On Traced Monoidal *Closed* Categories

Taking Higher-Order Spaghetti Seriously

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Thanks to: Paul-André Melliès, Shin-ya Katsumata

TMCNAA, 15 July 2007

Introduction (1/3)

We often talk about *traced monoidal categories* and *compact closed categories* (more generally *tortile monoidal categories*) cf. other talks of this workshop.

However, it seems that we have not paid much attention to traced monoidal *closed* categories...

Google search (14 July 2007) for

`"traced monoidal categories"`: 656 results

`"compact closed categories"`: 722 results

`"traced monoidal closed categories"`: 4 results (all related to *this* talk)

There are a few "reasons".

Introduction (2/3)

There are a few “reasons”:

1. lack of good examples: many important traced monoidal closed categories known so far are either
   (a) cartesian closed categories with fixed-point operator (via the trace-fixpoint correspondence), or
   (b) compact closed/tortile.
2. lack of motivations: trace and closedness seem to be rather orthogonal concepts, and can be discussed separatedly.
3. another reason: *Int-construction*. In any case one may obtain compact closed/trotile categories from traced categories for free via the Int-construction.

Introduction (3/3)

In this talk I will explain some observations on traced monoidal *closed* categories, and try to convince you that they are in fact interesting.

We will revisit the notions of traced monoidal categories and Int-construction, and give a (hopefully new) characterisation of traced monoidal *closed* categories as monoidal co-reflexive full subcategories of tortile monoidal categories.

Applications to models of linear logic, models of recursive computation, and program transformations will then be presented.

Caution: Whenever possible, I will talk about
{traced *balanced (braided)* monoidal categories, tortile monoidal categories}
rather than just about
{traced *symmetric* monoidal categories, compact closed categories}
following the original development by Joyal, Street and Verity.

Part I: Traced Monoidal Categories

Preliminaries: Monoidal Categories

A *monoidal category* (*tensor category*) $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ consists of a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $I \in \mathcal{C}$ and natural isomorphisms $a_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$, $l_A : I \otimes A \xrightarrow{\sim} A$ and $r_A : A \otimes I \xrightarrow{\sim} A$ such that the following two diagrams commute:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "a"] \arrow[d, "a \otimes D"'] & (A \otimes B) \otimes (C \otimes D) \arrow[dd, "a"] \\
(A \otimes (B \otimes C)) \otimes D \arrow[d, "a"'] & \\
A \otimes ((B \otimes C) \otimes D) \arrow[r, "A \otimes a"'] & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "a"] \arrow[dr, "r \otimes B"'] & & A \otimes (I \otimes B) \arrow[dl, "A \otimes l"] \\
& A \otimes B &
\end{tikzcd}$$

(In practice, one may forget a, ℓ, r, and identify $(A \otimes B) \otimes C$ with $A \otimes (B \otimes C)$ etc — thanks to the coherence theorem.)

Preliminaries: Braidings, Symmetries and Twists

A *braiding* is a natural isomorphism $c_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$ such that both c and c^{-1} satisfy the "bilinearity" diagrams (the case for c^{-1} is omitted):

$$
\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "a"] \arrow[d, "c \otimes C"'] & A \otimes (B \otimes C) \arrow[r, "c"] & (B \otimes C) \otimes A \arrow[d, "a"] \\
(B \otimes A) \otimes C \arrow[r, "a"'] & B \otimes (A \otimes C) \arrow[r, "B \otimes c"'] & B \otimes (C \otimes A)
\end{tikzcd}
$$

A *symmetry* is a braiding such that $c_{A,B} = c^{-1}_{B,A}$.

A *braided/symmetric monoidal category* is a monoidal category equipped with a braiding/symmetry.

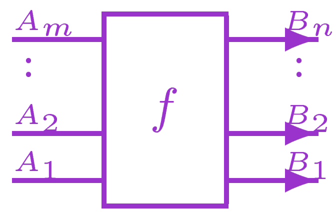
A *twist* or a *balance* for a braided monoidal category is a natural isomorphism $\theta_A : A \xrightarrow{\sim} A$ such that $\theta_I = id_I$ and $\theta_{A \otimes B} = c_{B,A} \circ (\theta_B \otimes \theta_A) \circ c_{A,B}$ hold.

A *balanced monoidal category* is a braided monoidal category with a twist.

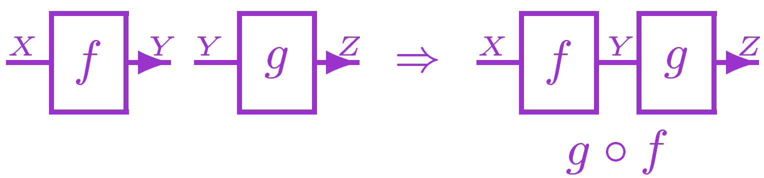
Geometry of Monoidal Categories

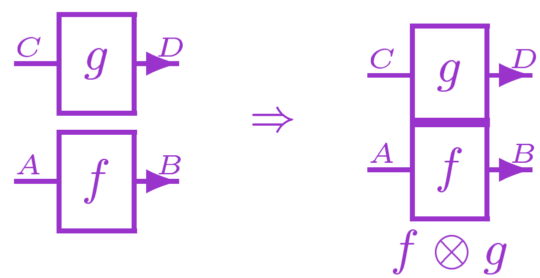
(Geometry of tensor calculus, Joyal and Street / string diagrams, Penrose)

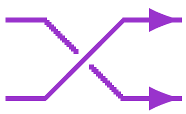
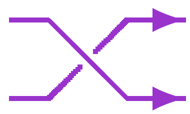
A morphism $f : A_1 \otimes A_2 \otimes \ldots \otimes A_m \to B_1 \otimes B_2 \otimes \ldots \otimes B_n$ can be drawn as:



Morphisms can be composed, either sequentially or in parallel:





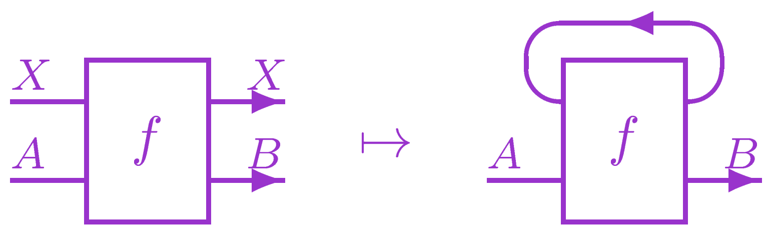
Braids: $c =$  $c^{-1} =$  Twists: $\theta =$  $\theta^{-1} =$ 

Traced Monoidal Categories (Joyal, Street and Verity, 1996)

Categorical structure for cyclic structures like knots and cyclic graphs

A *traced monoidal category* is a balanced monoidal category $\mathcal{C}$ equipped with a family of functions, called *trace* operator

$$Tr^X_{A,B} : \mathcal{C}(A \otimes X, B \otimes X) \longrightarrow \mathcal{C}(A, B)$$



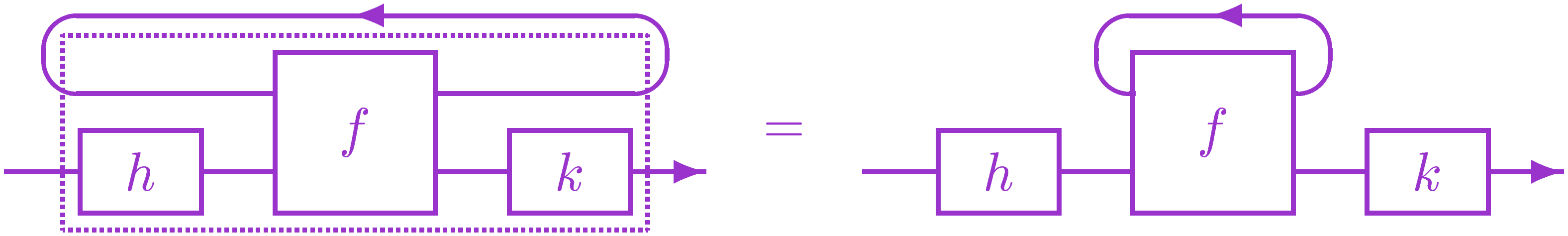
subject to a few coherence axioms.

Axioms for Trace (1/2)

(slightly simpler than the original version)

Tightening (Naturality)

$$Tr^X_{A',B'}((k \otimes id_X) \circ f \circ (h \otimes id_X)) = k \circ Tr^X_{A,B}(f) \circ h$$



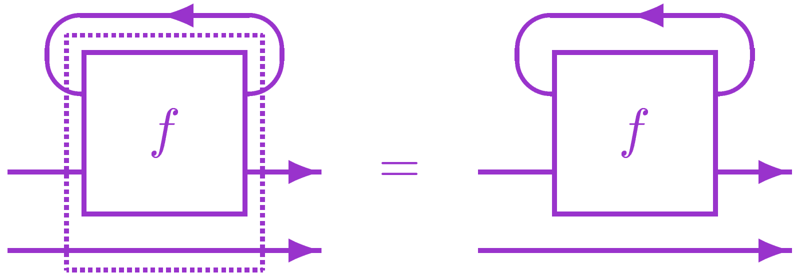
Yanking

$$Tr^X_{X,X}(c_{X,X}) \circ \theta_X^{-1} = id_X = Tr^X_{X,X}(c^{-1}_{X,X}) \circ \theta_X$$

Axioms for Trace (2/2)

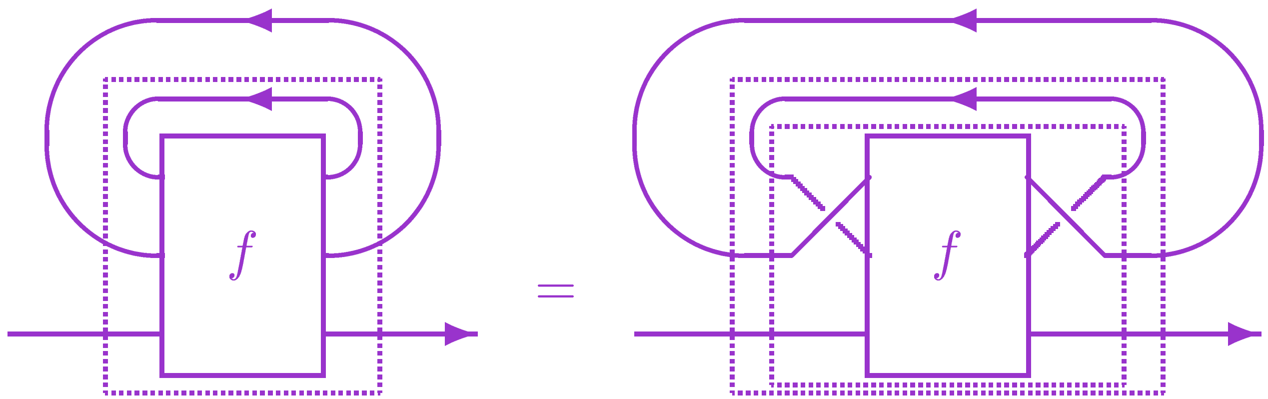
Superposing

$$Tr^{X}_{C\otimes A, C\otimes B}(id_C \otimes f) = id_C \otimes Tr^{X}_{A,B}(f)$$



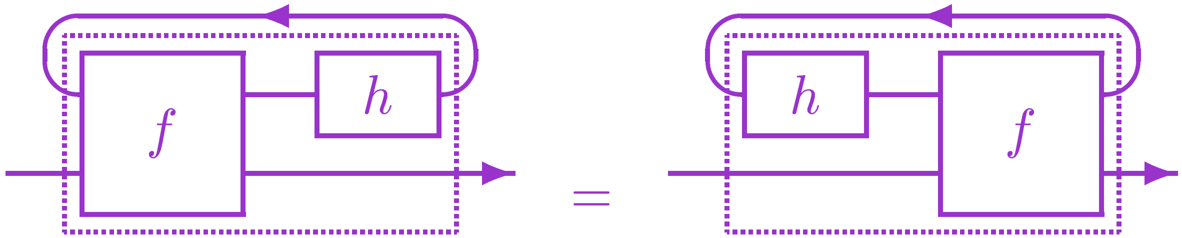
Exchange

$$Tr^{X}_{A,B}(Tr^{Y}_{A\otimes X, B\otimes X}(f)) =$$
$$Tr^{Y}_{A,B}(Tr^{X}_{A\otimes Y, B\otimes Y}((id_B \otimes c_{Y,X}) \circ f \circ (id_A \otimes c^{-1}_{Y,X})))$$

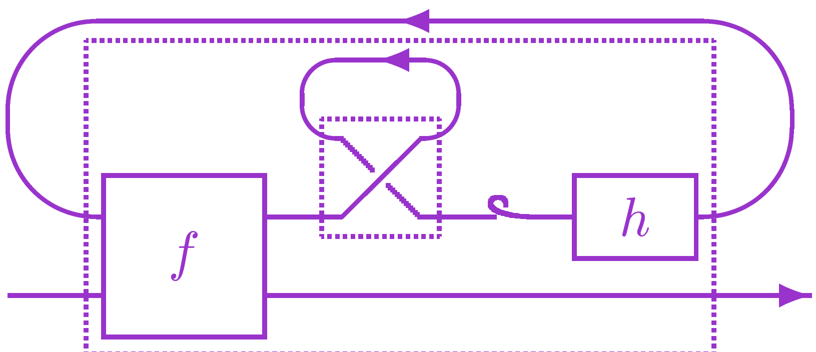


Exercise: Sliding (Dinaturality)

Proposition. The following equation is derivable.



Proof: By *Yanking*, LHS is equal to



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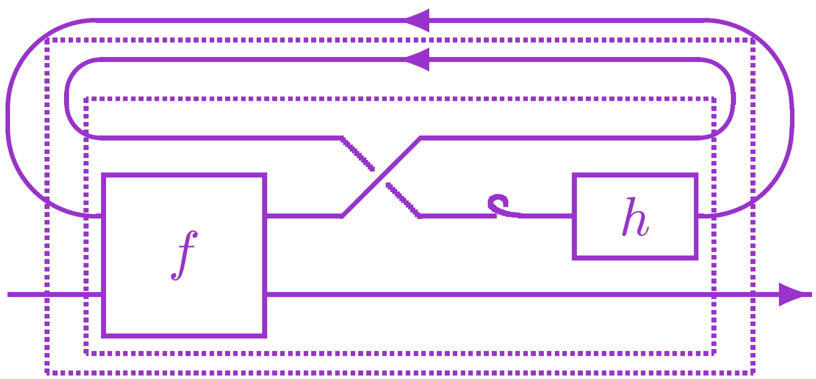
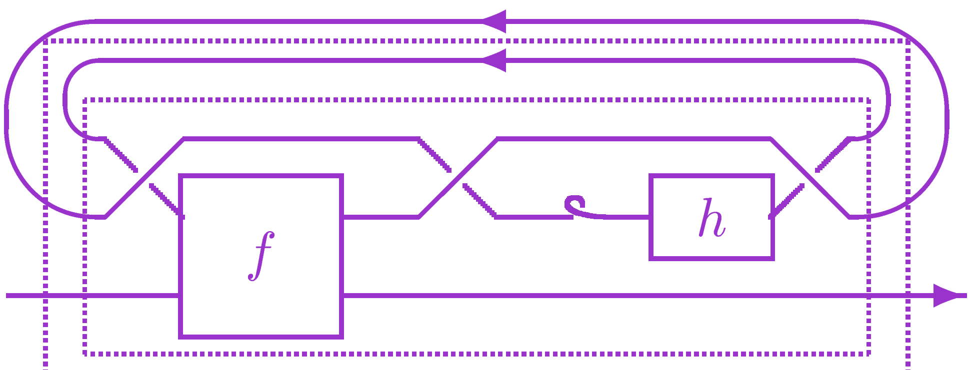
Using *Superposing* and *Tightening*, we have



Then we apply *Exchange*:



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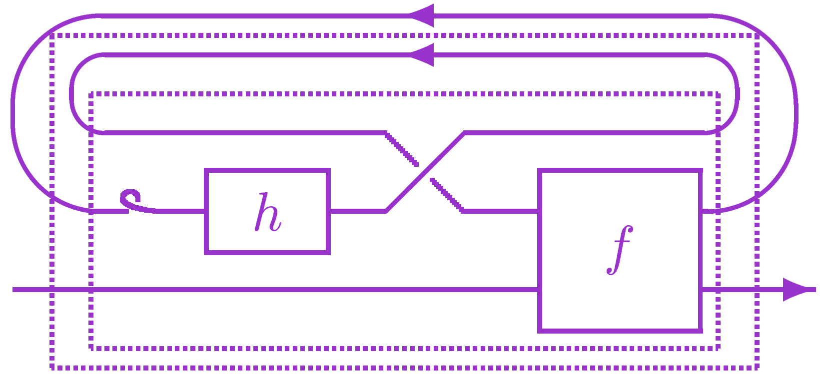
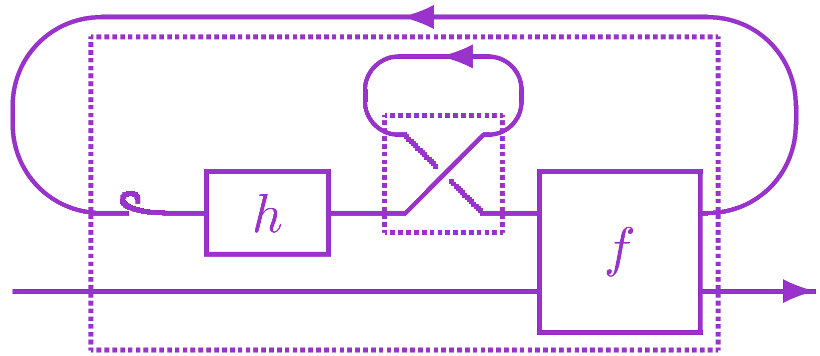
Thanks to the naturality of braidings, this is equal to the following.



Applying *Tightening* and *Superposing*, we obtain



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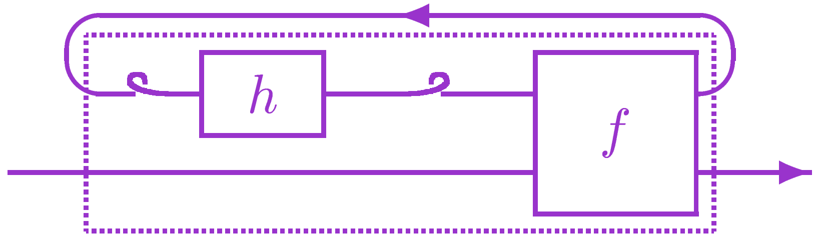
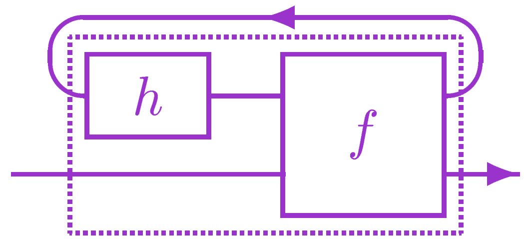
By *Yanking*



Now we have done, by the naturality of twisting.



(QED)

"Vanishing for tensor" is also derivable by a similar argument.
("Vanishing for unit" is redundant in the original axiomatisation.)
Cf. similar axioms used by Milner / Blute, Cockett and Seely

Examples of Traced Monoidal Categories

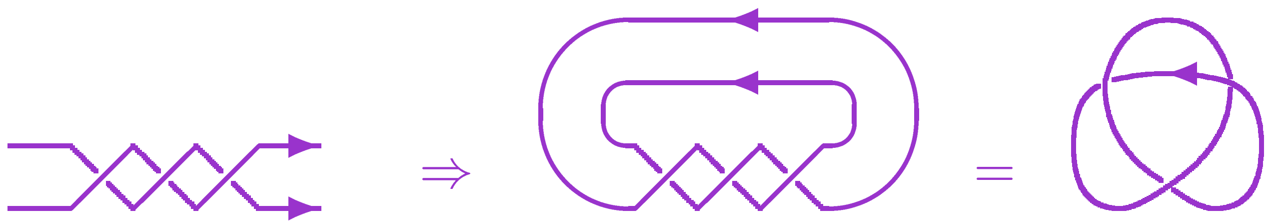
Linear Algebra (the classical example)

The category of fin. dim. vector spaces and linear maps. For a linear map $f : U \otimes_K W \to V \otimes_K W$, its trace $Tr^W_{U,V}(f) : U \to V$ is given by

$$(Tr^W_{U,V}(f))_{i,j} = \Sigma_k f_{i\otimes k, j\otimes k}$$

Quantum Invariants of Knots

The category of representations of a quasi-triangular Hopf algebra, which gives rise to the knot (ribbon) invariants.

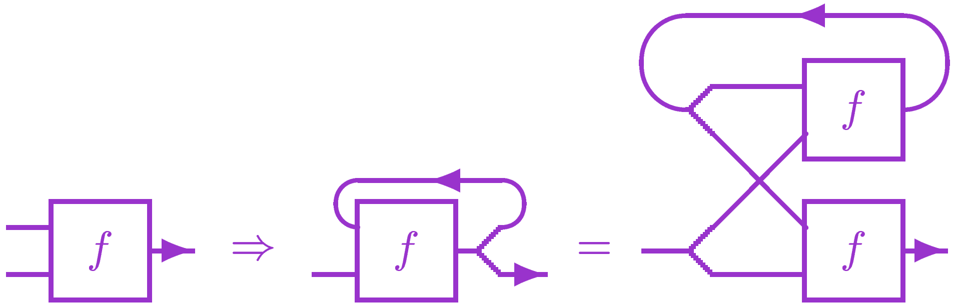


Various interesting knot invariants arise in this way.

Examples of Traced Monoidal Categories (cont.)

Trace-Fixpoint Correspondence (Hyland/Hasegawa)[a]

A category with finite products (products taken as the monoidal structure) is traced if and only if it has a fixed-point operator satisfying the dinaturality and diagonal property, or the Bekic property (Conway fixed-point operator in the sense of Bloom and Ésik).



Thus many categories in denotational/algebraic semantics are traced.

For example, the category Cppo of pointed cpo's and continuous functions is traced, where trace determined by the least fixed-points.

[a] Cf. mathematically equivalent observations by Bloom and Ésik / Ştefănescu

Part II: The Int-Construction

Tortile Monoidal Categories

A *tortile monoidal category*[a] is a balanced monoidal category with an A^* for each object A, unit $I \to A \otimes A^*$ and counit $A^* \otimes A \to I$ subject to a few equations (saying $(-) \otimes A \dashv (-) \otimes A^*$ and $\theta^*_A = \theta_{A^*}$; $(-)^*$ extends to a contravariant equivalence).

(Note: tortile symmetric monoidal categories = compact closed categories)

The following result is important for applications in knot theory:

> Theorem (Shum). The tortile category freely generated by a single object is equivalent to the category of framed tangles.

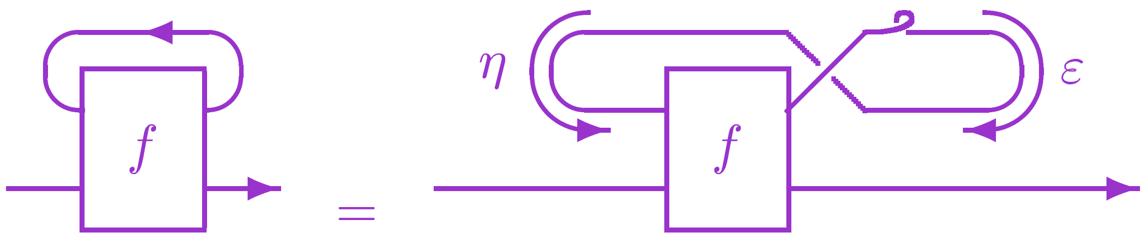
Therefore, tortile categories give rise to invariants for tangles.

"Abstract is concrete" (Hyland)

[a] other names: *ribbon category*, *braided compact closed category*, ...

Traced Categories vs Tortile Categories

Any tortile category has a unique trace (called *canonical trace*), hence is also a traced monoidal category.



It follows that a monoidal full subcategory of a totile category is traced.

In fact, every traced monoidal category arises in this way: given a traced monoidal category $\mathcal{C}$, we can construct a tortile category $\mathbf{Int}\,\mathcal{C}$ to which $\mathcal{C}$ fully faithfully embeds, via the Int-construction of Joyal, Street and Verity — an abstract version of "Geometry of Interaction" of Girard/Abramsky.

The Int Construction (Joyal, Street and Verity)

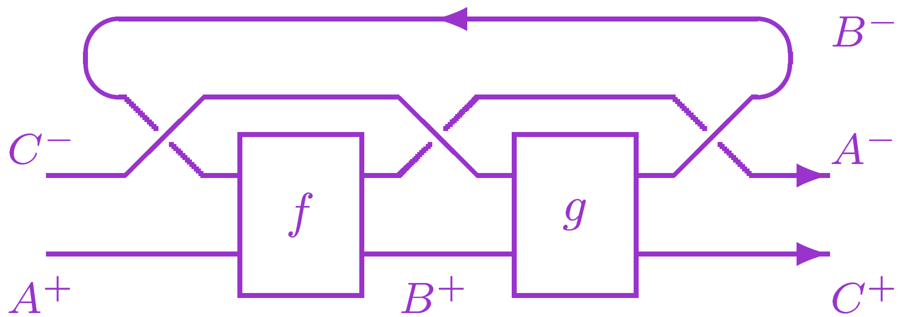
Given a traced monoidal category $\mathcal{C}$, we construct a category $\mathbf{Int}\,\mathcal{C}$ as follows.

Objects: pairs of objects of $\mathcal{C}$

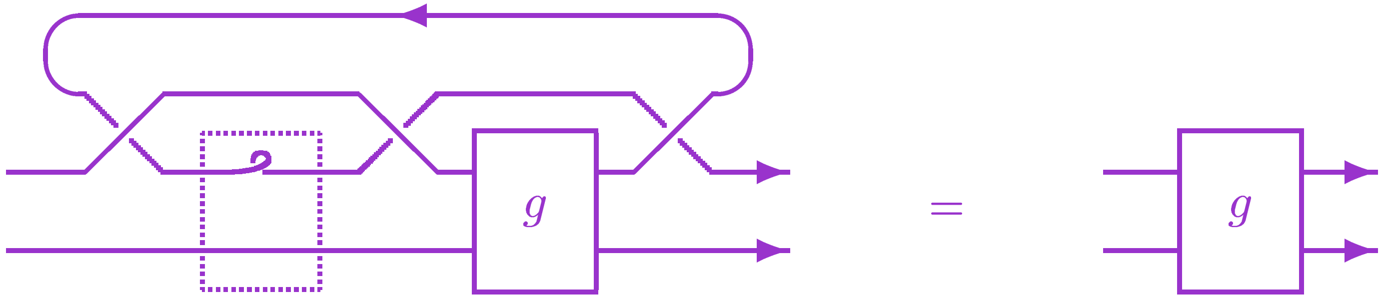
Arrows: $\mathbf{Int}\,\mathcal{C}((A^+, A^-), (B^+, B^-)) = \mathcal{C}(A^+ \otimes B^-, B^+ \otimes A^-)$

The identity on (A^+, A^-) is $id_{A^+} \otimes \theta^{-1}_{A^-} \in \mathcal{C}(A^+ \otimes A^-, A^+ \otimes A^-)$.

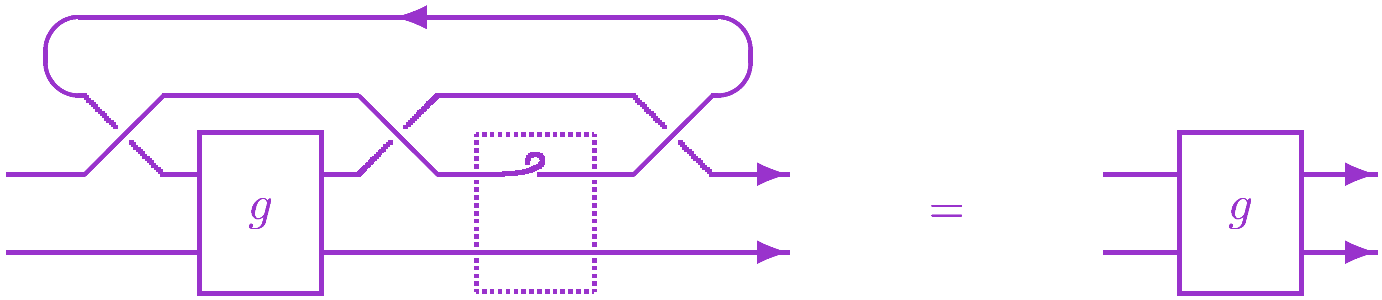
The composition of $f \in \mathbf{Int}\,\mathcal{C}((A^+, A^-), (B^+, B^-)) = \mathcal{C}(A^+ \otimes B^-, B^+ \otimes A^-)$ and $g \in \mathbf{Int}\,\mathcal{C}((B^+, B^-), (C^+, C^-)) = \mathcal{C}(B^+ \otimes C^-, C^+ \otimes B^-)$ is given by



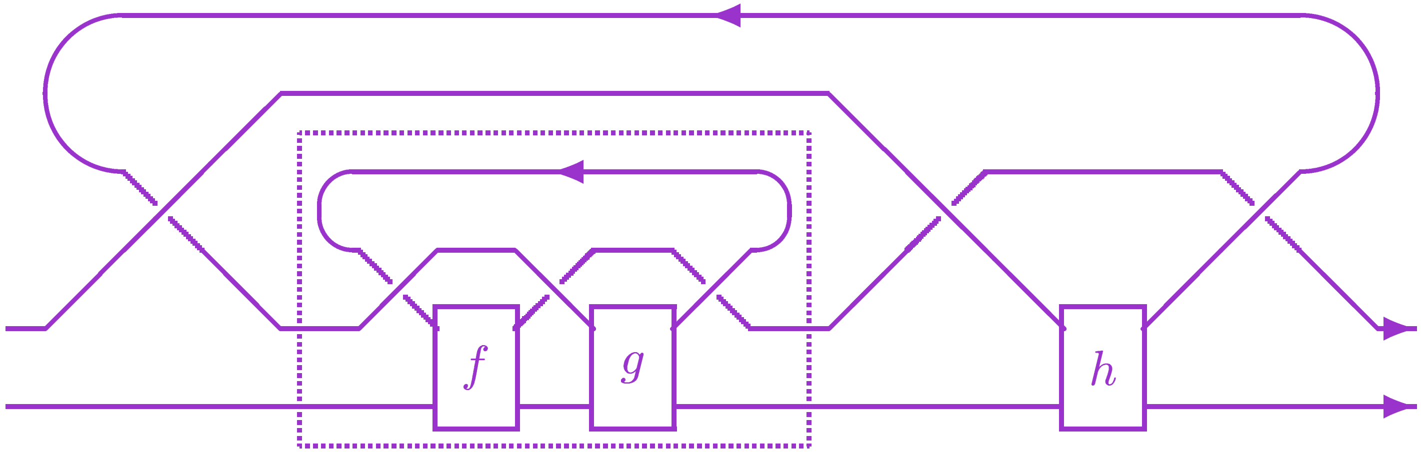
$g \circ id = g$:



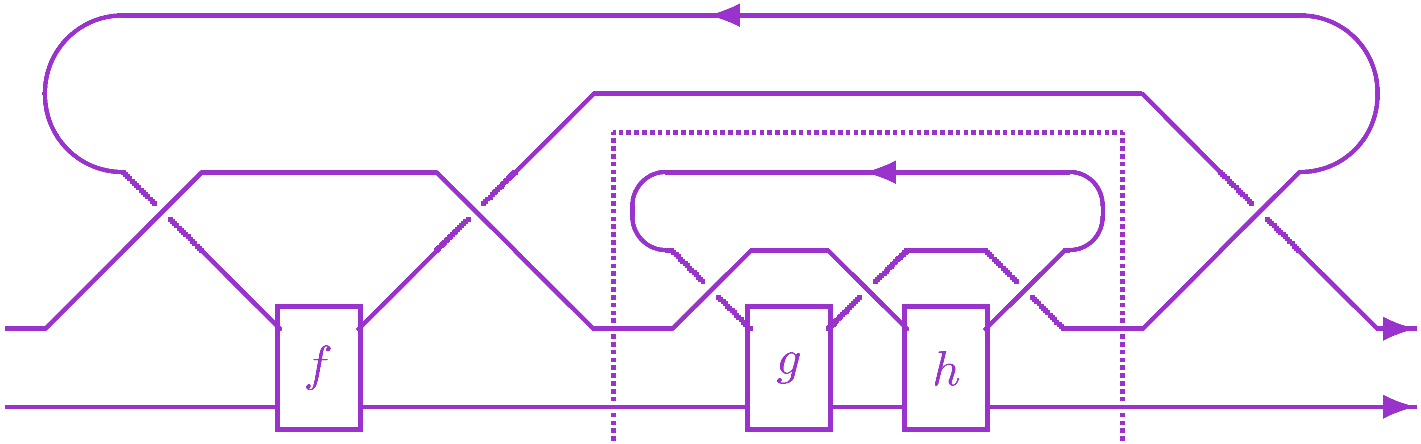
$id \circ g = g$:



$h \circ (g \circ f) = (h \circ g) \circ f$:



$=$

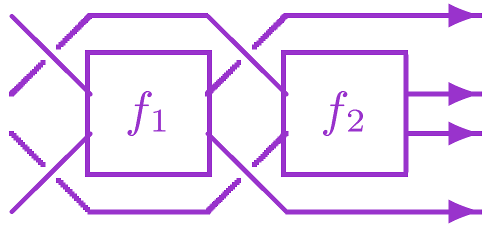


The Int Construction (cont.)

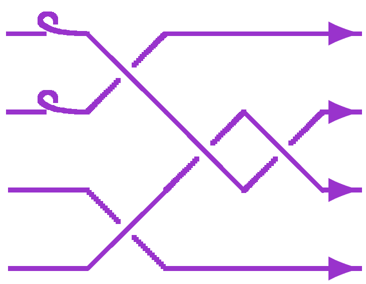
Monoidal structure: we define tensor and unit by

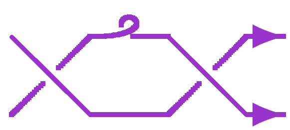
$(A^+, A^-) \otimes (B^+, B^-) = (A^+ \otimes B^+, B^- \otimes A^-)$ and $I = (I, I)$.

For $f_1 : (A_1^+, A_1^-) \to (B_1^+, B_1^-)$ and $f_2 : (A_2^+, A_2^-) \to (B_2^+, B_2^-)$, define $f_1 \otimes f_2$ by



Braids and twists (not quite obvious for non-symmetric case):

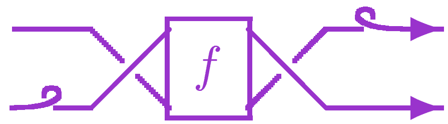
$c =$ 

$\theta =$ 

(Exercise: what are c^{-1} and θ^{-1} ?)

Duality: $(A^+, A^-)^* = (A^-, A^+)$

For $f : (A^+, A^-) \to (B^+, B^-)$ define $f^* : (B^+, B^-)^* \to (A^+, A^-)^*$ as



The unit $\eta_{(A^+,A^-)} : I \to (A^+, A^-) \otimes (A^+, A^-)^*$ is given by $id_{A^+} \otimes \theta^{-1}_{A^-}$.

The counit $\varepsilon_{(A^+,A^-)} : (A^+, A^-)^* \otimes (A^+, A^-) \to I$ is $id_{A^-} \otimes \theta_{A^+}$.

Theorem (Joyal, Street and Verity).

These data determine a tortile monoidal structure on $\mathbf{Int}\,\mathcal{C}$.

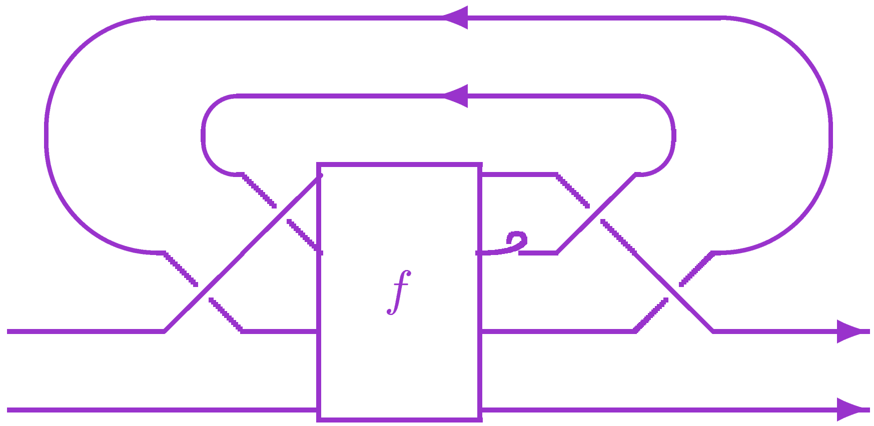
Moreover, the functor $\mathcal{N} : \mathcal{C} \to \mathbf{Int}\,\mathcal{C}$ sending A to (A, I) strongly preserves the traced monoidal structure, and is full faithful.

In fact, Int-construction is *universal*, as shown by JSV: it gives a left biadjoint to the forgetful 2-functor from the 2-category of tortile categories to that of traced monoidal categories.

The Canonical Trace on $\mathbf{Int}\,\mathcal{C}$

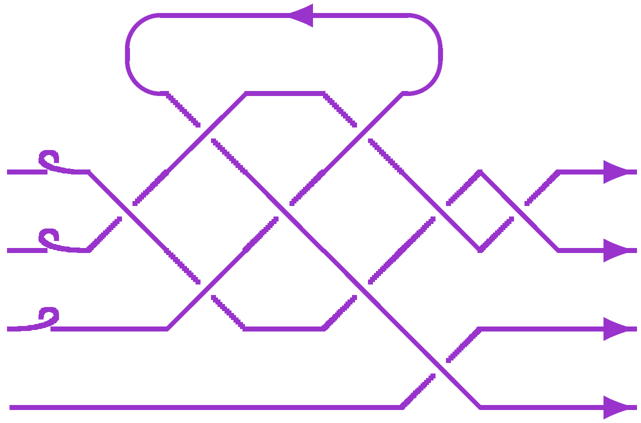
Explicitly, the canonical trace on $\mathbf{Int}\,\mathcal{C}$ can be given as follows.
(It is not entirely obvious for the non-symmetric case.)

For $f : (A^+, A^-) \otimes (X^+, X^-) \to (B^+, B^-) \otimes (X^+, X^-)$, its trace $Tr^{(X^+, X^-)} f : (A^+, A^-) \to (B^+, B^-)$ is

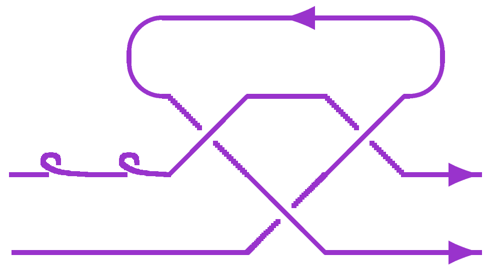


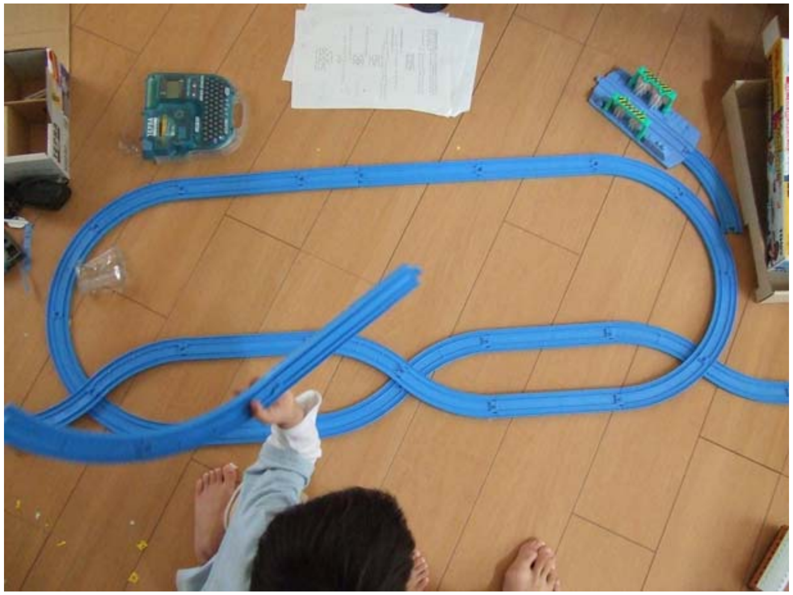
Answers to the exercise

c^{-1} :



θ^{-1} :





Trying to realize Int-construction with my son

Part III: Traced Monoidal *Closed* Categories

Closed Structure

So far we have not thought much about *closed structure*, or higher-types.

Recall that a monoidal category $\mathcal{C}$ is *closed* if $- \otimes A : \mathcal{C} \to \mathcal{C}$ has a right adjoint $A \multimap -$:

$$\mathcal{C}(X \otimes A, Y) \simeq \mathcal{C}(X, A \multimap Y)$$

In particular, tortile categories are closed, with $A \multimap B = B \otimes A^*$.

In the context of linear logic, being symmetric monoidal closed means that we can interpret the intuitionistic multiplicative fragment (tensor $\otimes$, unit $\top$, and linear implication $\multimap$) in $\mathcal{C}$.

In the rest of this talk, we will see that, for a traced monoidal category, closedness has yet another reading in terms of the Int-construction, related to the modality $!$ and linear decomposition $A \to B = !A \multimap B$.

A Folklore on Monoidal Closed Categories

It is known that a monoidal co-reflective full subcategory of a monoidal closed category is also closed (although the closed structure may not be preserved by the inclusion):

Lemma (folklore[a]). Let $\mathcal{C} \underset{\mathcal{U}}{\overset{\mathcal{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{D}$ be a monoidal adjunction.
If $\mathcal{F}$ is full faithful and $\mathcal{D}$ is closed, then $\mathcal{C}$ is also closed, with $A \multimap_{\mathcal{C}} B = \mathcal{U}(\mathcal{F}A \multimap_{\mathcal{D}} \mathcal{F}B)$.

$$
\begin{array}{llll}
\text{Proof: } \mathcal{C}(C \otimes A, B) & \simeq & \mathcal{D}(\mathcal{F}(C \otimes A), \mathcal{F}B) & \mathcal{F} \text{ is full faithful} \\
& \simeq & \mathcal{D}(\mathcal{F}C \otimes \mathcal{F}A, \mathcal{F}B) & \mathcal{F} \text{ is strong monoidal} \\
& \simeq & \mathcal{D}(\mathcal{F}C, \mathcal{F}A \multimap \mathcal{F}B) & \mathcal{D} \text{ is closed} \\
& \simeq & \mathcal{C}(C, \mathcal{U}(\mathcal{F}A \multimap \mathcal{F}B)) & \mathcal{F} \dashv \mathcal{U}
\end{array}
$$

Note: an adjunction $\mathcal{F} \dashv \mathcal{U}$ is a monoidal adjunction iff $\mathcal{F}$ is strong monoidal (Kelly).

[a] At the workshop, Hyland told me that this result was due to Day.

Main Observation

Below we present a variation for traced monoidal categories. It characterizes closedness in terms of an adjunction associated to the Int-construction.

Theorem.
Let $\mathcal{C}$ be a traced monoidal category, and $\mathcal{N} : \mathcal{C} \to \mathbf{Int}\,\mathcal{C}$ be the canonical inclusion from $\mathcal{C}$ into $\mathbf{Int}\,\mathcal{C}$ (i.e. $\mathcal{N}(A) = (A, I)$).
Then $\mathcal{C}$ is closed if and only if $\mathcal{N}$ has a right adjoint.

Thus every traced monoidal closed category arises as a monoidal co-reflexive full subcategory of a tortile monoidal category.

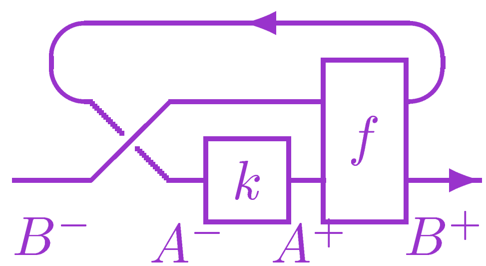
Proof: "if" follows from the previous folklore lemma, as $\mathcal{N}$ is full faithful and strong symmetric monoidal. Note that, by the lemma,

$$A \multimap_{\mathcal{C}} B \;\simeq\; \mathcal{U}(\mathcal{N}A \multimap_{\mathbf{Int}\,\mathcal{C}} \mathcal{N}B) \;=\; \mathcal{U}((A,I)^* \otimes (B,I)) \;\simeq\; \mathcal{U}(B,A)$$

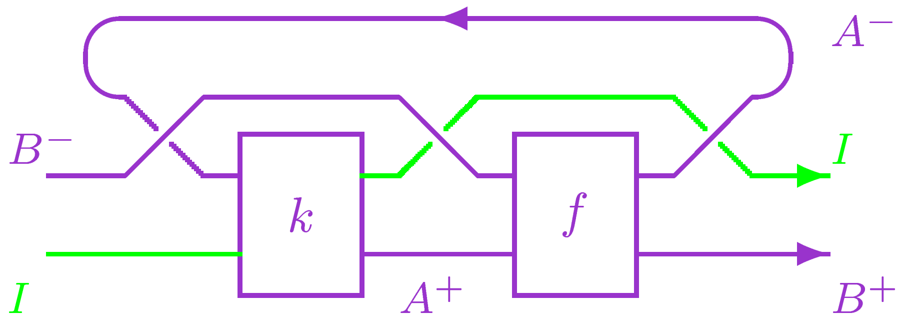
where $\mathcal{U}$ is right adjoint to $\mathcal{N}$. This suggests how we proceed to show the "only if" part.

Thus, if $\mathcal{C}$ is closed, define $\mathcal{U}(A^+, A^-) = A^- \multimap A^+$.

For $f : (A^+, A^-) \to (B^+, B^-)$, let $\mathcal{U}(f) : (A^- \multimap A^+) \to (B^- \multimap B^+)$ send $k : A^- \to A^+$ to $Tr^{A^-}_{B^-,B^+}(f \circ (k \otimes B^-) \circ c_{B^-,A^-}) : B^- \to B^+$.



In other words, $\mathcal{U}(f)$ describes "composition with f in $\mathbf{Int}\,\mathcal{C}$" in $\mathcal{C}$.



It is immediate to see that $\mathcal{U}$ is a functor.

Finally, it is easy to see the adjointness:

$$\begin{aligned}
\mathbf{Int}\,\mathcal{C}(\mathcal{N}(A),(B^+,B^-)) &= \mathbf{Int}\,\mathcal{C}((A,I),(B^+,B^-)) \\
&= \mathcal{C}(A\otimes B^-, B^+\otimes I) \\
&\simeq \mathcal{C}(A,\, B^- \multimap B^+) \\
&= \mathcal{C}(A, \mathcal{U}(B^+,B^-)).
\end{aligned}$$

Corollary: Monoidal Comonad

Below we consider only the case with symmetry.

Note that the adjunction in this theorem gives rise to an idempotent symmetric monoidal comonad $\mathcal{NU}$ on $\mathbf{Int}\,\mathcal{C}$ which sends (A^+, A^-) to $(A^- \multimap A^+, I)$. Thus we have:

Corollary. For any traced symmetric monoidal closed category $\mathcal{C}$, there is an idempotent symmetric monoidal comonad on $\mathbf{Int}\,\mathcal{C}$ such that its co-Kleisli category is equivalent to $\mathcal{C}$.

Let us apply this to models of linear logic, having in mind that linear decompostion amounts to taking co-Kleisli categoies of certain comonads corresponding to $!$.

Corollary: Linear Decomposition is an Inverse to Int

A symmetric monoidal adjunction between a category with finite products and a symmetric monoidal category gives rise to a comonad which models the exponential $!$ of linear logic (*linear exponential comonad* in the sense of Hyland and Schalk).

Corollary. For any traced cartesian closed category $\mathcal{C}$, there is an idempotent linear exponential comonad on $\mathbf{Int}\,\mathcal{C}$ such that its co-Kleisli category is equivalent to $\mathcal{C}$.

Explicitly, this comonad sends (A^+, A^-) to $(A^- \Rightarrow A^+, 1)$.

Together with the trace-fixpoint correspondence:

Corollary. Any CCC with a Conway fixed point operator is equivalent to one arising from a compact closed model of multiplicative exponential linear logic via the co-Kleisli construction.

Corollary: More Exponentials on $\mathbf{Int}\,\mathcal{C}$

Since monoidal adjunctions are closed under composition, Int-construction actually send a traced model for IMELL to a compact closed model for MELL:

Corollary. Let $\mathcal{C}$ be a traced symmetric monoidal closed category with a linear exponential comonad $!$ (i.e. a model of IMELL). Then $\mathbf{Int}\,\mathcal{C}$ is equipped with a linear exponential comonad $!'$ given by $!'(A^+, A^-) = \mathcal{N}(!(\mathcal{U}(A^+, A^-))) = (!(A^- \multimap A^+), I)$.

$$
\begin{tikzcd}
! \arrow[loop left] & \mathcal{C} \arrow[r, shift left=2, "\mathcal{N}"] \arrow[r, phantom, "\perp"] & \mathbf{Int}\,\mathcal{C} \arrow[l, shift left=2, "\mathcal{U}"]
\end{tikzcd}
$$

Int-construction in Program Transformations

Katsumata and Nishimura (ICFP'06) introduced a program transformation technique called *(semantic) higher-order removal*.

Roughly, their technique transforms a higher-order function $g : (A^- \Rightarrow A^+) \Rightarrow (B^- \Rightarrow B^+)$ (created in the process of dealing with fusions of functions with accumulating parameters, which involves certain bi-directional information flow) to a first-order function $f : A^+ \times B^- \Rightarrow B^+ \times A^-$ such that $\mathcal{U}(f) = g$, where $\mathcal{U}$ is right adjoint to $\mathcal{N}$.

They give a syntactic condition which ensures that g is in the image of $\mathcal{U}$ in their semantic models, and presented a procedure for identifying f such that $\mathcal{U}(f) = g$.

Int-construction and Attribute Grammars

More recently, Katsumata has shown that a substantial part of the theory of attribute grammars (Knuth 1968) can be carried out very cleanly in terms of traced monoidal categories and Int-construction.

In his work, the adjuntion $\mathcal{N} \dashv \mathcal{U}$ provides the equivalence between attribute grammars and synthesized attribute grammars.

(Ask Shin-ya Katsumata for more detail!)

Linear Fixed-Points

In my paper in TLCA'97, it is shown that

Theorem. Given a symmetric monoidal adjunction $\mathcal{C} \underset{U}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathcal{D}$ between a category $\mathcal{C}$ with finite products (taken as the monoidal structure) and a traced symmetric monoidal category $\mathcal{D}$, there exists a family of functions

$$(-)^\dagger : \mathcal{D}(FA \otimes X, X) \longrightarrow \mathcal{D}(FA, X)$$

which is natural in A and dinatural in X.

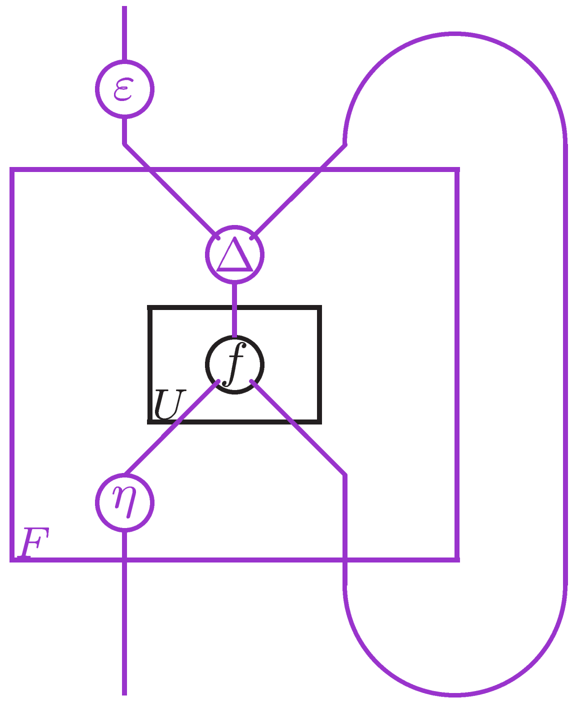
In particular, the fixed-point equation

$$f^\dagger = f \circ (1_{FA} \otimes f^\dagger) \circ n \circ F\Delta_A$$

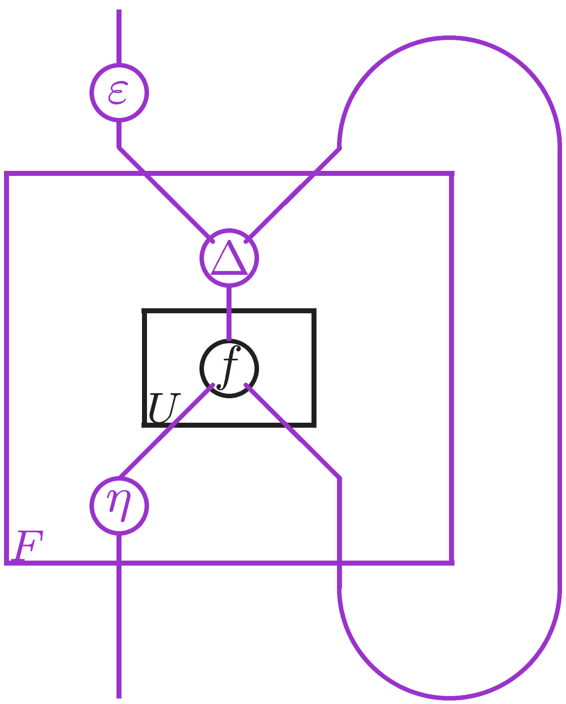
holds (where $n : F(A \times A) \xrightarrow{\sim} FA \otimes FA$ and $\Delta : A \to A \times A$).

This result has been used for providing semantics of recursion in lambda calculi with cyclic sharing ("recursion from cyclic sharing").

Using *functorial boxes* of Melliès (CSL'06)[a], which are a convenient notation for monoidal functors and monoidal natural transformations, this $f^\dagger$ can be expressed as follows (to be read from the bottom).



[a]First introduced by Cockett and Seely as "monoidal functor boxes" (JPAA 1999)



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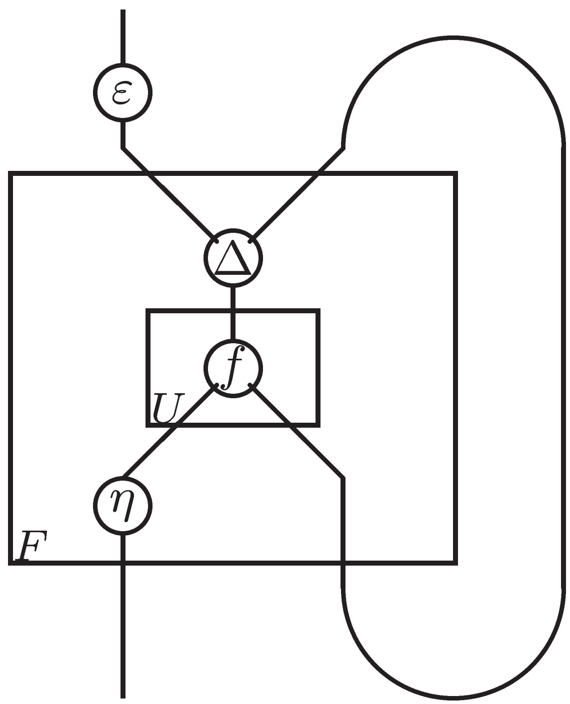

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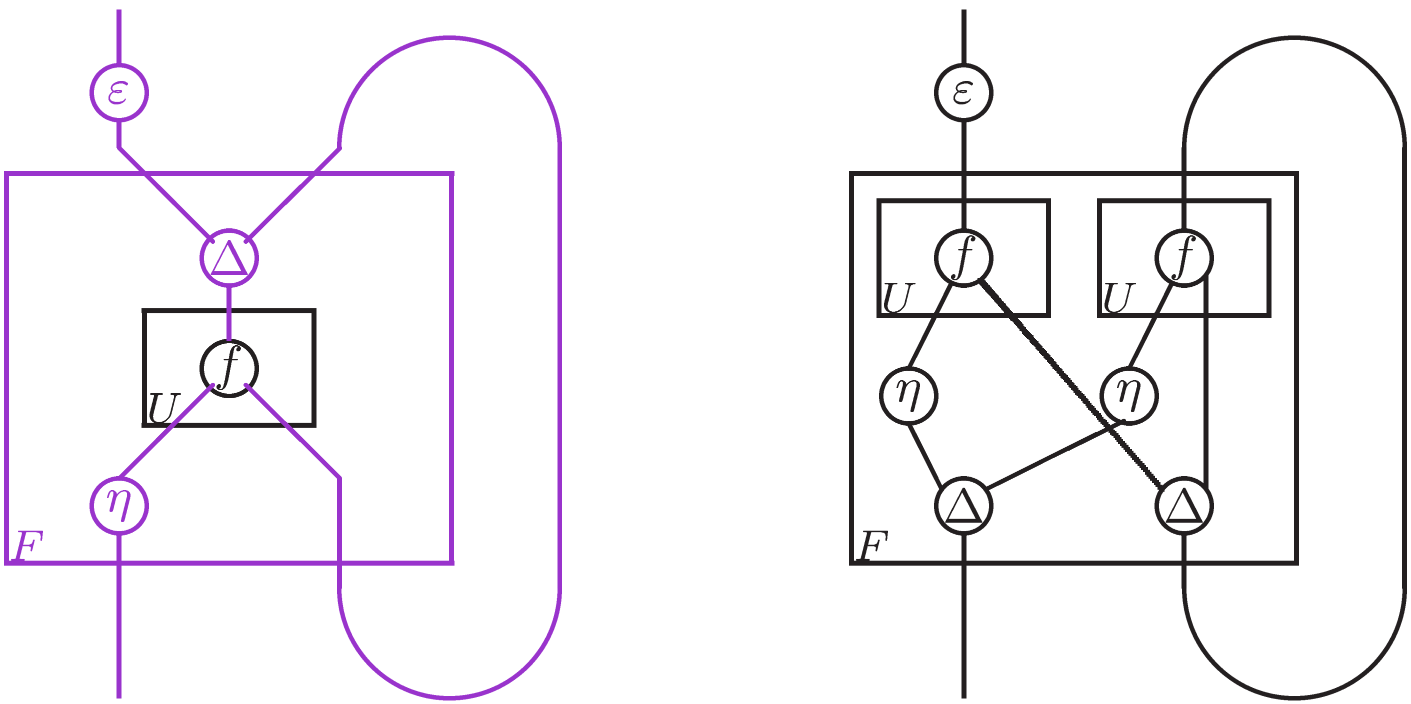
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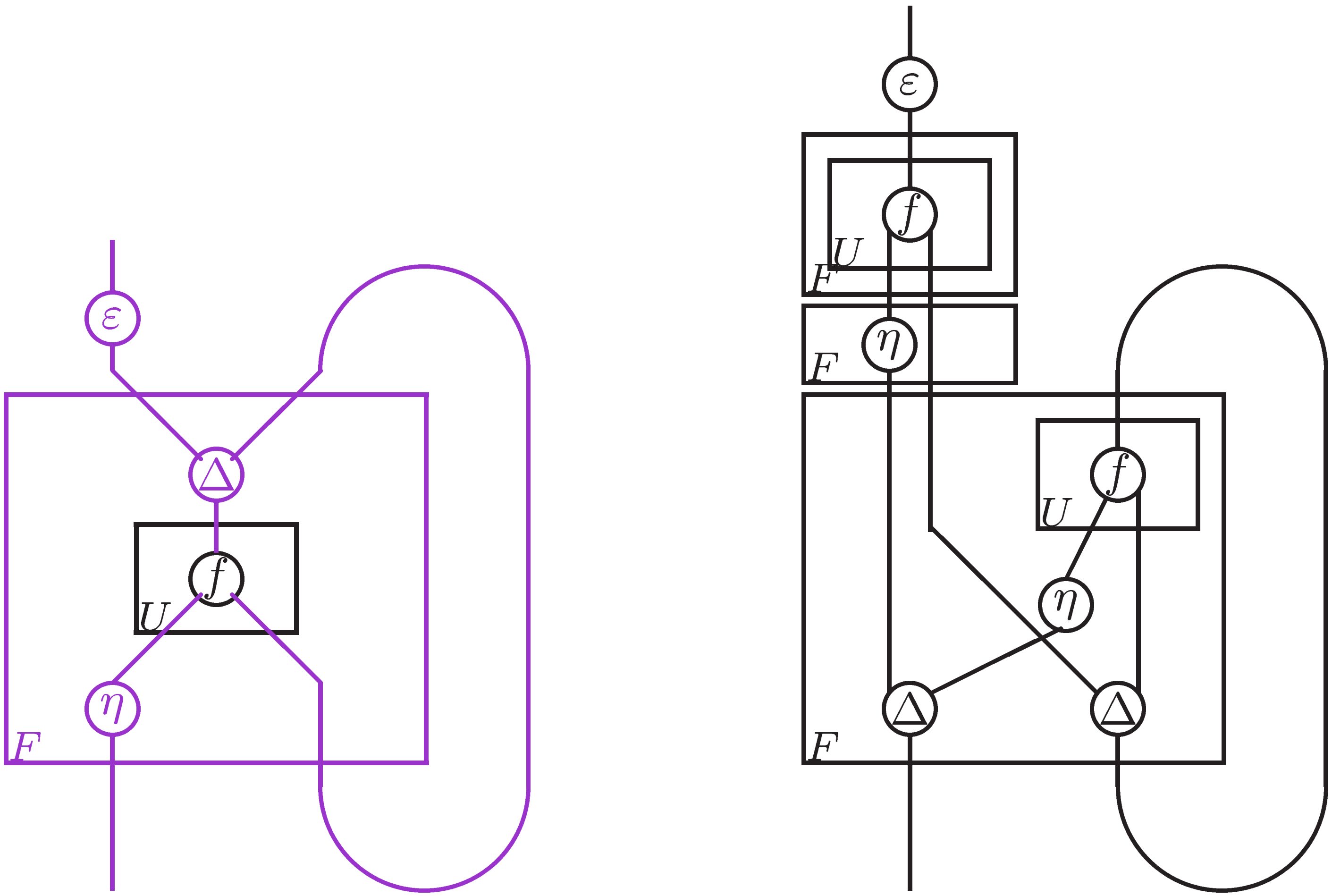
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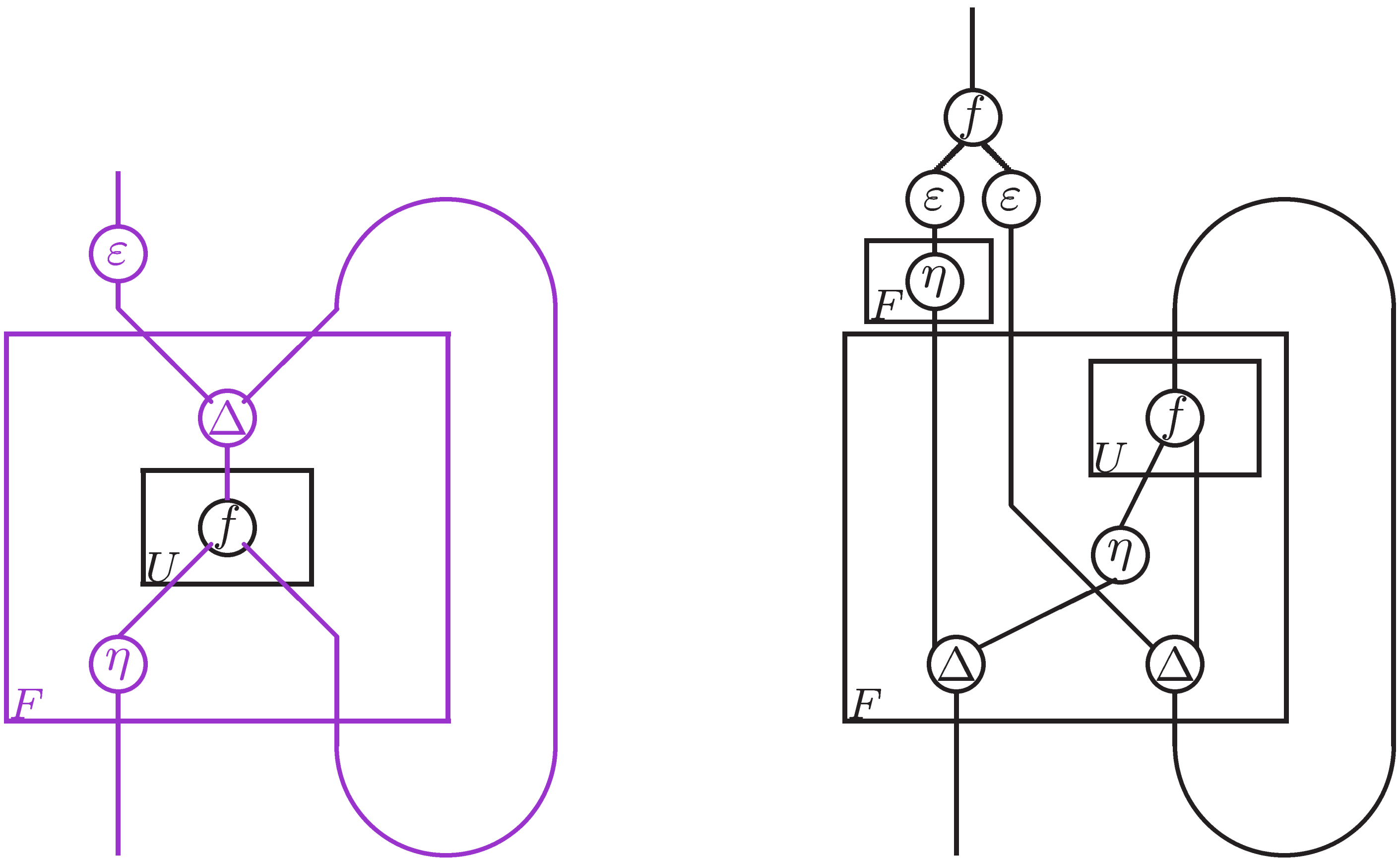

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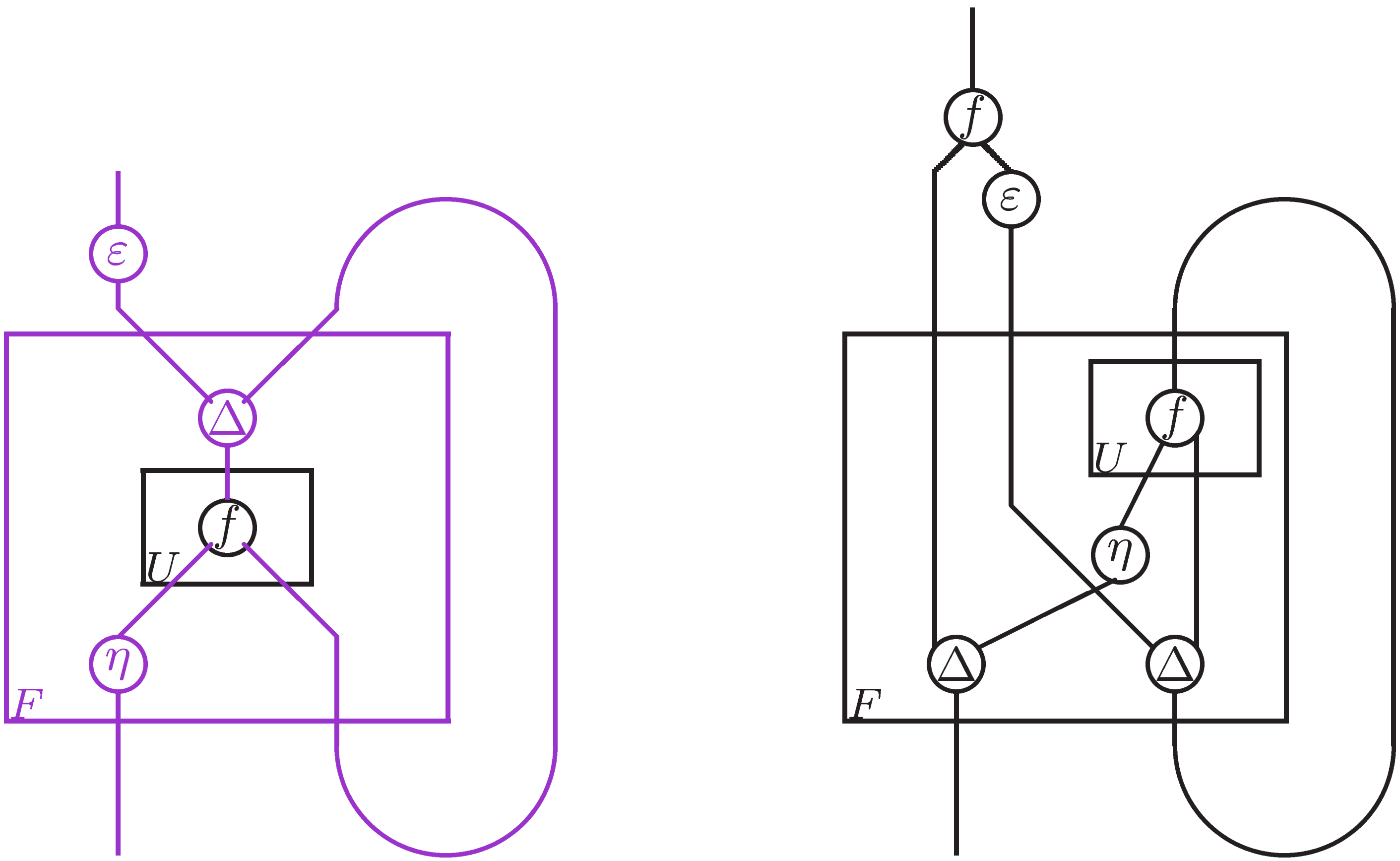

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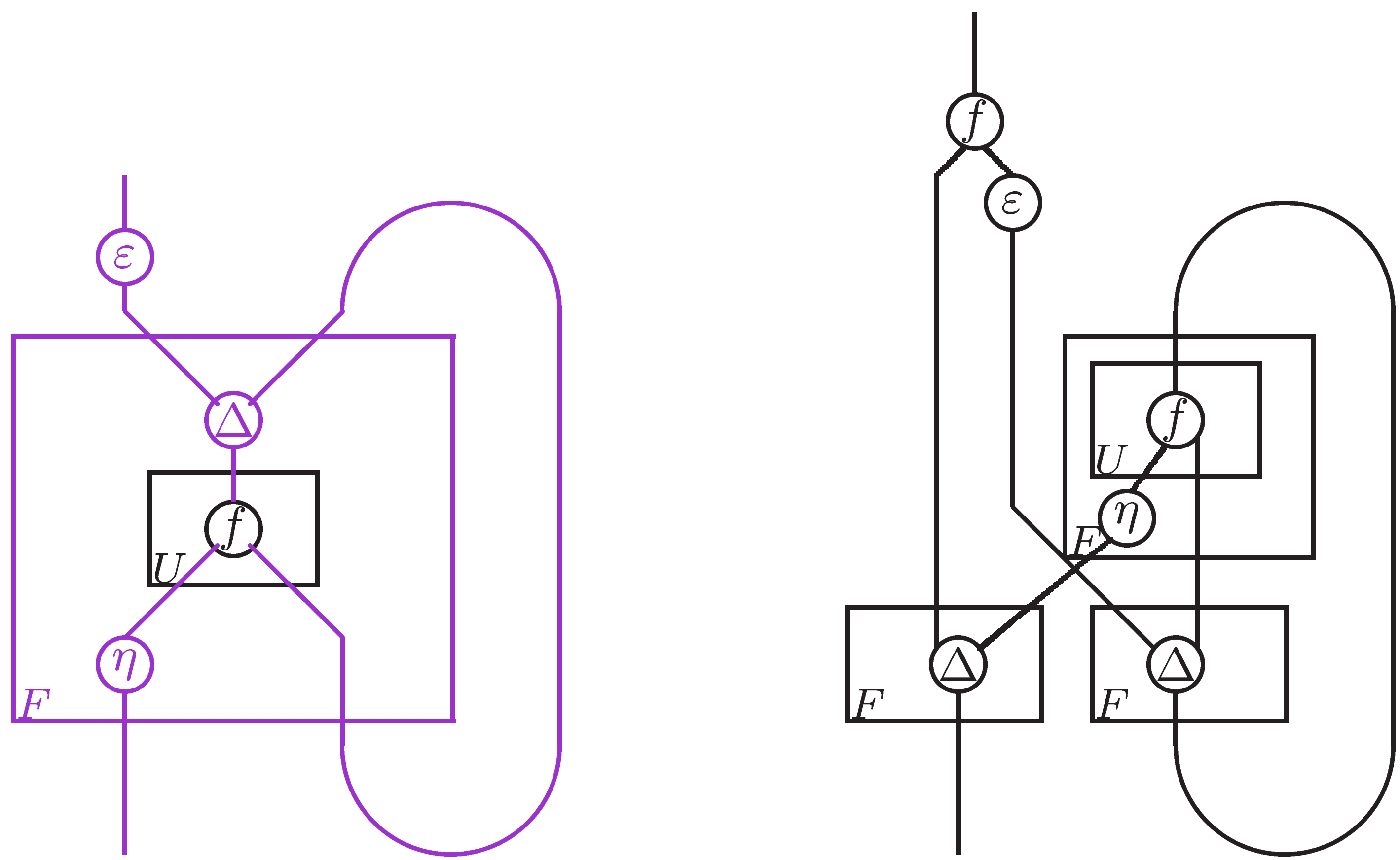

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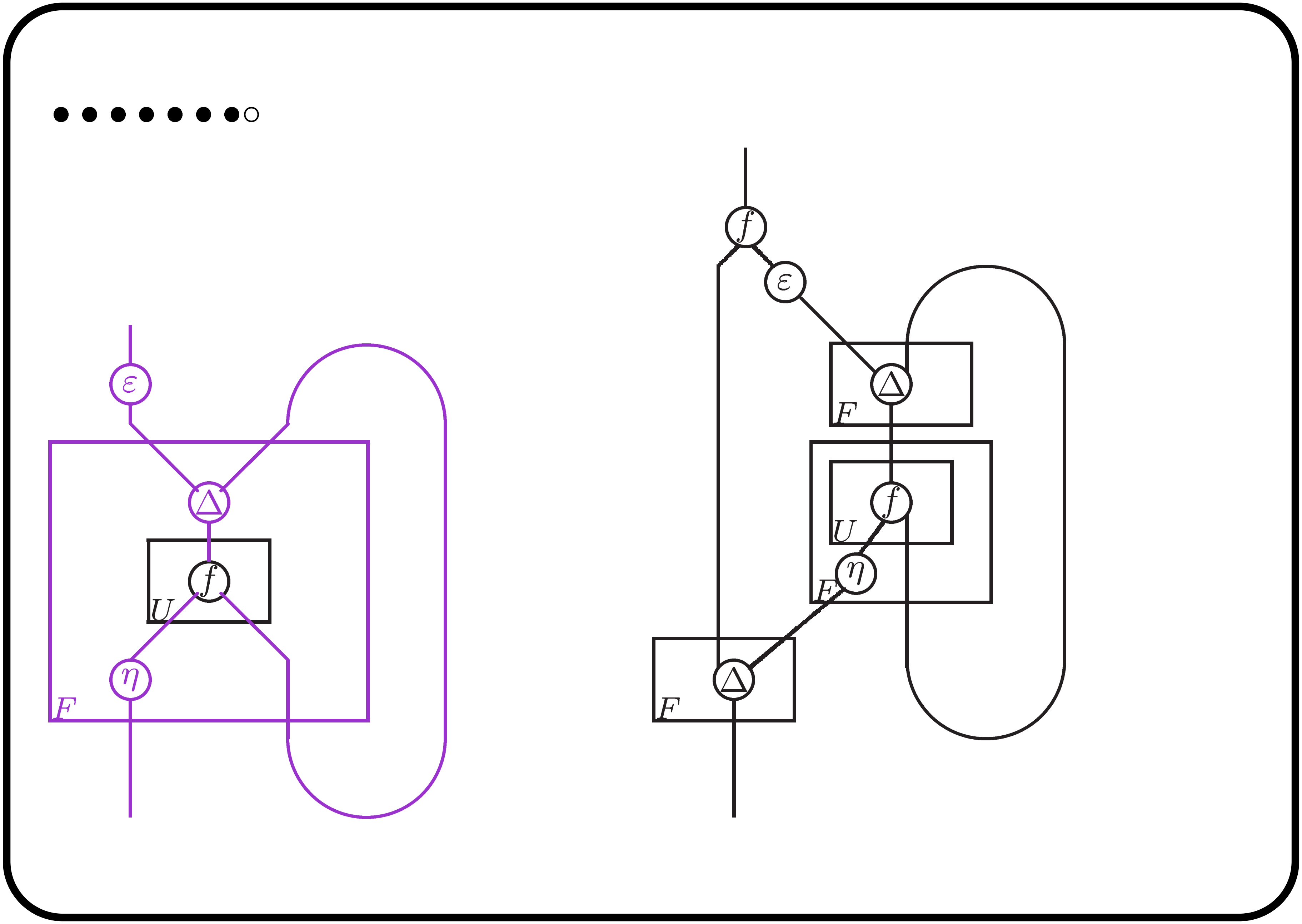
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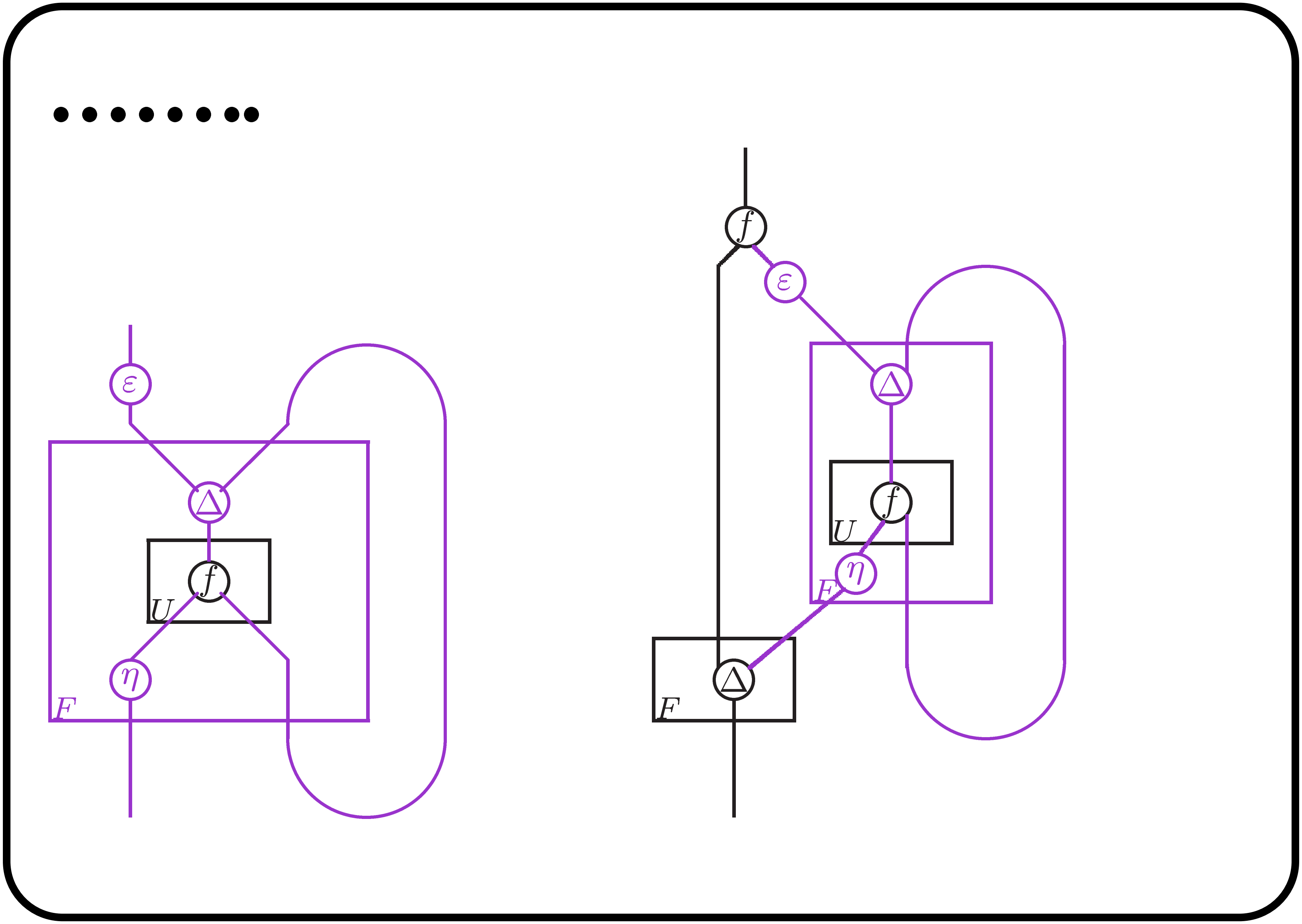

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Linear Fixed-Points (cont.)

In terms of linear logic (or traced symmetric monoidal closed categories with a linear exponential comonad):
there is a *linear* fixed-point operator $\mathsf{Y} : !(X \multimap X) \multimap X$ such that

$$\mathsf{Y}\,(!f) \;=\; f\,(\mathsf{Y}\,(!f)) \qquad (f : X \multimap X)$$

(Note: not $!(!X \multimap X) \multimap X$)

As we have seen, if $\mathcal{C}$ is a traced symmetric monoidal closed category with a linear exponential comonad (traced model of IMELL), then $\mathbf{Int}\,\mathcal{C}$ is a compact closed category with a linear exponential comonad (compact model of MELL).
Both $\mathcal{C}$ and $\mathbf{Int}\,\mathcal{C}$ admit interpretations of such linear fixed-points.

More Considerations — (Negative) Conway Games

An interesting (non)example:
the category $\mathcal{Y}^-$ of negative Conway games (Melliès, 2004) is a symmetric monoidal full subcategory of the compact closed category $\mathcal{Y}$ of Conway games (Joyal 1977). The inclusion from $\mathcal{Y}^-$ to $\mathcal{Y}$ has a right adjoint. Thus (by the folklore lemma) $\mathcal{Y}^-$ is a *traced symmetric monoidal closed category*. (Cf. multi-bracketed version in LICS'07 paper)

$\mathcal{Y}^-$ is one of very few interesting traced symmetric monoidal closed categories which are nether cartesian closed nor compact closed.

$\mathbf{Int}\,\mathcal{Y}^-$ is *not* equivalent to $\mathcal{Y}$ — thus it does not really fit in our result. But the difference is subtle, and deserves further study.

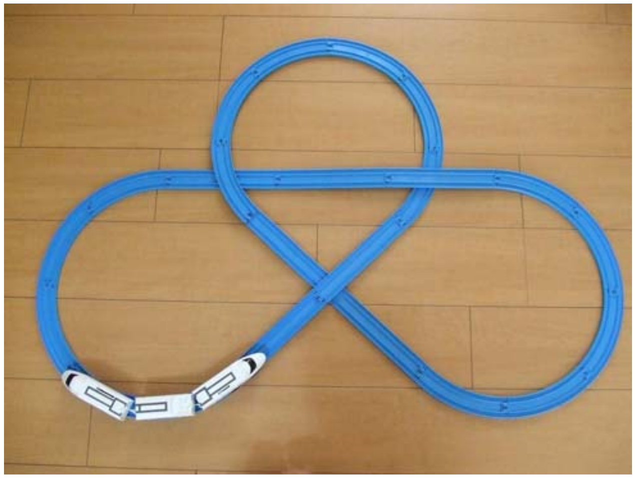
Conclusion

We looked at some elements of traced monoidal categories, and observed that closedness is equivalent to an adjointness associated to the Int-construction. This result has a number of applications.

I hope that these provide some good reasons/motivations to study traced monoidal *closed* categories.

Questions

- Is there a *good* concrete description of *free* traced monoidal closed categories? (Sort of "higher-order tangles"?)
- (Related to above) Good syntax for traced (symmetric) monoidal closed categories? (Term calculi? Proof nets?)
- Constructions on traced monoidal closed categories? (Uniformity?)
- Other traced monoidal closed categories of games?



End of Slides – Thank You.

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