

Matching Procedure for the Sixth Painlevé Equation

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Abstract

The matching procedure is a constructive way of using the isomonodromy deformation method, to obtain the critical behavior of Painlevé VI transcendents and solve the connection problem. This procedure yields two and one parameter families of solutions, including trigonometric and logarithmic behaviors, and three classes of solutions with Taylor expansion at a critical point.

1 Introduction

We present the results of our paper [11]. The sixth Painlevé equation is:

$$\frac{d^2y}{dx^2} = \frac{1}{2} \left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right] \left(\frac{dy}{dx} \right)^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right] \frac{dy}{dx} + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[\alpha + \beta \frac{x}{y^2} + \gamma \frac{x-1}{(y-1)^2} + \delta \frac{x(x-1)}{(y-x)^2} \right], \quad (\text{PVI}).$$

The generic solution has essential singularities and/or branch points in $0, 1, \infty$. Its behavior at these points will be called *critical*. The other singularities, which depend on the initial conditions, are poles. A solution of PVI can be analytically continued to a meromorphic function on the universal covering of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. For generic values of the integration constants and of the parameters $\alpha, \beta, \gamma, \delta$, it cannot be expressed via elementary or classical transcendental functions. For this reason, it is called a *Painlevé transcendent*. Solving (PVI) means: i) Determine the critical behavior of the transcendents at the *critical points* $x = 0, 1, \infty$. Such a behavior must depend on two integration constants. ii) Solve the *connection problem*, namely: find the relation between couples of integration constants at $x = 0, 1, \infty$.

We use a matching procedure to study the above two problems. The procedure allows us to compute the first leading terms of the critical behavior at a critical point and the associated monodromy data. This procedure is essentially the isomonodromy deformation method. The reason for our terminology is that we make particular use of the matching between local solutions of two different reductions of the linear system of ODE, associated to (PVI) by the isomonodromy deformation theory. This matching allows us to obtain the leading term(s) of the asymptotic behavior of a corresponding Painlevé transcendent $y(x)$. In this sense, we say that our approach is constructive. Namely, we don't assume any behavior of $y(x)$; rather, we obtain it from the matching condition. This differs from other authors' approach, who start by assuming a given asymptotics for $y(x)$ and then compute the corresponding monodromy data (and so they solve the connection problem). This kind of approach was successfully used for some of the Painlevé equations and allowed many progresses. Our approach is developed to tackle with the cases when we don't know - or we are not able to guess - the asymptotic

behavior. In the case of (PVI), we may say that most of the solutions are known. But for some points in the space of monodromy data, we still don't know the corresponding critical behaviors. Our work is motivated by the need to explore these remaining cases.

Once the local matching is done, we proceed with a global description of the solutions of the associate linear system of ODE, in order to compute its *monodromy data*. These are the monodromy data associated to the solution $y(x)$, of which the asymptotic behavior has been obtained by the precedent step. Again, this computation is done by a (global) matching, among solutions of the two reduced systems and that of the original one. This is the main powerful point of the isomonodromy deformation method. The monodromy data are computed in terms of the coefficients of the linear system of ODE, which are elementary functions of the parameters (namely, the integration constants) appearing in the leading term of the asymptotic behavior of $y(x)$. The inversion of the formulae expressing the monodromy data, gives the leading term of $y(x)$ in term of the monodromy data.

The procedure can be repeated at the other singularities $x = 1, \infty$. In case of (PVI), $x = 0, 1, \infty$ are equivalent by symmetry transformations. These facts allow to solve the connection problem ([16], [6], [7], [9], [3]).

The work of Jimbo [16] is the first on the subject. For generic values of $\alpha, \beta, \gamma, \delta$, PVI admits a 2-parameter class of solutions, with the following critical behaviors:

$$y(x) = ax^{1-\sigma}(1 + O(|x|^\epsilon)), \quad x \rightarrow 0, \quad (1)$$

$$y(x) = 1 - a^{(1)}(1-x)^{1-\sigma^{(1)}}(1 + O(|1-x|^\epsilon)), \quad x \rightarrow 1, \quad (2)$$

$$y(x) = a^{(\infty)}x^{\sigma^{(\infty)}}(1 + O(|x|^{-\epsilon})), \quad x \rightarrow \infty, \quad (3)$$

where ϵ is a small positive number, $a^{(i)}$ and $\sigma^{(i)}$ are complex numbers such that $a^{(i)} \neq 0$ and $0 < \Re\sigma < 1$, $0 < \Re\sigma^{(1)} < 1$, $0 < \Re\sigma^{(\infty)} < 1$. We remark that x converges to the critical points *inside a sector* with vertex on the corresponding critical point. The connection problem is to finding the relation among the three pairs (σ, a) , $(\sigma^{(1)}, a^{(1)})$, $(\sigma^{(\infty)}, a^{(\infty)})$. In [16] the problem is solved by the isomonodromy deformation method. In particular, the exponents are determined by the relations:

$$2 \cos(\pi\sigma) = \text{tr}(M_0M_x), \quad 2 \cos(\pi\sigma^{(1)}) = \text{tr}(M_1M_x), \quad 2 \cos(\pi\sigma^{(\infty)}) = \text{tr}(M_0M_1).$$

Here M_0, M_x, M_1 are monodromy matrices to be introduced below.

The above class of solutions was enlarged in [23] and [9], to the values $\sigma \in \mathbf{C}$, $\sigma \notin (-\infty, 0] \cup [1, +\infty)$ (here we consider $x \rightarrow 0$). When $\Re\sigma \geq 1$ or $\Re\sigma \leq 0$, the critical behavior is like the above, but it holds for $x \rightarrow 0$ in a spiral-shaped domain in the universal covering of a punctured neighborhood of $x = 0$, along a paths joining a point x_0 to $x = 0$. Along special paths which approach the movable poles, these solution may have behavior $y(x) \sim \sin^{-2}(\frac{i\sigma}{2} \ln x + \varphi(x, a))$, where $\varphi(x, a)$ is a phase depending on the parameter a . The transformation $\sigma \mapsto \pm\sigma + 2N$, $N \in \mathbf{Z}$, leaves the identity $\text{tr}(M_0M_x) = 2 \cos(\pi\sigma)$ invariant. Its effect on the solutions is studied in [9]. As a result, one can reduce to the values $0 \leq \Re\sigma \leq 1$, $\sigma \neq 0, 1$. The reader may find a synthetic description of these results in the review paper [10].

It is an open problem to determine the critical behavior, say at $x = 0$, for $\sigma = 0, 1$. To be more precise, the problem is encountered when $\text{tr}(M_iM_j) = \pm 2$. These are precisely the points of the space of monodromy data mentioned above, in correspondence of which we do not know the critical behavior. In addition, certain non-generic values of $\alpha, \beta, \gamma, \delta$ are not yet studied. The matching procedure is motivated by the need to explore these unknown cases.

As a result of the matching procedure, we obtain:

R1) A two-parameter family of solutions, of the type found by Jimbo [16]. Besides, we show that there are solutions with trigonometric behavior.

R2) One-parameter families of solutions, including a class of logarithmic solutions. Together with the results of [23] and [9], R1) and R2) will cover all cases $\text{tr}(M_i M_j) \neq -2$, namely $\sigma \neq 1$. [see Proposition 1]. By symmetry transformations, some of the cases $\text{tr}(M_i M_j) = -2$ can be obtained from the above results (for example, the *Chazy solutions* [20]).

R3) The solutions which admit a Taylor expansion at $x = 0$ [Proposition 2].

R4) We compute the corresponding monodromy data [Proposition 3].

In virtue of the symmetries of (PVI) (birational transformations of $(x, y(x))$), it can be shown that the solutions with Taylor expansion at $x = 0$, obtained by the matching procedure, are the representatives of three equivalent classes, which include all the solutions admitting a Taylor expansion at a critical point. If we define σ through the relation $\text{tr}(M_0 M_x) = 2 \cos(\pi\sigma)$, the representatives of three equivalent classes correspond to values $\sigma = 0$, $\sigma = \pm(\theta_1 \pm \theta_\infty)$ and $\sigma = 1$.

A further step in the study of PVI, is the problem of the systematic classification of all the solutions of (PVI) in terms of the monodromy data of the associated linear system. As we discussed above, the matching procedure is effective to produce new solutions, associated to monodromy data for which the connection problem has not yet been studied. Therefore, it is a tool to study the classification problem. This classification will be done in another paper.

A matching procedure, to obtain asymptotic behaviors and monodromy data in the framework of the isomonodromy deformation method, was suggested by Its and Novokshenov in [13], for the second and third Painlevé equations. The work by Jimbo [16] can be regarded as an implicit matching procedure. This method was further developed and used by Kapaev, Kitaev, Andreev, and Vartanian. Here we cite the case of the fifth Painlevé equation, in [2]. An analogous matching scheme is used in [1], for a different problem (limit PVI \rightarrow PV).

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2 Matching Procedure

PVI is the isomonodromy deformation equation of a Fuchsian system of differential equations [17]:

$$\frac{d\Psi}{d\lambda} = A(\lambda, x, \theta) \Psi, \quad A(\lambda, x, \theta) := \left[\frac{A_0(x, \theta)}{\lambda} + \frac{A_x(x, \theta)}{\lambda - x} + \frac{A_1(x, \theta)}{\lambda - 1} \right], \quad \lambda \in \mathbf{C}. \quad (4)$$

The 2×2 matrices $A_i(x, \theta)$ depend on x , in such a way that it is possible to find a fundamental solution $\Psi(\lambda, x)$ with monodromy independent of (local deformations of) x . They also depend on the parameters $\alpha, \beta, \gamma, \delta$ of PVI through more elementary parameters $\theta = (\theta_0, \theta_x, \theta_1, \theta_\infty)$ according to the following relations:

$$A_0 + A_1 + A_x = -\frac{\theta_\infty}{2} \sigma_3, \quad \text{Eigenvalues } (A_i) = \pm \frac{1}{2} \theta_i, \quad i = 0, 1, x;$$

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad -\beta = \frac{1}{2}\theta_0^2, \quad \gamma = \frac{1}{2}\theta_1^2, \quad \left(\frac{1}{2} - \delta\right) = \frac{1}{2}\theta_x^2, \quad \theta_\infty \neq 0. \quad (5)$$

Here σ_3 is the Pauli matrix. The equations of monodromy-preserving deformation (Schlesinger equations), can be written in Hamiltonian form and reduce to PVI, being the transcendent $y(x)$ solution of $A(y(x), x, \theta)_{1,2} = 0$. Namely:

$$y(x) = \frac{x (A_0)_{12}}{x [(A_0)_{12} + (A_1)_{12}] - (A_1)_{12}}, \quad (6)$$

The matrices $A_i(x, \theta)$, $i = 0, x, 1$, depend on $y(x)$, $\frac{dy(x)}{dx}$ and $\int y(x)$ through rational functions, which are given in [17]. In short, we will write $A_i = A_i(x)$.

The product of the monodromy matrices M_0, M_x, M_1 of a fundamental matrix solution Ψ at $\lambda = 0, x, 1$ respectively, is equal to the monodromy at $\lambda = \infty$. The order of the products depends on the choice of a basis of loops.

2.1 Leading Terms of $y(x)$ as a result of Matching

Since we are considering $x \rightarrow 0$, we divide the λ -plane into two domains. The “outside” domain is defined for λ sufficiently big:

$$|\lambda| \geq |x|^{\delta_{OUT}}, \quad \delta_{OUT} > 0. \quad (7)$$

Therefore, (4) can be written as:

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{A_x}{\lambda} \sum_{n=1}^{\infty} \left(\frac{x}{\lambda}\right)^n + \frac{A_1}{\lambda - 1} \right] \Psi. \quad (8)$$

The “inside” domain is defined for λ comparable with x , namely:

$$|\lambda| \leq |x|^{\delta_{IN}}, \quad \delta_{IN} > 0. \quad (9)$$

Therefore, $\lambda \rightarrow 0$ as $x \rightarrow 0$, and we rewrite (4) as:

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} - A_1 \sum_{n=0}^{\infty} \lambda^n \right] \Psi. \quad (10)$$

If the behavior of $A_0(x)$, $A_1(x)$ and $A_x(x)$ is sufficiently good, we expect that the higher order terms in the series of (8) and (10) are small corrections, which can be neglected when $x \rightarrow 0$. If this is the case, (8) and (10) reduce respectively to:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{A_x}{\lambda} \sum_{n=1}^{N_{OUT}} \left(\frac{x}{\lambda}\right)^n + \frac{A_1}{\lambda - 1} \right] \Psi_{OUT}, \quad (11)$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} - A_1 \sum_{n=0}^{N_{IN}} \lambda^n \right] \Psi_{IN}, \quad (12)$$

where N_{IN}, N_{OUT} are suitable integers. The simplest reduction is to Fuchsian systems:

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[\frac{A_0 + A_x}{\lambda} + \frac{A_1}{\lambda - 1} \right] \Psi_{OUT}, \quad (13)$$

$$\frac{d\Psi_{IN}}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} \right] \Psi_{IN}. \quad (14)$$

It is a feature of [11] that we used reduced non-Fuchsian systems. In the literature, the reduction to Fuchsian systems has been privileged, but in some relevant cases it cannot be used, being the reduction to non-Fuchsian systems necessary.

Generally speaking, we can parameterize the elements of $A_0 + A_x$ and A_1 of (13) in terms of θ_1 , the eigenvalues of $A_0 + A_x$ and the eigenvalues θ_∞ of $A_0 + A_x + A_1$. We also need an additional unknown function of x . In the same way, we can explicitly parameterize the elements of A_0 and A_x in (14) in terms of θ_0 , θ_x , the eigenvalues of $A_0 + A_x$ and another additional unknown function of x . When the reductions (11) and (12) are non-fuchsian, particular care must be paid [11]. Our purpose is to find the leading term of the unknown functions when $x \rightarrow 0$, in order to determine the critical behavior of $A_0(x)$, $A_1(x)$, $A_x(x)$ and (6). The leading term can be obtained as a result of two facts:

i) Systems (11) and (12) are isomonodromic. This imposes constraints on the form of the unknown functions. Typically, one of them must be constant.

ii) [Local Matching]. Two fundamental matrix solutions $\Psi_{OUT}(\lambda, x)$, $\Psi_{IN}(\lambda, x)$ must match in the region of overlap, provided this is not empty:

$$\Psi_{OUT}(\lambda, x) \sim \Psi_{IN}(\lambda, x), \quad |x|^{\delta_{OUT}} \leq |\lambda| \leq |x|^{\delta_{IN}}, \quad x \rightarrow 0 \quad (15)$$

This relation is to be intended in the sense that the leading terms of the local behavior of Ψ_{OUT} and Ψ_{IN} for $x \rightarrow 0$ must be equal. This determines a simple relation between the two functions of x appearing in A_0 , A_x , A_1 , $A_0 + A_x$. (15) also implies that $\delta_{IN} \leq \delta_{OUT}$.

To summarize, *matching* two fundamental solutions of the reduced *isomonodromic* systems (11) and (12), we obtain the leading term(s), for $x \rightarrow 0$, of the entries of the matrices of the original system (4). The only assumption about the asymptotic behavior is equation (15).

2.2 Computation of the Monodromy Data

Let Ψ be a fundamental matrix solution of (4), and let M_0 , M_x , M_1 , M_∞ be its monodromy matrices at $\lambda = 0, x, 1, \infty$ respectively (M_∞ is the product of M_0 , M_x , M_1 , the order depending on the choice of a basis of loops). As a consequence of isomonodromicity, there exists a fundamental solution Ψ_{OUT} of (11) such that

$$M_1^{OUT} = M_1, \quad M_\infty^{OUT} = M_\infty,$$

where M_1^{OUT} and M_∞^{OUT} are the monodromy matrices of Ψ_{OUT} at $\lambda = 1, \infty$. Moreover, $M_0^{OUT} = M_0 M_x$ or $M_x M_0$, depending on the order of loops. A detailed proof of these facts can be found in [7]. There also exists a fundamental solution Ψ_{IN} of (12) such that:

$$M_0^{IN} = M_0, \quad M_x^{IN} = M_x,$$

where M_0^{IN} and M_x^{IN} are the monodromy matrices of Ψ_{IN} at $\lambda = 0, x$.

The method is effective when the monodromy of the reduced systems (11), (12) can be explicitly computed. This is the case when the reduction is Fuchsian (namely (13), (14)), because Fuchsian systems with three singular points are equivalent to a Gauss hypergeometric equation (see Appendix 1 of [11]). For the reduction to non-Fuchsian systems, in general we can compute the monodromy when (11), (12) are solvable in terms of special or elementary functions.

In order for this procedure to work, the (locally) matching solutions Ψ_{OUT} and Ψ_{IN} of subsection 2.1, must match with a fundamental matrix solution Ψ of (4). Namely, we need to impose that Ψ_{OUT} matches with Ψ in some domain of the λ plane, and that Ψ_{IN} matches with *the same* Ψ in another domain of the λ plane. The standard choice of Ψ is as follows:

$$\Psi(\lambda) = \begin{cases} \left[I + O\left(\frac{1}{\lambda}\right) \right] \lambda^{-\frac{\theta_\infty}{2}\sigma_3} \lambda^{R_\infty}, & \lambda \rightarrow \infty; \\ \psi_0(x)[I + O(\lambda)] \lambda^{\frac{\theta_0}{2}\sigma_3} \lambda^{R_0} C_0, & \lambda \rightarrow 0; \\ \psi_x(x)[I + O(\lambda - x)] (\lambda - x)^{\frac{\theta_x}{2}\sigma_3} (\lambda - x)^{R_x} C_x, & \lambda \rightarrow x; \\ \psi_1(x)[I + O(\lambda - 1)] (\lambda - 1)^{\frac{\theta_1}{2}\sigma_3} (\lambda - 1)^{R_1} C_1, & \lambda \rightarrow 1; \end{cases} \quad (16)$$

Here $\psi_0(x)$, $\psi_x(x)$, $\psi_1(x)$ are the diagonalizing matrices of $A_0(x)$, $A_1(x)$, $A_x(x)$ respectively. They are defined by multiplication to the right by arbitrary diagonal matrices, possibly depending on x . C_κ , $\kappa = \infty, 0, x, 1$, are invertible *connection matrices*, independent of x [17]. Each R_κ , $\kappa = \infty, 0, x, 1$, is also independent of x , and:

$$R_\kappa = 0 \text{ if } \theta_\kappa \notin \mathbf{Z}, \quad R_\kappa = \begin{cases} \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, & \text{if } \theta_\kappa > 0 \text{ integer} \\ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, & \text{if } \theta_\kappa < 0 \text{ integer} \end{cases}$$

If $\theta_i = 0$, $i = 0, x, 1$, then R_i is to be considered the Jordan form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ of A_i . If $\theta_\infty = 0$, $R_\infty = 0$. Note that for the loop $\lambda \mapsto \lambda e^{2\pi i}$, $|\lambda| > \max\{1, |x|\}$, we immediately compute the monodromy at infinity:

$$M_\infty = \exp\{-i\pi\theta_\infty\} \exp\{2\pi i R_\infty\}.$$

Let Ψ_{OUT} and Ψ_{IN} be the solutions of (11) and (12) matching as in (15). We explain how they are matched with (16).

(*) Matching $\Psi \leftrightarrow \Psi_{OUT}$:

$\lambda = \infty$ is a Fuchsian singularity of (11), with residue $-A_\infty/\lambda$. Therefore, we can always find a fundamental matrix solution with behavior:

$$\Psi_{OUT}^{Match} = \left[I + O\left(\frac{1}{\lambda}\right) \right] \lambda^{-\frac{\theta_\infty}{2}\sigma_3} \lambda^{R_\infty}, \quad \lambda \rightarrow \infty.$$

This solution matches with Ψ . Also $\lambda = 1$ is a Fuchsian singularity of (11). Therefore, we have:

$$\Psi_{OUT}^{Match} = \psi_1^{OUT}(x)[I + O(\lambda - 1)] (\lambda - 1)^{\frac{\theta_1}{2}\sigma_3} (\lambda - 1)^{R_1} C_1^{OUT}, \quad \lambda \rightarrow 1;$$

Here C_1^{OUT} is a suitable connection matrix. $\psi_1^{OUT}(x)$ is the matrix that diagonalizes the leading terms of $A_1(x)$. Therefore, $\psi_1(x) \sim \psi_1^{OUT}(x)$ for $x \rightarrow 0$. As a consequence of isomonodromicity, R_1 is the same of Ψ .

As a consequence of the matching $\Psi \leftrightarrow \Psi_{OUT}^{Match}$, the monodromy of Ψ at $\lambda = 1$ is:

$$M_1 = C_1^{-1} \exp\{i\pi\theta_1\sigma_3\} \exp\{2\pi i R_1\} C_1, \quad \text{with } C_1 \equiv C_1^{OUT}.$$

We finally need an invertible connection matrix C_{OUT} to connect Ψ_{OUT}^{Match} with the solution Ψ_{OUT} appearing in (15). Namely, $\Psi_{OUT}^{Match} = \Psi_{OUT} C_{OUT}$.

(*) **Matching** $\Psi \leftrightarrow \Psi_{IN}$:

As a consequence of the matching $\Psi \leftrightarrow \Psi_{OUT}^{Match}$, we have to choose the IN-solution which matches with Ψ_{OUT}^{Match} . This is $\Psi_{IN}^{Match} := \Psi_{IN} C_{OUT}$.

Now, $\lambda = 0, x$ are Fuchsian singularities of (12). Therefore:

$$\Psi_{IN}^{Match} = \begin{cases} \psi_0^{IN}(x)[I + O(\lambda)] \lambda^{\frac{\theta_0}{2}\sigma_3} \lambda^{R_0} C_0^{IN}, & \lambda \rightarrow 0; \\ \psi_x^{IN}(x)[I + O(\lambda - x)] (\lambda - x)^{\frac{\theta_x}{2}\sigma_3} (\lambda - x)^{R_x} C_x^{IN}, & \lambda \rightarrow x; \end{cases}$$

The above hold for fixed small $x \neq 0$. Here C_0^{IN} and C_x^{IN} are suitable connection matrices. $\psi_0^{IN}(x)$ and $\psi_x(x)^{IN}$ are diagonalizing matrices of the leading terms of $A_0(x)$ and $A_x(x)$. For $x \rightarrow 0$ they match with $\psi_0(x)$ and $\psi_x(x)$ of Ψ in (16). On the other hand, as a consequence of isomonodromicity, the matrices R_0 and R_x are the same of Ψ .

By virtue of the matching $\Psi \leftrightarrow \Psi_{IN}^{Match}$, the connection matrices C_0 and C_x coincide with the x -independent connection matrices C_0^{IN} , C_x^{IN} respectively. As a result, we obtain the monodromy matrices for Ψ :

$$\begin{aligned} M_0 &= C_0^{-1} \exp\{i\pi\theta_0\sigma_3\} \exp\{2\pi i R_0\} C_0, & C_0 &\equiv C_0^{IN}, \\ M_x &= C_x^{-1} \exp\{i\pi\theta_x\sigma_3\} \exp\{2\pi i R_x\} C_x, & C_x &\equiv C_x^{IN}. \end{aligned}$$

Our reduction is useful if the connection matrices C_1^{OUT} , C_0^{IN} , C_x^{IN} can be computed explicitly.

3 Results

In the following, it is understood that $x \rightarrow 0$ inside a sector. Namely, $\arg(x)$ is bounded.

3.1 Results R1 and R2

When (4) can be reduced to the Fuchsian systems (13) and (14), the matching procedure yields the behaviors of Proposition 1. Let σ be a complex number defined, up to sign, by:

$$\text{tr}(M_0 M_x) = 2 \cos(\pi\sigma), \quad |\Re\sigma| \leq 1.$$

Actually, $\pm\sigma/2$ are the eigenvalues of $\lim_{x \rightarrow 0}(A_0 + A_x)$.

Proposition 1 *Let $r \in \mathbf{C}$ and σ be as above, with the restriction $|\Re\sigma| < 1$. (PVI) has a family of solutions depending on the two parameters r, σ . The leading terms of the critical behavior for $x \rightarrow 0$ may be parametrized as follows:*

For $\sigma \neq 0$:

$$y(x) \sim \begin{cases} \frac{1}{r} \frac{[\sigma^2 - (\theta_0 + \theta_x)^2][(\theta_0 - \theta_x)^2 - \sigma^2]}{16\sigma^3} x^{1-\sigma}, & \text{if } \Re\sigma > 0; \\ -\frac{r}{\sigma} x^{1+\sigma}, & \text{if } \Re\sigma < 0; \\ x \left\{ iA \sin(i\sigma \ln x + \phi) + \frac{\theta_0^2 - \theta_x^2 + \sigma^2}{2\sigma^2} \right\}, & \text{if } \Re\sigma = 0. \end{cases} \quad (17)$$

In the above formulae, $r \neq 0$ and

$$\phi := i \ln \frac{2r}{\sigma A}, \quad A := \left[\frac{\theta_0^2}{\sigma^2} - \left(\frac{\theta_0^2 - \theta_x^2 + \sigma^2}{2\sigma^2} \right)^2 \right]^{\frac{1}{2}}$$

For special values of $\sigma \neq 0$:

$$y(x) \sim \frac{\theta_0}{\theta_0 + \theta_x} x \mp \frac{r}{\theta_0 + \theta_x} x^{1+\sigma}, \quad \sigma = \pm(\theta_0 + \theta_x) \neq 0, \quad (18)$$

$$y(x) \sim \frac{\theta_0}{\theta_0 - \theta_x} x \mp \frac{r}{\theta_0 - \theta_x} x^{1+\sigma}, \quad \sigma = \pm(\theta_0 - \theta_x) \neq 0. \quad (19)$$

For $\sigma = 0$:

$$y(x) \sim \begin{cases} x \left\{ \frac{\theta_x^2 - \theta_0^2}{4} \left[\ln x + \frac{4r + 2\theta_0}{\theta_0^2 - \theta_x^2} \right]^2 + \frac{\theta_0^2}{\theta_0^2 - \theta_x^2} \right\}, & \theta_0 \neq \pm\theta_x, \\ x (r \pm \theta_0 \ln x), & \theta_0 = \pm\theta_x. \end{cases} \quad (20)$$

Comments:

1) r can be computed as a function of the monodromy data. See (36) and comments there. The branch of the square root appearing in A is arbitrary (its change does not affect $y(x)$). $x \rightarrow 0$ in a sector of width less than 2π .

2) Sub-cases of theorem 1.

i) When $\sigma \neq 0$, the result of the Theorem includes the sub-cases (18) and (19). If $r = 0$, $\theta_0 \neq 0$, $\theta_0 \pm \theta_x \notin \mathbf{Z}$, direct substitution into (PVI) gives the two Taylor expansions (28).

If $r \neq 0$, (18) and (19) are a 1-parameter family, with the restriction $|\Re\sigma| < 1$. The symmetry (27), to be introduced below, transforms them into the solutions (31), to be discussed later, the leading terms being respectively:

$$y(x) \sim \frac{\theta_\infty + \theta_1 - 1}{\theta_\infty - 1} \left(1 \pm \frac{r}{\theta_\infty - 1} x^\omega \right), \quad \omega = \pm(\theta_\infty + \theta_1 - 1) \neq 0,$$

$$y(x) \sim \frac{\theta_\infty - \theta_1 - 1}{\theta_\infty - 1} \left(1 \pm \frac{r}{\theta_\infty - 1} x^\omega \right), \quad \omega = \pm(\theta_\infty - \theta_1 - 1) \neq 0,$$

with the restriction $|\Re\omega| < 1$.

ii) The case $\sigma = 0$ includes the sub-case $y(x) \sim rx$, which occurs for $\theta_0 = \theta_x$, $\theta_0 = 0$. By direct substitution in (PVI) we obtain a series:

$$y(x) = r x + \sum_{n=3}^{\infty} b_n(r, \theta_1, \theta_\infty) x^n, \quad \theta_0 = \theta_x = 0, \quad r \neq 0, 1.$$

This is the solution (30), to be further discussed later. Note that the special sub-sub-case $\theta_0 = \theta_x = \theta_1 = 0$ has applications in the theory of *semi-simple Frobenius manifolds* of dimension three [5] [8].

3) The first two solutions in formula (17) were studied in [16]. Their existence was proved by assuming that the matrices A_0 , A_x , A_1 have a certain critical behavior for $x \rightarrow 0$, and proving that such matrices solve the Schlesinger equations. Then, the monodromy data were computed by a reduction of (4) to the 'out' and 'in' systems. These solutions were further studied in [6], [7], [9], [3]. These solutions can be obtained without any assumption by the matching procedure, together with the solutions (20) and the third solution in (17), which do not appear in [16].

The class of the first two solutions (17) was enlarged in [23] and [9], as already discussed in the introduction, to the values $\sigma \in \mathbf{C}$, $\sigma \notin (-\infty, 0] \cup [1, +\infty)$.

4) All solutions with expansion:

$$y(x) = x(A_1 + B_1 \ln x + C_1 \ln^2 x + D_1 \ln^3 x + \dots) + x^2(A_2 + B_2 \ln x + \dots) + \dots, \quad x \rightarrow 0.$$

are included in proposition 1 and 2. Actually, only the following cases are possible:

$$y(x) = \begin{cases} \frac{\theta_0}{\theta_0 \pm \theta_x} x + O(x^2) & \text{[Taylor expansion]}, \\ x \left(\frac{\theta_0^2 - B_1^2}{\theta_0^2 - \theta_x^2} + B_1 \ln x + \frac{\theta_x^2 - \theta_0^2}{4} \ln^2 x \right) + x^2(\dots) + \dots, \\ x (A_1 \pm \theta_0 \ln x) + x^2(\dots) + \dots, & \text{and } \theta_0 = \pm \theta_x. \end{cases} \quad (21)$$

A_1 and B_1 are parameters. We see that the higher orders in (20) are $O(x^2 \ln^m x)$, for some integer $m > 0$.

5) The symmetry (27) applied to solutions (20) gives:

$$y(x) = \frac{4}{[\theta_1^2 - (\theta_\infty - 1)^2] \ln^2 x} \left[1 + \frac{8r + 4(\theta_\infty - 1)}{\theta_1^2 - (\theta_\infty - 1)^2} \frac{1}{\ln x} + O\left(\frac{1}{\ln^2 x}\right) \right], \quad (22)$$

and

$$y(x) = \frac{\pm 1}{(\theta_\infty - 1) \ln x} \left[1 \mp \frac{r}{(\theta_\infty - 1) \ln x} + O\left(\frac{1}{\ln^2 x}\right) \right], \quad \theta_\infty \mp \theta_1 = 1.$$

The higher orders $O(1/\ln^2 x)$ include powers $x^n (\ln x)^{\pm m}$. The so called *Chazy solutions*, studied in [20] for the special case $\theta_0 = \theta_x = \theta_1 = 0$, $\theta_\infty = -1$, have the behavior (22).

6) In [4] it is proved that (PVI) has solutions with expansion at $x = \infty$, or $x = 0$, of the form $y = c_r x^r + \sum_s c_s x^s$, $c_r \in \mathbf{C}$. The c_s 's are either complex constants or polynomials in $\ln x$. r and s are integer or complex. If r is complex, the restriction $\Re r \in (0, 1)$ holds. The method used in [4] is a power geometry technique. The connection problem and the characterization of the associated monodromy data are not studied.

3.2 Result R3

When the matching procedure is applied to non-Fuchsian systems (11) and (12), we obtain all the solutions that admit a Taylor expansion

$$y(x) = b_0 + b_1 x + b_2 x^2 + \dots = \sum_{n=0}^{\infty} b_n x^n, \quad x \rightarrow 0.$$

Precisely, we obtain the representative solutions of three equivalence classes, the equivalence relation being the birational transformations [22].

Proposition 2 *The solutions of (PVI) with Taylor expansion at $x = 0$ are divided into four equivalent classes (one being that of singular solutions $y = 0, 1, x$). The representatives can be chosen as follows:*

1) Singular solution $y = 1$.

2) $\theta_\infty \neq 1$, $\theta_1 - \theta_\infty \notin \mathbf{Z}$ [representative of $\theta_1 \pm \theta_\infty \notin \mathbf{Z}$]:

$$y(x) = \frac{\theta_1 - \theta_\infty + 1}{1 - \theta_\infty} + \frac{\theta_1[(\theta_1 - \theta_\infty)(\theta_1 - \theta_\infty + 2) + \theta_x^2 - \theta_0^2]}{2(\theta_\infty - 1)(\theta_\infty - \theta_1)(\theta_\infty - \theta_1 - 2)} x + \sum_{n=3}^{\infty} b_n(\theta_1, \theta_\infty, \theta_0, \theta_x) x^n. \quad (23)$$

The coefficients are rational functions of $\theta_0, \theta_\infty, \theta_0, \theta_x$, that can be obtained in a recursive way by substitution of the series into the PVI equation.

3) $\theta_1 = \theta_\infty \neq 1, \theta_0 = \pm\theta_x$ [representative of $\theta_1 \pm \theta_\infty \in \mathbf{Z}, \theta_x \pm \theta_0 \in \mathbf{Z}$]:

$$y(x) = \frac{1}{1 - \theta_\infty} + ax + \sum_{n=2}^{\infty} b_n(a; \theta_0, \theta_\infty) x^n. \quad (24)$$

The coefficients are rational functions of θ_0, θ_∞ and a parameter $a \in \mathbf{C}$, which can be recursively obtained by substitution into PVI.

4) $\theta_\infty = 1, \theta_1 = 0$ [representative of $\theta_1 \pm \theta_\infty \in \mathbf{Z}, \theta_\infty \in \mathbf{Z} \setminus \{0\}$]:

$$y(x) = a + \frac{1-a}{2}(1 + \theta_0^2 - \theta_x^2) x + \sum_{n=2}^{\infty} b_n(a; \theta_0; \theta_x) x^n. \quad (25)$$

The coefficients are rational functions of θ_0, θ_x and a parameter $a \in \mathbf{C}$, which can be recursively obtained by substitution into PVI.

The monodromy data associated to the above solutions is given in proposition 3. The symmetry $\theta_1 \mapsto -\theta_1$, which leaves (PVI) invariant, transforms (23) into:

$$y(x) = \frac{\theta_1 + \theta_\infty - 1}{\theta_\infty - 1} + \frac{\theta_1[(\theta_1 + \theta_\infty)(\theta_1 + \theta_\infty - 2) + \theta_x^2 - \theta_0^2]}{2(1 - \theta_\infty)(\theta_\infty + \theta_1)(\theta_\infty + \theta_1 - 2)} x + \sum_{n=3}^{\infty} b_n(-\theta_1, \theta_\infty, \theta_0, \theta_x) x^n. \quad (26)$$

Here $\theta_\infty \neq 1, \theta_1 + \theta_\infty \notin \mathbf{Z}$. The coefficients b_n are the same of (23).

The convergence of the Taylor series can be proved by a Briot-Bouquet like argument. The reader can find the general procedure in [14] and an application to the fifth Painlevé equation in [19]

Comments:

1) Characterization of solutions $y(x) = \sum_{n=0}^{\infty} b_n x^n, b_0 \neq 0$.

(a) There always exists one solution (23) when $\theta_1 - \theta_\infty \notin \mathbf{Z}$; there always exists one solution (26) when $\theta_1 + \theta_\infty \notin \mathbf{Z}$. The coefficients b_n depend rationally on $\theta_\kappa, \kappa = 0, x, 1, \infty$. (b) There is a one-parameter family of solutions equivalent to (24), when $\theta_1 \pm \theta_\infty \in \mathbf{Z}$ and $\theta_0 \pm \theta_x$ has a particular integer value. The coefficients b_n depend rationally on a complex parameter a and θ_∞, θ_0 . (c) Finally, there is a one-parameter family of solutions equivalent to (25), when $\theta_1 \pm \theta_\infty \in \mathbf{Z}$, and θ_∞ has a particular integer value; the coefficients b_n depend rationally on a complex parameter a and θ_0, θ_x . The singular solutions $y = 0, 1, x$ are possibly obtained by birational transformations of (23), (24), (25). The coefficients b_n can always be computed recursively by direct substitution into (PVI).

2) Characterization of solutions $y(x) = \sum_{n=1}^{\infty} b_n x^n, b_1 \neq 0$.

These solutions are obtained from those of proposition 2 by the symmetry.

$$\theta_x \mapsto \theta_1, \quad \theta_0 \mapsto \theta_\infty - 1, \quad \theta_1 \mapsto \theta_x, \quad \theta_\infty \mapsto \theta_0 + 1; \quad y(x) \mapsto \frac{x}{y(x)}. \quad (27)$$

The solutions obtained from the singular solution $y = 1$ and (23), (24), (25) are respectively:

1) Singular solution $y(x) = x$.

2) $\theta_0 \neq 0, \theta_0 \pm \theta_x \notin \mathbf{Z}$:

$$y(x) = \frac{\theta_0}{\theta_0 \pm \theta_x} x \pm \frac{\theta_0 \theta_x [(\theta_0 \pm \theta_x)^2 + \theta_1^2 - \theta_\infty^2 + 2\theta_\infty - 2]}{2(\theta_0 \pm \theta_x)^2 [(\theta_0 \pm \theta_x)^2 - 1]} x^2 + \sum_{n=3}^{\infty} b_n(\theta_0, \theta_x, \theta_1, \theta_\infty) x^n. \quad (28)$$

3) $\theta_0 + \theta_x = 1, \theta_0 \neq 0, \theta_1 = \pm(\theta_\infty - 1)$:

$$y(x) = \theta_0 x + a x^2 + \sum_{n=3}^{\infty} b_n(a; \theta_0, \theta_\infty) x^n. \quad (29)$$

4) $\theta_x = \theta_0 = 0$.

$$y(x) = ax + \frac{a(a-1)}{2} (\theta_1^2 - (\theta_\infty - 1)^2 - 1) x^2 + \sum_{n=3}^{\infty} b_n(a; \theta_1, \theta_\infty) x^n. \quad (30)$$

(a) (PVI) has always one or both solutions (28) when $\theta_0 \pm \theta_x \notin \mathbf{Z}$. Also when $\theta_0 + \theta_x$ (or $\theta_0 - \theta_x$) is integer, (PVI) has a solution (28) corresponding to $\theta_0 - \theta_x$ not integer (or $\theta_0 + \theta_x$ not integer). (b) When $\theta_0 + \theta_x$ or $\theta_0 - \theta_x$ is integer, (PVI) has a one-parameter family of solutions equivalent (by birational transformations) to (29); this family exists provided that $\theta_1 \pm \theta_\infty$ has a particular integer value. (c) When $\theta_0 + \theta_x$ or $\theta_0 - \theta_x$ is integer and θ_0 has a particular integer value, there is a one parameter family of solutions equivalent to (30).

3) (PVI) has a one-parameter family of solutions of the type:

$$y(x) = y_0(x) + y_1(x) ax^\omega + y_2(x) (ax^\omega)^2 + \dots = \sum_{N=0}^{\infty} y_N(x) (ax^\omega)^N, \quad x \rightarrow 0; \quad (31)$$

where the parameter is $a \in \mathbf{C}$, and the $y_N(x)$'s are Taylor series:

$$y_N(x) = \sum_{k=0}^{\infty} b_{k,N}(\theta_1, \theta_\infty, \theta_0, \theta_x) x^k, \quad x \rightarrow 0.$$

Either $y_0(x)$ is (26) and $\omega = \pm(\theta_1 + \theta_\infty - 1)$, or $y_0(x)$ is (23) and $\omega = \pm(\theta_\infty - \theta_1 - 1)$. The conditions $|\Re \omega| < 1, \omega \neq 0$ hold. The coefficients $b_{k,N}(\theta_1, \theta_\infty, \theta_0, \theta_x)$ are certain rational functions that can be recursively determined by direct substitution into (PVI). These solutions are the images of solutions (18) and (19) respectively, through the symmetry (27). Taylor solutions (23), (26) are a special case of (31), when the parameter is zero. Solutions (24) and (25) – and their images by symmetry – are one parameters families of type (31), in non generic cases when $\omega \in \mathbf{Z}$.

4) Solutions (23) and the equivalent solutions (26), (28) were also derived in [18] by substitution of a Taylor expansion in (PVI). The corresponding monodromy was computed by coalescence of singularities of a Heun's type (scalar) equation.

3.3 Monodromy: Result R4

In [11], we computed the monodromy for the Taylor-expanded solutions, which correspond to a reductions of system (4) to non-Fuchsian systems. Because of the symmetries of (PVI), we can limit ourselves to the monodromy data for the representative solutions (23), (24) and (25).

Proposition 3 a) Let $\theta_\kappa \notin \mathbf{Z}$, $\kappa = 0, 1, x, \infty$. A representation for the monodromy matrices of the solution (23) is:

$$\begin{aligned} M_0 &= C_{0\infty} \exp\{i\pi\theta_0\sigma_3\} C_{0\infty}^{-1}, \\ M_x &= C_{0\infty} C_{01}^{-1} \exp\{i\pi\theta_x\sigma_3\} C_{01} C_{0\infty}^{-1}, \\ M_1 &= \exp\{-i\pi\theta_1\sigma_3\}, \quad M_\infty = \exp\{-i\pi\theta_\infty\sigma_3\}. \end{aligned}$$

The matrices $C_{0\infty}$ and C_{01} are:

$$C_{0\infty} := \begin{bmatrix} \frac{\Gamma(1+\frac{\theta_1-\theta_\infty}{2})\Gamma(1+\theta_0)e^{i\frac{\pi}{2}[\theta_0+\theta_x+\theta_\infty-\theta_1]}}{\Gamma(\frac{\theta_0+\theta_x+\theta_1-\theta_\infty}{2}+1)\Gamma(\frac{\theta_0-\theta_x+\theta_1-\theta_\infty}{2}+1)} & \frac{\Gamma(1+\frac{\theta_1-\theta_\infty}{2})\Gamma(1-\theta_0)e^{i\frac{\pi}{2}[\theta_x-\theta_0+\theta_\infty-\theta_1]}}{\Gamma(-\frac{\theta_0-\theta_x-\theta_\infty+\theta_1}{2}+1)\Gamma(\frac{\theta_x-\theta_0+\theta_1-\theta_\infty}{2}+1)} \\ -\frac{\Gamma(\frac{\theta_\infty-\theta_1-1}{2})\Gamma(1+\theta_0)e^{i\frac{\pi}{2}[\theta_0+\theta_x+\theta_1-\theta_\infty]}}{\Gamma(\frac{\theta_0+\theta_x+\theta_\infty-\theta_1}{2})\Gamma(\frac{\theta_0-\theta_x+\theta_\infty-\theta_1}{2})} & -\frac{\Gamma(\frac{\theta_\infty-\theta_1-1}{2})\Gamma(1-\theta_0)e^{i\frac{\pi}{2}[\theta_x-\theta_0+\theta_1-\theta_\infty]}}{\Gamma(-\frac{\theta_0-\theta_x-\theta_\infty+\theta_1}{2})\Gamma(\frac{\theta_x-\theta_0+\theta_\infty-\theta_1}{2})} \end{bmatrix}, \quad (32)$$

$$C_{01} := \begin{bmatrix} \frac{\Gamma(-\theta_x)\Gamma(1+\theta_0)}{\Gamma(\frac{\theta_0-\theta_x+\theta_1-\theta_\infty}{2}+1)\Gamma(\frac{\theta_0-\theta_x+\theta_\infty-\theta_1}{2})} & \frac{\Gamma(-\theta_x)\Gamma(1-\theta_0)}{\Gamma(-\frac{\theta_0-\theta_x-\theta_\infty+\theta_1}{2}+1)\Gamma(-\frac{\theta_0-\theta_x-\theta_1+\theta_\infty}{2})} \\ \frac{\Gamma(\theta_x)\Gamma(1+\theta_0)}{\Gamma(\frac{\theta_0+\theta_x+\theta_\infty-\theta_1}{2})\Gamma(\frac{\theta_0+\theta_x+\theta_1-\theta_\infty}{2})} & \frac{\Gamma(\theta_x)\Gamma(1-\theta_0)}{\Gamma(\frac{\theta_x-\theta_0+\theta_\infty-\theta_1}{2})\Gamma(\frac{\theta_x-\theta_0+\theta_1-\theta_\infty}{2}+1)} \end{bmatrix}, \quad (33)$$

The subgroup generated by M_0M_x and M_1 is reducible. As for the solution (26), we just need to change $\theta_1 \mapsto -\theta_1$.

b) It is convenient to re-parameterize the solution (24) by introducing a parameter s through the equality:

$$a = \frac{\theta_\infty(2s + \theta_x + 1)}{2(\theta_\infty - 1)}.$$

Let $\theta_x, \theta_\infty \notin \mathbf{Z}$. Then, a representation for the monodromy group is:

$$\begin{aligned} M_0 &= G \exp\{i\pi\theta_x\sigma_3\} G^{-1}, & M_1 &= \exp\{-i\pi\theta_\infty\sigma_3\} \\ M_x &= G \exp\{-i\pi\theta_x\sigma_3\} G^{-1}, & M_\infty &= \exp\{-i\pi\theta_\infty\sigma_3\} \end{aligned}$$

In particular, $M_1 = M_\infty$, $M_0M_x = I$. We can choose G as follows:

$$G = \begin{pmatrix} 1 & 1 \\ s+\theta_x & s \\ r & r \end{pmatrix}.$$

Conversely, we may express s as a function of the monodromy data:

$$s = \frac{\theta_x[2\cos(\pi(\theta_\infty + \theta_x)) - \text{tr}(M_1M_0)]}{2[\cos(\pi(\theta_\infty - \theta_x)) - \cos(\pi(\theta_\infty + \theta_x))]}.$$

c) We re-parameterize solution (25) introducing a new parameter s defined by $a =: (1-s)^{-1}$. Let $\theta_0, \theta_x \notin \mathbf{Z}$. Then, a monodromy representation for the solutions (25) is:

$$\begin{aligned} M_0 &= (C_{\infty 0})^{-1} \exp\{i\pi\theta_0\sigma_3\} C_{\infty 0}, & M_\infty &= \begin{pmatrix} -1 & 0 \\ 2\pi i(1-s) & -1 \end{pmatrix} \\ M_x &= (C_{\infty 0})^{-1}(C_{01})^{-1} \exp\{i\pi\theta_x\sigma_3\} C_{01}C_{\infty 0}, & M_1 &= \begin{pmatrix} 1 & 0 \\ 2\pi i s & 1 \end{pmatrix}. \end{aligned}$$

where $C_{\infty 0}$ and C_{01} are (34) and (35) given below. Conversely, we may express s as a function of the monodromy data:

$$s = \frac{\operatorname{tr}(M_1 M_0) - 2 \cos(\pi \theta_0)}{4\pi \sin(\pi \theta_0)} \frac{(C_{\infty 0})_{21}}{(C_{\infty 0})_{22}}.$$

The matrices $C_{\infty 0}$ and C_{01} are:

$$C_{\infty 0} = 2 \begin{pmatrix} 0 & \frac{\Gamma(-\theta_0) e^{-i\pi\{\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}\}}}{\Gamma\left(-\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{3}{2}\right) \Gamma\left(-\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}\right)} \\ \frac{\Gamma\left(-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{1}{2}\right) \Gamma\left(-\frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{1}{2}\right)}{\Gamma(1-\theta_0) e^{-i\pi\{\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{3}{2}\}}} & \frac{\Gamma(\theta_0) e^{-i\pi\{-\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}\}}}{\Gamma\left(\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{3}{2}\right) \Gamma\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}\right)} \end{pmatrix}. \quad (34)$$

$$C_{01} = \begin{pmatrix} \frac{\Gamma(-\theta_x) \Gamma(1+\theta_0)}{\Gamma\left(\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{3}{2}\right) \Gamma\left(\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{1}{2}\right)} & \frac{\Gamma(-\theta_x) \Gamma(1-\theta_0)}{\Gamma\left(-\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{3}{2}\right) \Gamma\left(-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{1}{2}\right)} \\ \frac{\Gamma(\theta_x) \Gamma(1+\theta_0)}{\Gamma\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}\right) \Gamma\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{1}{2}\right)} & \frac{\Gamma(\theta_x) \Gamma(1-\theta_0)}{\Gamma\left(-\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}\right) \Gamma\left(-\frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{1}{2}\right)} \end{pmatrix} \quad (35)$$

Comments.

1) The conditions $\theta_\kappa \notin \mathbf{Z}$ can be eliminated, and the computations can be repeated without conceptual changes, but with different results.

2) In the above theorem, the subgroups generated by $M_0 M_x$ and M_1 are *reducible*. This characterizes the monodromy associated to solutions which have a Taylor series at $x = 0$. The same characterization at $x = 1$ involves the subgroup generated by $M_1 M_x$ and M_0 . At $x = \infty$, it involves the subgroup generated by $M_0 M_1$ and M_x . In the appendix of [9], the reader may find explanations about how to obtain results at $x = 1, \infty$ from the results at $x = 0$. In another paper, we will consider again this characterization, together with the general problem of classification.

3) Let us define again σ by $\operatorname{tr}(M_0 M_x) = 2 \cos \pi \sigma$. Then, in case a), $\sigma = \pm(\theta_1 - \theta_\infty)$ [and $\pm(\theta_1 + \theta_\infty)$ for the change $\theta_1 \mapsto -\theta_1$]. In case b), $\operatorname{tr}(M_0 M_x) = 2$ and $\sigma = 0$. In case c), $\operatorname{tr}(M_0 M_x) = -2$, $\sigma = \pm 1$. The matching procedure is effective to produce solutions corresponding to monodromy data for which the connection problem is *so far* not well studied, such as the case $\operatorname{tr}(M_i M_j) = -2$.¹

4) Also the 1-parameter solutions (18) (19) and the second solution in (20) are characterized by a reducible subgroup generated by M_0, M_x .

5) The monodromy group for the solutions (28) was derived also in [18], by confluence of singularities of scalar equations (including a Heun's type equation). The result is equivalent to that in point a) of the above theorem.

6) The computation of the monodromy group of the fuchsian systems (13) and (14) is quite clear [16] [6] [9] [3]. It allows to express the parameter r of (17), (18), (19) and (20) as a

¹Here I remark that the formula (1.30), page 1293, of my paper [9] is wrong. The correct one is $\operatorname{tr}(M_i M_j) \notin (-\infty, -2]$. In [9] the connection problem is solved for $\operatorname{tr}(M_i M_j) \neq \pm 2$. The case $\operatorname{tr}(M_i M_j) = 2$ yields (20). For the special choice of the parameters $\theta_0 = \theta_x = \theta_1 = 0$, it was studied in [6] and [7] (no logarithmic terms appear in such a special case). The result (20) for the general (PVI), corresponding to $\operatorname{tr}(M_0 M_x) = 2$, appears in the present paper for the first time.

function of the monodromy data. We just report the result for (17), which can be found in [16] [9] [3]:

$$r = \frac{(\theta_0 - \theta_x + \sigma)(\theta_0 + \theta_x - \sigma)(\theta_\infty + \theta_1 - \sigma)}{4\sigma(\theta_\infty + \theta_1 + \sigma)} \frac{1}{\mathbf{F}}, \quad (36)$$

where

$$\begin{aligned} \mathbf{F} := & \frac{\Gamma(1 + \sigma)^2 \Gamma\left(\frac{1}{2}(\theta_0 + \theta_x - \sigma) + 1\right) \Gamma\left(\frac{1}{2}(\theta_x - \theta_0 - \sigma) + 1\right)}{\Gamma(1 - \sigma)^2 \Gamma\left(\frac{1}{2}(\theta_0 + \theta_x + \sigma) + 1\right) \Gamma\left(\frac{1}{2}(\theta_x - \theta_0 + \sigma) + 1\right)} \times \\ & \times \frac{\Gamma\left(\frac{1}{2}(\theta_\infty + \theta_1 - \sigma) + 1\right) \Gamma\left(\frac{1}{2}(\theta_1 - \theta_\infty - \sigma) + 1\right)}{\Gamma\left(\frac{1}{2}(\theta_\infty + \theta_1 + \sigma) + 1\right) \Gamma\left(\frac{1}{2}(\theta_1 - \theta_\infty + \sigma) + 1\right)} \frac{V}{U}, \end{aligned}$$

and:

$$\begin{aligned} U := & \left[\frac{i}{2} \sin(\pi\sigma) \operatorname{tr}(M_1 M_x) - \cos(\pi\theta_x) \cos(\pi\theta_\infty) - \cos(\pi\theta_0) \cos(\pi\theta_1) \right] e^{i\pi\sigma} + \\ & + \frac{i}{2} \sin(\pi\sigma) \operatorname{tr}(M_0 M_1) + \cos(\pi\theta_x) \cos(\pi\theta_1) + \cos(\pi\theta_\infty) \cos(\pi\theta_0) \end{aligned}$$

$$V := 4 \sin \frac{\pi}{2}(\theta_0 + \theta_x - \sigma) \sin \frac{\pi}{2}(\theta_0 - \theta_x + \sigma) \sin \frac{\pi}{2}(\theta_\infty + \theta_1 - \sigma) \sin \frac{\pi}{2}(\theta_\infty - \theta_1 + \sigma).$$

The above formula was computed with the assumption that $\sigma \pm (\theta_0 + \theta_x)$, $\sigma \pm (\theta_0 - \theta_x)$, $\sigma \pm (\theta_1 + \theta_\infty)$, $\sigma \pm (\theta_1 - \theta_\infty)$ are not even integers. ²

7) Reducible Monodromy. The monodromy groups in Theorem 3 are not reducible, but they have a reducible subgroup. If the entire group itself is completely reducible, the solutions of (PVI) are well known: they are classical solutions in the sense of Umemura [24]. We summarize them in the following proposition (the reader can see also [12]).

Proposition 4 *All the solutions of (PVI) corresponding to a reducible monodromy group are equivalent by birational canonical transformations to the following one-parameter family of solutions, with $\theta_\infty + \theta_1 + \theta_0 + \theta_x = 0$:*

$$y(x) = \frac{\theta_1 + \theta_\infty - 1 + x(1 + \theta_x)}{\theta_\infty - 1} - \frac{1}{\theta_\infty - 1} \frac{x(1-x)}{u(x; a)} \frac{du(x; a)}{dx}, \quad (37)$$

where $u(x; a) = u_1(x) + au_2(x)$; $a \in \mathbf{C}$, $u_1(x)$ and $u_2(x)$ are linear independent solutions of the hypergeometric equation:

$$x(1-x) \frac{d^2 u}{dx^2} + \{[2 - (\theta_\infty + \theta_1)] - (4 - \theta_\infty + \theta_x)x\} \frac{du}{dx} - (2 - \theta_\infty)(1 + \theta_x)u = 0$$

The monodromy matrices are upper triangular:

$$M_0 = \begin{pmatrix} \frac{\theta_0}{2} & * \\ 0 & -\frac{\theta_0}{2} \end{pmatrix}, \quad M_x = \begin{pmatrix} \frac{\theta_x}{2} & * \\ 0 & -\frac{\theta_x}{2} \end{pmatrix}, \quad M_1 = \begin{pmatrix} \frac{\theta_1}{2} & * \\ 0 & -\frac{\theta_1}{2} \end{pmatrix}.$$

Remark: The rational solutions of (PVI) are a special case of the above proposition. They were studied in [21]. Up to canonical birational transformations, they are realized for $\theta_\infty + \theta_1 + \theta_0 + \theta_x = 0$ and:

$$\theta_0 = 1 : \quad y(x) = \frac{\theta_\infty + \theta_1}{\theta_\infty} \frac{x-1}{x(1+\theta_1) - (\theta_1 + \theta_\infty)};$$

²In [9] there is a miss print in formula (A.30), which must be re-calculated. In [16], in formula (1.8) at the bottom of page 1141, the last sign is $\pm\sigma$ instead of $\mp\sigma$.

$$\theta_0 = -2 : \quad y(x) = \frac{(2 - (\theta_\infty + \theta_1) + \theta_1 x)^2 - 2 + \theta_\infty + \theta_1 - \theta_1 x^2}{(1 - \theta_\infty)(2 - (\theta_\infty + \theta_1) + \theta_1 x)}.$$

The computation of the expansion at $x = 0$ of (37) is just a consequence of the expansions of $u_1(x)$ and $u_2(x)$. The reader can find by himself a behavior $y \sim x(r(a) \pm \theta_x \ln(x))$ for $\theta_1 + \theta_\infty = \theta_0 + \theta_x = 0$, namely a sub-case of the second solution in (20). For $\theta_1 + \theta_\infty \notin \mathbf{Z}$, we find behaviors of the type (31) (and (23), (28) for $a = 0$).

This paper is a review of [11]. Therefore, we refer the reader to [11] for the derivation of the results.

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