

# Construction of the auxiliary functions for the value distribution of the fifth Painlevé transcendents in sectorial domains

Yoshikatsu SASAKI\*

Mathematical Subject Classification 2000: Primary 34M55; Secondary 30D35, 32A22

## 1 Introduction.

### 1.1 Value distribution of the fifth Painlevé transcendents in sectorial domains

Consider the fifth Painlevé equation  $P_V$

$$\frac{d^2y}{dx^2} = \left( \frac{1}{2y} + \frac{1}{y-1} \right) \left( \frac{dy}{dx} \right)^2 - \frac{1}{x} \frac{dy}{dx} + \frac{(y-1)^2}{x^2} \left( \alpha y + \frac{\beta}{y} \right) + \frac{\gamma y}{x} + \frac{\delta y(y+1)}{y-1}. \quad (1)$$

Note that the generic  $P_V$  has the parameters  $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$  satisfying

$$(\alpha, \beta) \neq (0, 0) \text{ and } (\gamma, \delta) \neq (0, 0); \quad (2)$$

when  $(\alpha, \beta) = (0, 0)$ ,  $P_V$  is reduced to  $P_{III}$  ([1], [2]Chap.8), and when  $(\gamma, \delta) = (0, 0)$ ,  $P_V$  is integrable ([3], [2]Chap.8).

We studied in [5] the value distribution of solutions of  $P_V$  in the sector

$$S(\phi, r, R) = \{x \mid \arg x < \phi < \pi, r < |x| < R\} \quad (3)$$

as  $r \rightarrow 0$  or  $R \rightarrow \infty$ , in the case where  $P_V$  is generic; namely,  $P_V$  under the condition (2). Defining

$$n(y, \phi, r, R) = \#\{x \in S(\phi, r, R) \mid y(x) = 1\},$$

where the number is counted with multiplicities, we settled in [5]

**Theorem.** *If  $(\alpha, \beta) \neq (0, 0)$  and  $(\gamma, \delta) \neq (0, 0)$ , then there exists a positive constant  $C$ , independent of  $(\alpha, \beta, \gamma, \delta)$ , such that for any solution  $y = y(x)$  of (1), one has*

$$\begin{aligned} n(y, \phi, r, R) &= O(r^{-C}) \text{ as } r \rightarrow 0, \\ n(y, \phi, r, R) &= O(R^C) \text{ as } R \rightarrow \infty. \end{aligned}$$

---

\*Graduate School of Mathematics, Kyushu University, 6-10-1, Hakozaki, Higashi-ku, Fukuoka, 812-8581 Japan. E-mail: sasaki@math.kyushu-u.ac.jp

## 1.2 Key lemma, cases and transformations.

To prove the theorem, in [5], we obtained the following lemma:

**Lemma 1.1.** *Let  $u(t)$  be a solution of*

$$\ddot{u} = 1 + g_0(t, u) + g_1(t, u)\dot{u} + g_2(t, u)\dot{u}^2 \quad (\cdot = \frac{d}{dt}), \quad (4)$$

around  $t = 0$ . Suppose  $g_j(t, u) (j = 0, 1, 2)$  is analytic in  $D_0 = \{(t, u) \in \mathbb{C}^2 \mid |t| < 1, |u| < R_0\}$ ,  $0 < R_0 < 1$ . Suppose  $|g_0(t, u)| < 1/200$ ,  $|g_1(t, u)| < K$ ,  $|g_2(t, u)| < K$  in  $D_0$ , where  $K$  is some positive number. Put  $\theta := \min\{R_0^{1/2}/4, (200K)^{-1/2}, (200K)^{-1}\}$ . If  $|u(0)| \leq \theta^2/8$ , then  $|u(t)| \leq 15\theta^2$  in the disk  $|t| < \rho_0$  and  $|u(t)| \geq \theta^2/4$  on the circle  $|t| = 3\rho_0/4$ , where

$$\rho_0 = \begin{cases} 4\theta & \text{if } |\dot{u}(0)| \leq \theta, \\ \frac{(4/3)\theta^2}{|\dot{u}(0)|} & \text{if } |\dot{u}(0)| > \theta. \end{cases}$$

This key lemma is an improvement of a similar lemma established in [6], and a basic idea of its proof is essentially due to M. Hukuhara (see [4]) as Shimomura wrote in [6]. Hukuhara's logic plays an essential role in my paper [5] as well. And, applying the lemma to  $P_V$  with  $\delta \neq 0$ , we obtain

**Lemma 1.2.** *For each  $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$  satisfying  $\delta \neq 0$ , there exists a quartet of positive numbers  $T_0, \mu \neq 1, \Delta$  and  $A_0$ , each of them independent of  $y(x)$ , with the properties: for  $x = a$  satisfying  $|a| > T_0$ , if  $|y(a) - \mu| \leq \Delta$ , then*

- (i)  $|y(x) - \mu| \geq 2\Delta$  on the circle  $|x - a| = \epsilon_a$ ;
- (ii)  $y(x) \neq 1$  in the disk  $|x - a| \leq \epsilon_a$ . Here,  $\epsilon_a > 0$  satisfies

$$\epsilon_a \leq A_0, \quad \epsilon_a^{-1} \leq A_0(1 + |y'(a)|). \quad (5)$$

By this lemma, we are able to construct a path on which the auxiliary function

$$\begin{aligned} \Psi(\mu, x) = & \frac{x^2 y'(x)^2}{y(x)(y(x) - 1)^2} - \frac{2(1 - \mu)xy'(x)}{(y(x) - 1)(y(x) - \mu)} \\ & - 2\alpha y(x) + \frac{2\beta}{y(x)} + \frac{2\gamma x}{y(x) - 1} + \frac{2\delta x^2 y(x)}{(y(x) - 1)^2} \end{aligned} \quad (6)$$

( $\mu \neq 0, 1, \infty$ ) is holomorphic. This function is used for global estimation of the derivative of a fifth Painlevé transcendent.

The aim of this article is to present a new method in order to handle the problem systematically, and to construct the auxiliary functions suitably for the new method.

In [5], we explained into details the behavior as  $R \rightarrow \infty$  under  $\delta \neq 0$ ; the behavior as  $R \rightarrow \infty$  under  $\delta = 0$  and as  $r \rightarrow 0$  are omitted in order to save the pages.

Now we present a new method: to study the value distribution of the function which has the same 1-points with multiplicities as those of a given fifth Painlevé transcendent. This method allows us to apply the key lemma systematically, and then all arguments progress in parallel to the case where  $R \rightarrow \infty$  under  $\delta \neq 0$ . And, if we use the new method, we have to construct a new auxiliary function suitably for a new function in each case.

We firstly show the list of cases, transformations and auxiliary functions. And secondly, we explain how to construct the auxiliary functions. For the study of behavior around  $x = \infty$ , we devide into 2 cases;

Case (i)  $\delta \neq 0$ : let  $y - 1 = \eta - 1$ ,  $\mu = 2$ ,  $u = \eta - \mu$ ,  $x = a + k^{-1/2}s$  and  $k = 6\delta$ ;  
Case (ii)  $\delta = 0$ : let  $y - 1 = (\eta - 1)/x$ ,  $\mu = 2$ ,  $u = \eta - \mu$ ,  $x = a + k^{-1/2}s$  and  $k = \gamma(\neq 0)$ .

The 1-points of  $y(x)$  coincide with the 1-points of  $\eta(x)$ , i.e.  $n(y, \phi, r, R) = n(\eta, \phi, r, R)$  in each case. That's all of the cases. Because  $(\gamma, \delta) = (0, 0)$  means  $P_V$  is integrable; namely each solution of  $P_V$  can be written concretely as a classical function. So in this case we need not use the value distribution theory.

And, for the study of behavior around  $x = 0$ , we make the change of variables  $x = 1/z$ . Then we are able to treat as behavior around  $z = \infty$  as well. That is convenient. So we add 2 cases as follows:

Case (iii)  $\delta \neq 0$ : let  $y - 1 = (\eta - 1)/z^2$ ,  $\mu = 2$ ,  $u = \eta - \mu$ ,  $z = a + k^{-1/2}s$  and  $k = 2\delta$ ;

Case (iv)  $\delta = 0$ : let  $y - 1 = (\eta - 1)/z^3$ ,  $\mu = 2$ ,  $u = \eta - \mu$ ,  $z = a + k^{-1/2}s$  and  $k = \gamma(\neq 0)$ .

Then, in each case, we obtain

$$\frac{d^2u}{ds^2} = 1 + g_1(u, s) \frac{du}{ds} + g_2(u, s) \left( \frac{du}{ds} \right)^2 + O(u) + O(1/x),$$

that is just in the form of Lemma 1.1.

## 2 List of the auxiliary functions.

Now we present the auxiliary function  $\Psi_{(j)}$  for Case  $(j)$  ( $j = i, ii, iii, iv$ ):

$$\begin{aligned} \Psi_{(i)}(\mu, x) &:= \frac{x^2(\eta')^2}{\eta(\eta-1)^2} - \frac{2(1-\mu)x\eta'}{(\eta-1)(\eta-\mu)} \\ &\quad - 2\alpha\eta + \frac{2\beta}{\eta} + \frac{2\gamma x}{\eta-1} + \frac{2\delta x^2\eta}{(\eta-1)^2} \quad (\mu \neq 0, 1, \infty), \end{aligned}$$

$$\begin{aligned} \Psi_{(ii)}(\mu, x) &:= \frac{x^2(\eta')^2}{(1+(\eta-1)/x)(\eta-1)^2} + \frac{1}{1+(\eta-1)/x} \\ &\quad - \frac{2x\eta'}{\eta-1} \left\{ \frac{1}{1+(\eta-1)/x} + 1 - \frac{\eta-1}{\eta-\mu} \right\} \\ &\quad - 2\alpha(1+(\eta-1)/x) + \frac{2\beta}{1+(\eta-1)/x} + \frac{2\gamma x^2}{\eta-1} \\ &\quad (\mu \neq 1-x_0, 1, \infty), \end{aligned}$$

$$\begin{aligned} \Psi_{(iii)}(\mu, z) &:= \frac{z^2(\eta')^2}{(1+(\eta-1)/z^2)(\eta-1)^2} + \frac{4}{1+(\eta-1)/z^2} \\ &\quad + \left\{ \frac{-2}{(1+(\eta-1)/z^2)(\eta-1)} + \frac{1}{\eta-1} + \frac{-1}{\eta-\mu} \right\} 2z\eta \\ &\quad - \frac{2\alpha\eta}{z^2} + \frac{2\beta}{1+(\eta-1)/z^2} + \frac{2\gamma z}{\eta-1} + \frac{2\delta z^2(1+(\eta-1)/z^2)}{(\eta-1)^2} \\ &\quad (\mu \neq 1-z_0^2, 1, \infty), \end{aligned}$$

$$\begin{aligned}\Psi_{(iv)}(\mu, z) &:= \frac{z^2(\dot{\eta})^2}{(1 + (\eta - 1)/z^3)(\eta - 1)^2} + \frac{9}{1 + (\eta - 1)/z^3} \\ &+ \left\{ \frac{-3}{(1 + (\eta - 1)/z^3)(\eta - 1)} + \frac{1}{\eta - 1} + \frac{-1}{\eta - \mu} \right\} 2z\dot{\eta} \\ &- 2\alpha \left( 1 + \frac{\eta - 1}{z^3} \right) + \frac{2\beta}{1 + (\eta - 1)/z^3} + \frac{2\gamma z}{\eta - 1} \\ &(\mu \neq 1 - z_0^3, 1, \infty),\end{aligned}$$

where  $' = d/dx$  and  $\dot{\phantom{x}} = d/dz$ .

In Case (iii), for example, the points satisfying  $\eta(z_0) = 1 - z_0^2, 1, \infty$  are singularities of  $P_V$  however, the auxiliary function  $\Psi_{(iii)}(\mu, z)$  is singular not at  $\eta(z_0) = 1 - z_0^2, 1, \infty$ , but at  $\eta(z_0) = \mu$ . So is in every other case, too.

Moreover,  $\Psi_{(j)}(\mu, x)$  satisfies the 1<sup>st</sup> order linear differential equation

$$\frac{d\Psi_{(j)}(\mu, x)}{dx} = P(x)\Psi_{(j)}(\mu, x) + Q(x),$$

where  $P(x)$  and  $Q(x)$  are singular only at  $\eta(x_0) = \mu$  as well. Then, by use of elementary calculus, we obtain

$$\Psi_{(j)}(\mu, x) = e^{\int P(x)dx} \left( \int Q(x)dx + \text{const.} \right)$$

and this integral is able to be estimated on the path on which  $\eta(x)$  does not coincide to  $\mu$ .

### 3 Construction of the auxiliary functions.

Shimomura investigated in [6] the value distribution of the modified  $P_V$ :

$$\frac{d^2w}{dv^2} = \left( \frac{1}{2w} + \frac{1}{w-1} \right) \left( \frac{dw}{dv} \right)^2 + (w-1)^2 \left( \alpha w + \frac{\beta}{w} \right) + \gamma e^v + \frac{\delta e^{2v} w(w+1)}{w-1}, \quad (7)$$

which is uniformization of  $P_V$  by  $x = e^v$ . Note that any solution of  $mP_V$  is meromorphic on  $\mathbb{C}$  ([3]).

The auxiliary function  $\Psi_{(i)}$  is obtained by that of  $mP_V$  given in [6]; by substituting  $e^v = x$  and  $y(e^v) = w(v)$ .

Or, we can obtain  $\Psi_{(i)}$  directly from the power series expansion of solutions of  $P_V$  by eliminating branches and higher order terms.

For example, in Case (iii),  $\eta(z)$  has series expansions at  $\eta(z_0) = 1 - z_0^2, 1, \infty$  w.r.t. local parameter  $Z := z - z_0$

$$\eta(z) = \begin{cases} 1 + \sqrt{-2\delta}Z - \frac{2\delta + (\gamma + \sqrt{-2\delta})z_0}{2z_0^2} + O(Z^3) & \text{at } \eta = 1, \\ 1 - z_0^2 + (-2 + \sqrt{-2\delta})z_0Z + hZ^2 + O(Z^3) & \text{at } \eta = 1 - z_0^2, \\ \frac{z_0^3/\sqrt{2\alpha}}{Z} + h + O(Z) & \text{at } \eta = \infty, \end{cases}$$

where  $h$  is an arbitrary constant called a *resonance*.

First, we move away  $\mu$  from the auxiliary function; we have  $\Psi_{(i)} + 2xy'/(y - \mu)$ . By the transformation  $y - 1 = (\eta - 1)/z^2$ ,  $\Psi_{(i)} + 2xy'/(y - \mu)$  yields the

following function:

$$\begin{aligned}\tilde{\Psi}_{(iii)}(z) &:= \frac{z^2(\dot{\eta})^2}{(1 + (\eta - 1)/z^2)(\eta - 1)^2} + \frac{4}{1 + (\eta - 1)/z^2} \\ &\quad + \frac{-4z\dot{\eta}}{(1 + (\eta - 1)/z^2)(\eta - 1)} + \frac{2z\dot{\eta}}{\eta - 1} \\ &\quad - 2\alpha \left(1 + \frac{\eta - 1}{z^2}\right) + \frac{2\beta}{1 + (\eta - 1)/z^2} + \frac{2\gamma z}{\eta - 1} + \frac{2\delta(1 + (\eta - 1)/z^2)}{(\eta - 1)^2} \\ &\equiv \begin{cases} \frac{z_0^2 - 1}{Z^2} + \frac{1 + z_0^2}{z_0} + \frac{-\gamma}{\sqrt{-2\delta}} \frac{1 - z_0^2}{z_0} & \text{at } \eta = 1, \\ \frac{z_0(-2 + \sqrt{2\alpha - z_0\sqrt{2\alpha}})}{Z} & \text{at } \eta = \infty, \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

where “ $\equiv$ ” means equivalence modulo  $O(1)$ .

Here we have

$$\begin{aligned}-\frac{2\alpha(z-1)\eta}{z^2} &\equiv \begin{cases} \frac{\sqrt{2\alpha z_0(1-z_0)}}{Z} & \text{at } \eta = \infty, \\ 0 & \text{otherwise,} \end{cases} \\ -\frac{2\delta(z^2-1)}{(\eta-1)^2} &\equiv \begin{cases} \frac{z_0^2-1}{Z} & \text{at } \eta = 1, \\ 0 & \text{otherwise,} \end{cases} \\ -\frac{2\delta(z^2-1)}{z^2(\eta-1)^2} &\equiv \begin{cases} \frac{z_0^2-1}{z_0^2} \frac{\sqrt{-2\delta}}{Z} & \text{at } \eta = 1, \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

and then

$$\tilde{\Psi}_{(iii)}(z) + \frac{2\alpha(z-1)\eta}{z^2} + \frac{2\delta(z^2-1)}{(\eta-1)^2} + \frac{2\delta(z^2-1)}{z^2(\eta-1)^2} \equiv \begin{cases} -\frac{2z_0}{Z} & \text{at } \eta = \infty, \\ 0 & \text{otherwise.} \end{cases}$$

And we also have

$$\frac{\dot{\eta}}{\eta-1} \equiv \begin{cases} 1/Z & \text{at } \eta = 1, \\ -1/Z & \text{at } \eta = \infty, \\ 0 & \text{otherwise.} \end{cases}$$

So, if we have a fractional linear transformation  $f = (a\eta + b)/(c\eta + d)$  which maps  $\eta = 1(\infty, \mu \text{ resp.}) \mapsto f = 0(-2, \infty \text{ resp.})$ , then  $\tilde{\Psi}_{(iii)}(z) + \frac{2\alpha(z-1)\eta}{z^2} + \frac{2\delta(z^2-1)}{(\eta-1)^2} + \frac{2\delta(z^2-1)}{z^2(\eta-1)^2} + \frac{z\dot{\eta}}{\eta-1}f$  is singular only at  $\mu$ . Such a transformation  $f = -2(\eta - 1)/(\eta - \mu)$  is easily obtained.

Now we have

$$\begin{aligned}\tilde{\Psi}_{(iii)}(z) + \frac{2\alpha(z-1)\eta}{z^2} + \frac{2\delta(z^2-1)}{(\eta-1)^2} + \frac{2\delta(z^2-1)}{z^2(\eta-1)^2} + \frac{z\dot{\eta}}{\eta-1}f \\ = \frac{z^2(\dot{\eta})^2}{(1 + (\eta - 1)/z^2)(\eta - 1)^2} + \frac{4}{1 + (\eta - 1)/z^2} \\ + \left\{ \frac{-2}{(1 + (\eta - 1)/z^2)(\eta - 1)} + \frac{1}{\eta - 1} + \frac{-1}{\eta - \mu} \right\} 2z\dot{\eta} \\ - 2\alpha \left( \frac{(z-1)\eta}{z^2} - \left(1 + \frac{\eta-1}{z^2}\right) \right) + \frac{2\beta}{1 + (\eta-1)/z^2} \\ + \frac{2\gamma z}{\eta-1} + 2\delta \left\{ \frac{1 + (\eta-1)/z^2}{(\eta-1)^2} + \frac{z^2-1}{(\eta-1)^2} + \frac{z^2-1}{z^2(\eta-1)} \right\},\end{aligned}$$

which is a function singular at  $\eta = \mu$  and holomorphic at elsewhere except for  $\eta = \mu$ , especially  $\eta = 1 - z_0^2, 1, \infty$ . And eliminating some holomorphic terms, we obtain the auxiliary function  $\Psi_{(iii)}(\mu, z)$  as above.

In any other cases, we obtain the auxiliary function  $\Psi_{(j)}$  in a similar way.

**Outline of calculation.** Once the auxiliary function is defined suitably, construction of an integral path, the estimation of the derivative of the fifth Painlevé transcendents and the estimation of the 1-points in a sectorial domain are similar to Case (i); that is precisely explained in [5].

## References

- [1] Gromak, V. I., *Reducibility of the Painlevé equations*, Differ. Uravn. **20**(1984), 1674-1683; Differential Equations **20**(1984), 1191-1198.
- [2] Gromak, V. I., Laine, I. and Shimomura, S., *Painlevé differential equations in the complex plane*, de Gruyter, Berlin, New York, 2002.
- [3] Hinkkanen, A. and Laine, I., *Solutions of a modified fifth Painlevé equation are meromorphic*, Papers on analysis, 133-146, Rep. Univ. Jyväskylä Dep. Math. Stat., **83**, Univ. Jyväskylä, Jyväskylä, 2001.
- [4] Okamoto, K. and Takano, K., *The proof of the Painlevé property by Masuo Hukuhara*, Funkcial. Ekvac. **44**(2001), 201-217.
- [5] Sasaki, Y., *Value distribution of the fifth Painlevé transcendents in sectorial domains*, J. Math. Anal. Appl. (2006), doi:10.1016/j.jmaa.2006.07.083; in press.
- [6] Shimomura, S., *Growth of modified Painlevé transcendents of the fifth and the third kind*, Forum Math., **16**(2004), 231-247.