OKAMOTO SYMMETRY OF PAINLEVÉ VI EQUATION AND ISOMONODROMIC DEFORMATION OF LAMÉ CONNECTIONS

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Abstract. Isomonodromic deformations of rank 2 logarithmic connections with singular points 0, 1, t and ∞ over the Riemann sphere are parametrized by the solutions $q(t)$ of Painlevé VI equation. Some discrete group of symetries of P_{VI} equation naturally arise from the birational geometry of logarithmic connections. An extra symmetry was found by Okamoto in [15] by direct computations. Here, we present a geometric interpretation of this symmetry. After lifting conveniently the connection over the elliptic curve $E_t : \{y^2 = x(x-1)(x-t)\}$, the variation of the underlying vector bundle (along isomonodromic deformation) provides a new solution $\tilde{q}(t)$ of P_{VI} equation, namely the Okamoto symetric of $q(t)$. In particular, isomonodromic deformations of the Lamé connection over E_t arise as a particular case of our construction and we recover in a natural way recent results of S. Kawai and Levin-Olshanetsky. All proofs will appear in a forthcoming paper

Introduction

Let ∇ be a rank 2 meromorphic connection over a compact Riemann surface X. In general, one cannot explicitly express the monodromy representation of ∇ in terms of the algebraic datas of ∇ by means of classical functions. This can be carried out essentially in two cases:

- $X = \mathbb{P}^1$ and ∇ has at most 3 poles counted with multiplicity,
- X is an elliptic curve and ∇ is regular.

The first case reduces to the Gauss hypergeometric equation (or its degeneracies); the monodromy can therefore be expressed in terms of the coefficients by means of trigonometric functions (see [8]). The second one, due to the fact that the fundamental group is abelian, reduces to the case of rank 1 connections over the elliptic curve and the monodromy can be expressed by means of elliptic functions (see [6]). Beyond these two cases, monodromy becomes no more computable in general, but still isomonodromic deformations satisfy an explicit algebraic differential equation.

1. Painlevé VI equation and isomonodromic deformations

The simplest case, where $X = \mathbb{P}^1$ and ∇ has 4 simple poles, has been carried out by R. Fuchs in 1907: it leads to the Painlevé VI equation. One can reduce this study to that of the $\mathrm{sl}(2,\mathbb{C})$-Fuchsian system

$$\frac{dY}{dx} = \left(\frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}\right) Y, \quad A_i \in \mathrm{sl}(2,\mathbb{C}) \tag{1}$$

corresponding to a trace free connection on the trivial vector bundle. Singular points are 0, 1, t and ∞ where $t \in \mathbb{P}^1 - \{0, 1, \infty\}$ is the deformation parameter. The residual matrix of the singular point at $x = \infty$ is given by

$$A_0 + A_1 + A_t + A_\infty = 0. \tag{2}$$

Let $\pm\frac{\theta_i}{2}$ denote the eigenvalues of A_i:

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \quad \text{with} \quad a_i^2 + b_i c_i = \frac{\theta_i^2}{4}, \quad i = 0, 1, t, \infty\ . \tag{3}$$

After change of variable, $Y := MY$ with $M \in \mathrm{SL}(2,\mathbb{C})$, we normalize

$$A_\infty = \begin{pmatrix} \frac{\theta_\infty}{2} & 0 \\ 0 & -\frac{\theta_\infty}{2} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ when } \theta_\infty = 0. \tag{4}$$

The zero $x = q$ of the $(1,2)$-coefficient of the system is given by

$$q = \frac{tb_0}{tb_0 + (t-1)b_1} \in \mathbb{P}^1 \tag{5}$$

Here, we exclude lower triangular systems for which q is not defined.

Theorem (Fuchs)**.** *Assume that system (1) has no apparent singular point. Then a small deformation $A_i(t)$ normalized by (4) is isomonodromic if, and only if, eigenvalues $\pm\frac{\theta_i}{2}$ are constant and $q(t) := \frac{tb_0(t)}{tb_0(t)+(t-1)b_1(t)}$ satisfies*

$$\frac{dq}{dt} = -2a_0\frac{q-1}{t-1} - 2a_1\frac{q}{t} + (1-\theta_\infty)\frac{q(q-1)}{t(t-1)} \tag{6}$$

and the Painlevé VI equation

$$\begin{aligned} \frac{d^2q}{dt^2} = {} & \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right) \\ & + \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{q^2} + \gamma\frac{t-1}{(q-1)^2} + \delta\frac{t(t-1)}{(q-t)^2}\right). \end{aligned} \tag{7}$$

with parameters $\alpha = \frac{(\theta_\infty-1)^2}{2}$, $\beta = -\frac{\theta_0^2}{2}$, $\gamma = \frac{\theta_1^2}{2}$ and $\delta = \frac{1-\theta_t^2}{2}$.

System (1) is uniquely determined up to conjugacy by initial datas $(q(t), q'(t)) \in \mathbb{C}^2$ of (7) by means of formulae (2), (3), (4), (5) and (6).

Recall that a singular point of a connection is said apparent when the connection becomes regular at this point after applying a convenient gauge transformation; for a simple pole, this is equivalent to say that the local monodromy is $\pm I$. When the initial system (1) has apparent singular points, there is a notion stronger than "isomonodromy" characterizing the deformation parametrized by the corresponding Painlevé VI solution.

Roughly speaking, Fuch's Theorem provides an almost one-to-one correspondence between local meromorphic solutions of the Painlevé VI equation and isomonodromic local deformations of trace free connections having 4 simple poles over the Riemann sphere. Poles of $q(t)$ arise from the fact that the underlying bundle of the connection ∇_t is no longer trivial, but accidentally $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ (see [3]).

2. Okamoto symetries

Given an isomonodromic deformation ∇_t, parametrized by $q(t)$, one can apply a permutation σ of the singular points and derive a new isomonodromic deformation $\tilde{\nabla}_t$, and thus a new solution $\tilde{q}(t)$ of Painlevé VI equation. For instance, the change of variable $\tilde{x} = x/t$ permutes the role of 1 and t and, via the correspondence above, provides the new solution $\tilde{q}(t) = \frac{q(t)}{t}$ of Painlevé VI equation with new parameters $(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty) = (\theta_0, \theta_t, \theta_1, \theta_\infty)$. The full permutation group S_4 acts as birational symetries of Painlevé equation. Other symetries arise from gauge transformations, that is a combination of a rational transformation of the bundle and a tensor product with a convenient rank 1 connection restoring the trace free property. Some of them preserve the polar divisor of the connection and have the effect to shift the eigenvalues by integers: starting from a solution $q(t)$ of (7), one derive for any given 4-uple $(n_0, n_1, n_t, n_\infty) \in \mathbb{Z}^4$, $n_0 + n_1 + n_t + n_\infty \in 2\mathbb{Z}$, a new solution $\tilde{q}(t)$ of Painlevé VI equation with new parameters $(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty) = (\theta_0 + n_0, \theta_1 + n_1, \theta_t + n_t, \theta_\infty + n_\infty)$.

Finally, an extra symmetry was derived by K. Okamoto in [15] by direct computations. This symmetry, denoted W_2 in [15] and $s_4 s_2 s_4$ in [14], is an involution and generates, together with the previous geometric ones, the full group of birational symetries of Painlevé VI equation. Here, we will replace $W_2 = s_4 s_2 s_4$ by its conjugate $s_1 s_2 s_1$ (see notations of [14]) which shares the same properties and is easier to describe.

Introducing the new variable

$$p(t) := \frac{(t-1)q' + \theta_0}{2q} + \frac{-tq' + \theta_1}{2(q-1)} + \frac{q' + \theta_t - 1}{2(q-t)} \quad (q' = \frac{dq}{dt}), \tag{8}$$

the Okamoto symetric $s_1 s_2 s_1$ of a solution $q(t)$ of (7) is

$$\tilde{q}(t) = q(t) - \frac{\theta_0 + \theta_1 + \theta_t - \theta_\infty}{2p(t)} \tag{9}$$

and is a solution of Painlevé VI equation with new parameters:
$(\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_t, \tilde{\theta}_\infty) = (\frac{\theta_0-\theta_1-\theta_t+\theta_\infty}{2}, \frac{-\theta_0+\theta_1-\theta_t+\theta_\infty}{2}, \frac{-\theta_0-\theta_1+\theta_t+\theta_\infty}{2}, \frac{\theta_0+\theta_1+\theta_t+\theta_\infty}{2})$.

In [2, 14], the Painlevé VI equation is derived from isomonodromic deformation of higher rank connections in which the full birational group (including Okamoto symmetry) arise from geometric transformations of the connection. In the rank 2 case, one easily sees from its action on parameters θ_i that even the Galois group of the connection is not invariant under Okamoto symmetry: both finiteness and irreducibility of the Galois group are not preserved.

3. Main result

Consider the isomonodromic deformation ∇_t of the Fuchsian system (1) with eigenvalues $\pm\frac{\theta_i}{2}$ and let $q(t)$ be the corresponding solution of the Painlevé VI equation. One can lift-up this connection to the Legendre elliptic curve $E_t : \{y^2 = x(x-1)(x-t)\}$ via the double cover $\pi : (x,y) \mapsto x$ as a logarithmic connection $\tilde{\nabla}_t$ having simple poles over order 2 periodic points $\omega_i = (i, 0) \in E_t$, $i = 0, 1, t, \infty$. The simplest way to do this is a base change, leading to a connection $\tilde{\nabla}_t^0$ on the trivial bundle $\mathcal{O}_{E_t} \oplus \mathcal{O}_{E_t}$ with eigenvalues $\pm\theta_i$ over ω_i. After a convenient gauge transformation, one can shift all eigenvalues by one half: we thus obtain a trace free logarithmic connection $\tilde{\nabla}_t$ over E_t with poles ω_i and eigenvalues $\pm(\theta_i - \frac{1}{2})$. Precisely, this is done after 4 elementary transformations (see [4] for definition) and then tensoring by a rank 1 bundle in order to restore the trace free property. This construction can be carried out along an isomonodromic deformation ∇_t, and $\tilde{\nabla}_t$ is isomonodromic as well. The underlying vector bundle V_t of $\tilde{\nabla}_t$ is not trivial anymore.

Following Atiyah (see [1, 5]), almost all rank 2 vector bundles on E_t with trivial determinant are decomposable, i.e. of the form

$$V = L \oplus L^{\otimes(-1)} \quad \text{with} \quad L \in \mathrm{Pic}(E_t); \tag{10}$$

to complete the list, one has to add 4 extra bundles V_i, $i = 0, 1, t, \infty$. Semistable bundles are those decomposable ones with $L \in \mathrm{Pic}^0(E_t)$, i.e. of the form

$$V = \mathcal{O}_{E_t}([\omega]-[\omega_\infty]) \oplus \mathcal{O}_{E_t}([-\omega]-[\omega_\infty]), \quad \omega=(x,y)\in E_t, \tag{11}$$

together with the 4 indecomposable ones above. The corresponding moduli space is $\mathbb{P}^1$ (see [17]) with quotient map given by

$$\begin{array}{ll} V \mapsto x & \text{under notation of (11)} \\ V_i \mapsto i & \qquad i=0,1,t,\infty \end{array} \tag{12}$$

In fact, V_i is, for $i=0,1,t,\infty$, the unique non trivial extension

$$0 \to L_i \to V_i \to L_i \to 0, \quad \text{where } L_i = \mathcal{O}_{E_t}([\omega_i]-[\omega_\infty]);$$

in the quotient, V_i is identified with the trivial extension $L_i \oplus L_i$.

Theorem 1. *Let ∇_t be the isomonodromic deformation of an irreducible rank 2 connection over $\mathbb{P}^1$ having simple poles at $i=0,1,t,\infty$ with eigenvalues $\pm\frac{\theta_i}{2}$; let $q(t)$ be the corresponding solution of Painlevé VI equation. Let $\tilde{\nabla}_t$ be a trace free lifting of ∇_t on the double cover $E_t : \{y^2 = x(x-1)(x-t)\}$ with simple poles over $i=0,1,t,\infty$ and eigenvalues $\pm(\theta_i-\frac{1}{2})$. Then, for a Zariski open set of parameter t, the underlying vector bundle of $\tilde{\nabla}_t$ is semistable and its invariant $x(t)\in\mathbb{P}^1$ defined by (12) is given by*

$$x(t) = \tilde{q}, \quad \frac{t}{\tilde{q}}, \quad \frac{\tilde{q}-t}{\tilde{q}-1} \quad \text{or} \quad t\frac{\tilde{q}-1}{\tilde{q}-t} \tag{13}$$

where $\tilde{q}(t)$ is the Okamoto symetric of $q(t)$ defined by (9).

In the statement above, the value of $x(t)$ depends on the choice of the lifting $\tilde{\nabla}_t$: it is well defined up to tensor product by a regular rank 1 connection whose $\otimes$-square is trivial; there are 4 possibilities. We have seen that $\tilde{q}$ is a solution of Painlevé VI equation; the other ones are obtained after composition by one of the symetries r_3, r_4 and r_1 respectively of [14], corresponding to permutations of the singular points: they are also solution of Painlevé VI equation.

In the proof of Theorem 1, we actually show that the vector bundle V_t is semistable and decomposable whenever $x(t)\neq 0$, 1, t or ∞. By "Zariski open", we just mean that exceptional values of t form a discrete subset of the parameter space, i.e. the universal cover of $\mathbb{P}^1-\{0,1,\infty\}$.

4. Lamé connections

Let us call Lamé connection any logarithmic trace free connection on the elliptic curve E_t with a single pole at ω_∞. In the special case $(\theta_0,\theta_1,\theta_t,\theta_\infty) = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}+\vartheta)$ of our construction, $\vartheta\in\mathbb{C}$, the lifted connection $\tilde{\nabla}_t$ is a Lamé connection with eigenvalues $\pm\vartheta$. It turns out that all irreducible Lamé connections arise by this way and we get

Corollary 2. *Let $\tilde{\nabla}_t$ be the isomonodromic deformation of a Lamé connection over the Legendre deformation E_t (for local parameter t) with eigenvalues $\pm\vartheta \notin \frac{1}{2}\mathbb{Z}$. Then, for a Zariski open set of parameter t, the underlying vector bundle of $\tilde{\nabla}_t$ is semistable and its invariant $x(t) \in \mathbb{P}^1$ defined by (12) is solution of Painlevé VI equation with parameters $(\alpha, \beta, \gamma, \delta) = (\frac{\vartheta^2}{8}, -\frac{\vartheta^2}{8}, \frac{\vartheta^2}{8}, \frac{1}{2} - \frac{\vartheta^2}{8})$.*

It is already known that isomonodromic deformation of Lamé connections are parametrized by Painlevé VI equation with above parameters: in [9, 11], isomonodromic deformation equations are directly computed on the elliptic curve, and the elliptic form of Painlevé VI equation (see [12]) is recognized.

Our approach of this result is quite different since we lift-up (isomonodromic deformation of) connections over $\mathbb{P}^1$ onto the elliptic curve instead of lifting Painlevé equation: our reduction of isomonodromic equations to Painlevé VI equation is therefore the classical one on $\mathbb{P}^1$ due to Fuchs. Moreover, isomonodromic deformations are parametrized in our case by Painlevé VI equation with parameters $(\alpha, \beta, \gamma, \delta) = (\frac{(2\vartheta-1)^2}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$ and parameters of [9, 11] are derived from the variation of the bundle; Okamoto symmetry is in between. Finally, when $\vartheta \notin \frac{1}{2}\mathbb{Z}$, the Lamé connection is irreducible; the fact that it can be pushed down to $\mathbb{P}^1$ reduces, via the Riemann-Hilbert correspondence, to the following:

two elements of $A, B \in SL(2, \mathbb{C})$ are simultaneously conjugated to their inverse A^{-1}, B^{-1} provided that A and B generate an irreducible subgroup.

When $\vartheta \in \frac{1}{2}\mathbb{Z}$, there are Lamé connections that do not come from our construction. For instance, when $\vartheta \in \mathbb{Z}$ and the singular point is not apparent, the monodromy is reducible, non abelian and is not invariant under the involution of the cover $E_t \to \mathbb{P}^1$: the corresponding connection cannot be pushed down. Nevertheless, when $\vartheta \in \frac{1}{2} + \mathbb{Z}$ and the singular point is not apparent, the connection is irreducible and the conclusion of Corollary (2) still holds. On the other hand, when the Lamé connection has an apparent singular point, the monodromy representation is abelian ($\vartheta \in \mathbb{Z}$) or quaternionic ($\vartheta \in \frac{1}{2} + \mathbb{Z}$) and is invariant under the elliptic involution; however, the monodromy representation fails to determine the connection in this case and the connection cannot always be pushed down.

Finally, in the very special case $\vartheta = 0$, the lifted connection $\tilde{\nabla}_t$ is regular and generically splits into the direct sum of regular rank 1 connections (there are also 4 codimension one sub-families of undecomposable connections having abelian parabolic monodromy). Our construction coincides in this case with that one of N. Hitchin in [6] and we retrieve

Corollary 3. *Let ∇_t be the isomonodromic deformation of a regular rank 1 connection on the elliptic curve E_t and let $\mathcal{O}_{E_t}([\omega(t)] - [\omega_\infty])$ be the underlying line bundle, $\omega(t) = (x(t), y(t)) \in E_t$. Then $x(t)$ is solution of Painlevé VI equation with parameters $(\alpha, \beta, \gamma, \delta) = (0, 0, 0, \frac{1}{2})$.*

This latter corollary, observed in [10], is just a modern translation of

Theorem (Picard [16], see [13])**.** *The general solution of Painlevé VI equation with parameters $(\alpha, \beta, \gamma, \delta) = (0, 0, 0, \frac{1}{2})$ is given by*

$$t \mapsto x(t) \quad \textit{where} \quad (x(t), y(t)) := \pi(c_0 \cdot \omega_0 + c_1 \cdot \omega_1), \;\; c_0, c_1 \in \mathbb{C}$$

(where $\pi : \mathbb{C} \to E_t$ is the universal cover and $\omega_i(t)$, half-periods of E_t).

In this sense, Corollary 2 may be viewed as a generalization of Picard Theorem.

For general parameters $(\alpha, \beta, \gamma, \delta)$, our lifting construction provides an isomonodromic deformation problem (a Lax pair) for the general elliptic form of Painlevé VI equation just by considering those rank 2 trace free connections over E_t with simple poles over ω_i, $i = 0, 1, t, \infty$, that moreover commute with the involution $\sigma : z \mapsto -z$ (compare [18]).

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