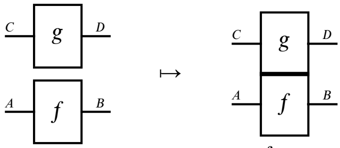


which is to be read from left to right.

Morphisms can be composed, sequentially:



or in parallel:



$f \otimes g$

A *braiding* (Joyal and Street 1993) is a natural isomorphism

$$c_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$$

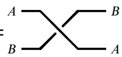
such that both c and c^{-1} satisfy the following 'bilinearity' or 'Hexagon Axiom' (the case for c^{-1} is omitted):

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "a_{A,B,C}"] \arrow[d, "c_{A,B} \otimes C"'] & A \otimes (B \otimes C) \arrow[r, "c_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "a_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "a_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "B \otimes c_{A,C}"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

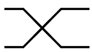
For braidings, we shall use the drawings

$c_{A,B} =$ 

and

$c^{-1}_{A,B} =$ 

A *symmetry* is a braiding such that $c_{A,B} = c^{-1}_{B,A}$. In that case we simply draw



hence

A *braided/symmetric monoidal category* is a monoidal category equipped with a braiding/symmetry.

A *twist* or a *balance* for a braided monoidal category is a natural isomorphism

$$\theta_A : A \xrightarrow{\sim} A$$

such that

$$\theta_{A \otimes B} = c_{B,A} \circ (\theta_B \otimes \theta_A) \circ c_{A,B}.$$

Twists are drawn as

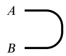
$\theta_A =$ 

$\theta_A^{-1} =$ 

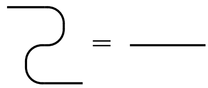
A *balanced monoidal category* is a braided monoidal category with a twist. Note that a symmetric monoidal category is precisely a balanced monoidal category with $\theta_A = id_A$ for every A.

In a monoidal category, a *dual pairing* between two objects A and B is given by a pair of morphisms $d : I \to A \otimes B$, called unit, and $e : B \otimes A \to I$, called counit, drawn as

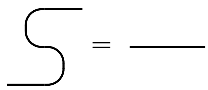
and



respectively, and satisfying



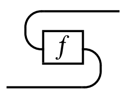
and



In such a dual pairing, B is called the *left dual* of A, and A is called the *right dual* of B. For an object, its left (or right) dual, if it exists, is uniquely determined up to isomorphism. A monoidal category is *left autonomous* or *left rigid* if every object A has a left dual A^* with unit $\eta_A : I \to A \otimes A^*$ and counit $\varepsilon_A : A^* \otimes A \to I$. In a left autonomous category,

$$I \cong I^*$$
$$A^* \otimes B^* \cong (B \otimes A)^*.$$

Also, $(-)^*$ extends to a contravariant functor, where, for a morphism $f : A \to B$, its dual $f^* : B^* \to A^*$ is given as:

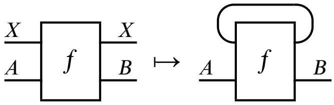


A *ribbon category* (Turaev 1994) or *tortile monoidal category* (Shum 1994) is a balanced monoidal category that is left autonomous and satisfies $(\theta_A)^* = \theta_{A^*}$. In a ribbon category, $(-)^*$ is a contravariant equivalence, and there is a natural isomorphism $A^{**} \cong A$ (hence the left dual of A and the right dual of A are isomorphic). Note that a ribbon category whose twist is the identity is a *compact closed category* (Kelly and Laplaza 1980).

A *traced monoidal category* (Joyal *et al.* 1996) is a balanced monoidal category $\mathscr{C}$ equipped with a trace operator

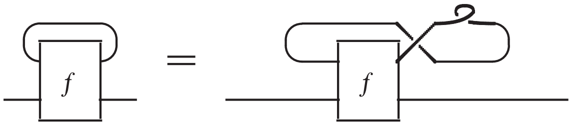
$$Tr^X_{A,B} : \mathscr{C}(A \otimes X, B \otimes X) \to \mathscr{C}(A, B),$$

which will be drawn as a 'feedback' operator



satisfying a few coherence axioms. Alternatively, by the structure theorem in Joyal *et al.* (1996), traced monoidal categories are characterised as monoidal full subcategories of ribbon categories. Any ribbon category has a unique trace, called its *canonical trace* (Joyal *et al.* 1996) – for uniqueness, see, for example, Hasegawa (2009). For a morphism $f : A \otimes X \to B \otimes X$ in a ribbon category, its trace $Tr^X_{A,B} f : A \to B$ is given by

$$Tr^X_{A,B} f = (id_B \otimes (\varepsilon_X \circ (id_{X^*} \otimes \theta_X) \circ c_{X,X^*})) \circ (f \otimes id_{X^*}) \circ (id_A \otimes \eta_X).$$



For monoidal categories $\mathscr{C} = (\mathscr{C}, \otimes, I, a, l, r)$ and $\mathscr{C}' = (\mathscr{C}', \otimes', I', a', l', r')$, a *monoidal functor* from $\mathscr{C}$ to $\mathscr{C}'$ is a tuple (F, m, m_I) where F is a functor from $\mathscr{C}$ to $\mathscr{C}'$, m is a natural transformation from $F(-) \otimes' F(=)$ to $F(- \otimes =)$ and $m_I : I' \to FI$ is an arrow in $\mathscr{C}'$, satisfying three coherence conditions. It is said to be *strong* if $m_{A,B}$ and m_I are all isomorphisms, and *strict* if they are all identities. A *balanced monoidal functor* from a balanced $\mathscr{C}$ to another $\mathscr{C}'$ is a monoidal functor (F, m, m_I) that also satisfies

$$m_{B,A} \circ c_{FA,FB} = Fc_{A,B} \circ m_{A.B}$$
$$F\theta_A = \theta_{FA}.$$

For monoidal functors (F, m, m_I) and (G, n, n_I) with the same source and target monoidal categories, a *monoidal natural transformation* from (F, m, m_I) to (G, n, n_I) is a natural transformation $\varphi : F \to G$ such that

$$\varphi_{A \otimes B} \circ m_{A,B} = n_{A,B} \circ \varphi_A \otimes \varphi_B$$
$$\varphi_I \circ m_I = n_I.$$

A *(balanced/symmetric) monoidal adjunction* between (balanced/symmetric) monoidal categories is an adjunction in which both of the functors are (balanced/symmetric) monoidal and the unit and counit are monoidal natural transformations.

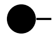
2.2. *Monoids, comonoids and (co)modules*

A *monoid* in a monoidal category $\mathscr{C} = (\mathscr{C}, \otimes, I, a, l, r)$ is an object A equipped with morphisms $m : A \otimes A \to A$, called the *multiplication*, and $1 : I \to A$, called the *unit*, such that the following diagrams commute.

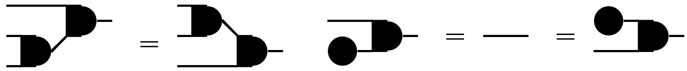
$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "A\otimes m"] \arrow[d, "m\otimes A"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "A\otimes 1"] \arrow[d, "1\otimes A"'] \arrow[dr, "id"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"'] & A \end{tikzcd}$$

With the notation

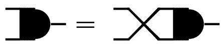
$m =$ 

$1 =$ 

these diagrams can be expressed as follows:



When $\mathscr{C}$ is symmetric and $m \circ c_{A,A} = m$, that is,

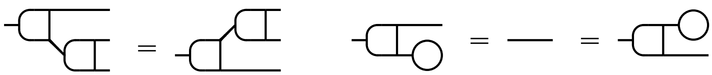


holds, we say A is *commutative*.

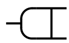
Dually, a *comonoid* in a monoidal category $\mathscr{C}$ is an object A equipped with morphisms $\Delta : A \to A \otimes A$, called the *comultiplication*, and $\epsilon : A \to I$, called the *counit*, satisfying

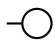
$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "A\otimes\Delta"] \\ A \otimes A \arrow[r, "\Delta\otimes A"'] & A \otimes A \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "id"] & A \otimes A \arrow[d, "A\otimes\epsilon"] \\ A \otimes A \arrow[r, "\epsilon\otimes A"'] & A \end{tikzcd}$$

They can be drawn as

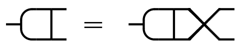


where

$\Delta =$ 

$\epsilon =$ 

We say A is *co-commutative* when $\mathscr{C}$ is symmetric and $c_{A,A} \circ \Delta = \Delta$, or, graphically,



Suppose $A = (A, m, 1)$ is a monoid. A gives rise to a monad $A \otimes (-)$ whose multiplication is

$$m \otimes X : A \otimes A \otimes X \to A \otimes X$$

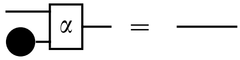
and unit is

$$1 \otimes X : X \to A \otimes X.$$

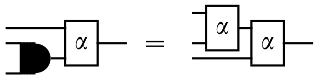
A (left) *A-module* is an Eilenberg–Moore algebra of this monad. More explicitly, an A-module consists of an object X and a morphism $\alpha : A \otimes X \to X$, called the action, satisfying

$$\begin{tikzcd} X \arrow[r, "1\otimes X"] \arrow[dr, "id"'] & A \otimes X \arrow[d, "\alpha"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \otimes X \arrow[r, "A\otimes\alpha"] \arrow[d, "m\otimes X"'] & A \otimes X \arrow[d, "\alpha"] \\ A \otimes X \arrow[r, "\alpha"'] & X \end{tikzcd}$$

or, in the graphical presentation,



and



A morphism of A-modules from (X, α) to (Y, β) is a morphism $f : X \to Y$ satisfying

$$\begin{tikzcd} A \otimes X \arrow[r, "A\otimes f"] \arrow[d, "\alpha"'] & A \otimes Y \arrow[d, "\beta"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

We will use **Mod**(A) to denote the category of A-modules and morphisms.

Dually, given a comonoid $A = (A, \Delta, \epsilon)$, a (left) A-*comodule* is an Eilenberg-Moore coalgebra of the comonad $A \otimes (-)$ whose comultiplication is

$$\Delta \otimes X : A \otimes X \to A \otimes A \otimes X$$

and counit is

$$\epsilon \otimes X : A \otimes X \to X.$$

Explicitly, an A-comodule consists of an object X and a morphism $\alpha : X \to A \otimes X$, called the coaction, satisfying the axioms dual to those of modules. A morphism of A-comodules from (X, α) to (Y, β) is then a morphism $f : X \to Y$ making the evident diagram commute. We will denote the category of A-comodules and morphisms by **Comod**(A).

2.3. *Bialgebras and Hopf algebras*

We now suppose that $\mathscr{C}$ is a symmetric monoidal category with a symmetry

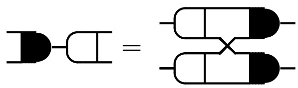
$$c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X.$$

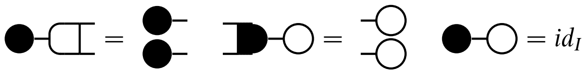
A *bialgebra* in $\mathscr{C}$ is given by a tuple $A = (A, m, 1, \Delta, \epsilon)$ where A is an object of $\mathscr{C}$ and $(A, m, 1)$ is a monoid in $\mathscr{C}$ while (A, Δ, ϵ) is a comonoid in $\mathscr{C}$, satisfying

$$\begin{tikzcd} A \otimes A \arrow[rr, "m"] \arrow[d, "\Delta\otimes\Delta"'] & & A \arrow[d, "\Delta"] \\ A \otimes A \otimes A \otimes A \arrow[r, "A\otimes c_{A,A}\otimes A"'] & A \otimes A \otimes A \otimes A \arrow[r, "m\otimes m"'] & A \otimes A \end{tikzcd}$$

$$\begin{tikzcd} A \otimes A \arrow[rr, "m"] \arrow[dr, "\epsilon\otimes\epsilon"'] & & A \arrow[dl, "\epsilon"] \\ & I & \end{tikzcd} \qquad \begin{tikzcd} & I \arrow[dl, "1"'] \arrow[dr, "1\otimes 1"] & \\ A \arrow[rr, "\Delta"'] & & A \otimes A \end{tikzcd} \qquad \begin{tikzcd} I \arrow[rr, "id"] \arrow[dr, "1"'] & & I \\ & A \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

Graphically,



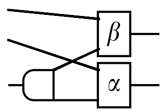


We say A is commutative (respectively, co-commutative) when it is commutative (respectively, co-commutative) as a monoid (respectively, comonoid). For a bialgebra A, we can consider the category of modules **Mod**(A) (for A as a monoid) as well as that of comodules **Comod**(A) (for A as a comonoid). The functor $A \otimes (-)$ is both monoidal and comonoidal. Moreover, as a monad $A \otimes (-)$ is comonoidal, while as a comonad it is monoidal. It follows that both **Mod**(A) and **Comod**(A) are monoidal categories (*cf.* Bruguieres and Virelizier (2006) and Pastro and Street (2009)). Explicitly, in **Mod**(A), the tensor unit is

$$(I, A \otimes I \cong A \xrightarrow{\epsilon} I)$$

and the tensor product of (X, α) and (Y, β) is

$$(X \otimes Y, A \otimes X \otimes Y \xrightarrow{\Delta \otimes X \otimes Y} A \otimes A \otimes X \otimes Y \xrightarrow{A \otimes c_{A,X} \otimes Y} A \otimes X \otimes A \otimes Y \xrightarrow{\alpha \otimes \beta} X \otimes Y).$$

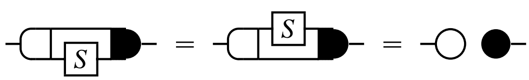


The monoidal structure of **Comod**(A) is given by dualising that of **Mod**(A).

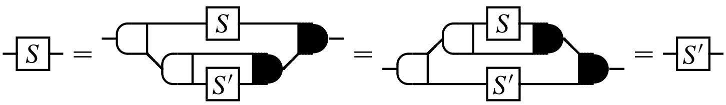
A *Hopf algebra* is a bialgebra $A = (A, m, 1, \Delta, \epsilon)$ equipped with a morphism $S : A \to A$, called an *antipode*, such that

$$\begin{tikzcd}
 & A \otimes A \arrow[r, "S \otimes A"] & A \otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & I \arrow[r, "1"] & A \\
 & A \otimes A \arrow[r, "A \otimes S"'] & A \otimes A \arrow[ur, "m"'] &
\end{tikzcd}$$

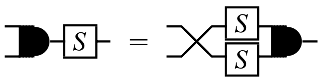
commutes - see the picture below:



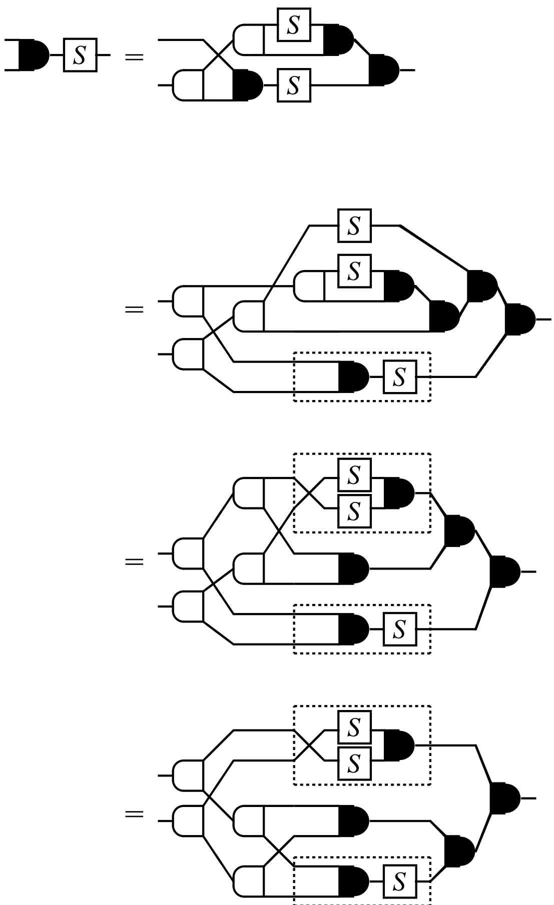
We shall now recall some basic results on Hopf algebras. First, the antipode of a Hopf algebra is unique – if S and S' are both antipodes, we have $S = S'$ because

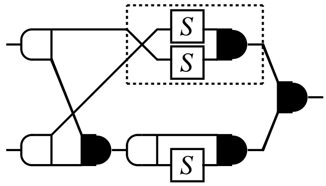


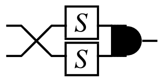
Lemma 2.1. For any Hopf algebra $A = (A, m, 1, \Delta, \epsilon, S)$, the equation $S \circ m = m \circ (S \otimes S) \circ c_{A,A}$ holds.



Proof. We give a graphical proof, in which each step follows from the axioms of bialgebras and antipode:



$=$ 

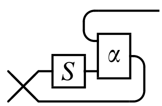
$=$ 

(Those familiar with group theory might notice that this is just a graphical reworking of the proof of $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$, *cf.* Example 2.7.) □

From Lemma 2.1, we can easily derive the well-known fact that the antipode S of any commutative or co-commutative Hopf algebra satisfies $S \circ S = id$, and is thus invertible. In general, an antipode does not have to be invertible – see Takeuchi (1971) for some examples. It is also known that any Hopf algebra in a compact closed category with equalisers has an invertible antipode (Takeuchi 1999), and this is the case for the category of finite dimensional vector spaces. All concrete examples considered below have an invertible antipode (see also Remark 3.5).

Lemma 2.2. If $\mathscr{C}$ is a compact closed category and A is a Hopf algebra in $\mathscr{C}$, then $\mathbf{Mod}(A)$ is left autonomous, where a left dual of a module (X, α) is

$$A \otimes X^* \xrightarrow{c} X^* \otimes A \xrightarrow{X^* \otimes S \otimes \eta} X^* \otimes A \otimes X \otimes X^* \xrightarrow{X^* \otimes \alpha \otimes X^*} X^* \otimes X \otimes X^* \xrightarrow{\varepsilon \otimes X^*} X^*.$$



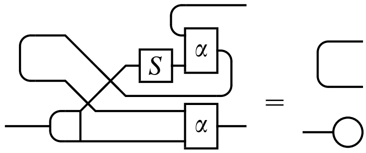
The unit and counit of the dual pairing are given by the unit and counit of the dual pairing of X and X^* in $\mathscr{C}$.

Proof. It suffices to show that

(i) the unit $\eta_X : I \to X \otimes X^*$ is a morphism of modules from (I, ϵ) to $(X, \alpha) \otimes (X, \alpha)^*$; and

(ii) the counit $\varepsilon_X : X^* \otimes X \to I$ is a morphism of modules of $(X, \alpha)^* \otimes (X, \alpha) \to (I, \epsilon)$.

The first of these amounts to the equation



which follows from the axioms of duality, modules and antipode. We can also show (ii) in a similar way. □

Remark 2.3. In this paper we only consider bialgebras and Hopf algebras in *symmetric* monoidal categories. However, it makes complete sense to think about bialgebras and Hopf algebras in *braided* monoidal categories, and this is the central topic in Majid (1994).

Remark 2.4. As noted in Cockett and Seely (1997), the category of modules of a bialgebra in a symmetric or braided linearly distributive category is a linearly distributive category. Similarly, the category of modules of a Hopf algebra in a symmetric or braided $*$-autonomous category is a $*$-autonomous category.

2.4. *Braiding and twists on modules of a bialgebra*

If a bialgebra A is co-commutative, the monoidal category $\mathbf{Mod}(A)$ has a symmetry inherited from the base symmetric monoidal category. However, whether A is co-commutative or not, there can be some non-trivial braiding and twist on $\mathbf{Mod}(A)$. We now suppose $\mathbf{Mod}(A)$ is braided with a braiding σ (while we use c for the symmetry of the base symmetric monoidal category). Since $A = (A, m)$ is an A-module, we have

$$\sigma_{A,A} : A \otimes A \to A \otimes A,$$

and

$$c_{A,A} \circ \sigma_{A,A} \circ (1 \otimes 1) : I \to A \otimes A,$$

which we shall denote by R. Conversely, for this $R : I \to A \otimes A$, it can be seen that

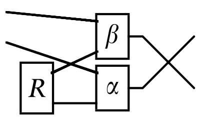
$$\sigma_{X,Y} \circ (f \otimes g) = c_{X,Y} \circ (\alpha \otimes \beta) \circ (A \otimes c_{A,X} \otimes Y) \circ (R \otimes X \otimes Y) \circ (f \otimes g)$$

holds for modules $X = (X, \alpha)$ and $Y = (Y, \beta)$ and morphisms $f : I \to X$ and $g : I \to Y$ in $\mathscr{C}$. So from R, we can recover

$$\sigma_{X,Y} : X \otimes Y \to Y \otimes X$$

as

$$\sigma_{X,Y} = c_{X,Y} \circ (\alpha \otimes \beta) \circ (A \otimes c_{A,X} \otimes Y) \circ (R \otimes X \otimes Y)$$

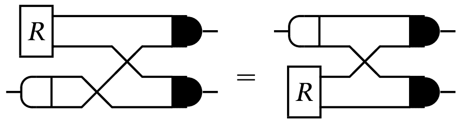


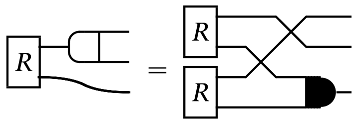
provided the base symmetric monoidal category $\mathscr{C}$ is closed and the global section functor $\mathscr{C}(I, -) : \mathscr{C} \to \mathbf{Set}$ is faithful, which is the case for all commonly used examples, including the category of vector spaces and linear maps, as well as **Rel**. In such cases, there is a bijective correspondence between braidings on **Mod**(A) and morphisms of $I \to A \otimes A$ satisfying certain equations (Kassel 1995; Majid 1995; Street 2007). Such a morphism of $I \to A \otimes A$ is called a *universal R-matrix* or a *braiding element*. Explicitly, a universal R-matrix is a morphism $R : I \to A \otimes A$ that:

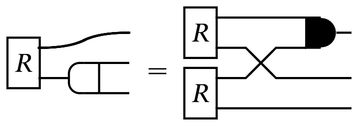
(i) is convolution-invertible, that is, there exists $R^\circ : I \to A \otimes A$ satisfying

$$\begin{aligned}(m \otimes m) \circ (A \otimes c_{A,A} \otimes A) \circ (R \otimes R^\circ) &= (m \otimes m) \circ (A \otimes c_{A,A} \otimes A) \circ (R^\circ \otimes R)\\ &= 1 \otimes 1;\end{aligned}$$

(ii) satisfies the following three equations:







The convolution-invertibility ensures the invertibility of the braid σ induced from R. These three graphically presented equations imply that σ is a morphism of modules, that σ is bilinear and that σ^{-1} is bilinear, respectively. A bialgebra equipped with a universal R-matrix is called a *quasi-triangular bialgebra*.

We next let A be a quasi-triangular Hopf algebra in a compact closed category $\mathscr{C}$ and suppose that **Mod**(A) is a ribbon category, that is, not just braided but also with a twist θ. We then have a morphism $v = \theta_A \circ 1 : I \to A$, which satisfies

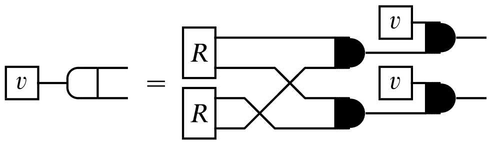
$$\theta_X \circ f = \alpha \circ (v \otimes X) \circ f$$

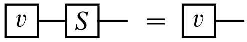
for a module $X = (X, \alpha)$ and a morphism $f : I \to X$ in $\mathscr{C}$. Thus, from this v we can recover θ_X as $\theta_X = \alpha \circ (v \otimes X)$ provided the global section functor $\mathscr{C}(I, -)$ is faithful. In such cases, we have a bijective correspondence between twists on **Mod**(A) and certain morphisms $v : I \to A$ satisfying a few axioms (Kassel 1995; Majid 1995; Turaev 1994). Such a v is called a *universal twist* or a *twist element*. Explicitly, a universal twist is a morphism $v : I \to A$ that:

(i) is convolution-invertible, that is, there exists $v^\circ : I \to A$ such that $m \circ (v \otimes v^\circ) = 1$;

(ii) is central $(m \circ (A \otimes v) = m \circ (v \otimes A))$; and

(iii) satisfies the following two equations:





The convolution-invertibility implies that θ induced from v is invertible, and centrality says that θ is a morphism of modules. The first equation amounts to the axiom for twists, and the second is required for the axiom $(\theta_X)^* = \theta_{X^*}$. A quasi-triangular Hopf algebra equipped with a universal twist is called a *ribbon Hopf algebra*. In summary, we have the following results.

Proposition 2.5 (Turaev 1994; Kassel 1995; Yetter 2001).

(1) If A is a quasi-triangular bialgebra in a symmetric monoidal category $\mathscr{C}$, then $\mathbf{Mod}(A)$ is a braided monoidal category.

(2) If A is a ribbon Hopf algebra in a compact closed category $\mathscr{C}$, then $\mathbf{Mod}(A)$ is a ribbon category.

Note that every co-commutative Hopf algebra is equipped with a universal R-matrix $R = 1 \otimes 1$ and a universal twist $v = 1$, giving rise to the symmetry and trivial twist on the category of modules. We will give a non-commutative non-co-commutative ribbon Hopf algebra in **Rel** in Section 4.

2.5. *Examples*

We shall now look at a few basic cases.

Example 2.6. As a classical example, consider the category $\mathbf{Vect}_k$ of vector spaces over a field k and linear maps. $\mathbf{Vect}_k$ is a symmetric monoidal category whose monoidal product is given by the tensor product of vector spaces, and k (the 1-dimensional space) serves as the tensor unit. Its full subcategory $\mathbf{Vect}_k^{\mathrm{fin}}$ of finite dimensional vector spaces is a compact closed category: for a finite dimensional V, its left (and right) dual is the dual vector space $V^* = \mathrm{hom}(V, k)$ of linear maps from V to k, with unit given by the dual basis and counit the evaluation map. A monoid in $\mathbf{Vect}_k$ is nothing but an algebra in the standard sense. Similarly, a comonoid in $\mathbf{Vect}_k$ is what is normally called a coalgebra. Modules, comodules, bialgebras and Hopf algebras in $\mathbf{Vect}_k$ and $\mathbf{Vect}_k^{\mathrm{fin}}$ are exactly those in the classical sense – a detailed account can be found in Kassel (1995).

Example 2.7. Let **Set** be the category of sets and functions. By taking finite products as tensor products, **Set** forms a symmetric monoidal category. A monoid in **Set** is just a monoid in the usual sense. For any set X, the diagonal map $X \to X \times X$ and the terminal map $X \to 1$ give a commutative comonoid structure on X – and this is the unique comonoid structure on X. Given a monoid M, its modules are just the M-sets, that is, sets on which M acts, and $\mathbf{Mod}(M)$ is isomorphic to the category M-**Set** of M-sets and functions respecting M-actions. For any set X, a comodule $(A, \alpha : A \to X \times A)$ of the unique comonoid $X = (X, \Delta, \epsilon)$ on X is determined by the function $\pi \circ \alpha : A \to X$, and $\mathbf{Comod}(X)$ is isomorphic to the slice category $\mathbf{Set}/X$. A bialgebra in **Set** is a monoid equipped with the unique comonoid structure. A Hopf algebra in **Set** is then a group G with the unique comonoid structure, where the antipode is given by the inverse $g \mapsto g^{-1} : G \to G$.

3. Bialgebras in Rel

We will now turn our attention to the category **Rel** of sets and binary relations. **Rel** is a compact closed (hence ribbon) category, where the tensor product of sets X and Y is given by the direct product $X \times Y$ of sets and the unit object is a singleton set $I = \{*\}$. For a set X, its left dual X^* is X itself, with unit and counit given by

$$\begin{aligned} \eta_X &= \{(*, (x, x)) \mid x \in X\} : I \to X \times X \\ \varepsilon_X &= \{((x, x), *) \mid x \in X\} : X \times X \to I. \end{aligned}$$

3.1. *Bialgebras and Hopf algebras inherited from* **Set**

The easiest cases of bialgebras and Hopf algebras in **Rel** are those arising from monoids and groups in **Set**, respectively. First, note that there is an identity-on-objects strict symmetric monoidal functor $J : \mathbf{Set} \to \mathbf{Rel}$ sending a set to itself and a function $f : X \to Y$ to a binary relation $\{(x, f(x)) \mid x \in X\}$ from X to Y. We also recall the following standard result.

Lemma 3.1. A strong symmetric monoidal functor preserves the structure of monoids, comonoids, bialgebras and Hopf algebras.

From this and Example 2.7, it follows that a monoid $M = (M, \cdot, e)$ (in **Set**) gives rise to a co-commutative bialgebra $\overline{M} = (M, m, 1, \Delta, \epsilon)$ in **Rel**, with

$$\begin{array}{rcllcl} m &=& J((a_1, a_2) \mapsto a_1 \cdot a_2) &=& \{((a_1, a_2), a_1 \cdot a_2) \mid a_1, a_2 \in M\} \\ 1 &=& J(* \mapsto e) &=& \{(*, e)\} \\ \Delta &=& J(a \mapsto (a, a)) &=& \{(a, (a, a)) \mid a \in M\} \\ \epsilon &=& J(a \mapsto *) &=& \{(a, *) \mid a \in M\}. \end{array}$$

$\overline{M}$ is commutative if M is commutative. Similarly, a group $G = (G, \cdot, e, (-)^{-1})$ gives rise to a co-commutative Hopf algebra

$$\overline{G} = (G, m, 1, \Delta, \epsilon, S)$$

in **Rel**, with an antipode

$$S = \{(g, g^{-1}) \mid g \in G\} : G \to G.$$

Let us examine the category $\mathbf{Mod}(\overline{G})$ for a group $G = (G, \cdot, e, (-)^{-1})$ – it makes sense to think about $\mathbf{Mod}(\overline{M})$ for a monoid M, but when M is not a group the description of $\mathbf{Mod}(\overline{M})$ can be rather complicated. A module of $\overline{G}$ is a set X equipped with a binary relation $\alpha : G \times X \to X$ subject to the two axioms given above. It is not hard to see that α is actually a function because, for each $g \in G$, the relation

$$\alpha \circ (g \times X) = \{(x, x') \mid ((g, x), x') \in \alpha\} : X \to X$$

is an isomorphism in **Rel** with inverse $\alpha \circ (g^{-1} \times X)$, and hence it is a bijective function. In fact, α is a G-action on X since for $g \in G$ and $x \in X$, by letting $g \bullet x$ be the unique $x' \in X$ such that $((g, x), x') \in \alpha$, we have $e \bullet x = x$ and

$$(g \cdot h) \bullet x = g \bullet (h \bullet x).$$

Therefore we can identify objects of $\mathbf{Mod}(\overline{G})$ with G-sets. A morphism from a G-set $(X, \bullet)$ to $(Y, \bullet)$ is then a binary relation $r : X \to Y$ such that $(x, y) \in r$ implies $(g \bullet x, g \bullet y) \in r$. Since $\overline{G}$ is a co-commutative Hopf algebra, $\mathbf{Mod}(\overline{G})$ is a compact closed category that is actually very similar to **Rel**. Explicitly, the tensor of $(X, \bullet)$ and $(Y, \bullet)$ is

$$(X \times Y, (g, (x, y)) \mapsto (g \bullet x, g \bullet y)),$$

while the tensor unit is $(\{*\}, (g, *) \mapsto *)$. A left dual of $(X, \bullet)$ is $(X, \bullet)$ itself.

We shall now look at $\mathbf{Comod}(\overline{M})$ for a monoid $M = (M, \cdot, e)$. A comodule of $\overline{M}$ is a set X with a binary relation $\alpha : X \to M \times X$ subject to the comodule axioms – but the axioms imply that α is a function whose second component is the identity on X. Hence an object of $\mathbf{Comod}(\overline{M})$ can be identified with a set X equipped with a function $|\_| : X \to M$. A morphism from $(X, |\_|)$ to $(Y, |\_|)$ is then a binary relation $r : X \to Y$ such that $(x, y) \in r$ implies $|x| = |y|$. $\mathbf{Comod}(\overline{M})$ is a monoidal category, with

$$(X, |\_|) \otimes (Y, |\_|) = (X \times Y, (x, y) \mapsto |x| \cdot |y|)$$
$$I = (\{*\}, * \mapsto e).$$

Note that the function $|\_|$ does not have to respect the monoid structure in any way: indeed, the only place where the monoid structure of M is used is in the definition of $\otimes$ and I.

Proposition 3.2.

(1) If M is a commutative monoid, $\mathbf{Comod}(\overline{M})$ is symmetric monoidal.
(2) If M is a left (respectively, right)-cancellable monoid, $\mathbf{Comod}(\overline{M})$ has a left (respectively, right) trace in the sense of Selinger (2011).
(3) If M is a commutative cancellable monoid, $\mathbf{Comod}(\overline{M})$ is a traced symmetric monoidal category.
(4) If G is a group, every object $(X, |\_|)$ of $\mathbf{Comod}(\overline{G})$ has a left dual $(X, |\_|^{-1})$, and $\mathbf{Comod}(\overline{G})$ is pivotal (Freyd and Yetter 1989).
(5) If G is an Abelian group, $\mathbf{Comod}(\overline{G})$ is a compact closed category.

This proposition makes a connection between structures on a monoid (commutativity, left/right cancellability, inverses) and the respective structures induced on the monoidal category of comodules (symmetry, left/right trace, pivotal structure).

Thus we can derive a number of monoidal categories with symmetry, duals, and trace as categories of (co)modules of (the associated bialgebra of) a monoid or a group. However, none of them have a non-symmetric braiding; we will give a Hopf algebra in **Rel** in Section 4 whose category of modules has a non-symmetric braiding and a non-trivial twist.

3.2. *Some constructions*

There are a number of ways of constructing bialgebras and Hopf algebras in **Rel** from the existing ones. Here we shall look at some basic constructions that make sense not only for **Rel** but also for general symmetric monoidal categories and compact closed categories.

Opposite bialgebras and Hopf algebras. Given a bialgebra

$$A = (A, m, 1, \Delta, \epsilon)$$

in a symmetric monoidal category, its *opposite bialgebra* is the bialgebra

$$A^{\mathrm{op}} = (A, m^{\mathrm{op}}, 1, \Delta, \epsilon)$$

where $m^{\mathrm{op}} = m \circ c_{A,A}$, that is,

If A is a Hopf algebra with invertible antipode S, then A^{op} is a Hopf algebra with antipode S^{-1}. The fact that S^{-1} is an antipode of A^{op} is an immediate consequence of Lemma 2.1.

This opposite construction makes sense in **Rel**. Concretely, given a bialgebra

$$A = (A, m, 1, \Delta, \epsilon)$$

in **Rel**, its *opposite bialgebra* is the bialgebra

$$A^{\mathrm{op}} = (A, m^{\mathrm{op}}, 1, \Delta, \epsilon)$$

where

$$m^{\mathrm{op}} = m \circ c_{A,A} = \{((x_2, x_1), y) \mid ((x_1, x_2), y) \in m\}.$$

If A is a Hopf algebra with invertible antipode S, then A^{op} is a Hopf algebra with antipode

$$S^{-1} = \{(y, x) \mid (x, y) \in S\}.$$

For a group G, the Hopf algebra $\overline{G}^{\mathrm{op}}$ is isomorphic to $\overline{G^{\mathrm{op}}}$ where G^{op} is the group obtained by reverting the multiplication of G.

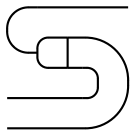
Dual bialgebras and Hopf algebras. Given a bialgebra

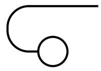
$$A = (A, m, 1, \Delta, \epsilon)$$

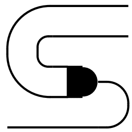
in a compact closed category, its *dual bialgebra* is the bialgebra

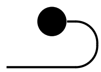
$$A^* = (A^*, \Delta^*, \epsilon^*, m^*, 1^*)$$

where†

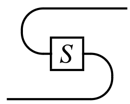
$\Delta^* =$ 

$\epsilon^* =$ 

$m^* =$ 

$1^* =$ 

If A is a Hopf algebra with antipode S, then A^* is a Hopf algebra with antipode

$S^* =$ 

In the case of **Rel**, for a bialgebra

$$A = (A, m, 1, \Delta, \epsilon),$$

its *dual bialgebra* is the bialgebra

$$A^* = (A, \Delta^*, \epsilon^*, m^*, 1^*)$$

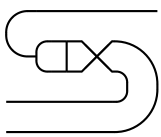
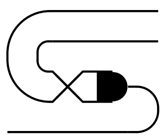
where

$$\begin{aligned}
\Delta^* &= \{((y_2, y_1), x) \mid (x, (y_1, y_2)) \in \Delta\} \\
\epsilon^* &= \{(*, x) \mid (x, *) \in \epsilon\} \\
m^* &= \{(y, (x_2, x_1)) \mid ((x_1, x_2), y) \in m\} \\
1^* &= \{(y, *) \mid (*, y) \in 1\}.
\end{aligned}$$

If A is a Hopf algebra with antipode S, then A^* is a Hopf algebra with antipode

$$S^* = \{(y, x) \mid (x, y) \in S\}.$$

Remark 3.3. Some other authors define the dual bialgebra (Hopf algebra) A^* as our $((A^{\mathrm{op}})^*)^{\mathrm{op}}$, whose multiplication and comultiplication are given by

 and 

† Strictly speaking, we need to include the isomorphisms $X^* \otimes Y^* \cong (Y \otimes X)^*$ and $I \cong I^*$ in some appropriate places in this definition.

See, for example, Kassel (1995) – our definition agrees with that of Majid (1994).

Tensor products. When

$$A_1 = (A_1, m_1, 1_1, \Delta_1, \epsilon_1)$$
$$A_2 = (A_2, m_2, 1_2, \Delta_2, \epsilon_2)$$

are bialgebras in a symmetric monoidal category, their *tensor product* is the bialgebra

$$A_1 \otimes A_2 = (A_1 \otimes A_2, m_{12}, 1_{12}, \Delta_{12}, \epsilon_{12})$$

where

$$\begin{aligned} m_{12} &= (m_1 \otimes m_2) \circ (A_1 \otimes c_{A_2,A_1} \otimes A_2) \\ 1_{12} &= 1_1 \otimes 1_2 \\ \Delta_{12} &= (A_1 \otimes c_{A_1,A_2} \otimes A_2) \circ (\Delta_1 \otimes \Delta_2) \\ \epsilon_{12} &= \epsilon_1 \otimes \epsilon_2. \end{aligned}$$

Note that the bialgebra $A_1 \otimes A_2$ is isomorphic to $A_2 \otimes A_1$, and $A_1^{\mathrm{op}} \otimes A_2^{\mathrm{op}}$ is isomorphic to $(A_1 \otimes A_2)^{\mathrm{op}}$. In the case of compact closed categories, the bialgebra $A_1^* \otimes A_2^*$ is isomorphic to $(A_2 \otimes A_1)^*$. When both A_1 and A_2 are Hopf algebras with antipodes S_1 and S_2, respectively, then $A_1 \otimes A_2$ is a Hopf algebra with antipode $S_{12} = S_1 \otimes S_2$. In **Rel**, they are

$$\begin{aligned} m_{12} &= \{(((x_1, x_2), (y_1, y_2)), (z_1, z_2)) \mid ((x_i, y_i), z_i) \in m_i\} \\ 1_{12} &= \{(*, (x_1, x_2)) \mid x_i \in 1_i\} \\ \Delta_{12} &= \{((x_1, x_2), ((y_1, y_2), (z_1, z_2))) \mid (x_i, (y_i, z_i)) \in \Delta_i\} \\ \epsilon_{12} &= \{((x_1, x_2), *) \mid (x_i, *) \in \epsilon_i\} \\ S_{12} &= \{((x_1, x_2), (y_1, y_2)) \mid (x_i, y_i) \in S_i\}. \end{aligned}$$

For groups G_1 and G_2, it is not hard to see that $\overline{G_1} \otimes \overline{G_2}$ is isomorphic to $\overline{G_1 \times G_2}$.

Using these constructions, we can construct non-commutative non-co-commutative bialgebras and Hopf algebras in **Rel**. For example, for a non-Abelian group G, we have $\overline{G} \otimes \overline{G}^*$ is a Hopf algebra that is neither commutative nor co-commutative. However, this Hopf algebra does not have an R-matrix – for which we need a more sophisticated construction, which is the topic of the next section.

Remark 3.4. Of course, there are lots of bialgebras and Hopf algebras in **Rel** which cannot be obtained by these constructions on $\overline{M}$s or $\overline{G}$s. (In fact, bialgebras derived in this way are isomorphic to $\overline{M_1} \otimes \overline{M_2}^*$ for some monoids M_1 and M_2.) For an easy example, let X be a set and $(MX, \oplus, 0)$ be the free commutative monoid on X, or, equivalently, let MX be the set of finite multisets of elements of X, $\oplus$ be the union of multisets and 0 be the empty multiset. Then there is a bialgebra

$$MX = (MX, m, 1, \Delta, \epsilon)$$

in **Rel** where

$$\begin{aligned} m &= \{((x_1, x_2), x_1 \oplus x_2) \mid x_i \in MX\} \\ 1 &= \{(*, 0)\} \\ \Delta &= \{(x_1 \oplus x_2, (x_1, x_2)) \mid x_i \in MX\} \\ \epsilon &= \{(0, *)\}. \end{aligned}$$

MX^{op} and MX^* are obviously isomorphic to MX.

Remark 3.5. At the time of writing this paper, we do not know if all Hopf algebras in **Rel** have an invertible antipode. Note that **Rel** does not have all equalisers, so the result in Takeuchi (1999) cannot be applied to **Rel**. On the other hand, it is not clear if the construction of a Hopf algebra with a non-invertible antipode in Takeuchi (1971) can be carried out in **Rel**.

4. A quantum double construction in Rel

In the previous section, we observed that every group $G = (G, \cdot, e, (-)^{-1})$ gives rise to a co-commutative Hopf algebra $\overline{G} = (G, m, 1, \Delta, \epsilon, S)$ in **Rel**. In this section we will obtain a quasi-triangular Hopf algebra by applying Drinfel'd's *quantum double construction* (Drinfel'd 1987; Majid 1990) to $\overline{G}$.

4.1. *Quantum double construction in compact and traced categories*

We shall use the quantum double construction given in terms of Hopf algebras in compact closed categories.

Proposition 4.1. (See Chen (2000), Kassel (1995) and Kassel and Turaev (1995).) Suppose $\mathscr{C}$ is a compact closed category and $A = (A, m, 1, \Delta, \epsilon, S)$ is a Hopf algebra in $\mathscr{C}$, where the antipode S is invertible. Then there exists a quasi-triangular Hopf algebra $D(A)$ on $A \otimes A^*$.

Before going into the technical details, we will first explain an outline of the construction and make some informal remarks. Given a Hopf algebra $A = (A, m, 1, \Delta, \epsilon, S)$ with S invertible, let

$$A^{\mathrm{op}*} = (A^*, \Delta^*, \epsilon^*, (m^{\mathrm{op}})^*, 1^*, (S^{-1})^*)$$

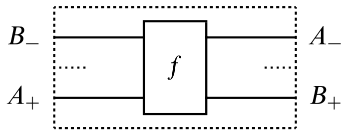
be the dual opposite Hopf algebra. There are suitable actions of A on $A^{\mathrm{op}*}$ and $A^{\mathrm{op}*}$ on A, and with them we can form a bicrossed product (Majid 1990; 1995) of A with $A^{\mathrm{op}*}$, which is the Hopf algebra $D(A)$. Note that $D(A)$ is almost like a tensor product of A and $A^{\mathrm{op}*}$ – apart from some clever adjustments to the multiplication and antipode. Also note that $\mathbf{Mod}(A^{\mathrm{op}*})$ is isomorphic to $\mathbf{Comod}(A)$ as a monoidal category, and $\mathbf{Mod}(D(A))$ can be regarded as a combination of $\mathbf{Mod}(A)$ and $\mathbf{Comod}(A)$, as we will soon see for the case of $\overline{G}$ in **Rel**.

Unfortunately, a direct description of $D(A)$ is rather complicated – see Chen (2000), for instance. Instead, we shall give an alternative, simpler description using the *Int-construction* of Joyal *et al.* (1996).

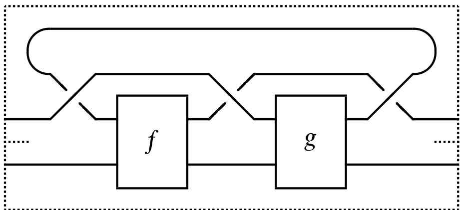
Recall that for a traced monoidal category $\mathscr{C}$, we can construct a ribbon category $\mathbf{Int}(\mathscr{C})$ whose objects are pairs of those of $\mathscr{C}$, and a morphism

$$f : (A_+, A_-) \to (B_+, B_-)$$

in $\mathbf{Int}(\mathscr{C})$ is a morphism from $A_+ \otimes B_-$ to $B_+ \otimes A_-$ in $\mathscr{C}$, which can be drawn as

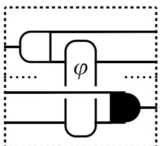


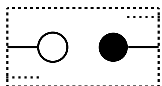
The composition of $f : (A_+, A_-) \to (B_+, B_-)$ and $g : (B_+, B_-) \to (C_+, C_-)$ is

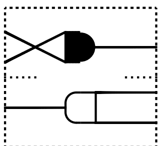


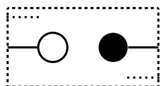
The tensor product of (A_+, A_-) and (B_+, B_-) is $(A_+ \otimes B_+, B_- \otimes A_-)$, while the unit object is (I, I) – see Joyal *et al.* (1996) and Hasegawa (2009) for further details of the structure of $\mathbf{Int}(\mathscr{C})$.

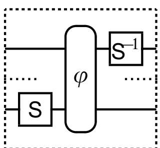
Proposition 4.2. For a Hopf algebra $A = (A, m, 1, \Delta, \epsilon, S)$ with an invertible antipode S in a traced symmetric monoidal category $\mathscr{C}$, there is a quasi-triangular Hopf algebra $((A, A), m^{\mathrm{d}}, 1^{\mathrm{d}}, \Delta^{\mathrm{d}}, \epsilon^{\mathrm{d}}, S^{\mathrm{d}})$ with a universal R-matrix R in $\mathbf{Int}(\mathscr{C})$ given as follows:

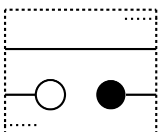
$m^{\mathrm{d}} : (A, A) \otimes (A, A) \to (A, A)$ 

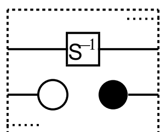
$1^{\mathrm{d}} : I \to (A, A)$ 

$\Delta^{\mathrm{d}} : (A, A) \to (A, A) \otimes (A, A)$ 

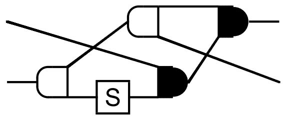
$\epsilon^{\mathrm{d}} : (A, A) \to I$ 

$S^{\mathrm{d}} : (A, A) \to (A, A)$ 

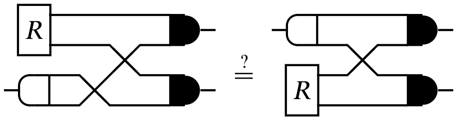
$R : I \to (A, A) \otimes (A, A)$ 

$R^{\circ} : I \to (A, A) \otimes (A, A)$ 

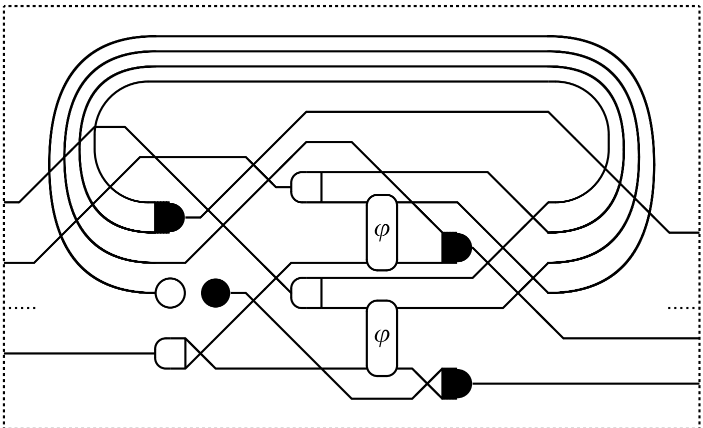
where $\varphi : A \otimes A \to A \otimes A$ is



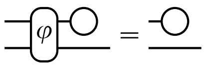
Proof (outline). We just need to check that all axioms of quasi-triangular Hopf algebras hold – this is perhaps best done by equational reasoning on the graphical presentations using sufficiently large sheets of paper. Checking the axioms that do not involve m^d and S^d is fairly straightforward since there is no interaction between the first (positive, lower) component and the second (negative, upper) component. Cases with m^d or S^d, and thus φ, do need some work. Here we shall just show one of the more complex cases in the form of the first axiom for universal R-matrices:



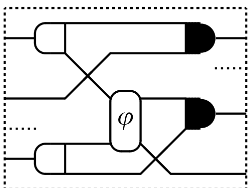
From the definition, the left-hand side of this axiom is equal to



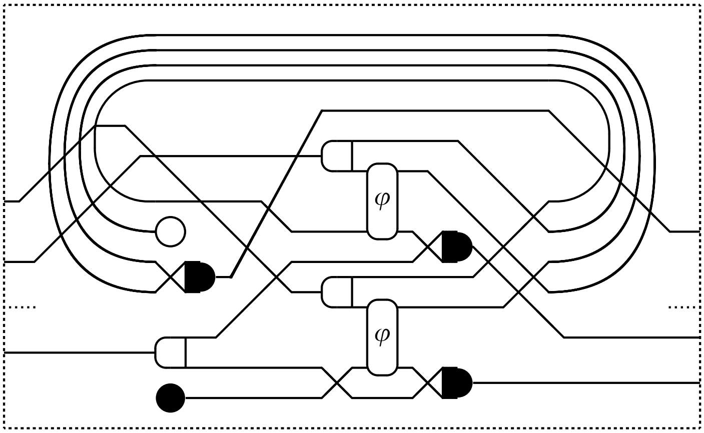
After some simplifications, using the easily derivable equation



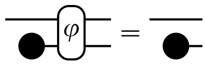
This is equal to



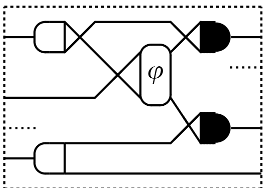
Similarly, the right-hand side is



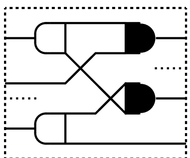
which, making use of the easy equation



turns out to be equal to



By expanding the definition of φ and using some further simplifications, both of these finally agree with



□

When $\mathscr{C}$ itself is a compact closed category, there is a strong symmetric monoidal equivalence $F : \mathbf{Int}(\mathscr{C}) \to \mathscr{C}$ sending (A_+, A_-) to $F(A_+, A_-) = A_+ \otimes A_-^*$, with the obvious isomorphism from

$$F(A_+, A_-) \otimes F(B_+, B_-) = A_+ \otimes A_-^* \otimes B_+ \otimes B_-^*$$

to

$$F((A_+, A_-) \otimes (B_+, B_-)) = A_+ \otimes B_+ \otimes (B_- \otimes A_-)^*.$$

Through this equivalence, this quasi-triangular Hopf algebra on (A, A) in $\mathbf{Int}(\mathscr{C})$ is sent to a quasi-triangular Hopf algebra on $A \otimes A^*$ in $\mathscr{C}$, as claimed in Proposition 4.1.

4.2. *Quantum double of* $\overline{G}$ *in* **Rel**

We will now turn our attention to the Hopf algebra $\overline{G}$ in **Rel**. Since the antipode S of $\overline{G}$ is invertible, we can apply the quantum double construction to $\overline{G}$ and obtain a quasi-triangular (in fact, ribbon) Hopf algebra $D(\overline{G})$.

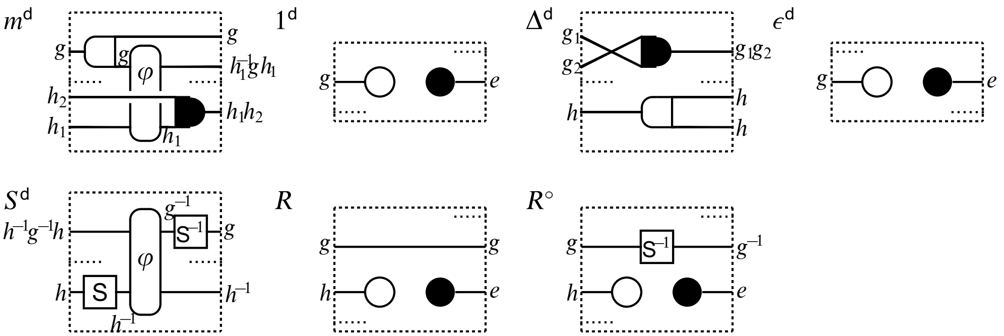
By Proposition 4.2, the quantum double of $\overline{G}$ in **Int**(**Rel**) is

$$((G,G), m^{\mathrm{d}}, 1^{\mathrm{d}}, \Delta^{\mathrm{d}}, \epsilon^{\mathrm{d}}, S^{\mathrm{d}}, R)$$

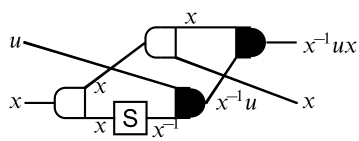
where

$$\begin{aligned}
m^{\mathrm{d}} &= \{(((h_1,h_2),g),(h_1h_2,(h_1^{-1}gh_1,g))) \mid g,h_1,h_2 \in G\} \\
1^{\mathrm{d}} &= \{((*,g),(e,*)) \mid g \in G\} \\
\Delta^{\mathrm{d}} &= \{((h,(g_2,g_1)),((h,h),g_1g_2)) \mid g_1,g_2,h \in G\} \\
\epsilon^{\mathrm{d}} &= \{((g,*),(*,e)) \mid g \in G\} \\
S^{\mathrm{d}} &= \{((h,h^{-1}g^{-1}h),(h^{-1},g)) \mid g,h \in G\} \\
R &= \{((*,(h,g)),((e,g),*)) \mid g,h \in G\} \\
R^{\circ} &= \{((*,(h,g^{-1})),((e,g),*)) \mid g,h \in G\}
\end{aligned}$$

Graphically:



where

$\varphi =$ 

Moreover, $\overline{G}$ has a universal twist

$$\{((*,g),(g,*)) \mid g \in G\} : I \to (G,G),$$

so it is a ribbon Hopf algebra. We can then obtain the following theorem through the strong symmetric monoidal equivalence from **Int**(**Rel**) to **Rel**.

Theorem 4.3. Suppose $G = (G, \cdot, e, (-)^{-1})$ is a group. There is a ribbon Hopf algebra

$$D(\overline{G}) = (G \times G, m^{\mathrm{d}}, 1^{\mathrm{d}}, \Delta^{\mathrm{d}}, \epsilon^{\mathrm{d}}, S^{\mathrm{d}}, R, v)$$

in **Rel**, with

$$\begin{aligned}
m^{\mathrm{d}} &= \{(((g,h_1),(h_1^{-1}gh_1,h_2)),(g,h_1h_2)) \mid g,h_1,h_2 \in G\}\\
1^{\mathrm{d}} &= \{(*,(g,e)) \mid g \in G\}\\
\Delta^{\mathrm{d}} &= \{((g_1g_2,h),((g_1,h),(g_2,h)) \mid g_1,g_2,h \in G\}\\
\epsilon^{\mathrm{d}} &= \{((e,g),*) \mid g \in G\}\\
S^{\mathrm{d}} &= \{((g,h),(h^{-1}g^{-1}h,h^{-1})) \mid g,h \in G\}\\
R &= \{(*,((g,e),(h,g))) \mid g,h \in G\}\\
v &= \{(*,(g,g)) \mid g \in G\}
\end{aligned}$$

where R is the universal R-matrix and v is the universal twist.

When G is not Abelian, $D(\overline{G})$ is neither commutative nor co-commutative. In the next section, we shall see that modules of $D(\overline{G})$ can be identified with the *crossed G-sets* (Freyd and Yetter 1989; Whitehead 1949).

5. A ribbon category of crossed G-sets

5.1. *Crossed G-sets*

Let $G = (G, \cdot, e, (-)^{-1})$ be a group. A *crossed G-set* $X = (X, \bullet, |\_|)$ is given by a set X together with a group action $\bullet : G \times X \to X$ and a function $|\_|$ from X to G such that, for any $g \in G$ and $x \in X$, we have $|g \bullet x| = g \cdot |x| \cdot g^{-1}$. For instance, G itself can be seen to be a crossed G-set with $g \bullet h = g \cdot h \cdot g^{-1}$ and $|h| = h$. Another trivial example is a G-set with $|x| = e$.

Proposition 5.1. For any set X, there is a bijective correspondence between $D(\overline{G})$-modules on X and crossed G-sets on X.

Proof. If $\alpha : G \times G \times X \to X$ is a $D(\overline{G})$-module, for any $g \in G$ and $x \in X$, there are unique $h \in G$ and $y \in X$ such that $(((h,g),x),y) \in \alpha$, and X carries the structure of a crossed G-set where $g \bullet x$ is this uniquely determined y and $|x|$ is the unique h such that $(((h,e),x),x) \in \alpha$. Conversely, a crossed G-set $(X, \bullet, |\_|)$ gives rise to a module

$$\{(((|g \bullet x|, g), x), g \bullet x) \mid g \in G, x \in X\} : G \times G \times X \to X.$$

It is not hard to see that this is a bijective correspondence. □

A morphism of crossed G-sets from $(X, \bullet, |\_|)$ to $(Y, \bullet, |\_|)$, corresponding to the morphism of $D(\overline{G})$-modules, is a binary relation $r : X \to Y$ such that $(x,y) \in r$ implies $(g \bullet x, g \bullet y) \in r$ as well as $|x| = |y|$. The identity and composition of morphisms are just the same as those for binary relations. We will use **XRel**(G) to denote the category of crossed G-sets and morphisms, which is isomorphic to **Mod**$(D(\overline{G}))$. Note that the category G-$\mathbb{XS}\mathbf{f}$ of crossed G-sets due to Freyd and Yetter (Freyd and Yetter 1989) is the subcategory of **XRel**(G) for which the morphisms are restricted to functions and the objects are finite. A variant of **XRel**(G) where G is not a group but a commutative monoid has appeared in Abramsky *et al.* (1999).

For any set X, the free crossed G-set over X is given by

$$\mathscr{F}(X) = (G \times G \times X, \bullet, |\_|)$$

with

$$g \bullet (h_1, h_2, x) = (g \cdot h_1 \cdot g^{-1}, g \cdot h_2, x)$$

and

$$|(h_1, h_2, x)| = h_1.$$

$\mathscr{F}$ extends to a functor from **Rel** to $\mathbf{XRel}(G)$ which is left adjoint to the forgetful functor $\mathscr{U} : \mathbf{XRel}(G) \to \mathbf{Rel}$ that sends $(X, \bullet, |\_|)$ to X.

5.2. *The ribbon structure on* $\mathbf{XRel}(G)$

By Proposition 2.5, $\mathbf{Mod}(D(\overline{G}))$, and thus $\mathbf{XRel}(G)$, is a ribbon category. In $\mathbf{XRel}(G)$, the tensor unit is $I = (\{*\}, (g, *) \mapsto *, * \mapsto e)$, and the tensor product of $X = (X, \bullet, |\_|)$ and $Y = (Y, \bullet, |\_|)$ is

$$X \otimes Y = (X \times Y, (g, (x, y)) \mapsto (g \bullet x, g \bullet y), (x, y) \mapsto |x| \cdot |y|).$$

The tensor product of morphisms, as well as the coherence isomorphisms a, l and r, are inherited from **Rel**. For this monoidal structure we have a braiding $\sigma_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$ induced by the universal R-matrix R given by

$$\sigma_{X,Y} = \{((x, y), (|x| \bullet y, x)) \mid x \in X, y \in Y\}.$$

There is also a twist $\theta_X : X \xrightarrow{\cong} X$ induced by the universal twist v given by

$$\theta_X = \{(x, |x| \bullet x) \mid x \in X\}.$$

For a crossed G-set $X = (X, \bullet, |\_|)$, its left dual is $X^* = (X, \bullet, |\_|^{-1})$, with unit

$$\eta_X = \{(*, (x, x)) \mid x \in X\} : I \to X \otimes X^*$$

and counit

$$\varepsilon_X = \{((x, x), *) \mid x \in X\} : X^* \otimes X \to I.$$

Note that the canonical trace on $\mathbf{XRel}(G)$ is just given like that on **Rel**, so for

$$f : A \otimes X \to B \otimes X,$$

the trace $Tr^X_{A,B} f : A \to B$ is

$$Tr^X_{A,B} f = \{(a, b) \in A \times B \mid \exists x \in X((a, x), (b, x)) \in f\}.$$

5.3. *Interpreting tangles in* $\mathbf{XRel}(G)$

Since the category of (oriented, framed) tangles is equivalent to the ribbon category freely generated by a single object (Shum 1994), by specifying a ribbon category and an object, we always obtain a structure-preserving functor from the category of tangles to the ribbon category, which determines an invariant of tangles (Yetter 2001). This is also the case for **XRel**(G).

In order to understand how a crossed G-set gives rise to an invariant of tangles, it is helpful to consider the *rack* (Fenn and Rourke 1992) associated with the crossed G-set†. Given a crossed G-set $(X, \bullet, |\_|)$, we define operators $\triangleright, \triangleright^{-1} : X \times X \to X$ by

$$x \triangleright y = |y| \bullet x$$
$$x \triangleright^{-1} y = |y|^{-1} \bullet x.$$

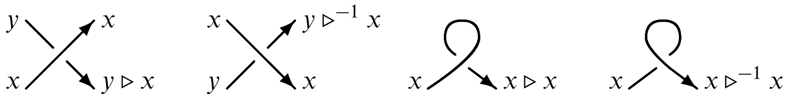
Then $(X, \triangleright, \triangleright^{-1})$ forms a rack: that is, the following equations hold‡:

$$(x \triangleright y) \triangleright^{-1} y = x = x \triangleright^{-1} y) \triangleright y \qquad \text{(bijectivity of } (-) \triangleright y)$$
$$(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z). \qquad \text{(self-distributivity)}$$

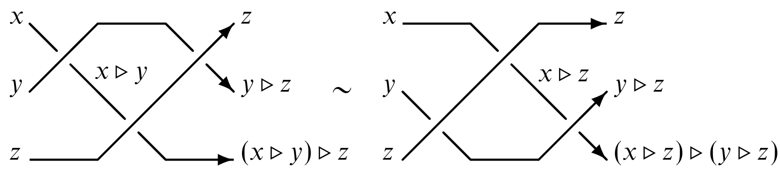
We can now describe braiding and twist in terms of this rack:

$$\sigma_{X,Y} = \{((x, y), (y \triangleright x, x)) \mid x \in X, y \in Y\}$$
$$\theta_X = \{(x, x \triangleright x) \mid x \in X\}.$$

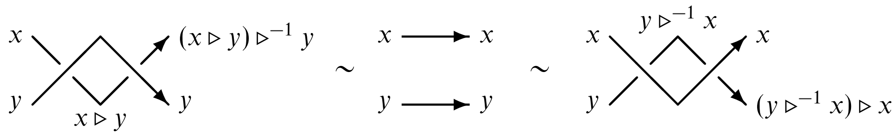
The interpretation of a tangle diagram in **XRel**(G) with a crossed G-set X is then determined by all possible X-labellings of the segments from an underpass to the next underpass satisfying 'y under x from the left gives $y \triangleright x$' and 'y under x from the right gives $y \triangleright^{-1} x$':



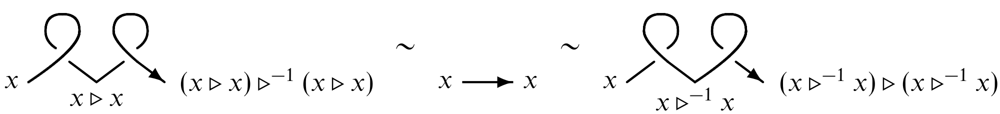
For instance, the self-distributivity justifies the Reidemeister move III:



Similarly, the Reidemeister move II is justified by the bijectivity:



The framed version of the Reidemeister move I is also justified by the self-distributivity and bijectivity:



† Indeed, another name for crossed G-sets is *augmented racks*, which was coined by Fenn and Rourke, who showed that every rack arises from an augmented rack, and hence a crossed G-set.

‡ However, this does not have to be a *quandle* in the sense of Joyce (1982), since the idempotency $x \triangleright x = x$ does not hold in general.

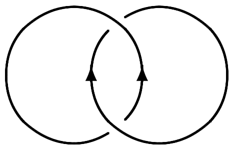
In the above, the equation $(x \triangleright x) \triangleright^{-1} (x \triangleright x) = x$ is derivable as

$$\begin{aligned}
(x \triangleright x) \triangleright^{-1} (x \triangleright x) &= (((x \triangleright^{-1} x) \triangleright x) \triangleright x) \triangleright^{-1} (x \triangleright x) && \text{(bijectivity)}\\
&= (((x \triangleright^{-1} x) \triangleright x) \triangleright (x \triangleright x)) \triangleright^{-1} (x \triangleright x) && \text{(self-distributivity)}\\
&= (x \triangleright^{-1} x) \triangleright x && \text{(bijectivity)}\\
&= x. && \text{(bijectivity)}
\end{aligned}$$

The equation $(x \triangleright^{-1} x) \triangleright (x \triangleright^{-1} x) = x$ also follows from a similar reasoning – consider

$$(((x \triangleright^{-1} x) \triangleright (x \triangleright^{-1} x)) \triangleright x) \triangleright^{-1} x.$$

Example 5.2. Consider the following link:



Its interpretation in $\mathbf{XRel}(G)$ with a crossed G-set X takes a value in

$$\mathbf{XRel}(G)(I, I) = \{id_I, \varnothing\},$$

and it is the identity relation id_I if there exist $x, y \in X$ such that

$$x = x \triangleright y$$
$$y = y \triangleright x$$

hold; otherwise it is the empty relation $\varnothing$.

These invariants are far from complete. For example, the links

  and 

always have the same interpretation for any crossed G-set.

6. A model of braided linear logic

In this section we outline the notion of models of (fragments of) braided linear logic, and see how $\mathbf{XRel}(G)$ in the previous section gives such a model. For a detailed exposition on categorical models of linear logic, see Melliès (2009). Some considerations on the proof theory of braided linear logic can be found in Bellin and Fleury (1998).

6.1. *Models of braided linear logic*

By a model of braided multiplicative linear logic (braided MLL), we mean a braided $*$-autonomous category (Barr 1995); note that a ribbon category is braided $*$-autonomous, and is thus a model of braided MLL. A model of braided multiplicative additive linear logic (braided MALL) is then a braided $*$-autonomous category with finite products.

For the exponential, we employ the following generalisation of the notion of linear exponential comonads (Hyland and Schalk 2003) on symmetric monoidal categories: by a linear exponential comonad on a braided monoidal category, we mean a braided monoidal comonad whose category of coalgebras is a category of commutative comonoids. A model of braided MELL is then a braided $*$-autonomous category with a linear exponential comonad. (An implication of this definition is that braiding becomes symmetry on exponential objects: $\sigma^{-1}_{!X,!Y} = \sigma_{!Y,!X}$.) A model of braided LL is a model of braided MALL with a linear exponential comonad (or a model of MELL with finite products).

6.2. **XRel**(G) *as a model of braided linear logic*

XRel(G) is a ribbon category with finite products, and is thus a model of braided MALL.

There is a strict balanced monoidal functor $F : \mathbf{Rel} \to \mathbf{XRel}(G)$ that sends a set X to

$$FX = (X,\ (g,x) \mapsto x,\ x \mapsto e).$$

F has a right adjoint $U : \mathbf{XRel}(G) \to \mathbf{Rel}$ that sends $X = (X, \bullet, |\_|)$ to

$$UX = \{x \in X \mid |x| = e\}/_{\sim}$$

where $x \sim y$ if and only if $g \bullet x = y$ for some g. By composing F and U with a linear exponential comonad ! on **Rel** (for example, the finite multiset comonad), we obtain a linear exponential comonad $F!U$ on **XRel**(G) whose category of coalgebras is equivalent to that of !. Hence, **XRel**(G) is a model of braided LL.

As a result, there exists a linear fixed-point operator on **XRel**(G) derived from the trace and the linear exponential comonad on **XRel**(G) as given in Hasegawa (2009), which can be used for interpreting a linear fixed-point combinator $\mathsf{Y}_X :!(X \multimap X) \multimap X$. (In Hasegawa (2009), we constructed such a linear fixed-point operator on traced *symmetric* monoidal categories with a linear exponential comonad. While the braiding of **XRel**(G) is not symmetric, the construction given there works without any change, essentially because braiding becomes symmetry on exponential objects, as noted above.)

XRel(G) is degenerate as a model of LL in the sense that it cannot distinguish tensor from par. As an easy remedy, one may apply the simple self-dualisation construction (Hyland and Schalk 2003) to obtain a 'non-compact' model. For a braided monoidal closed category $\mathscr{C}$ with finite products, there is a braided $*$-autonomous structure on $\mathscr{C} \times \mathscr{C}^{\mathrm{op}}$ whose tensor unit is $(I, 1)$ (where 1 is a terminal object and should not be confused with the unit element of a monoid) and tensor product is given by

$$(U, X) \otimes (V, Y) = (U \otimes V, U \multimap Y \times V \multimap X),$$

while the duality is given by $(U, X)^{\perp} = (X, U)$. By applying the simple self-dualisation construction to **XRel**(G), we obtain a 'non-compact' model **XRel**$(G)\times$**XRel**$(G)^{\mathrm{op}}$ of braided LL. Alternatively, **XRel**$(G) \times$ **XRel**$(G)^{\mathrm{op}}$ arises as the category of modules of $D(\overline{G})$ (or $(D(\overline{G}), \varnothing)$ to be more precise) in the $*$-autonomous category **Rel** $\times$ **Rel**$^{\mathrm{op}}$ obtained by the simple self-dualisation on **Rel**.

7. Concluding remarks

We have demonstrated that there are many non-trivial Hopf algebras in the category of sets and binary relations. In particular, by applying the quantum double construction, we have constructed a non-commutative non-co-commutative Hopf algebra with a universal R-matrix and a universal twist, and the ribbon category of its modules turns out to be a category of crossed G-sets.

Technically, most of our results are variations or instances of the already established theory of quantum groups, and we do not claim much novelty in this regard. What is much more important in this work, we believe, is that our results show that it is indeed possible to carry out a substantial part of quantum group theory in a category used for the semantics of computation and logic. Although we have spelled out just a particular case of **Rel**, we expect that the same can be done meaningfully in various other settings, including:

- the $*$-autonomous category of coherent spaces and linear maps (Girard 1987), and its variations used as models of linear logic;
- various categories of games, in particular the compact closed category of Conway games (Joyal 1977; Melliès 2004); and
- the category of sets (or presheaves on discrete categories) and linear normal functors (Hasegawa 2002), as well as the bicategory of small categories and profunctors.

The first two would lead to models of braided linear logic and some braided variants of game semantics. The third should be a direct refinement of our work on **Rel**, in that we replace binary relations $X \times Y \to 2$ with **Set**-valued functors $X \times Y \to \mathbf{Set}$ (which amount to linear normal functors from $\mathbf{Set}^X$ to $\mathbf{Set}^Y$).

Finally, we must admit that the computational significance of braided monoidal structure is yet to be examined. As far as we know, $\mathbf{XRel}(G)$ is the first non-symmetric ribbon category featuring a linear exponential comonad, which allows non-trivial interpretations of braidings as well as recursive programs at the same time. If we are to develop a sort of braided variant of denotational semantics in future, $\mathbf{XRel}(G)$ might be a good starting point. A potentially related direction would be the area of *topological quantum computation* (Freedman *et al.* 2002; Kitaev 2003; Wang 2010; Panangaden and Paquette 2011), in which modular tensor categories† (Turaev 1994; Bakalov and Kirilov 2001) play the central role. Although $\mathbf{XRel}(G)$ is not modular, it might be possible to develop a toy (and suitably simplified) model of topological quantum computation in it.

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† In other words, semisimple ribbon categories with finite simple objects satisfying an extra condition

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