Rational transformations of confluent hypergeometric equations and algebraic solutions of the Painlevé equations: P1 to P5

Yousuke Ohyama
Graduate School of Information Science and Technology,
Osaka University, Machikaneyama-machi 1-1, Toyonaka, 560-0043, Japan

1 Introduction

In this note, we review our recent result with S. Okumura that algebraic solutions of the Painlevé equations from the second to the fifth are obtained by pullback transformations of confluent hypergeometric equations.

It is useful to study special solutions to understand the Painlevé equations. Especially, algebraic solutions and the Riccati solutions (hypergeometric-type solutions) are studied by many researchers. Now all of algebraic solutions of the Painlevé equations are classified except for the sixth equation. The Riccati solutions are completely classified for all the Painlevé equations.

The Painlevé equations can be obtained by isomonodromic deformations of the linear ordinary equations. But special solutions are mainly studied without isomonodromic deformation method. In this note we will show that algebraic solutions of the Painlevé equations from the first to the fifth can be computed by using pullback transformations of confluent hypergeometric equations.

Many algebraic solutions of the sixth Painlevé equation can be computed by using pullback transformations of the Gauss hypergeometric equations by Kitaev [15]. Such pullback transformations were used by R. Fuchs [4] at first. He proposed the following problem:

> When can we transform a linearization of a Painlevé function $y(t)$
>
> $$\frac{d^2v}{dz^2} = Q(t, y(t), z)v$$
>
> into an equation without the deformation parameter t
>
> $$\frac{d^2u}{dx^2} = \tilde{Q}(x)u$$
>
> by a suitable transformation $x = x(z,t), v = \sqrt{dz/dx}\ u$?

He showed that the linear equation can be transformed to the Gauss hypergeometric equation for three, four and six divided points of Picard's solutions. See also [5].

While R. Fuchs and Kitaev studied only the sixth Painlevé equation, we study other types of Painlevé equations. We classify pullback transformations of the confluent hypergeometric equations, because the linear equations corresponding to the Painlevé equations from the first to the fifth have irregular singularities. We remark that we use not only the standard Whittaker confluent hypergeometric equation but also a degenerate confluent hypergeometric equation.

In section two, we review isomonodromic deformations associated with the Painlevé equations. In [16], Okamoto presented a coalescent diagram of the Painlevé equations using confluence of singularities. We extend his coalescent diagram to include irregular singularities whose Poincaré rank are half-integers. We call linear equations in our coalescent diagram *equations of the Painlevé type*.

In section three, we list up all of rational transformations of the confluent hypergeometric equations to linear equations of the Painlevé type. Such rational transformations give most of all algebraic solutions of the Painlevé equations from the first to the fifth and symmetric solutions, which are non-algebraic solutions of the first, second and fourth Painlevé equations [14], [9]. Although two of the algebraic solutions are not obtained by rational transformations of the confluent hypergeometric equations, they can also be obtained by non-rational pullback transformations. Thus we can obtain all of algebraic solutions of the Painlevé equations except for the sixth equation by pullback transformations of confluent hypergeometric equations.

Since the monodromy representations for these two algebraic solutions are completely reducible, they cannot be obtained by rational transformations of confluent hypergeometric equations.

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2 Coalescent diagram of the Painlevé equations

In this section, we extend the coalescent diagram of the Painlevé equations given by Okamoto [16]. This section is a review of [18].

We list the Painlevé equations:

$$
\begin{aligned}
\text{P1)}\quad y'' &= 6y^2 + t, \\
\text{P2)}\quad y'' &= 2y^3 + ty + \alpha, \\
\text{P4)}\quad y'' &= \frac{1}{2y}y'^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \\
\text{P3)}\quad y'' &= \frac{1}{y}y'^2 - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y}, \\
\text{P5)}\quad y'' &= \left(\frac{1}{2y} + \frac{1}{y-1}\right) y'^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma \frac{y}{t} + \delta \frac{y(y+1)}{y-1}, \\
\text{P6)}\quad y'' &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right) y'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right) y' \\
&\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right].
\end{aligned}
$$

Here $\alpha, \beta, \gamma, \delta$ are complex parameters.

As usual, we write the coalescent diagram of the Painlevé equations as follows:

$$
\begin{array}{ccccccccc}
 & & & \nearrow & (2+2) & \searrow & & & \\
(1+1+1+1) & \longrightarrow & (1+1+2) & & & & (4) & \longrightarrow & (7/2) \\
 & & & \searrow & (1+3) & \nearrow & & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & \nearrow & \text{P3} & \searrow & & & \\
\text{P6} & \longrightarrow & \text{P5} & & & & \text{P2} & \longrightarrow & \text{P1} \\
 & & & \searrow & \text{P4} & \nearrow & & &
\end{array}
$$

This diagram is well-known since Painlevé [21]. But from the viewpoint of the isomonodromic deformations, it is more natural to extend the diagram so that it includes irregular singularities whose Poincare rank are half-integers.

Let

$$\frac{d^2u}{dx^2} + p_1(x)\frac{du}{dx} + p_2(x)u = 0, \tag{1}$$

be a second-order linear equation where $p_1(x)$ and $p_2(x)$ have the expansion

$$p_1(x) = c_0x^k + c_1x^{k-1} + \cdots, \; p_2(x) = d_0x^l + d_1x^{l-1} + \cdots,$$

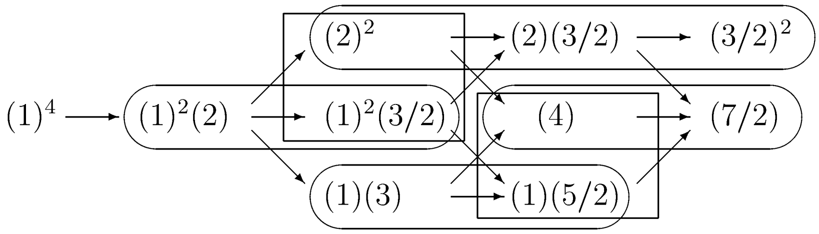
around $x = \infty$ with non-zero constants c_0 and d_0. If

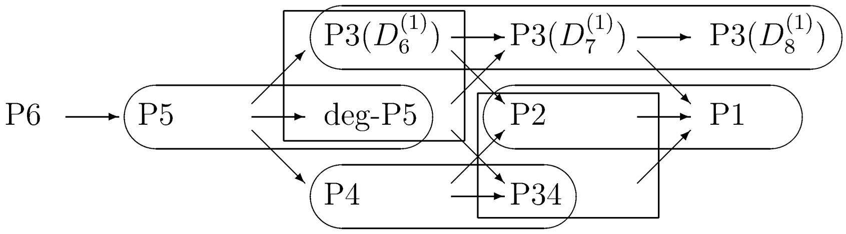
$$r = \max\,(k+1, (l+2)/2)$$

is positive, $x=\infty$ is an irregular singularity of (1) and the number r is called the Poincaré rank of (1) at $x=\infty$. The Poincaré rank r may be a half integer. If $x=\infty$ is an irregular singularity with the Poincaré rank r, (1) has a solution with asymptotics

$$u_j \sim \exp\left(\kappa_j x^r\right).$$

We list all types of the singularities and corresponding Painlevé equations:





Here the symbol (1) means a regular singularity and the symbol (n) means an irregular singularity with the Poincaré rank $n-1$. The symbol $(1)^4$ means four regular singularities. The singularity type $(1)^4$ gives P6, which is shown in [3].

We explain the label of Painlevé equations. For the third Painlevé equation

$$y'' = \frac{1}{y}y'^2 - \frac{y'}{t} + \frac{\alpha y^2+\beta}{t} + \gamma y^3 + \frac{\delta}{y}$$

we divide into four types:

(P3-A) $\gamma \neq 0, \delta \neq 0$
(P3-B) $\gamma \neq 0, \delta = 0$ or $\gamma = 0,\ \delta \neq 0$
(P3-C) $\gamma = 0,\ \delta = 0$
(P3-D) $\alpha = 0,\ \gamma = 0$ or $\beta = 0,\ \delta = 0$.

(P3-A), (P3-B) and (P3-C) are called P3($D_6^{(1)}$), P3($D_7^{(1)}$) and P3($D_8^{(1)}$), respectively. We exclude (P3-D) from a family of the Painlevé equations, since it is quadrature. In the usual setting we fix $\gamma = 4, \delta = -4$ for P3($D_6^{(1)}$), $\alpha = 2, \gamma = 0, \delta = -4$ for P3($D_7^{(1)}$) and $\alpha = 4, \beta = -4, \gamma = 0, \delta = 0$ for P3($D_8^{(1)}$). These three different types of the third equations were noticed by Painlevé [20]. For P3($D_7^{(1)}$) and P3($D_8^{(1)}$), see also [19].

We use another form of the third Painlevé equation P3′$(\alpha, \beta, \gamma, \delta)$

$$q'' = \frac{1}{q}q'^2 - \frac{q'}{x} + \frac{\alpha q^2 + \gamma q^3}{4x^2} + \frac{\beta}{4x} + \frac{\delta}{4q},$$

since P3′ is is suited to isomonodromic deformations better than P3. We can change P3 to P3′ by $x = t^2, ty = q$.

For the fifth Painlevé equation

$$y'' = \frac{1}{y}y'^2 - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y},$$

we divide into three types:

(P5-A) $\delta \neq 0$
(P5-B) $\gamma \neq 0, \delta = 0$
(P5-C) $\gamma = 0,\ \delta = 0.$

The case (P5-A) is a generic P5 and we denote (P5-B) as deg-P5, which is equivalent to P3$(D_6^{(1)})$ [7]. We exclude (P5-C) from a family of the Painlevé equations, since it is quadrature. In the usual setting we fix $\delta = -1/2$ for (P5-A) and $\gamma = -2, \delta = 0$ for (P5-B).

2.1 The Flaschka-Newell form

We have obtained ten different types of singularities. But it occurs that two different types of singularities give the same Painlevé equations. There are two such examples; $(2)^2$ gives standard P3 while $(1)^2(3/2)$ gives degenerate P5 with $\delta = 0$, but standard P3 and degenerate P5 are equivalent. Similarly (4) gives standard P2 while (1)(5/2) gives degenerate P4, but they are also equivalent. The degenerate P4 is nothing but P34 in Gambier's list [6]. It is known that P34 is equivalent to P2. We also show that (1)(5/2) is equivalent to the Flaschka-Newell form of P2 [2], [12], [13].

There are two different isomonodromic deformations of P2: $y'' = 2y^3 + ty + \alpha$. One is by Miwa-Jimbo [8] and the other is by Flaschka-Newell [2]. In the matrix form

$$\frac{\partial Y}{\partial x} = A(x,t)Y, \quad \frac{\partial Y}{\partial t} = B(x,t)Y,$$

the Flaschka-Newell form (FN) is

$$\begin{aligned} A^{FN}(x,t) &= -4\begin{pmatrix} x^2 & yx \\ yx & -x^2 \end{pmatrix} + \begin{pmatrix} t+2y^2 & -2z \\ 2z & -t-2y^2 \end{pmatrix} - \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}\frac{1}{x}, \\ B^{FN}(x,t) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}, \end{aligned} \tag{2}$$

while the Miwa-Jimbo form (MJ) is

$$\begin{aligned} A^{MJ}(x,t) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x^2 + \begin{pmatrix} 0 & u \\ \frac{-2}{u}z & 0 \end{pmatrix} x + \begin{pmatrix} z+\frac{t}{2} & -uy \\ \frac{-2}{u}(\theta+yz) & -z-\frac{t}{2} \end{pmatrix}, \\ B^{MJ}(x,t) &= \frac{x}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & u \\ -\frac{2}{u}z & 0 \end{pmatrix}. \end{aligned} \tag{3}$$

$A^{FN}(x,t)$ has an irregular singularity of the Poincaré rank three at $x=\infty$ and a regular singularity at $x=0$. $A^{MJ}(x,t)$ has an irregular singularity of the Poincaré rank three but has no other singularities. They are not connected by any rational transform of the independent variable.

Proposition 1 *The Flaschka-Newell form of P2 is a double cover of the linear equation of the singularity type* (1)(5/2)*. If we write the equation of the type* (1)(5/2) *as a single equation, the apparent singularity satisfies P34(α)*

$$y'' = \frac{y'^2}{2y} + 2y^2 - ty - \frac{\alpha}{2y}.$$

Remark. The name *the thirty-fourth Painlevé equation* comes from Gambier's classification [6].

Proof. We consider the following deformation equation.

$$\begin{aligned} \frac{dZ}{dw} &= \left[\begin{pmatrix} 0 & 2w \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -2y & -y^2-z-t/2 \\ 2 & 2y \end{pmatrix} + \begin{pmatrix} -\alpha+1/2 & 0 \\ -2y^2+2z-t & \alpha-1/2 \end{pmatrix} \frac{1}{2w} \right] Z, \\ \frac{\partial Z}{\partial t} &= \begin{pmatrix} y & -w \\ -1 & -y \end{pmatrix} Z. \end{aligned} \tag{4}$$

By the compatibility condition, we obtain P2(α)

$$y' = z, \quad z' = 2y^3 + ty + \alpha.$$

If we change $w = x^2$ and $Z = RY$ with

$$R = \begin{pmatrix} \sqrt{x} & \sqrt{x} \\ -1/\sqrt{x} & 1/\sqrt{x} \end{pmatrix},$$

we obtain the FN form (2). Since the exponents of (4) at $w=\infty$ coincide, the Poincaré rank at $w=\infty$ of the equation (4) is 3/2.

We will rewrite (4) as a single equation of the second order. We set

$$Z = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1 = w^{1/4-\alpha/2}u,$$

and change the variables

$$w \to \frac{w}{2}, \ z \to p^2 + q - \frac{t}{2}, \ y \to -p.$$

Eliminating u_2, we get a single equation for $u = u_1$:

$$\begin{aligned} \frac{d^2u}{dw^2} + p_1(w,t)\frac{du}{dw} + p_2(w,t)u = 0, \\ \frac{\partial u}{\partial t} = a(w,t)\frac{\partial u}{\partial w} + b(w,t)u, \end{aligned} \tag{5}$$

where

$$\begin{aligned} p_1(w,t) &= -\frac{1}{w-q} + \frac{1/2 - \alpha}{w}, \quad p_2(w,t) = -\frac{w}{2} + \frac{t}{2} + \frac{\mathcal{H}_{34}}{w} + \frac{pq}{w(w-q)}, \\ a(w,t) &= -\frac{w}{w-q}, \quad b(w,t) = \frac{pq}{w-q}, \\ \mathcal{H}_{34} &= -qp^2 + \left(\alpha + \frac{1}{2}\right)p + \frac{q^2}{2} - \frac{1}{2}tq. \end{aligned}$$

The isomonodromic deformation (5) is described by the Hamiltonian system with the Hamiltonian $\mathcal{H}_{34}$. If we eliminate p from the Hamiltonian system, we obtain P34($(\alpha + 1/2)^2$) for q.

The first equation of (5) has a regular singularity $w = 0$ and an irregular singularity of the Poincaré rank 3/2 at $w = \infty$. It also has an apparent singularity at $w = q$. When we write isomonodromic deformation of a linear equation associated with the Painlevé equation in the form of a single equation of the second order, it has an apparent singularity, which gives the Painlevé function. Moreover

$$p = \mathrm{Res}_{w=q} p_2(w,t)$$

is a canonical coordinate [16]. In the Flaschka-Newell case, the apparent singularity q satisfies P34 but not P2. □

3 Pullback of confluent hypergeometric equations

In this section we show that algebraic solutions of the Painlevé equations from the first to the fifth can be obtained by rational transformations of confluent hypergeometric equations. This section is a survey of [17]. In subsection 3.1, we review classical solutions (in the sense of Umemura [22]) of the Painlevé equations from the first to the fifth. We also explain symmetric solutions of P1, P2 and P4. In subsection 3.2, we review confluent hypergeometric equations. We also use a degenerate form of confluent hypergeometric equations which have an irregular singularity with the Poincaré rank

1/2 at the infinity. In subsection 3.2, we list up all rational transformations of the confluent hypergeometric equations that give linear equations in the extended coalescent diagram. We call a linear equation in our extended coalescent diagram *a linear equation of the Painlevé type*.

By using rational transformations of confluent hypergeometric equations, we obtain almost all algebraic solutions of the Painlevé equations, but not all of them. In addition we obtain some non-algebraic solutions of the Painlevé equations, which are called *symmetric solutions* [14], [9]. The linear equations of the symmetric solutions are reduced to pullback of confluent hypergeometric equations only for a special initial value.

Although two of the algebraic solutions are not obtained by rational transformations of the confluent hypergeometric equations, they can be obtained by non-rational pullback transformations. Thus we can obtain all of algebraic solutions of the Painlevé equations except for the sixth equation by pullback transformations of confluent hypergeometric equations. Kitaev and Vidunas constructed many algebraic solutions of the sixth Painlevé equation by pullback of hypergeometric equations.

3.1 Special solutions of the Painlevé equations

We study special solutions of the Painlevé equations from the first to the fifth. Classical solutions in the sense of Umemura are either algebraic or the Riccati type solutions. They are all classified for the first to the fifth Painlevé equations.

Theorem 2 *1) All solutions of P1 are transcendental.*

2) P2(0) *has a rational solution* $y = 0$. *P2*$(-1/2)$ *has a Riccati type solution* $y = -u'/u$. *Here* u *is any solution of the Airy equation* $u'' + tu/2 = 0$.

3) P34$((\alpha + 1/2)^2)$ *is equivalent to P2*(α). *P34*$(1/4)$ *has a rational solution* $y = t/2$. *P34*(1) *has Riccati type solutions.*

4) P4$(0, -2/9)$ *has a rational solution* $y = -2t/3$. *P4*$(1 - s, -2s^2)$ *has a Riccati type solution* $y = -u'/u$. *Here* u *is any solution of the Hermite-Weber equation* $u'' + 2tu' + 2su = 0$. *If* $s = 1$, *P4*$(0, -2)$ *has a rational solution* $y = -2t$, *which is reduced to the Hermite polynomial.*

5) P3'$(D_6)(a, -a, 4, -4)$ *has an algebraic solution* $y = -\sqrt{t}$. *P3'*$(D_6)(4h, 4(h+1), 4, -4)$ *has a Riccati type solution* $y = u'/u$. *Here* u *is any solution of* $tu'' + (2h+1)u' - 4tu = 0$.

6) P3'$(D_7)(\alpha, \beta, \gamma, 0)$ *does not have a Riccati type solution. P3'*$(D_7)(0, -2, 2, 0)$ *has an algebraic solution* $y = t^{1/3}$.

7) P3'$(D_8)(\alpha, \beta, 0, 0)$ *does not have a Riccati type solution. P3'*(D_8) $(8h, -8h, 0, 0)$ *has an algebraic solution* $y = -\sqrt{t}$.

8) P5$(a, -a, 0, \delta)$ *has a rational solution* $y = -1$. *P5*$((\kappa_0+s)^2/2, -\kappa_0^2/2, -(s+1), -1/2)$ *has Riccati type solutions* $y = -tu'/(\kappa_0 + s)u$. *Here* u *is any solution of* $t^2u'' + t(t -$

$s - 2\kappa_0 + 1)u' + \kappa_0(\kappa_0 + s)u2 = 0$. If $\kappa_0 = 1$, P5($(s+1)^2/2, -1/2, -(s+1), -1/2$) has a rational solution $y = t/(s+1) + 1$, which is reduced to the Laguerre polynomial.

9) deg-P5($\alpha_1^2/2, -\beta_1^2/2, -2, 0$) is equivalent to P3($D_6$)($4(\alpha_1-\beta_1), -4(\alpha_1+\beta_1-1), 4, -4$). deg-P5($h^2/2, -8, -2, 0$) has an algebraic solution $y = 1 + 2\sqrt{t}/h$. deg-P5($\alpha, 0, \gamma, 0$) has Riccati type solutions.

10) All of the classical solutions of P1 to P5 are equivalent to the above solutions up to the Bäcklund transformations.

It is known that the first, second and fourth Painlevé equations have a simple symmetry:

$$\begin{array}{lll} \text{P1} & y \to \zeta^3 y, \quad t \to \zeta t, & (\zeta^5 = 1) \\ \text{P2} & y \to \omega y, \quad t \to \omega^2 t, & (\omega^3 = 1) \\ \text{P4} & y \to -y, \quad t \to -t, & \end{array}$$

There exist symmetric solutions invariant under the action of the simple symmetry above. The symmetric solutions are studied by Kitaev [14] for P1 and P2 and by Kaneko [9] [10] for P2 and P4. Since these symmetric solutions exist for any parameter of the Painlevé equations, they are not algebraic for generic parameters.

Theorem 3 *1) For P1, we have two symmetric solutions*

$$\begin{aligned} y &= \frac{1}{6}t^3 + \frac{1}{336}t^8 + \frac{1}{26208}t^{13} + \frac{95}{224550144}t^{18} + \cdots, \\ y &= t^{-2} - \frac{1}{6}t^3 + \frac{1}{264}t^8 - \frac{1}{19008}t^{13} + \cdots. \end{aligned}$$

2) For P2(α), we have three symmetric solutions

$$\begin{aligned} y &= \frac{\alpha}{2}t^2 + \frac{\alpha}{40}t^5 + \frac{10\alpha^3 + \alpha}{2240}t^8 + \cdots, \\ y &= t^{-1} - \frac{\alpha+1}{4}t^3 + \frac{(\alpha+1)(3\alpha+1)}{112}t^5 + \cdots, \\ y &= -t^{-1} - \frac{\alpha-1}{4}t^3 - \frac{(\alpha-1)(3\alpha-1)}{112}t^5 + \cdots. \end{aligned}$$

They are equivalent to each other by the Bäcklund transformations.

3) For P4($\alpha, -8\theta_0^2$), we have four symmetric solutions

$$\begin{aligned} y &= \pm 4\theta_0 \left(t - \frac{2\alpha}{3}t^3 + \frac{2}{15}(\alpha^2 + 12\theta_0^2 \pm \theta_0 + 1)t^5 + \cdots \right), \\ y &= \pm t^{-1} + \frac{2}{3}(\pm\alpha - 2)t \mp \frac{2}{45}(-7\alpha^2 \pm 16\alpha + 36\theta_0^2 - 4)t^3 + \cdots. \end{aligned}$$

They are equivalent to each other by the Bäcklund transformations.

Symmetric solutions exist for P6 with special parameters [11]. But we do not treat P6 in this paper.

3.2 Transformations of linear equations

In this subsection we review confluent hypergeometric equations.

The confluent hypergeometric equation has two standard forms. One is the Kummer type and the other is the Whittaker type. In this paper we use the Whittaker type of confluent hypergeometric equations.

$$W_{k,m}: \quad \frac{d^2u}{dx^2} = \left(\frac{1}{4} - \frac{k}{x} + \frac{m^2 - \frac{1}{4}}{x^2}\right) u, \tag{6}$$

$$DW_m: \quad \frac{d^2u}{dx^2} = \left(\frac{1}{x} + \frac{m^2 - \frac{1}{4}}{x^2}\right) u. \tag{7}$$

The second equation DW_m is a degeneration of the Whittaker equation. The author does not know the standard name of DW_m, although the degeneration of the Kummer equation

$$x\frac{d^2u}{dx^2} + c\frac{du}{dx} - u = 0, \tag{8}$$

was studied by Kummer himself [1]. The solutions of (8) is

$$y = C\ {}_0F_1(c; x) + D\ x^{1-c}{}_0F_1(2 - c; x).$$

3.3 Pullback of $W_{k,m}$ and DW_m

The following lemma is well-known but it is useful to construct pullback transformations.

Lemma 4 *For an equation*

$$\frac{d^2u}{dx^2} = Q(x)u,$$

we set

$$x = x(z), \ \ u(x) = \sqrt{\frac{dx}{dz}}v(z).$$

Then v satisfies

$$\frac{d^2v}{dz^2} = \left(Q(x(z))(x'(z))^2 - \frac{1}{2}\{x, z\}\right) v. \tag{9}$$

Here $\{x, z\}$ is the Schwarzian derivative

$$\{x, z\} = \frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'}\right)^2.$$

The following theorem is our main result. We list up all of rational transformations $x = x(z)$ which transform Whittaker or degenerate Whittaker equations into linear equations of the Painlevé type or confluent hypergeometric equations of Weber, Bessel or Airy type.

Theorem 5 *By a rational transform $x = x(z)$, $W_{k,m}$ or DW_m is transformed to a linear equation of the Painlevé type or a confluent hypergeometric equation if and only if one of the following cases occurs.*

1) Double cover

$W_{k,m}$	$(2\|2)$	$(0)(2)$	*P4-sym*
$W_{k,1/4}$	$(2\|2)$	(2)	*Weber*
$W_{k,1/4}$	$(2\|1+1)$	$(1)^2$	*D6-alg*
$W_{0,1/2}$	$(1+1\|2)$	$(0)(2)$	*P4-Her*
DW_m	$(2\|2)$	$(0)(1)$	*Bessel*
DW_m	$(1+1\|2)$	$(1)^2(2)$	*P5-rat*
$DW_{1/4}$	$(2\|1+1)$	$(1/2)^2$	*D8-alg*

2) Cubic cover

$W_{k,1/3}$	$(3\|3)$	(3)	*P2-sym*
DW_m	$(3\|3)$	$(1)(3/2)$	*P34-sym*
$DW_{1/6}$	$(3\|3)$	$(3/2)$	*Airy*
$DW_{1/4}$	$(2+1\|3)$	$(0)(3/2)$	*P34-rat*
$DW_{1/6}$	$(3\|2+1)$	$(1)(1/2)$	*D7-alg*

3) Quartic cover

$DW_{1/6}$	$(3+1\|4)$	(3)	*P4-rat*

4) Quintic cover

$DW_{1/5}$	$(5\|5)$	$(5/2)$	*P1-sym*
$DW_{1/10}$	$(5\|5)$	$(5/2)$	*P1-sym*

5) Sextic cover

$DW_{1/6}$	$(3+3\|6)$	(3)	*P2-rat*

Here the first column is the starting linear equation. The second column is the type of a rational transform. The third column is the singularity type of the transformed linear equation. The fourth column is the solution of the Painlevé equation.

Remark. The labels *Weber, Bessel* and *Airy* in the fourth column mean well-known relations between special functions [1]:

$$\begin{aligned}
\text{Weber:} \quad & D_{2k-1/2}(z) &&= 2^k z^{-1/2} W_{k,-1/4}\left(z^2/2\right), \\
\text{Bessel:} \quad & {}_0F_1\left(c; x^2/16\right) &&= e^{-x/2}\, {}_1F_1(c-1/2, 2c-1; x), \\
\text{Airy:} \quad & \mathrm{Ai}(x) &&= \frac{1}{3^{2/3}\Gamma(\frac{2}{3})} {}_0F_1\left(\frac{2}{3}; \frac{z^3}{9}\right) - \frac{x}{3^{1/3}\Gamma(\frac{1}{3})} {}_0F_1\left(\frac{4}{3}; \frac{z^3}{9}\right).
\end{aligned}$$

We explain the case of D_7-alg in Theorem 5. The symbol $(3|2+1)$ means that the inverse image of $x = 0$ consists of one branch point of order 3 and the inverse image of $x=\infty$ consists of one branch point of order 2 and one non-branched point. The map $x=\left(z+2t^{1/3}\right)^3/32z$ is one of such pullbacks. See Figure 1. $x=0$ is a regular singularity and the difference of the local exponents is 1/3. Since the branch point $z=-2t^{1/3}$ is order 3, $z=-2t^{1/3}$ is an apparent singularity. $x=\infty$ is an irregular singularity with the Poincaré rank 1/2. Since the branch point $z=\infty$ is order 2, $z=-2t^{1/3}$ is an irregular singularity with the Poincaré rank 1. $z=0$ is not a branch point.

The pullback of $DW_{1/6}$ is

$$\frac{\partial^2 u}{\partial z^2}=V(z,t)u,$$

where

$$V(z,t)=\frac{t}{4z^3}-\frac{16+27t^{2/3}}{72z^2}+\frac{2}{3t^{1/3}z}+\frac{1}{8}+\frac{3}{4(z-t^{1/3})^2}-\frac{2}{3t^{1/3}(z-t^{1/3})}.$$

This gives an algebraic solution $q(t)=t^{1/3}$ of P3(0, −2, 2, 0).

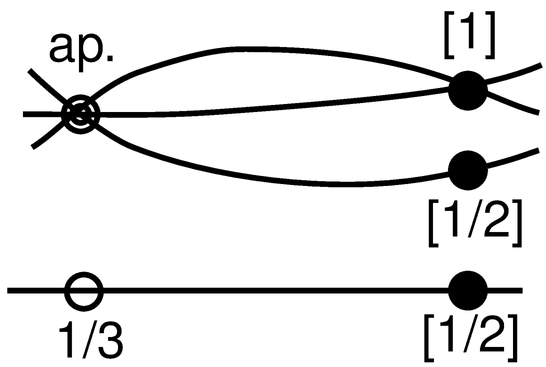


Figure 1: D_7-alg (3|2 + 1) from $DW_{1/6}$

For the Laguerre type solution of P5, we start from

$$\frac{d^2u}{dx^2}=\frac{h^2-1}{4x^2}u. \tag{10}$$

The pullback of (10) by $x=e^{t/(h(z-1))}(z-1)$ is

$$\frac{d^2u}{dz^2}=V(z,t)u(z), \tag{11}$$

where

$$V(z,t)=\frac{t^2}{4(z-1)^4}-\frac{ht}{2(z-1)^3}+\frac{h^2/4-1}{(z-1)^2}-\frac{3}{4(z-t/h-1)^2}-\frac{ht}{(z-1)^2(z-t/h-1)}.$$

This gives a rational solution $y = t/h + 1$ for P5$(h^2/2, -1/2, -h, -1/2)$. Since the monodromy group of (11) is diagonal, it cannot be reduced to $W_{k,m}$ nor DW_m.

The algebraic solution of deg-P5, which does not appear in the list of Theorem 5, is also obtained by a similar transformation from (10).

Theorem 5 gives an answer to R. Fuchs' problem for the first to the fifth Painlevé equations. For algebraic solutions of the Painlevé equations, we can take a suitable transformation $x = x(z)$ such that the corresponding linear equation (9) is either $W_{k,m}$, DW_m or (10). Conversely, if we change the independent variable of $W_{k,m}$ or DW_m so that the singularity type is the same as one of the Painlevé equations, we obtain all of the algebraic solutions and symmetric solutions except for the Laguerre type solution of P5 and algebraic solutions of deg-P5.

The linear equations of symmetric solutions are obtained by pullback only for $t = 0$. We remark that Kaneko and Okumura also showed that a similar result holds for symmetric solutions of P6 [11].

References

[1] Erdélyi, A. ed., *Higher transcendental functions. Vols. I*, McGraw-Hill, 1953.

[2] Flaschka, H., Newell, A. C. "Monodromy- and spectrum-preserving deformations. I", *Commun. Math. Phys.* **76** (1980), 65–116.

[3] Fuchs, R., "Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegene wesentlich singulären Stellen", *Math. Ann.* **63** (1906), 301–321.

[4] Fuchs, R., "Ueber lineare homogene Differentialgleichungen zweiter Ordnung mit vier wesentlich singulären Stellen", Nachr. d. Kgl. Ges. d. Wiss. zu Gẗtingen, math.-phys. Klasse (1910) 146–153.

[5] Fuchs, R., "Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegene wesentlich singulären Stellen", *Math. Ann.* **70** (1911), 525–549.

[6] Gambier, B. "Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixés," *Acta Math.* **33** (1909), 1–55.

[7] Gromak, V. I., "Theory of Painlevé's equations," *Differential Equations* **11** (1975), 285–287.

[8] Jimbo, M. and Miwa, T., "Monodromy preserving deformations of linear ordinary differential equations with rational coefficients," II. *Phys.* **D 2** (1981), 407–448.

[9] Kaneko, K., "A new solution of the fourth Painlevé equation with a solvable monodromy", Proc. Japan Acad., **81**, Ser. A (2005), 75–79.

[10] Kaneko, K., "A new solution of the fourth Painlevé equation with a solvable monodromy" (Japanese), Master thesis in Osaka University (2003).

[11] Kaneko, K. and Okumura, Y., "Special Solutions of the Sixth Painlevé Equation with Solvable Monodromy", `math.CA/0610673`.

[12] Kapaev, A. A., "Lax pairs for Painlevé equations," *CRM Proc. Lecture Notes*, vol. **31**, Amer. Math. Soc., (2002), 37–48.

[13] Kapaev, A. A. and Hubert, E., "A note on the Lax pairs for Painlevé equations," *J. Phys. A: Math. Gen.* **32** (1999) 8145–8156.

[14] Kitaev, A. V., "Symmetric solutions for the first and the second Painleve equation," (Russian) *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **187** (1991), 129–138; translation in *J. Math. Sci.* **73** (1995), 494–499.

[15] Kitaev, A. V., "Dessins d'Enfants, Their Deformations and Algebraic the Sixth Painlevé and Gauss Hypergeometric Functions," *St. Petersburg Math. J.* **17** (2006), 169–206.

[16] Okamoto, K. "Isomonodromic deformation and Painlevé equations and the Garnier system," *J. Fac. Sci. Univ. Tokyo, Sect. IA Math.* **33** (1986), 575–618.

[17] Ohyama, Y. and Okumura, S., "R. Fuchs' problem of the Painleve equations from the first to the fifth", `math.CA/0512243`.

[18] Ohyama, Y. and Okumura, S., "A coalescent diagram of the Painleve equations from the viewpoint of isomonodromic deformations", *J. Phys. A: Math. Gen.* **39** (2006), 12129–12151.

[19] Ohyama, Y., Kawamuko, H., Sakai, T. and Okamoto, K., "Studies on the Painlevé equations V, third Painlevé equations of special type $P_{III}(D_7)$ and $P_{III}(D_8)$", *J. Math. Sci. Univ. Tokyo* **13** (2006), 145–204.

[20] Painlevé, P., Sur les équations différentielles du second ordre à points critiques fixes, *C. R.* **127** (1898) 945–948; Œuvres III 35–38.

[21] Painlevé, P., "Sur les équations différentielles du second ordre à points critiques fixes", *C. R. Acad. Sci. Paris* **143** (1906), 1111–1117.

[22] Umemura, H., "Birational automorphism groups and differential equations". *Nagoya Math. J.* **119** (1990), 1–80.