Problem : Discrete Painlevé equations and their Lax forms

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Abstract

In the talk of the conference, a Lax formalism of q-Painlevé equation associated to $A_2^{(1)}$-surface was presented. We can see this result in the paper, [21]. In this article, we see a rough picture around this problem.

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1 Introduction

Theory of the Painlevé differential equations has developed through two important aspects. One is the classification of 2nd order algebraic ordinary differential equations of normal type which satisfy the Painlevé property. The other one is a deformation theory of linear ordinary differential equations. P. Painlevé and B. Gambier completed the first one and obtained the six Painlevé differential equations. On the other hand, R. Fuchs reached the sixth Painlevé equation from completely different problem, deformation theory of linear equations. Going into detail, we see that the sixth equation appears as the condition that we move the coefficients of the 2nd order Fuchsian equation having four regular singularities without changing its monodromy ([4]).

This result of R. Fuchs was generalized afterwards by R. Garnier and L. Schlesinger. A result of R. Garnier is connected to deformation theory of 2nd order linear equation with irregular singularities. He obtained the other five Painlevé equations from this consideration ([5]). L. Schlesinger consider the isomonodromic deformation of $m \times m$-linear system of 1st order differential equations with regular singularities ([22]). At a later time M. Jimbo, T. Miwa, and K. Ueno established a general theory of monodromy preserving deformation for the matrix systems of 1st order differential equations with regular and irregular singularities ([7, 8]). In their theory the Painlevé equations are written in the form of a compatibility condition between a 2×2-linear system and an associated deformation system. We call this description “Lax form” of the Painlevé equations.

We see some merits that we could express the Painlevé equations in their Lax form. First of all, linear differential equations are easily identified with their data of singularities; in particular, the classification of the Painlevé equations corresponds to a coalescence of singularities of linear differential equations. Besides particular solutions of Riccati type appear where the monodromy of linear equations is reducible; we obtain a key for particular solutions from studies of associated linear equations.

We can consider these two important aspects on the discrete Painlevé equations. Singularity confinement, which was presented by A. Ramani and B. Grammaticos et al. is a discretization of the Painlevé property. Then, how about the other one, Lax form? That is our problem.

The text is organized as follows: in the following section, we see a classification of discrete Painlevé equations. In Section 3, we consider Lax form of the additive (difference) case, and in Section 4, the multiplicative (q-difference) case.

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2 Classification of the discrete Painlevé equations

A classification of discrete Painlevé equations with a view of theory of rational surfaces is known ([19]). While the author prefers to call equations by the types of surfaces because of uniqueness of their correspondence, there are many researchers who call them by their symmetries. Hence we write down the both of the lists.

surface	$A_0^{(1)}$	$A_0^{(1)*}$	$A_1^{(1)}$	$A_2^{(1)}$	$A_3^{(1)}$	$A_4^{(1)}$	$A_5^{(1)}$	$A_6^{(1)}$	$A_7^{(1)}$	$A_7^{(1)\prime}$	$A_8^{(1)}$
symmetry	$E_8^{(1)}$	$E_8^{(1)}$	$E_7^{(1)}$	$E_6^{(1)}$	$D_5^{(1)}$	$A_4^{(1)}$	$(A_2+A_1)^{(1)}$	$(A_1+A_1)^{(1)}$	$A_1^{(1)}$	$A_1^{(1)}$	–

$A_0^{(1)**}$	$A_1^{(1)*}$	$A_2^{(1)*}$	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$
$E_8^{(1)}$	$E_7^{(1)}$	$E_6^{(1)}$	$D_4^{(1)}$	$A_3^{(1)}$	$(A_1+A_1)^{(1)}$	$A_1^{(1)}$	–	$A_2^{(1)}$	$A_1^{(1)}$	–

Here we look at the expression of each discrete Painlevé equations. We will briefly get onto historical notes afterwards.

The most complicated equation is the only one elliptic-difference equation, from which we can obtain all of the discrete Painlevé equations as degenerations.

$$\textit{ell-}P(A_0) \quad : \quad \begin{pmatrix} b & b_1 & b_2 & b_3 & b_4 \\ \tau & b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \Bigg) \mapsto \begin{pmatrix} b+\delta/2 & b_1 & b_2 & b_3 & b_4 \\ \tau & b_5 & b_6 & b_7 & b_8 \end{pmatrix}; \overline{f}, \overline{g} \Bigg),$$

$$(2.1) \qquad \det\left(v(f,g), v_1, \ldots, v_8, v_c\right) \det\left(v(\overline{f}, g), \check{v}_1, \ldots, \check{v}_8, \check{v}_c\right) = P_+(f-f_c)(\overline{f}-\overline{f_c}) \prod_{i=1}^{8} (g-g_i),$$

$$(2.2) \qquad \det\left(v(g,\overline{f}), \hat{u}_1, \ldots, \hat{u}_8, \hat{u}_c\right) \det\left(v(\overline{g}, \overline{f}), \overline{u_1}, \ldots, \overline{u_8}, \overline{u_c}\right) = \overline{P_-}(g-g_c)(\overline{g}-\overline{g_c}) \prod_{i=1}^{8} (\overline{f}-\overline{f_i}),$$

$$
\begin{aligned}
\text{where}\quad & v(f,g)={}^t\left(g^4f,g^3f,g^2f,gf,f,g^4,g^3,g^2,g,1\right),\\
& v_i=v(f_i,g_i),\quad \check{v}_i=v(\overline{f_i},g_i),\quad \hat{u}_i=v(g_i,\overline{f_i}),\quad \overline{u_i}=v(\overline{g_i},\overline{f_i}),\quad i=1,\dots,8,c,\\
& f_i=\wp(b-b_i),\quad g_i=\wp(b+b_i),\quad 1\le i\le 8,\quad f_c=\wp\left(b+\frac{b^2}{\delta}\right),\quad g_c=\wp\left(b-\frac{b^2}{\delta}\right),\\
& P_\pm=\frac{\sigma(4b)^4\sigma(4b\pm\delta)^4\prod_{1\le i<j\le 8}\sigma(b_i-b_j)^2}{\sigma(b\mp\frac{b^2}{\delta})^{16}}\prod_{i=1}^{8}\frac{\sigma(\frac{b^2}{\delta}-b_i)\sigma(2b\pm\frac{b^2}{\delta}\pm b_i)}{\sigma(b\pm b_i)^{14}\sigma(b\mp b_i)^2\sigma(b\mp b_i\pm\frac{\delta}{2})^2},\\
& (\delta=b_1+\cdots+b_8).
\end{aligned}
$$

This expression was obtained by M. Murata ([12]), and is easier to write down than before. Although he considered systematic way to express each discrete Painlevé equations as above, we look at the other equations in the ordinary expression according to Y.Ohta, A.Ramani, B.Grammaticos, *et al.*, which can be obtained from Murata's expression by expanding determinants.

q-$P(A_0^*)$:
$$\begin{pmatrix} b & b_1 & b_2 & b_3 & b_4 \\ & b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f,g \mapsto \begin{pmatrix} b\lambda & b_1 & b_2 & b_3 & b_4 \\ & b_5 & b_6 & b_7 & b_8 \end{pmatrix}; \overline{f},\overline{g},$$

(2.3)
$$\frac{(fb^2-g)(\overline{f}b^2\lambda-g)-(b^4\lambda^2-1)(b^4-1)}{\left(\frac{f}{b^2}-g\right)\left(\frac{\overline{f}}{b^2\lambda}-g\right)-\left(1-\frac{1}{b^4\lambda^2}\right)\left(1-\frac{1}{b^4}\right)}=\frac{\lambda^2P(g,b,\sigma_0,\sigma_1,\dots,\sigma_8)}{P(g,1/b,g,b,\sigma_8,\dots,\sigma_1,\sigma_0)},$$

(2.4)
$$\frac{(gb^2\lambda-\overline{f})(\overline{g}b^2\lambda^2-\overline{f})-(b^4\lambda^2-1)(b^4\lambda^4-1)}{\left(\frac{g}{b^2\lambda}-\overline{f}\right)\left(\frac{\overline{g}}{b^2\lambda^2}-\overline{f}\right)-\left(1-\frac{1}{b^4\lambda^2}\right)\left(1-\frac{1}{b^4\lambda^4}\right)}=\frac{P(\overline{f},b\lambda,\sigma_8,\dots,\sigma_1,\sigma_0)}{\lambda^2P(\overline{f},1/(b\lambda),\sigma_0,\sigma_1,\dots,\sigma_8)},$$

where
$$
\begin{aligned}
P(g,b,\sigma_0,\sigma_1,\dots,\sigma_8)&=\sigma_0g^4-\sigma_1bg^3+(-\sigma_8b^8+\sigma_2b^2-3\sigma_0)g^2\\
&\quad+(\sigma_7b^7-\sigma_3b^3+2\sigma_1b)g+\sigma_8b^8-\sigma_6b^6+\sigma_4b^4-\sigma_2b^2+\sigma_0,
\end{aligned}
$$
$\big(\lambda^2=b_1b_2b_3b_4b_5b_6b_7b_8$, and σ_i is the i-th elementary symmetric function of b_j, $(j=1,\dots,8)$, $\sigma_0=1\big)$.

q-$P(A_1)$:
$$\begin{pmatrix} b & b_1 & b_2 & b_3 & b_4 \\ & b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f,g \mapsto \begin{pmatrix} qb & b_1 & b_2 & b_3 & b_4 \\ & b_5 & b_6 & b_7 & b_8 \end{pmatrix}; \overline{f},\overline{g},$$

(2.5)
$$\frac{(gf-b^2)(g\overline{f}-qb^2)}{(gf-1)(g\overline{f}-1)}=\frac{(g-b_1b)(g-b_2b)(g-b_3b)(g-b_4b)}{(g-b_5)(g-b_6)(g-b_7)(g-b_8)},$$

(2.6)
$$\frac{(g\overline{f}-qb^2)(\overline{g}\overline{f}-q^2b^2)}{(g\overline{f}-1)(\overline{g}\overline{f}-1)}=\frac{(\overline{f}-qb/b_1)(\overline{f}-qb/b_2)(\overline{f}-qb/b_3)(\overline{f}-qb/b_4)}{(\overline{f}-1/b_5)(\overline{f}-1/b_6)(\overline{f}-1/b_7)(\overline{f}-1/b_8)},$$
$$\left(q=\frac{b_1b_2b_3b_4}{b_5b_6b_7b_8}\right).$$

$$q\text{-}P(A_2)\ :\ \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \mapsto \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ qb_5 & qb_6 & qb_7 & qb_8 \end{pmatrix}; \overline{f}, \overline{g},$$

(2.7) $$\frac{(gf-1)(g\overline{f}-1)}{qb_7b_8} = \frac{(g-b_1)(g-b_2)(g-b_3)(g-b_4)}{(g-b_5)(g-b_6)},$$

(2.8) $$\frac{(g\overline{f}-1)(\overline{g}\overline{f}-1)}{qb_5b_6} = \frac{(\overline{f}-1/b_1)(\overline{f}-1/b_2)(\overline{f}-1/b_3)(\overline{f}-1/b_4)}{(\overline{f}-qb_7)(\overline{f}-qb_8)},$$

$$\left(q = \frac{b_5b_6}{b_1b_2b_3b_4b_7b_8}\right).$$

$$q\text{-}P(A_3)\ :\ \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \mapsto \begin{pmatrix} qb_1 & qb_2 & b_3 & b_4 \\ qb_5 & qb_6 & b_7 & b_8 \end{pmatrix}; \overline{f}, \overline{g},$$

(2.9) $$\frac{f\overline{f}}{b_7b_8} = \frac{g-b_1}{g-b_3}\frac{g-b_2}{g-b_4},$$

(2.10) $$\frac{g\overline{g}}{b_3b_4} = \frac{\overline{f}-qb_5}{\overline{f}-b_7}\frac{\overline{f}-qb_6}{\overline{f}-b_8},$$

$$\left(q = \frac{b_1b_2b_7b_8}{b_3b_4b_5b_6}\right).$$

$$q\text{-}P(A_4)\ :\ \begin{pmatrix} a_1 & a_2 & a_3 \\ & a_4 & a_0 \end{pmatrix}; f, g \mapsto \begin{pmatrix} a_1 & a_2 & a_3 \\ & a_4/q & qa_0 \end{pmatrix}; \overline{f}, \overline{g},$$

(2.11) $$f\overline{f} = a_3a_0\frac{(1+g)(1+a_1g)}{g+a_0},$$

(2.12) $$g\overline{g} = a_2a_0\frac{(\overline{f}-a_3)(\overline{f}-1)}{a_1a_2a_3a_0-\overline{f}},$$

$(q = a_1a_2a_3a_4a_0)$.

$$q\text{-}P(A_5)\ :\ \begin{pmatrix} a_1 & a_2 & a_0 \\ & b_1 & b_0 \end{pmatrix}; f, g \mapsto \begin{pmatrix} a_1 & a_2 & a_0 \\ & b_1/q & qb_0 \end{pmatrix}; \overline{f}, \overline{g},$$

(2.13) $$f\overline{f} = \frac{b_0}{a_1a_2}(1-g)(g-a_1)/g,$$

(2.14) $$g\overline{g} = \frac{a_1}{b_0}\frac{\overline{f}+b_0}{\overline{f}+1},$$

$(q = a_1a_2a_0 = b_1b_0)$.

$$q\text{-}P(A_5)^\sharp\ :\ \begin{pmatrix} a_1 & a_2 & a_0 \\ & b_1 & b_0 \end{pmatrix}; f, g \mapsto \begin{pmatrix} a_1 & a_2/q & qa_0 \\ & b_1 & b_0 \end{pmatrix}; \overline{f}, \overline{g},$$

(2.15) $$f\overline{f} = \frac{b_1}{a_2}\frac{g-a_0}{g(1-g)},$$

(2.16) $$g\overline{g} = a_0b_1\frac{\overline{f}-a_1a_0}{\overline{f}(1-\overline{f})}.$$

$$q\text{-}P(A_6) \quad : \quad \begin{pmatrix} a_1 & & a_0 \\ & b & \end{pmatrix}; f, g \mapsto \begin{pmatrix} a_1 & & a_0 \\ & qb & \end{pmatrix}; \overline{f}, \overline{g},$$

(2.17) $$f\overline{f} = a_1 b(g-1),$$

(2.18) $$g\overline{g} = \frac{b\overline{f}}{a_1 b - \overline{f}},$$

$$(q = a_1 a_0).$$

$$q\text{-}P(A_6)^\sharp \quad : \quad \begin{pmatrix} a_1 & & a_0 \\ & b & \end{pmatrix}; f, g \mapsto \begin{pmatrix} qa_1 & & a_0/q \\ & b & \end{pmatrix}; \overline{f}, \overline{g},$$

(2.19) $$f\overline{f} = \frac{bg(g-a_1)}{g-1},$$

(2.20) $$g\overline{g} = \overline{f}(a_1 - \overline{f}/b),$$

$$q\text{-}P(A_7) \quad : \quad (a_1, a_0; f, g) \mapsto \left(qa_1, a_0/q; \overline{f}, \overline{g}\right),$$

(2.21) $$f\overline{f} = a_1(1-g),$$

(2.22) $$g\overline{g} = \overline{f},$$

$$(q = a_1 a_0).$$

$$q\text{-}P(A_7') \quad : \quad (a_1, a_0; f, g) \mapsto \left(qa_1, a_0/q; \overline{f}, \overline{g}\right),$$

(2.23) $$f\overline{f} = \frac{g(1-a_1 g)}{a_1(g-1)},$$

(2.24) $$g\overline{g} = \overline{f}^2,$$

$$(q = a_1 a_0).$$

$$d\text{-}P(A_0^{**}) \quad : \quad \begin{pmatrix} b & b_1 & b_2 & b_3 & b_4 \\ & b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \mapsto \begin{pmatrix} b+\delta/2 & b_1 & b_2 & b_3 & b_4 \\ & b_5 & b_6 & b_7 & b_8 \end{pmatrix}; \overline{f}, \overline{g},$$

(2.25) $$\frac{(g-f+4b^2)(g-\overline{f}+(2b+\delta/2)^2)+4g(2b+\delta/2)2b}{2b(g-\overline{f}+(2b+\delta/2)^2)+(2b+\delta/2)(g-f+4b^2)} = 2\frac{g^4+S_2g^3+S_4g^2+S_6g+S_8}{S_1g^3+S_3g^2+S_5g+S_7},$$

(2.26) $$\frac{(\overline{f}-g+(2b+\delta/2)^2)(\overline{f}-\overline{g}+(2b+\delta)^2)+4\overline{f}(2b+\delta/2)(2b+\delta)}{(2b+\delta)(\overline{f}-g+(2b+\delta/2)^2)+(2b+\delta/2)(\overline{f}-\overline{g}+(2b+\delta)^2)} = 2\frac{f^4+\Sigma_2f^3+\Sigma_4f^2+\Sigma_6f+\Sigma_8}{\Sigma_1f^3+\Sigma_3f^2+\Sigma_5f+\Sigma_7},$$

($\delta = b_1+b_2+b_3+b_4+b_5+b_6+b_7+b_8$ and S_i (resp. Σ_i) is the i-th elementary symmetric function of $b+b_j$ (resp. $b-b_j$), $(j=1,\ldots,8)$).

$$d\text{-}P(A_1^*) \quad : \quad \begin{pmatrix} b & b_1 & b_2 & b_3 & b_4 \\ & b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \mapsto \begin{pmatrix} b-\delta & b_1 & b_2 & b_3 & b_4 \\ & b_5 & b_6 & b_7 & b_8 \end{pmatrix}; \overline{f}, \overline{g},$$

$$(2.27) \qquad \frac{(g+f-2b)(g+\overline{f}-2b+\delta)}{(g+f)(g+\overline{f})} = \frac{(g+b_1-b)(g+b_2-b)(g+b_3-b)(g+b_4-b)}{(g-b_5)(g-b_6)(g-b_7)(g-b_8)},$$

$$(2.28) \qquad \frac{(g+\overline{f}-2b+\delta)(\overline{g}+\overline{f}-2b+2\delta)}{(g+\overline{f})(\overline{g}+\overline{f})} = \frac{(\overline{f}-b_1-b+\delta)(\overline{f}-b_2-b+\delta)(\overline{f}-b_3-b+\delta)(\overline{f}-b_4-b+\delta)}{(\overline{f}+b_5)(\overline{f}+b_6)(\overline{f}+b_7)(\overline{f}+b_8)},$$

$$(\delta = b_1+b_2+b_3+b_4+b_5+b_6+b_7+b_8).$$

$$d\text{-}P(A_2^*) \quad : \quad \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \mapsto \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5+\delta & b_6+\delta & b_7-\delta & b_8-\delta \end{pmatrix}; \overline{f}, \overline{g},$$

$$(2.29) \qquad (g+f)\left(g+\overline{f}\right) = \frac{(g+b_1)(g+b_2)(g+b_3)(g+b_4)}{(g-b_5)(g-b_6)},$$

$$(2.30) \qquad \left(g+\overline{f}\right)\left(\overline{g}+\overline{f}\right) = \frac{(\overline{f}-b_1)(\overline{f}-b_2)(\overline{f}-b_3)(\overline{f}-b_4)}{(\overline{f}+b_7-\delta)(\overline{f}+b_8-\delta)},$$

$$(\delta = b_1+b_2+b_3+b_4+b_5+b_6+b_7+b_8).$$

$$d\text{-}P(D_4) \quad : \quad \begin{pmatrix} a_1 & a_2 & a_3 \\ & a_4 & a_0 \end{pmatrix}; s; f, g \mapsto \begin{pmatrix} a_1+\delta & a_2-\delta & a_3 \\ & a_4 & a_0+\delta \end{pmatrix}; s; \overline{f}, \overline{g},$$

$$(2.31) \qquad f+\overline{f} = a_3 + \frac{a_1}{g+1} + \frac{a_0}{sg+1},$$

$$(2.32) \qquad g\overline{g} = \frac{(\overline{f}+a_2-\delta)(\overline{f}+a_2+a_4-\delta)}{s\overline{f}(\overline{f}-a_3)},$$

$$(\delta = a_1+2a_2+a_3+a_4+a_0).$$

$$d\text{-}P(D_5) \quad : \quad \begin{pmatrix} a_1 & a_2 \\ a_3 & a_0 \end{pmatrix}; s; f, g \mapsto \begin{pmatrix} a_1 & a_2 \\ a_3-\delta & a_0+\delta \end{pmatrix}; s; \overline{f}, \overline{g},$$

$$(2.33) \qquad f+\overline{f} = a_2+a_0-g+\frac{(a_1+a_0)s}{g+s},$$

$$(2.34) \qquad g\overline{g} = s(\overline{f}+a_3-\delta)(\overline{f}-a_0-\delta)/\overline{f},$$

$$(\delta = a_1+a_2+a_3+a_0).$$

$$d\text{-}P(D_6) \quad : \quad \begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix}; s; f, g \mapsto \begin{pmatrix} a_1 & a_0 \\ b_1-\delta & b_0+\delta \end{pmatrix}; s; \overline{f}, \overline{g},$$

$$(2.35) \qquad f+\overline{f} = b_0-a_0-g-\frac{s}{g},$$

$$(2.36) \qquad g\overline{g} = s-\frac{b_0 s}{\overline{f}},$$

$$(\delta = a_1+a_0 = b_1+b_0).$$

$$d\text{-}P(D_7) \ : \ (a_1,a_0;s;f,g) \mapsto \left(a_1-\delta,a_0+\delta;s;\overline{f},\overline{g}\right),$$

$$f+\overline{f}=a_0s-\frac{s^2}{g}, \tag{2.37}$$

$$g\overline{g}=\overline{f}, \tag{2.38}$$

$$(\delta=a_1+a_0).$$

$$d\text{-}P(E_6) \ : \ (a_1,a_2,a_0;s;f,g) \mapsto \left(a_1,a_2-\delta,a_0+\delta;s;\overline{f},\overline{g}\right),$$

$$f+\overline{f}=s-g+\frac{a_0}{g}, \tag{2.39}$$

$$g+\overline{g}=s-\overline{f}-\frac{a_2-\delta}{\overline{f}}, \tag{2.40}$$

$$(\delta=a_1+a_2+a_0).$$

$$d\text{-}P(E_7) \ : \ (a_1,a_0;s;f,g) \mapsto \left(a_1-\delta,a_0+\delta;s;\overline{f},\overline{g}\right),$$

$$f+\overline{f}=s+g^2, \tag{2.41}$$

$$g+\overline{g}=-\frac{a_0}{\overline{f}}, \tag{2.42}$$

$$(\delta=a_1+a_0).$$

The parameters a_i, b_i, b belong to $\mathbb{C}$ in the elliptic and the additive (difference) case, and belong to $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ in the multiplicative (q-difference) case. The parameter s belongs to the following domain:

type of surface	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$
domain	$\mathbb{C}\setminus\{0,1\}$	$\mathbb{C}^*$	$\mathbb{C}^*$	$\mathbb{C}^*$	$\mathbb{C}^*$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

We can normalize f and g as $\delta=1$ except *ell*-$P(A_0)$. Moreover we can reduce parameters by normalization for some cases. In fact, we can set $b_5b_6b_7b_8=1$ for q-$P(A_1)$; $b_1b_2b_3b_4=1$ for q-$P(A_2)$; $b_3b_4=b_7b_8=1$ for q-$P(A_3)$; and $b_5+b_6+b_7+b_8=0$ for d-$P(A_1^*)$; $b_1+b_2+b_3+b_4=0$ for d-$P(A_2^*)$.

Remark 2.1. The discrete equations, q-$P(A_6)$, q-$P(A_7)$, d-$P(D_6)$, d-$P(D_7)$, and d-$P(E_7)$, can be written in the form of the single equations:

$$q\text{-}P(A_6) \ : \ \left(f\overline{f}+a_1b\right)\left(\overline{f}\,\overline{\overline{f}}+qa_1b\right)=\frac{qa_1^2b^3\overline{f}}{a_1b-\overline{f}}, \tag{2.43}$$

$$q\text{-}P(A_7) \ : \ \underline{g}g^2\overline{g}=a_1(1-g), \tag{2.44}$$

$$d\text{-}P(D_6) \ : \ \frac{(b_0-\delta)s}{s-\underline{g}g}+\frac{b_0s}{s-g\overline{g}}=b_0-a_0+g+\frac{s}{g}, \tag{2.45}$$

$$d\text{-}P(D_7) \ : \ \overline{g}+\underline{g}=\frac{a_0s}{g}-\frac{s^2}{g^2}, \tag{2.46}$$

$$d\text{-}P(E_7) \ : \ \frac{a_0-\delta}{\underline{g}+g}+\frac{a_0}{g+\overline{g}}+g^2+s=0. \tag{2.47}$$

□

***Remark* 2.2.** Generally speaking, it is difficult to tell the type of a given discrete Painlevé equation. The following equations is, actually, $d\text{-}P(D_6)$:

$$\begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix}; s; F, G \Big) \mapsto \begin{pmatrix} a_1 & a_0 \\ b_1 - \delta & b_0 + \delta \end{pmatrix}; s; \overline{F}, \overline{G} \Big),$$

$$F\overline{F} = \frac{G(a_1 - G)}{s}, \tag{2.48}$$

$$G + \overline{G} = a_1 + \frac{b_0 \overline{F}}{1 - \overline{F}}, \tag{2.49}$$

$$(\delta = a_1 + a_0 = b_1 + b_0).$$

The correspondence between the variables are as follows:

$$g = \frac{G}{F}, \quad f = F + G - b_1.$$

For your information, this equations can be written in the form of the single equation:

$$\left(1 + \frac{b_0 - \delta}{\underline{G} + G - b_1 - a_1}\right)\left(1 + \frac{b_0}{G + \overline{G} + b_0 - a_1}\right) = \frac{G(a_1 - G)}{s}.$$

A reliable method to know the types of the equations is to construct the space of initial conditions. We can identify each dynamical system with an action on Picard group of the space of the initial conditions.

However, even if two dynamical systems have the same space, we can not correspond one to the other by a change of variables when two actions on the Picard group are not conjugate with each other. For example, the following dynamical system has $D_5^{(1)}$-surface as its space of initial conditions; but it is a different system from $d\text{-}P(D_5)$ above. (It is a composition of two (differently directed) $d\text{-}P(D_5)$.)

$$d\text{-}P(D_5)^{[2]} \quad : \quad \left(\begin{matrix} a_1 & a_2 \\ a_3 & a_0 \end{matrix}; s; F, G\right) \mapsto \left(\begin{matrix} a_1 & a_2 - \delta \\ a_3 & a_0 + \delta \end{matrix}; s; \overline{F}, \overline{G}\right),$$

$$F + \overline{F} = s - \frac{a_1 + a_0}{1 - G} + \frac{a_3 + a_0}{G}, \tag{2.50}$$

$$G + \overline{G} = 1 - \frac{a_0 + \delta}{s - \overline{F}} - \frac{a_2 - \delta}{\overline{F}}. \tag{2.51}$$

In such a sense, we have infinitely many discrete Painlevé equations on each surfaces. □

Now we briefly mention how discrete Painlevé equations were discovered. While discrete dynamical systems were initially studied in connection with chaos, many interesting results on discrete integrable systems appeared. Among them, several mappings that naturally appear in physical applications turned out to be discrete analog of the Painlevé equations. For example, the calculation of a certain partition function of a 2-dimensional quantum gravity model led to the following equation (see [2]):

$$f_{n+1} + f_n + f_{n-1} = (an + b)/f_n + c.$$

This is d-$P(E_6)$ in the above list, whose parameters are restricted, and is called d-P_{I}. Another 2-dimensional gravity model led to the following ([16]):

$$f_{n-1} + f_{n-1} = \frac{(an+b)f_n + c}{1 - f_n^2}.$$

Simirality reduction of a discrete analog of the modified Korteweg-de Vries equation led to the same equation ([13]). This is a certain restriction of d-$P(D_5)^{[2]}$ and is called d-P_{II}.

***Remark* 2.3.** At the early phase of studies, the equations were not given in the form of the system with full-parameters, but given in the form of the single equations as these equations above. However there are many cases in which we should consider all parameters, for example, the case that we study symmetry of the equations. Hence the author thinks that we should study the system in general, and the single equations should be regarded as a certain restriction of the parameters. (In many cases, they are recognized as different systems. The single equations are called the symmetric type and the systems are called the asymmetric type.)

There is also a delicate matter in the names of systems. The name, like d-P_{I}, is based on the existence of a continuous limit. But there exist many different dynamical systems which have the same differential equation as a continuous limit, and a continuous limit is not unique for each discrete system. In the case of d-P_{I} and d-P_{II}, there are a limit to P_{II} and P_{III} respectively from the viewpoint of surface theory. When we mention the more direct connection with differential equations, these are Bäcklund transformations of P_{IV} and P_{V} respectively. In order to avoid confusions and misleadings, the author calls the systems by their surfaces in the list.

□

The crucial step in the study of discrete Painlevé equations was made by B. Grammaticos and A. Ramani, *et al.* They proposed the singularity confinement test as a discrete counter part of the Painlevé property ([6]) and this test has led to discovery of several discrete Painlevé equations ([17]). Let's see the correspondence between their equations in the paper and our list:

d-P_{I}	d-P_{II}	$d(q)$-P_{III}	d-P_{IV}	$d(q)$-P_{V}
d-$P(E_6)$	d-$P(D_5)^{[2]}$	q-$P(A_3)$	d-$P(A_2^*)$	q-$P(A_2)$

A. Ramani and B. Grammaticos and their coworkers found almost all discrete Painlevé equations in the years that were to follow. We refer to only two other papers, although there are many papers of theirs and earlier works exist (see [18, 14]). The author added the elliptic-difference Painlevé equation, ell-$P(A_0)$, and finished the classification ([19]).

***Remark* 2.4.** K. Kajiwara, M. Noumi, and Y. Yamada proposed a series of discrete dynamical systems from a birational representation of the extended Weyle groups $\widetilde{W}(A^{(1)}_{m-1} \times A^{(1)}_{n-1})$. In the case of $m = 2$ and $n = 3, 4$, these difine two dimensional dynamical systems (see [11]).

In the case of $m = 2$ and $n = 3$, this system is called q-P_{IV} and coincides with q-$P(A_5)$, as is shown in their paper, [10].

In the case of $m = 2$ and $n = 4$, this system is called q-$P_{\rm V}$ and can be expressed as the following system:

$$q\text{-}P(A_3)^{[2]} \ : \ \begin{pmatrix} c_1 & c_2 & c_3 & c_0 \\ & c_4 & c_5 & \end{pmatrix}; f, g \mapsto \begin{pmatrix} c_1 & c_2 & c_3 & c_0 \\ & c_4/\lambda & c_5/\lambda & \end{pmatrix}; \overline{f}, \overline{g},$$

$$\text{(2.52)} \quad \overline{f} = \frac{\lambda c_0 c_1 f g + c_4(c_5 + c_0 c_5 f + \lambda c_0 c_1 c_2 g)}{\lambda f(1 + c_0(f + c_1 c_2 c_4 g + c_1 f g))},$$

$$\text{(2.53)} \quad \overline{g} = \frac{c_4(c_5 + c_0 c_5 f + c_0 c_1 c_5 f g + \lambda c_0 c_1 c_2 g)}{g(\lambda c_0 f(1 + c_1 g) + c_4(c_5 + q c_0 c_1 c_2 g))}.$$

$$\left(\begin{pmatrix} b_1, b_2, b_3, b_4 \\ b_5, b_6, b_7, b_8 \end{pmatrix} = \begin{pmatrix} -c_3 c_5 & -c_1 & -c_5/c_3 & -1/c_1 \\ -1/c_0 & -c_4/c_2 & -c_2 c_4 & -c_0 \end{pmatrix}, \qquad \lambda = c_1 c_2 c_3 c_0, \quad \lambda^2 = q \right).$$

T. Takenawa has shown that this system is a composition of two mappings, each of which is conjugate to q-$P(A_3)$ ([23]).

3 Differential equations and difference equations

It is well known that the Painlevé differential equations are obtained from deformation theory of linear differential equations. The Painlevé equations are written in the form of compatibility,

$$\text{(3.1)} \qquad \frac{\partial}{\partial t} A - \frac{\partial}{\partial x} B + [A, B] = 0,$$

between associated linear equation $\frac{\partial}{\partial x} Y = A(x,t) Y$ and its deformation equation $\frac{\partial}{\partial t} Y = B(x,t) Y$.

Associated linear equation is not unique, but there is a correspondence between the Painlevé equations and certain 2×2-linear systems, which are characterized by their singularities. The following table expresses the Poincaré rank+1 for the each singular points of associated linear equation, $\frac{d}{dx} Y = A(x) Y$.

	$P_{\rm VI}$	$P_{\rm V}$	$P_{\rm III}(D_6)$	$P_{\rm III}(D_7)$	$P_{\rm III}(D_8)$	$P_{\rm IV}$	$P_{\rm II}$	$P_{\rm I}$
Poincaré rank +1	(1,1,1,1)	(2,1,1)	(2,2)	(2,3/2)	(3/2,3/2)	(3,1)	(4)	(7/2)

On the other hand, the difference Painlevé equations of $D_l^{(1)}$ and $E_l^{(1)}$ types possess the same rational surfaces that the Painlevé equations have as a compactification of their space of initial conditions. Hence difference equations of $D_l^{(1)}$ and $E_l^{(1)}$ can be regarded as contiguity relations, *i.e.,* Bäcklund transformations of the Painlevé differential equations. (It is because Bäcklund transformations of Painlevé coincide with Cremona actions of the surfaces ([19]) and they generate discrete Painlevé equations.) We can lift up these relations to associated linear equations; we see them as discrete deformation (Schlesinger transformation) of linear differential equations. (Although a certain Bäcklund transformation might not be a Schlesinger transformation, the discrete Painlevé of $D_l^{(1)}$ and $E_l^{(1)}$ turns out to be Schlesinger from the move of parameters.) Schlesinger transformation is written in the form of the compatibility,

$$\text{(3.2)} \qquad \overline{A} = R A R^{-1} + \frac{dR}{dx} R^{-1},$$

between associated linear equation, $\frac{d}{dx}Y = A(x)Y$, and its deformation equation, $\overline{Y} = RY$. Additionally, if we have two distinct discrete deformations of the differential equation, then we have another compatibility,

$$\widetilde{R_1}R_2 = \overline{R_2}R_1,$$

between two deformation equations, $\overline{Y} = R_1Y$ and $\widetilde{Y} = R_2Y$.

Therefore the discrete Painlevé equations of type $D_l^{(1)}$ and $E_l^{(1)}$ can be characterized by the same linear differential equations as the Painlevé differential equations:

	$d\text{-}P(D_4)$	$d\text{-}P(D_5)$	$d\text{-}P(D_6)$	$d\text{-}P(D_7)$	$d\text{-}P(E_6)$	$d\text{-}P(E_7)$
Poincaré rank +1	(1,1,1,1)	(2,1,1)	(2,2)	(2,3/2)	(3,1)	(4)

Although the difference equations of types $A_0^{(1)**}$, $A_1^{(1)*}$, and $A_2^{(1)*}$ do not correspond to any Painlevé differential equation, the author believes that they should correspond to the Garnier system or degenerated Garnier systems; they should be written in the framework of Schlesinger transformations, which is generally studied in M. Jimbo and T. Miwa's paper ([8]). Recently, D. Arinkin and A. Borodin calculated a Lax pair of difference Painlevé equation of $A_2^{(1)*}$ type, which is in the form of compatibility of two linear difference equations, and in fact, they show that the system can be regarded as a discrete deformation of a linear differential equation, though they did not give explicit form of this linear differential equation ([1]).

Hence the remained problem is as follows:

Problem A. Write down the discrete Painlevé equations, $d\text{-}P(A_0^{**})$, $d\text{-}P(A_1^{*})$, $d\text{-}P(A_2^{*})$, in the form of the Schlesinger transformations. Furthermore, characterize them by the data of singularities of the linear differential equations.

4 q-difference equations

The generalized Riemann problem of q-difference equations was studied in the paper of G. D. Birkhoff ([3]). Hence, a next step was a q-analog of the deformation theory. In the paper, [9], we consider the Lax pair in the terms of deformation theory of linear q-difference equations, and characterize $q\text{-}P(A_3)$ by the data of the associated linear q-difference equation. We also refer to the earlier result of V. G. Papageorgiou, F. W. Nijihoff, B. Grammaticos, and A. Ramani, [15]. They constructed 4×4 Lax pair of $q\text{-}P(A_3)$.

Consider a $m \times m$ matrix system with polynomial coefficients

$$Y(qx) = A(x)Y(x). \tag{4.1}$$

More general case of a rational $A(x)$ can be reduced to this case by solving scalar q-difference equations. Namely, if function $f(x)$ satisfies $f(qx) = (1/\prod_{i=1}^{M}(x-c_i))f(x)$, then the q-difference equation

$$\tilde{Y}(qx) = \frac{A(x)}{\prod_{i=1}^{M}(x-c_i)}\tilde{Y}(x)$$

has a solution $\tilde{Y}(x) = f(x)Y(x)$.

In the theory of the monodromy preserving deformation of Fuchsian equations, an extra parameter $t = (t_j)$ is introduced denoting the position of regular singular points. In the formulation, in terms of q-difference equations, we put the (discrete) deformation parameters at zeros of $\det A(x)$, the eigen values of the leading term, and the eigen values of the constant term.

The connection preserving deformation of the linear q-difference equation, which is a discrete counterpart of monodromy preserving deformation, is equivalent to the existence of a linear deformation equation whose coefficients are rational in x. We express the deformation equation in the form

$$\overline{Y(x)} = B(x)Y(x), \tag{4.2}$$

and can express the q-Schlesinger equation in the form

$$\overline{A(x)}B(x) = B(qx)A(x) \tag{4.3}$$

by the compatibility of the deformation equation and the original linear q-difference equation.

In the previous paper, [20], we studied closely in the case of 2×2-matrix system. We assumed the leading coefficient and the constant term to be invertible and semi-simple. In the case of degree $N+1$, 2×2 q-Shlesinger equation defines a nonlinear discrete dynamical system on $2N$-dimensional space; we call it q-Garnier system (of $2N$ dimensional). This has the original Garnier system as its continuous limit.

In differential case, two dimensional Garnier system coincides with the sixth Painlevé equation, the most generic Painlevé differential equation. On the other hand, two dimensional q-Garnier system coincides with q-$P(A_3)$; the more generic equations, q-$P(A_2)$, q-$P(A_1)$, and q-$P(A_0^*)$ do not appear. This is the problem.

Problem B. Write down the q-Painlevé equations, q-$P(A_2)$, q-$P(A_1)$, q-$P(A_0^*)$, in the form of the q-Schlesinger equations. Furthermore, characterize them by the data of the associated linear q-difference equations.

Recently, the author constructed a Lax pair of q-$P(A_2)$. This is a special case of the four dimensional q-Garnier system ([21]). The same problem for q-$P(A_1)$ and q-$P(A_0^*)$ still remains open.

The following problem, which seems to be easier, also remains.

Problem C. Write down the q-Painlevé equations, q-$P(A_4)$, q-$P(A_5)$, q-$P(A_5)^\sharp$, q-$P(A_6)$, q-$P(A_6)^\sharp$, q-$P(A_7)$, q-$P(A_7')$, in the form of the q-Schlesinger equations, through degeneration from q-$P(A_3)$. Furthermore, characterize them by the data of the associated linear q-difference equations.

5 Elliptic-difference equation

We do not know anything about Lax pair of the elliptic-difference Painlevé equation, *ell-*$P(A_0)$.

References

[1] D.Arinkin and A.Borodin, Moduli space of d-connection and difference Painlevé equations, math.AG/0411584.

[2] E. Brézin and V. A. Kazakov, Exactly solvable field theories of closed strings, *Phys. Lett. B* **236** (1990), no. 2, 144–150.

[3] G. D. Birkhoff, The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations, *Proc. Am. Acad. Arts and Sciences* **49** (1914) 521–568.

[4] R.Fuchs, Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegene wesentlich singulären Stellen, *Math. Ann.* **63** (1907) 301–321.

[5] R.Garnier, Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes, *Ann. Sci. Ecole Norm. Sup.* **29** (1912) 1–126.

[6] B. Grammaticos, A. Ramani, and V. G. Papageorgiou, Do integral mappings have the Painlevé property?, *Phys. Rev. Lett.* **67** (1991) 1825–1828.

[7] M. Jimbo, T. Miwa, and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I, *Physica* **2D** (1981) 306 – 352.

[8] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II, *Physica* **2D** (1981) 407–448.

[9] M. Jimbo and H. Sakai, A q-analog of the sixth Painlevé equation, *Lett. Math. Phys.* **38** (1996) 145–154.

[10] K. Kajiwara, M. Noumi, and Y. Yamada, A study on the fourth q-Painlevé equation, *J. Phys. A.: Math. Gen.* **41** (2001), 8563–8581.

[11] K. Kajiwara, M. Noumi, and Y. Yamada, Discrete dynamical systems with $W(A^{(1)}_{m-1} \times A^{(1)}_{n-1})$ symmetry, *Lett. Math. Phys.* **60** (2002), 211–219.

[12] M. Murata, New expressions for discrete Painlevé equations, *Funkcial. Ekvac.* **47** (2004) 291–305.

[13] F. W. Nijhoff and V. G. Papaeorigiou, Similarity reductions of integrable lattices and discrete analogues of the Painlevé II equation, *Phys. Lett. A* **153** (1991), no.6-7, 337–344.

[14] Y. Ohta, A. Ramani, and B. Grammaticos, An affine Weyl group approach to the eight-parameter discrete Painlevé equation, *J. Phys. A.: Math. Gen.* **34** (2001) 10523–10532.

[15] V. G. Papageorgiou, F. W. Nijhoff, B. Grammaticos, and A. Ramani, Isomonodromic deformation problem for discrete analogeus of Painlevé equations, *Phys. Lett. A* **164** (1992) 57–64.

[16] V. Periwal and D. Shevitz, Exactly solvable unitary matrix models: multicritical potentials and correlations, *Nuclear Phys. B* **344** (1990), no. 3, 731–746.

[17] A. Ramani, B. Grammaticos, and J. Hietarinta, Discrete versions of the Painlevé equations, *Phys. Rev. Lett.* **67** (1991) 1829–1832.

[18] A. Ramani, B. Grammaticos, T. Tamizhmani, and K. M. Tamizhmani, Special function solutions of the discrete Painlevé equations, *Comput. Math. Appl.* **42** (2001),no.3–5, 603–614.

[19] H. Sakai, Rational surfaces associated with affine root systems and geometry of the Painlevé equations, *Comm. Math. Phys.* **220** (2001) 165–229.

[20] H.Sakai, A q-analog of the Garnier system, *Funkcial. Ekvac.* **48** (2005) 273–297.

[21] H.Sakai, Lax form of the q-Painlevé equation associated to the $A_2^{(1)}$ surface, *J. Phys. A: Math. Gen.* **39** (2006) 12203–12210.

[22] L.Schlesinger, Über eine Klasse von Differentialsystemen beliebliger Ordnung mit festen kritischen Punkten, *J. für Math.***141** (1912) 96–145.

[23] T. Takenawa, Weyl group symmetry of type $D_5^{(1)}$ in the q-Painlevé V equation *Funkcial. Ekvac.* **46** (2003) 173–186.