Chaos in the Sixth Painlevé Equation*

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Abstract

In our previous paper [10] an ergodic theory of Painlevé VI is developed and the chaotic nature of its Poincaré return map is discovered. This article outlines the main contents of that work and describes the principal ideas leading to its main results. An announcement of new results is also given along with some open problems to be discussed in the future.

1 Introduction

In [10] the authors developed an ergodic theory of the sixth Painlevé equation and discovered the chaotic nature of its Poincaré return map along almost every loop in the space of a time variable. As a résumé of [10], this article outlines the main contents of that work and describes the principal ideas leading to its main results, presenting a few remarks and discussions which could not be included in [10]. An announcement of some advances made after the completion of [10] is also given along with some open problems to be discussed in the near future.

The work [10] is built upon two foundations; one is the algebraic geometry of the sixth Painlevé equation [7, 8, 9], especially its moduli-theoretical formulation based on geometric invariant theory [12]; the other is the ergodic theory of birational maps on surfaces recently developed in [1, 3, 4, 5]. These two ingredients are combined fruitfully via a Riemann-Hilbert correspondence to reveal the chaotic nature of the sixth Painlevé dynamics. Here our main objective is to construct an invariant measure which is mixing, hyperbolic and of maximal entropy and to count the number of periodic points of the Poincaré return map.

2 The Sixth Painlevé Equation

The sixth Painlevé equation $\mathrm{P_{VI}}(\kappa)$ is a Hamiltonian system of differential equations

$$\frac{dq}{dx} = \frac{\partial H(\kappa)}{\partial p}, \qquad \frac{dp}{dx} = -\frac{\partial H(\kappa)}{\partial q}, \tag{1}$$

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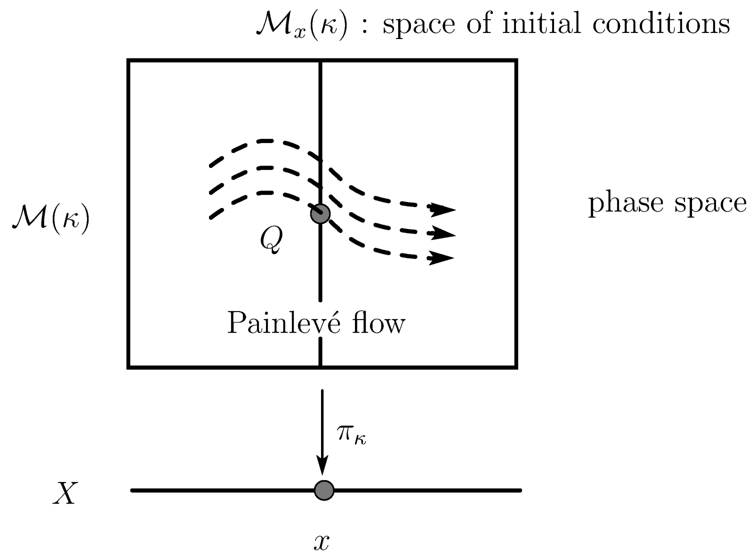

Figure 1: A schematic picture of the sixth Painlevé dynamics

with a time variable $x \in X := \mathbb{P}^1 - \{0, 1, \infty\}$ and unknown functions $q = q(x)$, $p = p(x)$, depending on complex parameters $\kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ in the 4-dimensional affine space

$$\mathcal{K} := \{\, \kappa = (\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4) \in \mathbb{C}^5 \;:\; 2\kappa_0 + \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 = 1 \,\},$$

where the Hamiltonian $H(\kappa) = H(q, p, x; \kappa)$ is given by

$$x(x-1)H(\kappa) \;=\; (q_0 q_1 q_x)p^2 - \{\kappa_1 q_1 q_x + (\kappa_2 - 1) q_0 q_1 + \kappa_3 q_0 q_x\} p + \kappa_0(\kappa_0 + \kappa_4) q_x,$$

with $q_\nu := q - \nu$ for $\nu \in \{0, 1, x\}$. It is known that $\mathrm{P_{VI}}(\kappa)$ has the analytic Painlevé property, that is, any meromorphic solution germ to equation (1) at a base point $x \in X$ admits a unique global analytic continuation along any path emanating from x as a meromorphic function.

3 Algebraic Geometry of Painlevé VI

The equation (1) is only a fragmentary appearance of a more intrinsic object constructed algebro-geometrically [7, 8, 9], where $\mathrm{P_{VI}}(\kappa)$ is formulated as a holomorphic, uniform, transversal foliation on a fibration of certain smooth quasi-projective rational surfaces

$$\pi_\kappa : \mathcal{M}(\kappa) \to X := \mathbb{P}^1 - \{0, 1, \infty\},$$

whose fiber $\mathcal{M}_x(\kappa) := \pi_\kappa^{-1}(x)$ over $x \in X$, called the space of initial conditions at time x, is realized as a moduli space of stable parabolic connections (see Figure 1). In this formulation the uniformity, namely, the geometric Painlevé property of the Painlevé foliation is a natural consequence of a solution to the Riemann-Hilbert problem, especially of the properness of the Riemann-Hilbert correspondence [7]. Then equation (1) is just a coordinate expression of the foliation on an affine open chart of $\mathcal{M}(\kappa)$ and the analytic Painlevé property for equation (1) is an immediate consequence of the geometric Painlevé property and the algebraicity of the phase space $\mathcal{M}(\kappa)$. Moreover, there exists a natural compactification $\mathcal{M}_x(\kappa) \hookrightarrow \overline{\mathcal{M}}_x(\kappa)$ of the moduli space $\mathcal{M}_x(\kappa)$ into a moduli space $\overline{\mathcal{M}}_x(\kappa)$ of stable parabolic phi-connections.

Here we include a very sketchy explanation of the terminology used in the last paragraph. A *stable parabolic connection* is a (rank 2) vector bundle with a Fuchsian connection and a parabolic structure, satisfying a sort of stability in geometric invariant theory. On the other hand, a *stable parabolic phi-connection* is a variant of a stable parabolic connection allowing a "matrix-valued Planck constant" called a phi-operator ϕ such that the generalized Leibniz rule

$$\nabla(fs) = df \otimes \phi(s) + f\nabla(s)$$

is satisfied, where the field ϕ may be degenerate or simi-classical. Then the moduli space $\mathcal{M}_x(\kappa)$ can be compactified by adding some semi-classical objects, that is, some stable parabolic phi-connections with degenerate phi-operator ϕ.

There is the following characterization of our moduli spaces (see Figure 2).

Theorem 1 ([7, 8, 9])

(1) *The compactified moduli space $\overline{\mathcal{M}}_x(\kappa)$ is isomorphic to an 8-point blow-up of the Hirzebruch surface $\Sigma_2 \to \mathbb{P}^1$ of degree 2.*

(2) *$\overline{\mathcal{M}}_x(\kappa)$ has a unique effective anti-canonical divisor $\mathcal{Y}_x(\kappa)$, which is given by*

$$\mathcal{Y}_x(\kappa) = 2E_0 + E_1 + E_2 + E_3 + E_4, \tag{2}$$

where E_0 is the strict transform of the section at infinity and E_1, E_2, E_3, E_4 are the strict transforms of the fibers over the points $0, 1, x, \infty \in \mathbb{P}^1$ of the Hirzebruch surface $\Sigma_2 \to \mathbb{P}^1$.

(3) *The support of the divisor $\mathcal{Y}_x(\kappa)$ is exactly the locus where the phi-operator ϕ is degenerate, with the coefficients of formula (2) being the ranks of degeneracy of ϕ. In particular,*

$$\mathcal{M}_x(\kappa) = \overline{\mathcal{M}}_x(\kappa) - \mathcal{Y}_x(\kappa).$$

This theorem implies that $\overline{\mathcal{M}}_x(\kappa)$ is a generalized Halphen surface of type $D_4^{(1)}$ in [14] and $(\overline{\mathcal{M}}_x(\kappa), \mathcal{Y}_x(\kappa))$ is an Okamoto-Painlevé pair of type $\widetilde{D}_4$ in [13].

4 Poincaré Return Map

Since the Painlevé foliation is uniform (the geometric Painlevé property [7]), each loop $\gamma \in \pi_1(X, x)$ admits global horizontal lifts along the foliation and induces an automorphism

$$\gamma_* : \mathcal{M}_x(\kappa) \to \mathcal{M}_x(\kappa) \quad Q \mapsto Q', \tag{3}$$

called the Poincaré return map along the loop γ (see Figure 2). Then the main issues discussed in [10] are the following.

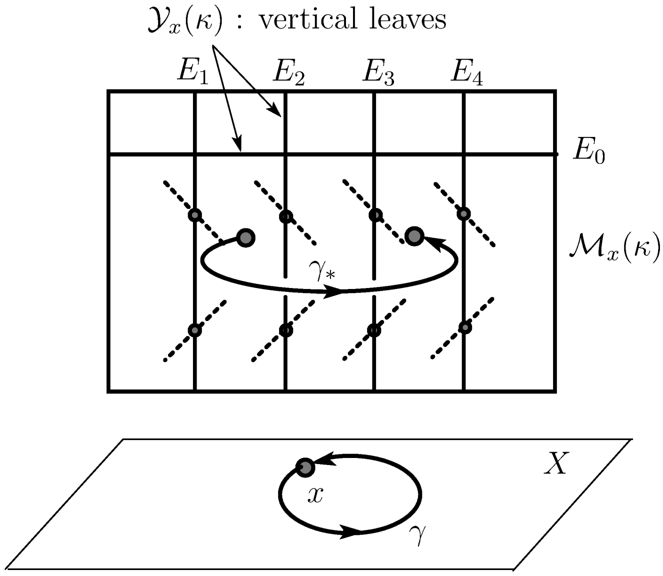

Figure 2: Poincaré return map $\gamma_* : \mathcal{M}_x(\kappa) \circlearrowleft$ along a loop $\gamma \in \pi_1(X, x)$

Problem 2 Given a loop $\gamma \in \pi_1(X, x)$,

(1) explore the dynamical nature of the Poincaré return map $\gamma_* : \mathcal{M}_x(\kappa) \circlearrowleft$; is it *chaotic* ?

(2) Count the number of periodic solutions of period $N \in \mathbb{N}$ along γ, that is, the cardinality of the set $\mathrm{Per}_N(\gamma;\kappa) := \{ Q \in \mathcal{M}_x(\kappa) \,:\, \gamma_*^N Q = Q \}$ of all initial conditions that come back to the original positions after the N-fold iterations of the Poincaré return map γ_*. In particular, find the *growth rate* of $\#\mathrm{Per}_N(\gamma;\kappa)$ as the period N tends to infinity.

Roughly speaking, our main results in [10] can be stated as follows.

$\boxed{\text{The Poincaré return map (3) is } \textit{chaotic} \text{ along every } \textit{non-elementary} \text{ loop } \gamma \in \pi_1(X, x)}$

This statement will be made precise in Theorem 6. Here the meaning of the adjective "chaotic" will be explained in §5, while the term "non-elementary loop" is used in the following sense.

- Non-elementary loop: Let γ_1, γ_2, γ_3 be loops as in Figure 3. Since $X = \mathbb{P}^1 - \{0, 1, \infty\}$, the fundamental group of X with base point at x is represented as

$$\pi_1(X, x) = \langle\, \gamma_1, \gamma_2, \gamma_3 \,|\, \gamma_1\gamma_2\gamma_3 = 1 \,\rangle. \tag{4}$$

Definition 3 A loop $\gamma \in \pi_1(X, x)$ is said to be *elementary* if γ is conjugate to the loop γ_i^m for some index $i \in \{1, 2, 3\}$ and some integer $m \in \mathbb{Z}$. Otherwise, γ is said to be *non-elementary*.

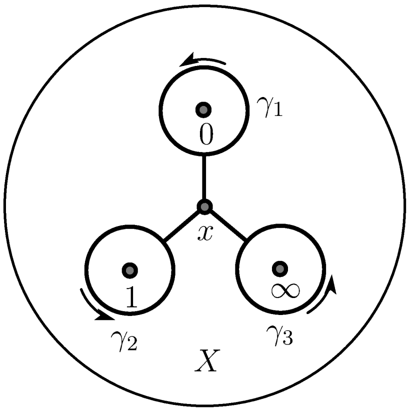

Figure 3: Three basic loops γ_1, γ_2, γ_3 in $X = \mathbb{P}^1 - \{0, 1, \infty\}$

5 Chaos in Surface Dynamics

Let $f : S \to S$ be a holomorphic map on a complex surface S (in our case, $S = \mathcal{M}_x(\kappa)$ and $f = \gamma_*$). By the word "chaos" we mean the following.

Definition 4 The dynamical system $f : S \to S$ is said to be *chaotic* if there exists an f-invariant Borel probability measure μ on S such that the following conditions are satisfied:

(C1) f has a *positive entropy* $h_\mu(f) > 0$ with respect to the measure μ.

(C2) f is *mixing* with respect to the measure μ, that is, $\mu(f^{-n}(A) \cap B) \to \mu(A)\mu(B)$ as $n \to \infty$ for any Borel subsets A, B of S. In particular, f is *ergodic* with respect to μ.

(C3) The ergodic measure μ is a *hyperbolic measure* of saddle type, that is, $L_-(f) < 0 < L_+(f)$, where $L_\pm(f)$ are the Lyapunov exponents of f with respect to μ. Moreover, μ has a product structure with respect to local stable and unstable manifolds.

(C4) hyperbolic periodic points of f are *dense* in the support of μ.

Remark 5 (Three requirements for "chaos") While there are many possible definitions of "chaos" (see [2, 11, 15]), the definition adopted here is a typical one possessing the following three ingredients usually required for "chaos".

(1) *unpredictability*: sensitive dependence on initial values $\cdots$ (C1) and (C3);

(2) *indecomposability*: ergodicity and its related properties $\cdots$ (C2);

(3) *an element of regularity*: periodic points which are dense $\cdots$ (C4).

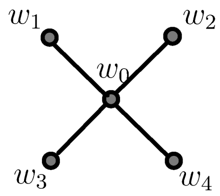

Figure 4: Dynkin diagram of type $D_4^{(1)}$

6 Affine Weyl Group

In order to state the main results of [10] we review the affine Weyl group structure acting on the parameter space $\mathcal{K}$. To this end we note that the affine space $\mathcal{K}$ can be identified with the linear space $\mathbb{C}^4$ by the isomorphism

$$\mathcal{K} \xrightarrow{\sim} \mathbb{C}^4, \quad \kappa \mapsto (\kappa_1, \kappa_2, \kappa_3, \kappa_4),$$

through which the standard (complex) Euclidean inner product on the latter space $\mathbb{C}^4$ is transferred to the former space $\mathcal{K}$. For each $i \in \{0, 1, 2, 3, 4\}$, let $w_i \curvearrowright \mathcal{K}$ be the orthogonal affine reflection in the hyperplane $\{\kappa \in \mathcal{K} : \kappa_i = 0\}$. Then the group generated by w_0, w_1, w_2, w_3, w_4 is an affine Weyl group of type $D_4^{(1)}$ corresponding to the Dynkin diagram in Figure 4:

$$W(D_4^{(1)}) := \langle w_0, w_1, w_2, w_3, w_4 \rangle.$$

Let **Wall** be the union of the reflecting hyperplanes of all reflections in the group $W(D_4^{(1)})$. Explicitly these hyperplanes are given by affine linear relations

$$\kappa_i = m, \quad \kappa_1 \pm \kappa_2 \pm \kappa_3 \pm \kappa_4 = 2m + 1 \qquad (m \in \mathbb{Z}, i \in \{1, 2, 3, 4\}).$$

In [10], Problem 2 was discussed only under the condition that κ is generic, i.e., only when $\kappa \in \mathcal{K} - \mathbf{Wall}$, and the nongeneric case $\kappa \in \mathbf{Wall}$ was not treated. See Theorem 6 for the results in the generic case. As to the nongeneric case, some progress was made after the completion of [10], which will be announced in the final section of this article (see Theorem 13).

7 Chaos in Painlevé VI

Under the set-up mentioned above the main results of [10] are stated in the following manner.

Theorem 6 ([10]) *Assume that* $\kappa \in \mathcal{K} - \mathbf{Wall}$. *For any non-elementary loop* $\gamma \in \pi_1(X, x)$, *the Poincaré return map* $\gamma_* : \mathcal{M}_x(\kappa) \circlearrowleft$ *is chaotic, that is, there exists a natural* γ_**-invariant Borel probability measure* μ_γ *such that all the conditions* (C1)–(C4) *in Definition* 4 *are satisfied. Moreover, there is a real number* $\lambda(\gamma) > 1$, *called the dynamical degree along* γ, *such that*

(1) **measure-theoretic entropy:** *the measure-theoretic entropy of the Poincaré return map* $\gamma_* : \mathcal{M}_x(\kappa) \circlearrowleft$ *with respect to the invariant measure* μ_γ *is given by*

$$h_{\mu_\gamma}(\gamma_*) = \log \lambda(\gamma),$$

(2) **number of periodic points:** *the cardinality of the set* $\mathrm{Per}_N(\gamma;\kappa)$ *is given by*

$$\#\mathrm{Per}_N(\gamma;\kappa) = \lambda(\gamma)^N + \lambda(\gamma)^{-N} + 4.$$

In particular the number $\#\mathrm{Per}_N(\gamma;\kappa)$ *grows exponentially with the growth rate* $\lambda(\gamma)$.

(3) *There exists an algorithm to calculate* $\lambda(\gamma)$ *in terms of the reduced word for a minimal representative of* γ *in the alphabet* γ_1, γ_2, γ_3 *(see Algorithm 10). Moreover the dynamical degree* $\lambda(\gamma)$ *is a quadratic unit admitting a lower bound*

$$\lambda(\gamma) \geq 3 + 2\sqrt{2} \tag{5}$$

where the equality holds if and only if γ *is an eight-loop introduced in Example 7.*

We present two examples to illustrate Theorem 6. In what follows we write $h(\gamma) := h_{\mu_\gamma}(\gamma_*)$.

Example 7 We put $x_1 = 0$, $x_2 = 1$, $x_3 = \infty$ (see Figures 3 and 5).

(1) An *eight-loop* is a loop conjugate to $\gamma_i \gamma_j^{-1}$ for some indices $\{i, j, k\} = \{1, 2, 3\}$. For any eight-loop γ, one has

$$h(\gamma) = \log(3 + 2\sqrt{2}), \qquad \#\,\mathrm{Per}_N(\gamma;\kappa) = (3 + 2\sqrt{2})^N + (3 + 2\sqrt{2})^{-N} + 4.$$

The eight-loop is the most "elementary" loop among all non-elementary loops in the sense that the lower bound is attained in (5).

(2) A *Pochhammer loop* is a loop conjugate to the commutator $[\gamma_i, \gamma_j^{-1}] = \gamma_i \gamma_j^{-1} \gamma_i^{-1} \gamma_j$ for some indices $\{i, j, k\} = \{1, 2, 3\}$. For any Pochhammer loop $\wp$, one has

$$h(\wp) = \log(9 + 4\sqrt{5}), \qquad \#\,\mathrm{Per}_N(\wp;\kappa) = (9 + 4\sqrt{5})^N + (9 + 4\sqrt{5})^{-N} + 4.$$

Remark 8 We showed that the Poincaré return map is chaotic along every non-elementary loop. It is natural to ask how it is along an *elementary* loop. The answer is that it is *integrable!* The moduli space $\mathcal{M}_x(\kappa)$ has a natural symplectic structure and, for every $\gamma \in \pi_1(X, x)$, the Poincaré return map $\gamma_* : \mathcal{M}_x(\kappa) \circlearrowleft$ is a symplectic automorphism. Now, if γ is elementary, then it turns out that γ_* preserves a Lagrangian fibration. In this sense it is Liouville integrable.

8 Algorithm to Calculate Dynamical Degree

The algorithm to calculate the dynamical degree $\lambda(\gamma)$ is given in terms of the reduced word for a minimal representative of the conjugacy class of the loop $\gamma \in \pi_1(X, x)$ and also in terms of the universal Coxeter group of rank 3 and its geometric representation.

- Minimal representative of a loop. Any loop $\gamma \in \pi_1(X, x)$ admits an expression

$$\gamma = \gamma_{j_1}^{\varepsilon_{j_1}} \gamma_{j_2}^{\varepsilon_{j_2}} \cdots \gamma_{j_m}^{\varepsilon_{j_m}}, \tag{6}$$

with some positive integer $m \in \mathbb{N}$, some indices $(j_1, \dots, j_m) \in \{1, 2, 3\}^m$ and some signs $(\varepsilon_{j_1}, \dots, \varepsilon_{j_m}) \in \{\pm 1\}^m$. The expression (6) is said to be *reduced* if the length m is minimal among all feasible expressions. The *length* $\ell_{\pi_1}(\gamma)$ of the loop γ is defined to be the length m of the reduced expression (6) of γ.

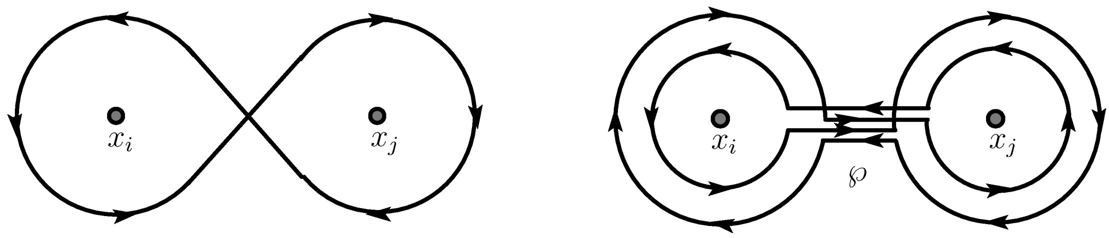

Figure 5: An eight-loop (left) and a Pochhammer loop (right)

Definition 9 A loop $\gamma \in \pi_1(X, x)$ is said to be *minimal* if it has the minimal length among all loops conjugate to γ.

- Universal Coxeter group of rank 3: Consider the universal Coxeter group of rank 3, that is, the free product of three copies of $\mathbb{Z}_2$,

$$G = \mathrm{UCG}(3) := \langle\, \sigma_1,\, \sigma_2,\, \sigma_3 \,|\, \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1 \,\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2. \tag{7}$$

Any element $\sigma \in G$ is uniquely represented as

$$\sigma = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n}, \tag{8}$$

with some $n \in \mathbb{N}$ and some indices $(i_1, \dots, i_n) \in \{1, 2, 3\}^n$ such that every neighboring indices i_ν and $i_{\nu+1}$ are distinct. The expression (8) is called the *reduced* expression of σ and the number $\ell_G(\sigma) := n$ is called the *length* of σ. An element $\sigma \in G$ is said to be *even* if the length $\ell_G(\sigma)$ is an even integer. Let $G(2)$ be the subgroup of all even elements in G. Then the change of alphabets $\{\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \gamma_3^{\pm 1}\} \to \{\sigma_1, \sigma_2, \sigma_3\}$ according to the translation rule

$$\gamma_1^{\pm 1} \mapsto \left\{ \begin{array}{l} \sigma_1 \sigma_2 \\ \sigma_2 \sigma_1 \end{array} \right. \qquad \gamma_2^{\pm 1} \mapsto \left\{ \begin{array}{l} \sigma_2 \sigma_3 \\ \sigma_3 \sigma_2 \end{array} \right. \qquad \gamma_3^{\pm 1} \mapsto \left\{ \begin{array}{l} \sigma_3 \sigma_1 \\ \sigma_1 \sigma_3 \end{array} \right. \tag{9}$$

induces an isomorphism of groups

$$\pi_1(X, z) \xrightarrow{\sim} G(2) \subset G, \qquad \gamma \mapsto \sigma.$$

If the expression (6) is reduced in $\pi_1(X, x)$, then the resulting word (8) is also reduced in G, and hence the reduced expression (6) is unique for a given loop γ and one has $\ell_G(\sigma) = 2\ell_{\pi_1}(\gamma)$, where $\sigma \in G(2)$ is the element corresponding to the loop γ.

- Gemetric representation. Any Coxeter group admits its geometric representation [6]. We apply this construction to our particular group G. Let $V := \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$ be the 3-dimensional vector space endowed with the nondegenerate symmetric bilinear form

$$B(e_i, e_j) = \left\{ \begin{array}{cl} 1 & \quad (i = j), \\ -1 & \quad (i \neq j). \end{array} \right.$$

For each index $i \in \{1, 2, 3\}$, we define a reflection $r_i : V \to V$ by

$$r_i(v) := v - 2B(e_i, v)\, e_i \qquad (v \in V).$$

Then there exists a faithful representation $\rho : G \to GL(V)$ such that $\rho(\sigma_i) = r_i$ for each $i \in \{1, 2, 3\}$, which is referred to as the *geometric representation* of G. Through this representation the group G may be identified with the reflection group $\langle r_1, r_2, r_3 \rangle$ acting on (V, B).

For each $i \in \{1, 2, 3\}$, we define an endomorphism $s_i : V \to V$ by the mean of the identity and the i-th basic reflection r_i, namely,

$$s_i(v) := \{v + r_i(v)\}/2 = v - B(e_i, v)\, e_i \qquad (v \in V).$$

Then the algorithm to calculate the dynamical degree $\lambda(\gamma)$ is given in the following manner.

Algorithm 10 ([10]) Given a non-elementary loop $\gamma \in \pi_1(X, x)$,

(1) choose a minimal representative of the conjugacy class of γ and call it γ again.

(2) Take the reduced expression of γ as in (6).

(3) Change alphabets $\{\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \gamma_3^{\pm 1}\} \to \{\sigma_1, \sigma_2, \sigma_3\}$ according to the rule (9) to obtain the corresponding element $\sigma \in G(2)$, together with its reduced expression as in (8).

(4) To the indices $(i_1, \dots, i_n)$ in (8), associate the endomorphism $s_\gamma := s_{i_n} \cdots s_{i_2} s_{i_1} \in \operatorname{End} V$.

(5) Take its trace $\alpha(\gamma) = \operatorname{Tr}[\, s_\gamma : V \to V\,]$, which turns out to be an integer ≥ 6.

(6) Finally, let $\lambda(\gamma)$ be the largest root of the quadratic equation $\lambda^2 - \alpha(\gamma)\lambda + 1 = 0$.

9 Riemann-Hilbert Correspondence

As is mentioned in the Introduction, the work [10] is based on an interplay between the algebraic geometry of the sixth Painlevé equation and the ergodic theory of birational maps on complex surfaces, connected via a Riemann-Hilbert correspondence. Following [7, 8, 9] we review the formulation of it together with a solution to the Riemann-Hilbert problem.

Generally speaking, a Riemann-Hilbert correspondence is the map from a moduli space of flat connections to a moduli space of monodromy representations, sending a connection to its monodromy. In our case, the moduli spaces of monodromy representations (with fixed local monodromy data) are realized as affine cubic surfaces $\mathcal{S}(\theta) = \{x \in \mathbb{C}^3_x : f(x, \theta) = 0\}$ with

$$f(x, \theta) := x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1 x_1 - \theta_2 x_2 - \theta_3 x_3 + \theta_4,$$

parametrized by the 4-dimensional affine space $\Theta := \mathbb{C}^4_\theta$. There exists a holomorphic map

$$\mathrm{rh} : \mathcal{K} \to \Theta,$$

called the Riemann-Hilbert correspondence in the parameter level [7]. It is a $W(D_4^{(1)})$-covering ramifying along **Wall** and mapping **Wall** onto the discriminant locus of the

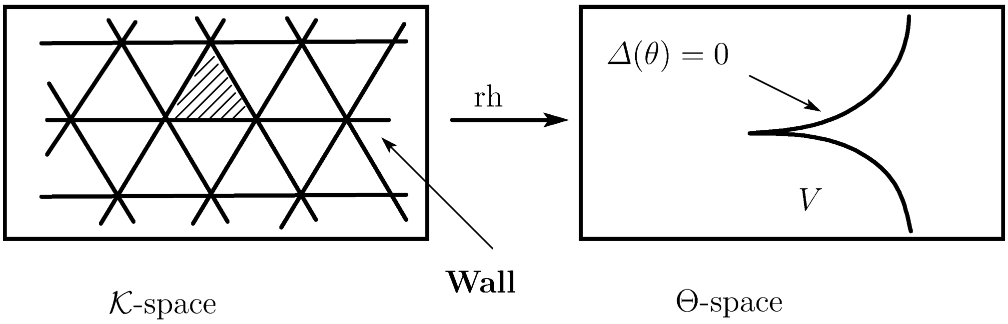

Figure 6: Riemann-Hilbert correspondence in the parameter level

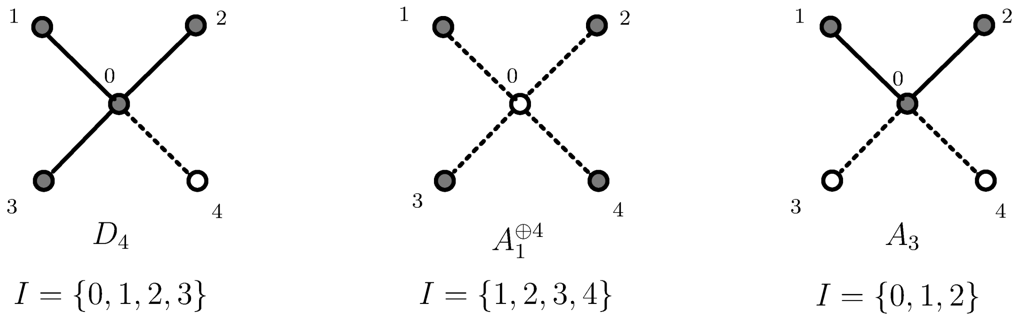

Figure 7: Stratification by Dynkin subdiagrams

cubic surfaces (see Figure 6). Then our Riemann-Hilbert correspondence is formulated as a holomorphic map

$$\mathrm{RH}_{x,\kappa} : \mathcal{M}_x(\kappa) \to \mathcal{S}(\theta) \qquad \text{with} \quad \theta = \mathrm{rh}(\kappa). \tag{10}$$

The singularity structure of the cubic surfaces $\mathcal{S}(\theta)$ can be described in terms of the stratification of $\mathcal{K}$ by Dynkin subdiagrams. For each *proper* subset $I \subset \{0, 1, 2, 3, 4\}$, we put

$$\begin{aligned} \mathcal{K}_I &= W(D_4^{(1)})\text{-translates of the subset } \{\, \kappa_i = 0 \ (i \in I),\ \kappa_i \neq 0 \ (i \notin I) \,\}, \\ D_I &= \text{Dynkin subdiagram of } D_4^{(1)} \text{ that has nodes } \bullet \text{ exactly in } I. \end{aligned}$$

For example one has the *big open* $\mathcal{K}_\emptyset = \mathcal{K} - \mathbf{Wall}$ when $I = \emptyset$. Other examples are given in Figure 7. Then there is a very neat solution to the Riemann-Hilbert problem.

Theorem 11 ([7, 8, 9]) *Given any $\kappa \in \mathcal{K}$, put $\theta = \mathrm{rh}(\kappa) \in \Theta$. Then,*

(1) *if $\kappa \in \mathcal{K}_I$ then $\mathcal{S}(\theta)$ has Kleinian singularities of Dynkin type D_I,*

(2) *the Riemann-Hilbert correspondence* (10) *is a proper surjective map that is an analytic minimal resolution of singularities.*

For example, on the big open, namley, if $\kappa \in \mathcal{K} - \mathbf{Wall}$ then $\mathcal{S}(\theta)$ is smooth and $\mathrm{RH}_{x,\kappa}$ is biholomorphic, while if $\kappa = (0, 0, 0, 0, 1)$ then $\mathcal{S}(\theta)$ has a Kleinian singularity of type D_4 and $\mathrm{RH}_{x,\kappa}$ is an analtytic minimal resolution as depicted in Figure 8.

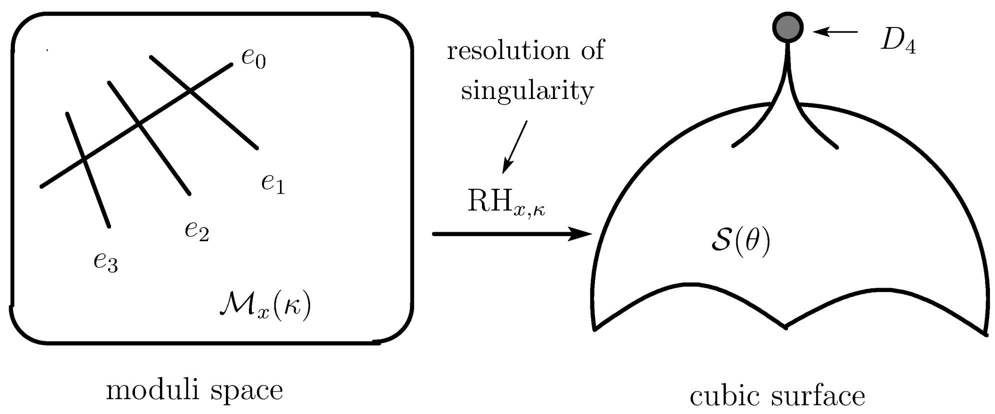

Figure 8: Resolution of singularities by Riemann-Hilbert correspondence

10 Invariant Measure

In this section a brief account of how to establish Theorem 6 is given with emphasis on how to construct the invariant measure μ_γ. The main strategy consists of the following procedures:

(1) to recast the Poincaré return map $\gamma_* : \mathcal{M}_x(\kappa) \circlearrowleft$ to an automorphism on $\mathcal{S}(\theta)$ arising as an action of braids on the moduli space of monodromy representations;

(2) to extend the automorphism to a birational map on the projective cubic surface $\overline{\mathcal{S}}(\theta)$ which is a compactification of the affine cubic surface $\mathcal{S}(\theta)$;

(3) to apply the ergodic theory of birational maps on complex surfaces;

(4) to pull back the obtained result to the moduli space $\mathcal{M}_x(\kappa)$ and the Poincaré return map on it via the Riemann-Hilbert correspondence to reach our final goal.

Some of the ingredients in these procedures are explained below. At this stage an overview of the sixth Painlevé dynamics as in Figure 9 may be helpful in grasping their total images.

First we explain how to recast the Poincaré return map $\gamma_* : \mathcal{M}_x(\kappa) \circlearrowleft$ to an automorphism $\sigma : \mathcal{S}(\theta) \circlearrowleft$. We begin by the case where γ is one of the basic loops γ_1, γ_2, γ_3 in (4). This case is closely related to the $(2,2,2)$-structure of the affine cubic surface $\mathcal{S}(\theta)$, namely, to the fact that its defining equation $f(x,\theta) = 0$ is a quadratic equation in each variable x_i, $i \in \{1,2,3\}$. This implies that the line through a point $x \in \mathcal{S}(\theta)$ parallel to the x_i-axis passes through a unique second point $x' \in \mathcal{S}(\theta)$ (see Figure 10), which defines three involutions

$$\sigma_i : \mathcal{S}(\theta) \to \mathcal{S}(\theta), \quad x \mapsto x', \qquad (i = 1,2,3).$$

It is shown in [10] that the group generated by σ_1, σ_2, σ_3 is a universal Coxeter group of rank 3, and hence it may be thought of as a concrete realization of the abstract group in (7). Then via the Riemann-Hilbert correspondence (10), the basic Poincaré return maps are transferred to automorphisms on $\mathcal{S}(\theta)$ in such a manner as

$$\gamma_{1*}^{\pm 1} \mapsto \begin{cases} \sigma_1\sigma_2 \\ \sigma_2\sigma_1 \end{cases} \qquad \gamma_{2*}^{\pm 1} \mapsto \begin{cases} \sigma_2\sigma_3 \\ \sigma_3\sigma_2 \end{cases} \qquad \gamma_{3*}^{\pm 1} \mapsto \begin{cases} \sigma_3\sigma_1 \\ \sigma_1\sigma_3 \end{cases} \tag{11}$$

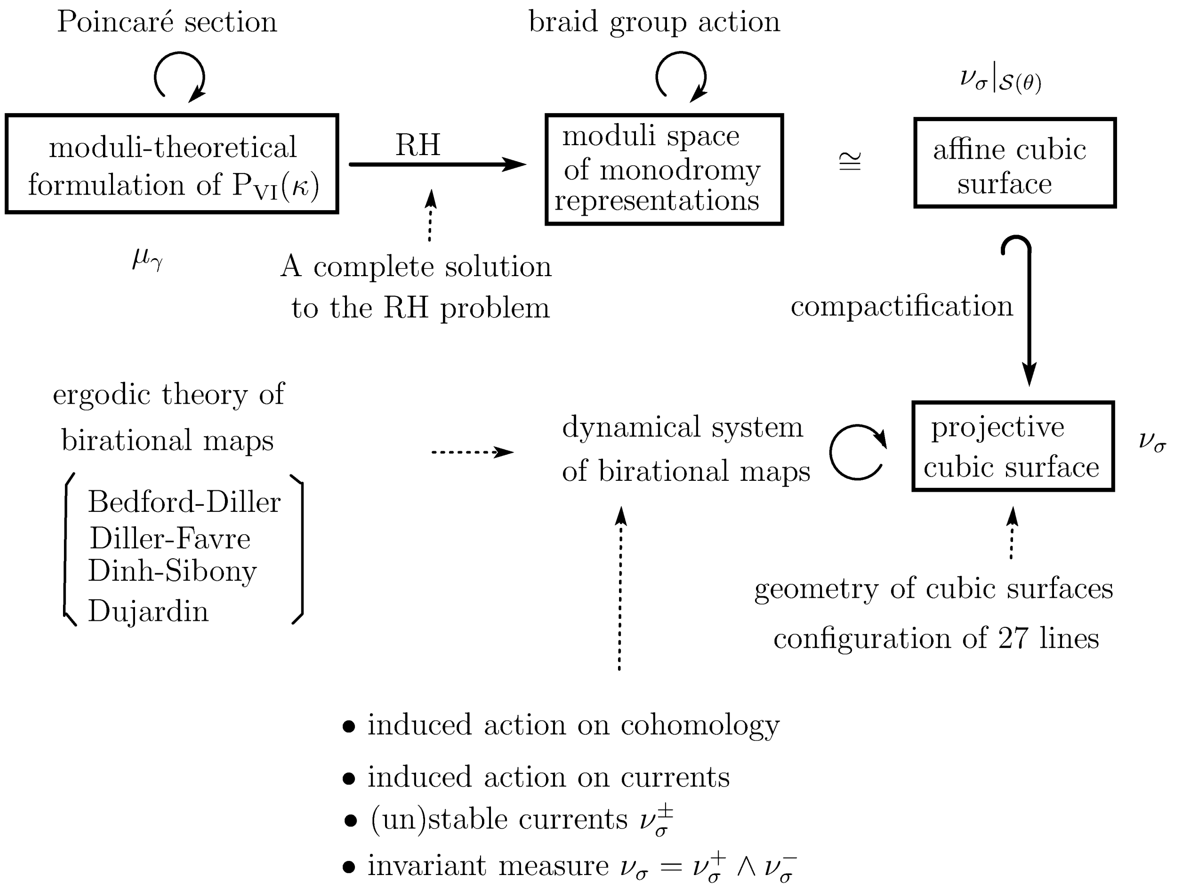

Figure 9: An overview of the sixth Painlevé dynamics

which agrees with the translation rule (9). Then for a general loop γ, the passage from the Poincaré return map $\gamma_* : \mathcal{M}_x(\kappa) \circlearrowleft$ to an automorphism $\sigma : \mathcal{S}(\theta) \circlearrowleft$ proceeds just as in the procedure from formula (6) to formula (8) in Algorithm 10.

The next step is to compactify the affine cubic surface $\mathcal{S}(\theta)$ by the standard embedding

$$\mathcal{S}(\theta) \hookrightarrow \overline{\mathcal{S}}(\theta) \subset \mathbb{P}^3, \qquad x = (x_1, x_2, x_3) \mapsto [1 : x_1 : x_2 : x_3],$$

where $\overline{\mathcal{S}}(\theta) = \{\, X \in \mathbb{P}^3 \,:\, F(X, \theta) = 0 \,\}$ is the projective cubic surface with defining equation

$$F(X, \theta) = X_1 X_2 X_3 + X_0(X_1^2 + X_2^2 + X_3^2) - X_0^2(\theta_1 X_1 + \theta_2 X_2 + \theta_3 X_3) + \theta_4 X_0^3.$$

The projective surface $\overline{\mathcal{S}}(\theta)$ is obtained from the affine surface $\mathcal{S}(\theta)$ by adding the tritangent lines $L = L_1 \cup L_2 \cup L_3$ at infinity (see Figure 11), where the line L_i is defined by

$$L_i = \{\, X \in \mathbb{P}^3 \,:\, X_0 = X_i = 0 \,\} \qquad (i = 1, 2, 3).$$

Now a crucial fact is that any element σ of $G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$, which is an automorphisms of $\mathcal{S}(\theta)$, extends to a *birational* map on $\overline{\mathcal{S}}(\theta)$. On th other hand, it cannot be expected that the Poincaré return map γ_*, which is an (analytic) automorphism of the quasi-projective surface $\mathcal{M}_x(\kappa)$, does extend to a bimeromorphic map on the projective surface $\overline{\mathcal{M}}_x(\kappa)$,

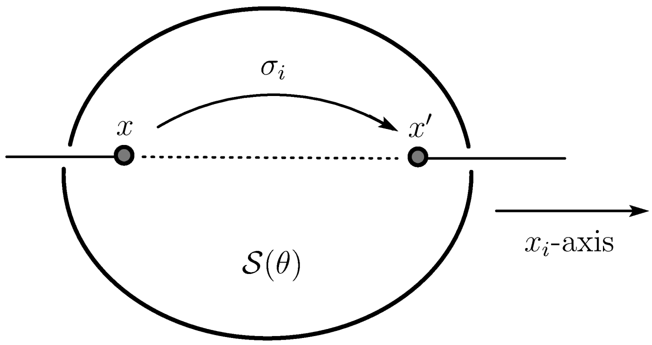

Figure 10: Involutions of the $(2,2,2)$-surface $\mathcal{S}(\theta)$

since the Painlevé flow and its Poincaré return map are too transcendental to admit such an extension. Thus one of the important roles of the Riemann-Hilbert correspondence is that it reduces the highly transcendental Poincaré return map γ_* into a more tractable birational map σ on $\overline{\mathcal{S}}(\theta)$, to which some recent advances [1, 3, 4, 5] in the ergodic theory of birational maps are applicable.

We procced to the ergodic theory. As in [10], assume that κ is generic, i.e., $\kappa \in \mathcal{K} - \mathbf{Wall}$. Theorem 11 then implies that $\mathcal{S}(\theta)$ is smooth and the Riemann-Hilbert correspondence (10) is biholomorphic, and further it is not hard to see that the projective cubic surface $\overline{\mathcal{S}}(\theta)$ is also smooth. Applying the methods in [1, 3, 4, 5] to our situation enables us to calculate the induced action of the birational map $\sigma : \overline{\mathcal{S}}(\theta) \circlearrowleft$ on the closed positive $(1,1)$-currents and also on the $(1,1)$-cohomology group. Moreover, it enables us to think of the stable and unstable currents $\nu_\sigma^\pm$ for the birational map σ and to legitimate their wedge product as a measure

$$\nu_\sigma = \nu_\sigma^+ \wedge \nu_\sigma^- . \tag{12}$$

It should be pointed out that the famous twenty-seven lines and related geometry on a smooth projective cubic surface also play an important part in these arguments.

Theorem 12 ([10]) *Assume that $\kappa \in \mathcal{K} - \mathbf{Wall}$. For any non-elementary loop $\gamma \in \pi_1(X,x)$, let $\sigma : \overline{\mathcal{S}}(\theta) \circlearrowleft$ be the birational map corresponding to the Poincaré return map $\gamma_* : \mathcal{M}_x(\kappa) \circlearrowleft$ via the Riemann-Hilbert correspondence* (10). *Then after a suitable renormalization, the wedge product* (12) *yields a σ-invariant Borel probability measure on $\overline{\mathcal{S}}(\theta)$ such that*

(1) *all the conditions* (C1)–(C4) *in Definition* 4 *are satisfied,*

(2) *ν_σ puts no mass on any algebraic curve on $\overline{\mathcal{S}}(\theta)$,*

(3) *the measure-theoretic entropy of σ with respect to ν_σ is given by $h_{\nu_\sigma}(\sigma) = \log \lambda(\sigma)$, where $\lambda(\sigma)$ is the spectral radius of the induced map σ^* on the cohomology group $H^{1,1}(\overline{\mathcal{S}}(\theta))$.*

Since the tritangent lines at infinity, $L = L_1 \cup L_2 \cup L_3$, are an algebraic curve on $\overline{\mathcal{S}}(\theta)$, it follows from property (2) of Theorem 12 that the ν_σ-measure of L is zero. Hence the

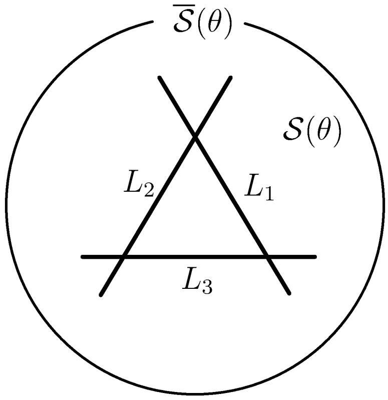

Figure 11: Tritangent lines at infinity on $\overline{\mathcal{S}}(\theta)$

probability measure ν_σ can be restricted to the affine cubic surface $\mathcal{S}(\theta) = \overline{\mathcal{S}}(\theta) - L$ without losing any mass. Then the resulting measure $\nu_\sigma|_{\mathcal{S}(\theta)}$, which is a probability measure on $\mathcal{S}(\theta)$, can be pulled back to the moduli space $\mathcal{M}_x(\kappa)$ to yield a probability measure

$$\mu_\gamma = \mathrm{RH}^*_{x,\kappa}(\nu_\sigma|_{\mathcal{S}(\theta)}) \qquad \text{on} \quad \mathcal{M}_x(\kappa) \tag{13}$$

via the Riemann-Hilbert correspondence (10), since it is a biholomorphism. This last measure is exactly what we have mentioned in Theorem 6. From property (3) of Theorem 12 one has $h_{\mu_\gamma}(\gamma_*) = \log \lambda(\sigma)$ and the latter quantity $\lambda(\sigma)$ can be calculated according to Algorithm 10.

11 Some Open Problems

In the study of the sixth Painlevé equation as a chaotic dynamical system, there remain many open problems yet to be discussed, some of which are presented in the end of this article.

- Nongeneric case. In [10], Theorem 6 was established only under the condition that κ is generic, that is, only when $\kappa \in \mathcal{K} - \mathbf{Wall}$. Now it is natural to ask what happens if $\kappa \in \mathbf{Wall}$. The latter case is more difficult to treat, where the difficulty lies in the fact that the Riemann-Hilbert correspondence (10) is not a biholomorphism but only an analytic resolution of Kleinian singularities (see Theorem 11) and hence it does not serve as a strict conjugacy. In order to get a strict conjugacy we should take a standard *algebraic* minimal resolution of singularities $\varphi : \widetilde{\mathcal{S}}(\theta) \to \mathcal{S}(\theta)$ and lift the Riemann-Hilbert correspondence (10) so as to induce the commutative diagram in Figure 12. Then the lifted Riemann-Hilbert correspondence

$$\widetilde{\mathrm{RH}}_{x,\kappa} : \mathcal{M}_x(\kappa) \to \widetilde{\mathcal{S}}(\theta)$$

is a biholomorphism and the Poincaré return map on $\mathcal{M}_x(\kappa)$ is strictly conjugated to an automorphism of $\widetilde{\mathcal{S}}(\theta)$ which can be extended to a birational map on the compactification of $\widetilde{\mathcal{S}}(\theta)$. In this manner we are still able to show that the chaotic nature of the Poincaré return map carries over to the nongeneric case, as is announced in the following:

$$\begin{tikzcd} \mathcal{M}_x(\kappa) \arrow[r, "\widetilde{\mathrm{RH}}_{x,\kappa}"] \arrow[d, equal] & \widetilde{\mathcal{S}}(\theta) \arrow[d, "\varphi"] \\ \mathcal{M}_x(\kappa) \arrow[r, "\mathrm{RH}_{x,\kappa}"] & \mathcal{S}(\theta) \end{tikzcd}$$

Figure 12: Lift of the Riemann-Hilbert correspondence

Theorem 13 *Even when* $\kappa \in \mathbf{Wall}$ *Theorem* 6 *remains valid except for assertion* (2).

However the issue treated in assertion (2) of Theorem 6, namely, calculating the number $\#\mathrm{Per}_N(\gamma;\kappa)$ of periodic points becomes subtle and yet to be explored. The subtlety comes from the existence of the exceptional locus $\mathcal{E}_x(\kappa) \subset \mathcal{M}_x(\kappa)$ for the resolution of singularities by the Riemann-Hilbert correspondence $\mathrm{RH}_{x,\kappa}$. It may happen that an irreducible component of $\mathcal{E}_x(\kappa)$ is a periodic curve of the Poincaré return map $\gamma_* : \mathcal{M}_x(\kappa) \circlearrowleft$. In that case the cardinality of $\mathrm{Per}_N(\gamma;\kappa)$ becomes infinite and therefore the problem should be replaced by the following:

Problem 14 Find the cardinality of $\mathrm{Per}_N^{\circ}(\gamma;\kappa) := \{\, Q \in \mathcal{M}_x(\kappa) - \mathcal{E}_x(\kappa) \,:\, \gamma_*^N Q = Q \,\}$.

The finiteness of $\#\mathrm{Per}_N^{\circ}(\gamma;\kappa)$ is already shown but its concrete value is yet to be calculated.

- Properties of invariant measures. The Poincaré return map $\gamma_* : \mathcal{M}_x(\kappa) \circlearrowleft$ admits (at least) two invariant measures; one is the *geometric measure*, that is, the symplectic volume form $vol_x(\kappa)$ constructed geometrically, associated to the Hamiltonian structure of $\mathrm{P_{VI}}(\kappa)$; the other is the *dynamical measure*, that is, the Borel probability measure μ_γ constructed dynamically as the "final state" of the infinitely many iterations of the Poincaré return map γ_*. Then it is interesting to discuss the relation between these two measures. For example one may pose:

Problem 15 Is the dynamical measure μ_γ absolutely continuous with respect to the geometric measure $vol_x(\kappa)$?

- Random Poincaré map. So far, the Poincaré return map $\gamma_* : \mathcal{M}_x(\kappa) \circlearrowleft$ has been considered for each *individual* loop $\gamma \in \pi_1(X,x)$. A next step would be to discuss the interaction of *plural* Poincaré return maps, namely, to consider the Poincaré return maps $\gamma_* : \mathcal{M}_x(\kappa) \circlearrowleft$ along various loops $\gamma \in \pi_1(X,x)$ together. A stochastic approach might be effective in such a question.

Problem 16 Explore statistical properties of the Poincaré return map over the random walks on the fundamental group $\pi_1(X,x)$ or on the universal Coxeter group $G = \mathrm{UCG}(3)$.

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