The Galois groupoid of Picard-Painlevé VI equation

Guy Casale

The sixth Painlevé equation for special values of classical parameters ($\alpha = \beta = \gamma = 0, \delta = 1/2$) was discovered by E.Picard in [10] as an exemple of order two non-linear equation without movable singularities. The usual form of this equation is called in this paper PP_6:

$$y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)(y')^2 + \left(\frac{1}{x-y} + \frac{1}{1-x} - \frac{1}{x}\right)y' + \frac{y(y-1)}{2x(x-1)(y-x)}.$$

Among all the P_6 equations this one has the property to be solved by a formula :

$$y(x) = \wp\left(a\omega_1(x) + b\omega_2(x); \omega_1(x), \omega_2(x)\right) \tag{1}$$

with a and b two constants, $\omega_{1,2}$ a basis of periods of $t^2 = y(y-1)(y-x)$ and $\wp(\,\cdot\,;\omega_1,\omega_2)$ the corresponding Weierstass function. For rational a and b, the solution is algebraic but for other values the solution is not even a classical function in the sense of H.Umemura [12] despite the formula to express it as it is proved by H. Watanabe in [15]. For a complete study of this equation, see the article of M.Mazzocco [8].

This is a common belief that this kind of property must be explained by a non-linear Galois theory. Two essentially equivalent very general differential Galoisian theories have been proposed in the last ten years by H.Umemura [13, 14] and B.Malgrange [6]. Because of its geometric flavor, we will focus on Malgrange's Galois groupoid to explain the existence of a formula to solve PP_6. The computation of the Galois groupoid of PP_6 can be reduced to computation already done by P.Painlevé in ([9] pp 501–517). Painlevé remarked that PP_6 is irreducible in his sense (which is very close to Nishioka-Umemura definition [11]) but it admits a system of first integrals in a Picard-Vessiot extension of the partial differential field $(\mathbb{C}(x, y, y'); \partial/\partial x, \partial/\partial y, \partial/\partial y')$. The aim of this article is to integrate this remark from Painlevé in the framework of Malgrange's Galois groupoid. Solutions of Painlevé equations describe isomonodromy deformations of rank two Fuchsian systems with four singularities on $\mathbb{CP}^1$. Algebraic properties of the monodromy data are related to the transcendence nature of the corresponding Painlevé solution [4]. In the Picard-Painlevé case, the monodromy data are not special but the space of monodromy data and the non-linear monodromy are special [8]

[0]This work was done when the author was supported by Marie Curie Fellowship (MEIF-CT-2005-025116) and invited participant of the PEM program of the Newton Institute, Cambridge, UK. The author is partially supported by the ANR project no. JC05 41465 and by GIFT NEST-Adventure Project no. 5006. The author thanks F.Loray for fruitful discussions.

Address: IRMAR, Université de Rennes 1, Campus de Beaulieu 35042 Rennes cedex, France
E-mail address: `guy.casale@univ-rennes1.fr`
URL: `http://www.perso.univ-rennes1.fr/guy.casale/`

1 The Galois groupoid of a vector field in $\mathbb{C}^3$

Let X be a rational vector field in $\mathbb{C}^3$. In general it is not complete and its flows are only defined on open set small enough. All the dynamic of this vector field is contained in the pseudogroup of transformations of $\mathbb{C}^3$ generated by these local flows. By keeping only the germs of diffeomorphisms from this pseudogroup one gets a groupoid, $TanX$, acting on $\mathbb{C}^3$.

The Galois groupoid of X is the Zariski closure of $TanX$ for a (nearly) obvious embedding of $TanX$ in an infinite dimensional algebraic variety.

This variety is the space J^* of formal diffeomorphisms of $\mathbb{C}^3$, *i.e.* the set of formal invertible maps $\varphi : \widehat{\mathbb{C}^3, a} \to \widehat{\mathbb{C}^3, b}$ and the embedding is the Taylor expansion of elements of $TanX$. The space J^* can be presented as the projective limit of the spaces J_q^* of order q jets of diffeomorphisms. These ones are isomorphic to

$$\mathbb{C}^3 \times \mathbb{C}^3 \times \mathrm{Gl}(\mathbb{C}^3) \times_{\substack{\alpha \in \mathbb{N}^3 \\ |\alpha| \leq q}} \mathbb{C}^{3|\alpha|}$$

and their coordinates rings

$$O(J_q^*) = \mathbb{C}\left[x, y, y', \overline{x}^\alpha, \overline{y}^\alpha, \overline{y'}^\alpha, \frac{1}{\det(\overline{x}^{\epsilon_i}, \overline{y}^{\epsilon_i}, \overline{y'}^{\epsilon_i})} \middle| \; \alpha \in \mathbb{N}^3; |\alpha| \leq q \right]$$

are the rings of partial differential equations of order q in three functions of three arguments with non vanishing jacobian. Futhermore these varieties get natural groupoid structures given by the computation rules for the Taylor expansion of the composition of formal diffeomorphisms. An algebraic subgroupoid of J_q^* is an algebraic subvariety whose ideal satisfies some stability conditions under inversion and composition.

The space J^* is $\varprojlim J_q^*$ and its ring $O(J^*) = \varinjlim O(J_q^*)$ is the commutative differential ring of non-linear partial differential equations on germs of diffeomorphisms. The derivations are given by the natural actions of $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial y'$ on partial differential equations.

Definition 1.1 (Malgrange [6]) *Let G be a subvariety of J^* described by a differential ideal. It is an algebraic $\mathcal{D}$-groupoid on $\mathbb{C}^3$ if there is a subvariety $Z \subset \mathbb{C}^3$ such that the projections of G on the finite order jet spaces defined algebraic subgroupoids on $\mathbb{C}^3 - Z$.*

Definition 1.2 (Malgrange [6]) *The Galois groupoid of X is the smallest algebraic $\mathcal{D}$-groupoid on $\mathbb{C}^3$ containing $TanX$.*

Roughly speaking, the Galois groupoid of X is the set of all the germs of diffeomorphisms of $\mathbb{C}^3$ solutions of all the PDE's vanishing on $TanX$. Because $L_X X = 0$, the Galois groupoid is a subgroupoid of the groupoid of transformations preserving X *i.e.* germs φ such that $\varphi_* X = X$. When the vector field is divergence free, one can add the equations given by the coordinates of $\varphi^*(dx \wedge dy \wedge dy') = dx \wedge dy \wedge dy'$.

Near a regular point of X one can choose a flowbox given by two transversal coordinates t_1, t_2 and a tangent one z. A germ of diffeomorphism preserving X can be written

$$\varphi : \begin{cases} \overline{t_1}(t_1, t_2) \\ \overline{t_2}(t_1, t_2) \\ \overline{z}(z, t_1, t_2). \end{cases}$$

By looking at the PDE's of the Galois groupoid in these coordinates, two type of PDE can be distinguished. The PDE's vanishing on the germs such that $\overline{z} = z$ are called tangential equations. The others are the transversal ones. For example, the transversal equations of the set of PDE : $L_X X = 0$ are given by $L_X X \wedge X = 0$.

Using Lie-Cartan local classification of pseudogroups acting on $\mathbb{C}^2$ established in [1], one has the following proposition

Proposition 1.3 ([3]) *If X is divergence free and γ is the closed 2-form vanishing on X, i.e. $\iota_X \gamma = 0$, then one of the following situations occurs:*

- $Gal(X)$ is imprimitive in codimension one*: there exists an algebraic 1-form θ such that $\theta \wedge d\theta = 0$ and $\theta(X) = 0$,*
- $Gal(X)$ is transversally affine*: there exists two algebraic independent 1-forms $\theta_{1,2}$ vanishing on X and a traceless matrix of 1-form $(\theta_{1,2}^{1,2})$ such that $d\theta_i = \sum_j \theta_i^j \wedge \theta_j$ and $d\theta_i^j = \sum_k \theta_i^k \wedge \theta_k^j$,*
- *the only transversal equations of $Gal(X)$ are those of invariance of γ.*

The result presented in this paper is the following.

Theorem 1.4 (Painlevé [9]) *The Galois groupoid of PP_6 is transversally affine.*

Transversally affine vector fields admit very special first integrals given by the a ffine structure out of a codimension one subvariety. These are described in the next section. The vector field over $\mathbb{C}(x, y, y')$ of PP_6 is not divergence free for the usual 'volume' form $dx \wedge dy \wedge dy'$ but for the canonical one. All the Painlevé equation can be express as time dependent Hamiltonians [7]. For PP_6, the Hamiltonian is :

$$\begin{cases} \frac{dq}{dx} = \frac{\partial K}{\partial p} \\ \frac{dp}{dx} = -\frac{\partial K}{\partial q} \\ K = \frac{1}{x(x-1)}[q(q-1)(q-x)p^2 - pq(q-1) + \frac{1}{4}(q-x)] \\ q = y \text{ and } p = \left(\frac{x(x-1)}{2y(y-1)(y-x)}\right) y' + \frac{1}{2(y-x)}. \end{cases}$$

Let θ_1 and θ_2 be the forms $dq - (\partial K/\partial p)dx$ and $dp + (\partial K/\partial q)dx$, $\theta_1 \wedge \theta_2 = dq \wedge dp + dH \wedge dx$ is a closed 2-form vanishing on PP_6.

2 Classical first integrals and Galois groupoid

The classical functions over $\mathbb{C}$ were introduced in [12] by H.Umemura. A function of one variable is said to be classical if one can find it in an ordinary differential fied extension of the rational functions field $\mathbb{C}(x)$ built by successive strongly normal extensions [5] or algebraic extensions. In general two successive strongly normal extensions fail to be a strongly normal one. Umemura's definition of classical functions of n variables is the following.

Definition 2.1 (Umemura [12]) *Let $\mathbb{C}(x_1, \dots, x_n)$ be the partial differential field of rational functions of n arguments with derivations $\partial_{x_1} \dots \partial_{x_n}$ and $\mathcal{K}$ be a differential extension of $\mathbb{C}(x_1, \dots, x_n)$. It is said to be classical if one can find a tower of differential extensions*

$$\mathbb{C}(x_1, \dots, x_n) = \mathcal{K}_0 \subset \mathcal{K}_1 \dots \subset \mathcal{K}_p = \mathcal{K}$$

such that $K_i \subset K_{i+1}$ is one of the following

- *algebraic,*
- *Picard-Vessiot:* $K_{i+1} = K_i(H_p^q)$ *where the H's are entries of a fundamental solution of linear equations* $\partial_{x_\ell} H_p^q = A_{\ell,p}^m H_m^q$ *with A's matrices with entries in* K_i*,*
- *'Abelian': There exists an Abelian function* A *of* m *arguments and* $f_1, \ldots, f_m \in K_i$*,* $K_{i+1} = K_i < A(f_1, \ldots, f_m) >$*.*

Most of the studies of Painlevé equations focus on their classical solutions ($n = 1$ in the definition) but it fails to explain the 'solvability' of the Picard-Painlevé equations. Nevertheless it is well known that this equation has classical first integrals ($n = 3$ in the definition) ([9]).

One of the basics properties of this type of extension is to have a finite transcendence degree. From the lemmas 4.4.5 and 4.4.6 of [2] and proposition 1.3 of this article, one deduces the particular form of the Galois groupoid of a vector field with classical first integrals.

Proposition 2.2 *Let* X *be a divergence free vector field on* $\mathbb{C}^3$ *with two independent classical first integrals. Then its Galois groupoid is transversally affine or imprimitive in codimension one.*

In fact all the imprimitive in codimension one cases cannot occur. It is possible to give a more precise statement for this proposition but it is not needed in this paper.

Conversely any transversally affine vector field has classical first integrals. They are built by solving the following linear system:

$$dL_i^j = \sum_k L_i^k \theta_k^j \qquad dH_i = \sum_j L_i^j \theta_j.$$

These functions are more than classical, they are in a Picard-Vessot extension of the field of rationnal functions of $\mathbb{C}^3$.

3 The first integrals of PP_6

In this section, computation of special first integrals of PP_6 is done to prove theorem 1.4. This computation follows P.Painlevé [9].

The PP_6 equations was discovered by E.Picard as the pull-back of a linear order two equation by a transcendantal function. For this reason it is solvable by the formula 1. This formula and the pull-back are given by a integral with x as parameter

$$\int_0^{y(x)} \frac{d\xi}{\sqrt{\xi(\xi-1)(\xi-x)}} = a\omega_1(x) + b\omega_2(x),$$

$\omega_{1,2}$ form a basis of periods of $t^2 = y(y-1)(y-x)$ i.e. the right hand side of the equality is the general solution of the Picard-Fuchs (PF) equation $4x(x-1)w'' - 4(2x-1)w' - w = 0$. Let X_{PP} and X_{PF} be the vector fields corresponding to the PP_6 and PF equations on their phase spaces. X_{PP} is the pull-back of X_{PF} by the following map:

$$\begin{cases} x = x \\ w = \displaystyle\int_0^y \frac{d\xi}{\sqrt{\xi(\xi-1)(\xi-x)}} \\ w' = \frac{y'}{\sqrt{y(y-1)(y-x)}} + \displaystyle\int_0^y \frac{d\xi}{2(\xi-x)\sqrt{\xi(\xi-1)(\xi-x)}}. \end{cases}$$

To get two first integrals for PP_6, one pull-backs two first integrals of PF. Because it is a linear equation, it has first integrals linear on the 'fibers' *i.e.* $H = \alpha(x)w + \beta(x)w'$ where (α, β) is a solution of the PF's adjoint equation :

$$\begin{cases} \alpha' = \beta\frac{-1}{4x(x-1)} \\ \beta' = \beta\frac{1-2x}{x(x-1)} - \alpha. \end{cases}$$

Let $H_{\alpha,\beta}$ be the function

$$\frac{y'\beta}{\sqrt{y(y-1)(y-x)}} + \int_0^y (\alpha + \frac{\beta}{2(\xi - x)})\frac{d\xi}{\sqrt{\xi(\xi-1)(\xi-x)}}.$$

It turns out that this function is a primitive of a closed 1-form with coefficients in the field $\mathbb{C}(x, \alpha, \beta, y, y'\sqrt{y(y-1)(y-x)})$:

$$\begin{aligned} H_{\alpha,\beta} = & \frac{y'\beta}{\sqrt{y(y-1)(y-x)}} \\ & + \int (\alpha + \frac{\beta}{2(y-x)})\frac{dy}{\sqrt{y(y-1)(y-x)}} - \frac{\beta y(y-1)}{2x(1-x)(y-x)}\frac{dx}{\sqrt{y(y-1)(y-x)}}. \end{aligned}$$

The transversally affine structure is given by a sequence of 1-forms. It is derived from two first integrals $H_1 = H_{\alpha_1,\beta_1}$ and $H_2 = H_{\alpha_2,\beta_2}$ with $\alpha_{1,2}$ and $\beta_{1,2}$ the entries of a fundamental solution F of the PF's adjoint equation. The derivatives give

$$dH_i = L_i^1\theta_1 + L_i^2\theta_2$$

where the θ's are the 1-forms given by the Hamiltonian form of PP_6. By construction, H_1 and H_2 are linear in $\alpha_{1,2}$ and $\beta_{1,2}$ and the matrix L equals $(\sqrt{y(y-1)(y-x)})^{-1}MF$ for a matrix M with entries in $\mathbb{C}(x, y, y')$.

The matrix F satisfies a linear equation $dF = F\Omega$, Ω a matrix of 1-forms with coefficients in $\mathbb{C}(x, y, y')$. This implies that $dL = L\underline{\Omega}$ with $\underline{\Omega} = -(1/2)(1/y + 1/(y-1) + 1/(y-x))dy\mathrm{Id} + dMM^{-1} + M\Omega M^{-1}$, Id is the identity matrix. Because $d\Omega = 0$, one gets $d\underline{\Omega} = \underline{\Omega} \wedge \underline{\Omega}$. The 1-forms $\theta_{1,2}$ and the coefficients $\theta_{1,2}^{1,2}$ of $\underline{\Omega}$ give the transversally affine structure.

References

[1] Cartan, É. - *Les sous-groupes des groupes continus de transformations*, Ann. Sci. École Normale Sup. **25** (1908) 57–194

[2] Casale, G. - *Sur le groupoïde de Galois d'un feuilletage*, available on `http://www.perso.univ-rennes1.fr/guy.casale/`

[3] Casale, G. - *Le groupoïde de Galois de P_1 et son irréductibilité*, to appear in Commentarii Mathematici

[4] Dubrovin B., Mazzocco M. *On the Reductions and Classical Solutions of the Schlesinger equations.* `math.DG/0610327`

[5] Kolchin, E. R. - *Differential Algebra and Algebraic Groups*, Pure and Applied Mathematics, Vol.**54** Academic Press, New York-London (1973)

[6] Malgrange, B. - *Le groupoïde de Galois d'un feuilletage*, Monographie **38** vol **2** de L'enseignement mathématique (2001)

[7] Okamoto, K. - *Studies on the Painlevé equations. I. Sixth Painlevé equation P_{VI}*, Ann. Mat. Pura Appl. (4) **146** (1987) 337–381.

[8] Mazzocco, M. - *Picard and Chazy solutions to the Painlevé VI equation*, Math. Ann. **321** (2001) 157–195

[9] Painlevé, P. - *Leçons de Stockholm* (1895), Oeuvres complètes Tome **1**, éditions du CNRS (1972)

[10] Picard, E. - *Mémoire sur la théorie des fonctions algébriques de deux variables*, Journal de Liouville **5** (1889) 135–319

[11] Umemura, H. - *On the irreducibility of the first differential equation of Painlevé*, Alg. Geom. and Com. Alg. in honor of M. Nagata (1987) 771–789

[12] Umemura, H. - *Painlevé equations and classical functions* [translation of Sūgaku **47** (1995)] Sugaku Expositions. **11** (1998) 77–100

[13] Umemura, H. - *Galois theory of algebraic and differential equation*, Nagoya Math. J. vol **144** (1996) 1–58

[14] Umemura, H. - *Differential Galois theory of infinite dimension*, Nagoya Math. J. vol **144** (1996) 59–135

[15] Watanabe, H. - *Birrational canonical transformations ans classical solutions of the sixth Painlevé equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci (4) Vol **XXVII** (1998) 379–425