SYMMETRY AND HOLOMORPHY OF PAINLEVÉ TYPE SYSTEMS

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Abstract. Recent developments on some higher order generalizations of Painlevé equations are summarized from the points of view of their symmetry and holomorphy properties.

1. Introduction

A series of differential equations with affine Weyl group symmetry of type $A_n^{(1)}$ ($n \geq 2$) was introduced in [3]. First two members of the series with $n = 2, 3$ are the fourth and fifth Painlevé equation respectively and remaining ones can be considered as a higher order generalization of them. Since the sixth Painlevé equation has affine Weyl group symmetry of type $D_4^{(1)}$[5], the generalization to $D_n^{(1)}$ (and $E_n^{(1)}, \cdots$) cases has been naturally expected.

The Bäcklund transformations of the Painlevé equations have the universal description to any root systems[4]. Since this universal Bäcklund transformation has Lie theoretic origin, similarity reduction of Drinfeld-Sokolov hierarchy admits such Bäcklund symmetry. The explicit construction of such a differential system is, however, very complicated in general.

The above situation was conquered in a recent work[6], where a series of differential systems with symmetry of type $D_n^{(1)}$ was constructed. The crucial idea of this work was to use the "holomorphy" characterization of Painlevé equations, which can be considered as a generalization of Takano's theory[8]. The aim of this note is to explain why and how the holomorphy approach works well for the above problem.

The paper is organized as follows. In section 2, we explain the term "symmetry" and "holomorphy" in case of the sixth Painlevé equation. In section 3, the case of $A_n^{(1)}$ is reviewed from the symmetry and holomorphy points of view. The coupled $P_{\rm V}$ and $P_{\rm VI}$ systems with $W(D_n^{(1)})$ symmetry [6] are described in section 4. A relation between the holomorphy and symmetry conditions are discussed in section 5. Holomorphy conditions of the Garnier system and the $E_6^{(1)}$-system[10] are discussed in sections 6 and 7. In section 8, we show the coupled $P_{\rm VI}$ system in terms of general form where the positions of four time variables are not specialized. The results in sections 5,6,7,8 are new.

2. The case of Painlevé VI

The sixth Painlevé equation $P_{\rm VI}$ can be written as the Hamiltonian system

$$\begin{aligned} &\frac{dq}{dt}=\frac{\partial H_{\rm VI}}{\partial p}, \quad \frac{dp}{dt}=-\frac{\partial H_{\rm VI}}{\partial q}, \\ &t(t-1)H_{\rm VI}=q(q-1)(q-t)p^2-\big\{(\alpha_0-1)q(q-1) \\ &\qquad +\alpha_3 q(q-t)+\alpha_4(q-1)(q-t)\big\}p+\alpha_2(\alpha_1+\alpha_2)q, \end{aligned} \tag{1}$$

where parameters $\alpha_0,\dots,\alpha_4$ are normalized as $\alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4=1$.

Let $A=(a_{ij})_{0\le i,j\le 4}$ be the Cartan matrix of type $D_4^{(1)}$:

$$A=\begin{pmatrix} 2 & & -1 & & \\ & 2 & -1 & & \\ -1 & -1 & 2 & -1 & -1 \\ & & -1 & 2 & \\ & & -1 & & 2 \end{pmatrix}. \tag{2}$$

The affine Weyl group $W(D_4^{(1)})$ is defined as

$$W(D_4^{(1)})=\langle s_0,\dots,s_4 \mid s_i^2=(s_is_j)^{m_{ij}}=1\ \rangle, \tag{3}$$

where $m_{ij}=2$ (for $a_{ij}=0$), $m_{ij}=3$ (for $a_{ij}=-1$).

By the work of Okamoto[5], it is known that the $P_{\rm VI}$ equation has the affine Weyl group symmetry $W(D_4^{(1)})$:

$$\begin{aligned} &s_0(\alpha_i)=\alpha_i-a_{i0}\alpha_0, \quad s_0(p)=p-\frac{\alpha_0}{q-t}, \\ &s_1(\alpha_i)=\alpha_i-a_{i1}\alpha_1, \\ &s_2(\alpha_i)=\alpha_i-a_{i2}\alpha_2, \quad s_2(q)=q+\frac{\alpha_2}{p}, \\ &s_3(\alpha_i)=\alpha_i-a_{i3}\alpha_3, \quad s_3(p)=p-\frac{\alpha_3}{q-1}, \\ &s_4(\alpha_i)=\alpha_i-a_{i4}\alpha_4, \quad s_4(p)=p-\frac{\alpha_4}{q}. \end{aligned} \tag{4}$$

Here and in the followings, we omit identical transformations $s(x)=x$. The symmetry can be extended to $\widetilde{W}(D_4^{(1)})$ by including the diagram automorphisms. For explicit formulas, see [2] for example.

The Hamiltonian $H_{\rm VI}$ (1) is a polynomial in the canonical variables p,q. In this sense we call the system (1) as a polynomial Hamiltonian system. Consider the following birational symplectic transformations r_i $(i=0,\dots,4)$:

$$\begin{aligned} &r_0(p)=\frac{1}{p}, \quad r_0(q)=t-p(p(q-t)-\alpha_0), \\ &r_1(q)=\frac{1}{q}, \quad r_1(p)=-q(qp+\alpha_1+\alpha_2), \\ &r_2(q)=\frac{1}{q}, \quad r_2(p)=-q(qp+\alpha_2), \\ &r_3(p)=\frac{1}{p}, \quad r_3(q)=1-p(p(q-1)-\alpha_3), \\ &r_4(p)=\frac{1}{p}, \quad r_4(q)=-p(pq-\alpha_4). \end{aligned} \tag{5}$$

Since the transformation r_i is symplectic, the system (1) is transformed into a Hamiltonian system whose Hamiltonian may have poles. It is remarkable that the transformed system is again a polynomial system for any $i = 0, \dots, 4$. Furthermore, this holomorphy property uniquely characterizes the $P_{\rm VI}$ equation[8].

Remark 2.1. If we look for a polynomial Hamiltonian system which admits the *symmetry* (4), we have to consider huge polynomial in variables q, p, t, α_i. On the other hand, in the *holomorphy* requirement (5), we only need to consider polynomials in q, p. This reduces the number of unknown coefficients drastically.

3. Systems of type $A_n^{(1)}$

Let f_i and α_i ($i \in \mathbb{Z}/(n+1)\mathbb{Z}$) be variables and parameters. We set $\displaystyle\sum_{i=0}^{n} \alpha_i = 1$ and

$$
\begin{aligned}
&\sum_{i=0}^{n} f_i = t, \quad (n = 2k)\\
&\sum_{r=0}^{k} f_{2r} = -t, \quad \sum_{r=0}^{k} f_{2r+1} = -s. \quad (n = 2k+1)
\end{aligned}
\tag{6}
$$

The following systems of differential equations were introduced in [3][1].

$$
\frac{d}{dt} f_i = f_i \sum_{r=1}^{k} (f_{i+2r-1} - f_{i+2r}) + \alpha_i. \quad (n = 2k) \tag{7}
$$

$$
\begin{aligned}
\frac{d}{dt} f_i = f_i &\sum_{1 \le r \le s \le k} (f_{i+2r-1} f_{i+2s} - f_{i+2r} f_{i+2s+1})\\
&+ (-1)^i f_i \sum_{r=0}^{k} \alpha_{2r+1} + \alpha_i \sum_{r=0}^{k} f_{i+2r}.
\end{aligned}
\qquad (n = 2k+1) \tag{8}
$$

Affine Weyl group $W(A_n^{(1)})$ is defined similarly as (3). Corresponding Cartan matrix $A = (a_{ij})_{0 \le i,j \le n}$ is given by

$$
A = \begin{pmatrix}
2 & -1 & & & -1\\
-1 & 2 & -1 & & \\
& \ddots & \ddots & \ddots & \\
& & -1 & 2 & -1\\
-1 & & & -1 & 2
\end{pmatrix}. \tag{9}
$$

The systems (7)(8) have the extended affine Weyl group symmetry $\widetilde{W}(A_n^{(1)})$[3]. The action of the generators s_i ($i \in \mathbb{Z}/(n+1)\mathbb{Z}$) and π is given as follows:

$$
\begin{aligned}
&s_i(\alpha_j) = \alpha_j - a_{ji}\alpha_i, \quad \pi(\alpha_i) = \alpha_{i+1},\\
&s_i(f_{i+1}) = f_{i+1} + \frac{\alpha_i}{f_i}, \quad s_i(f_{i-1}) = f_{i-1} - \frac{\alpha_i}{f_i},\\
&\begin{cases}
\pi(f_i) = f_{i+1}, \quad (n = 2k)\\
\pi(f_{2r}) = \dfrac{t}{s} f_{2r+1}, \quad \pi(f_{2r-1}) = \dfrac{s}{t} f_{2r}. \quad (n = 2k+1)
\end{cases}
\end{aligned}
\tag{10}
$$

[1] In case of $n = 2k+1$, the normalization of f_i are slightly modified from [3].

Due to the constraints (6), both the systems (7)(8) are of degree $2k$ effectively. In fact that the $A_n^{(1)}$ systems can be written as Hamiltonian systems

$$\frac{dq_i}{dt} = \frac{\partial H^{A_n^{(1)}}}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H^{A_n^{(1)}}}{\partial q_i}, \quad (i = 1, \ldots, k) \tag{11}$$

in terms of canonical variables

$$p_i = f_{2i}, \quad q_i = -\sum_{j=1}^{i} f_{2j-1}. \tag{12}$$

The non-vanishing Poisson brackets of f_i are $\{f_i, f_{i\pm1}\} = \pm 1$.

The following fact has been observed by the first author a few years ago.

Proposition 3.1. *The $A_{2k}^{(1)}$ [$A_{2k+1}^{(1)}$ resp.] system is written as coupled P_{IV} [P_{V} resp.] equations as follows:*

$$\begin{aligned} H^{A_{2k}^{(1)}} &= \sum_{i=1}^{k} H_{\mathrm{IV}}(q_i, p_i; \alpha_{2i}, \sum_{j=1}^{i} \alpha_{2j-1}) + 2 \sum_{1\le i<j\le k} q_i p_i p_j, \\ tH^{A_{2k+1}^{(1)}} &= \sum_{i=1}^{k} H_{\mathrm{V}}(q_i, p_i; \alpha_{2i}, \sum_{j=1}^{i} \alpha_{2j-1}, \sum_{j=1}^{k+1} \alpha_{2j-1}) + 2 \sum_{1\le i<j\le k} p_i p_j q_i (q_j - s), \end{aligned} \tag{13}$$

where

$$\begin{aligned} H_{\mathrm{IV}}(q, p; a, b) &= pq(p - q - t) - aq - bp, \\ H_{\mathrm{V}}(q, p; a, b, c) &= p(p + t)q(q - s) + atq + bsp - cpq. \end{aligned} \tag{14}$$

Define birational transformations r_i $(i \in \mathbb{Z}/(n+1)\mathbb{Z})$ as

$$\begin{aligned} r_i(f_{i-1}) &= f_{i-1} - \frac{1}{f_{i+1}} + f_{i+1}, \\ r_i(f_i) &= -f_{i+1}(f_{i+1} f_i + \alpha_i), \\ r_i(f_{i+1}) &= \frac{1}{f_{i+1}}, \\ r_i(f_{i+2}) &= f_{i+2} + f_i + f_{i+1}(f_{i+1} f_i + \alpha_i). \end{aligned} \tag{15}$$

In terms of the canonical variables, r_1 is simply written as

$$r_1(p_1) = \frac{1}{p_1}, \quad r_1(q_1) = -p_1(p_1 q_1 - \alpha_1). \tag{16}$$

The transformations r_i are symplectic.

Proposition 3.2. *Each of the transforms preserves the holomorphy of the system (7)(8). The system (7) [(8) resp.] is unique one of degree 3 [4 resp.] satisfying this holomorphy property.*

Remark 3.3. For $A_4^{(1)}$ case, the transformations r_i are essentially the same as that in [11].

4. Systems of type $D_n^{(1)}$

Introduce root parameters α_i $(i=0,\dots,n)$: $\alpha_0+\alpha_1+2(\alpha_2+\cdots+\alpha_{n-2})+\alpha_{n-1}+\alpha_n=1$. The $D_n^{(1)}$ Cartan matrix $(a_{ij})_{0\le ij\le n}$ is

$$\begin{pmatrix}
2 & & -1 & & & & & & \\
 & 2 & -1 & & & & & & \\
-1 & -1 & 2 & -1 & & & & & \\
 & & -1 & 2 & -1 & & & & \\
 & & & \ddots & \ddots & \ddots & & & \\
 & & & & -1 & 2 & -1 & & \\
 & & & & & -1 & 2 & -1 & -1 \\
 & & & & & & -1 & 2 & \\
 & & & & & & -1 & & 2
\end{pmatrix}. \tag{17}$$

The following discussion is separated into two cases: $n=2k+1$ and $n=2k+2$. In both cases, we introduce canonical variables q_i,p_i $(i=1,\dots,k)$.

4.1. **The case of** $n=2k+1$**.** : Let us put

$$\begin{aligned}
&f_0=p_1+t, \quad f_1=p_1,\\
&f_{2i}=q_i-q_{i+1}, \quad f_{2i+1}=p_{i+1}, \quad (1\le i\le k-1)\\
&f_{n-1}=q_k-1, \quad f_n=q_k.
\end{aligned} \tag{18}$$

Following [4], we define the actions s_i $(i=0,\dots,n)$ as

$$s_i(\alpha_j)=\alpha_j-a_{ji}\alpha_i, \quad s_i(g)=g+\frac{\alpha_i}{f_i}\{g,f_i\}, \quad (g=q_k,p_k), \tag{19}$$

where $\{\ ,\ \}$ is the Poisson bracket such that $\{q_i,p_j\}=\delta_{ij}$. These actions define the representation of affine Weyl group $W(D_n^{(1)})$.

Define birational symplectic transformations r_i $(i=0,\dots,n)$ as follows:

$$\begin{aligned}
&r_0(q_1)=\frac{1}{q_1}, \quad r_0(p_1)=-t-q_1(q_1(p_1+t)+\alpha_0),\\
&r_1(q_1)=\frac{1}{q_1}, \quad r_1(p_1)=-q_1(q_1p_1+\alpha_1),\\
&r_{2i}(p_i)=\frac{1}{p_i}, \quad r_{2i}(q_i)=q_{i+1}-p_i(p_i(q_i-q_{i+1})-\alpha_{2i}),\\
&r_{2i}(p_{i+1})=p_{i+1}+p_i-\frac{1}{p_i}, \quad (i=1,\dots,k-1)\\
&r_{2i+1}(q_{i+1})=\frac{1}{q_{i+1}},\\
&r_{2i+1}(p_{i+1})=-q_{i+1}(q_{i+1}p_{i+1}+\alpha_{2i+1}), \quad (i=1,\dots,k-1)\\
&r_{n-1}(p_k)=\frac{1}{p_k}, \quad r_{n-1}(q_k)=1-p_k(p_k(q_k-1)-\alpha_{n-1}),\\
&r_n(p_k)=\frac{1}{p_k}, \quad r_n(q_k)=-p_k(p_kq_k-\alpha_n).
\end{aligned} \tag{20}$$

Theorem 4.1. [6] *There exists a unique polynomial Hamiltonian system of degree 4, which is holomorphic in each coordinates r_i $(i=0,\dots,n)$. The system is invariant under the Weyl group $W(D_n^{(1)})$ and given by the following Hamiltonian (the* *coupled $P_{\rm V}$):*

$$\begin{aligned} tH = &\sum_{i=1}^{k} H_{\rm V}(q_i,p_i;\alpha_n+\sum_{j=1}^{k-1}\alpha_{2j},\alpha_{2i-1},\alpha_{n-1}+\alpha_n+2\sum_{j=1}^{k-1}(\alpha_{2j}+\alpha_{2j+1}))\\ &+2\sum_{1\le i<j\le k} p_iq_j(p_j(q_j-1)+\alpha_{2j-1}), \end{aligned} \tag{21}$$

where

$$H_{\rm V}(q,p;a,b,c) = q(q-1)p(p+t)+ap+btq-cpq. \tag{22}$$

4.2. **The case of $n=2k+2$.** : Let us put

$$\begin{aligned} &f_0 = q_1 - t, \quad f_1 = q_1 - \infty,\\ &f_{2i} = p_i, \quad f_{2i+1} = q_i - q_{i+1}, \quad (1\le i\le k-1)\\ &f_{n-1} = q_k - 1, \quad f_n = q_k. \end{aligned} \tag{23}$$

The Weyl group representation $s_i \in W(D_n^{(1)})$ is defined by the same formulas (19).

Define birational symplectic transformations r_i $(i=0,\dots,n)$ as follows:

$$\begin{aligned} &r_0(p_1) = \frac{1}{p_1}, \quad r_0(q_1) = t - p_1(p_1(q_1-t)-\alpha_0),\\ &r_1(q_1) = \frac{1}{q_1}, \quad r_1(p_1) = -q_1(q_1p_0+\alpha_1+\alpha_2),\\ &r_{2i}(q_i) = \frac{1}{q_i}, \quad r_{2i}(p_i) = -q_i(q_ip_i+\alpha_{2i}), \quad (i=1,\dots,k)\\ &r_{2i+1}(p_i) = \frac{1}{p_i}, \quad r_{2i+1}(q_i) = q_{i+1} - p_i(p_i(q_i-q_{i+1})-\alpha_{2i+1}),\\ &r_{2i+1}(p_{i+1}) = p_{i+1}+p_i-\frac{1}{p_i}, \quad (i=1,\dots,k-1)\\ &r_{n-1}(p_k) = \frac{1}{p_n}, \quad r_{n-1}(q_k) = 1 - p_k(p_k(q_k-1)-\alpha_{n-1}),\\ &r_n(p_k) = \frac{1}{p_n}, \quad r_n(q_k) = -p_k(p_kq_k-\alpha_n). \end{aligned} \tag{24}$$

Theorem 4.2. [6] *There exists a unique polynomial Hamiltonian system of degree 5, which is holomorphic in each coordinates r_i $(i=0,\dots,n)$. The system is invariant under the Weyl group $W(D_n^{(1)})$ and given by the following Hamiltonian (the coupled $P_{\rm VI}$):*

$$\begin{aligned} t(t-1)H = &\sum_{i=1}^{k} H_{\rm VI}(q_i,p_i;a_i,b_i,c_i,d_i)\\ &+2\sum_{1\le i<j\le k} p_i(q_i-t)q_j\Big[p_j(q_j-1)+\alpha_{2j}\Big], \end{aligned} \tag{25}$$

where

$$\begin{aligned} &H_{\rm VI}(q,p;a,b,c,d) = q(q-1)(q-t)p^2\\ &-\big\{(a-1)q(q-1)+bq(q-t)+c(q-1)(q-t)\big\}p+dq, \end{aligned} \tag{26}$$

and

$$
\begin{aligned}
&a_i = \alpha_0 + \sum_{j=1}^{i-1} \alpha_{2j+1}, \quad \{ \begin{matrix} b_i \\ c_i \end{matrix} \} = \{ \begin{matrix} \alpha_{n-1} \\ \alpha_n \end{matrix} \} + \sum_{j=i}^{k-1} \alpha_{2j+1}, \\
&d_i = \alpha_{2i}\big(\alpha_1 + \alpha_{2i} + 2\sum_{j=1}^{i-1} \alpha_{2j} + \sum_{j=1}^{i-1} \alpha_{2j+1}\big).
\end{aligned} \tag{27}
$$

5. A relation between r_i and s_i

In all the previous examples, there exist as many transformations r_i as symmetries s_i. They enjoy the relations $r_i^2 = 1$ and $r_i s_i = s_i r_i$. We have seen that the Hamiltonian system characterized by the holomorphy condition w.r.t. r_i's has the Bäcklund symmetry under s_i's. Here, we will explain a reason of these fortunate phenomena. The following argument is also applicable in case of many variables $q_1, p_1, q_2, p_2, \cdots$ and many parameters $a, b, c, \cdots$, we consider a Hamiltonian system with Hamiltonian $H = H(p, q, t; a, b)$ for simplicity.

Proposition 5.1. *If the polynomial Hamiltonian system has a t-independent Bäcklund symmetry*

$$s : (p, q, t; a, b) \mapsto (p - \frac{a}{q}, q, t; -a, b'), \tag{28}$$

then $r(H)$ is also polynomial in p, q, where

$$r : (p, q, t; a, b) \mapsto (\frac{1}{p}, -p(pq - a), t, a, b). \tag{29}$$

Proof. Since the transformation s is t-independent, the symmetry requires the polynomiality of

$$s(H) = H(p - \frac{a}{q}, q, t, -a, b'). \tag{30}$$

By changing the parameters $a \mapsto -a$, $b' \mapsto b$, this condition is equivalent to the polynomiality of

$$H(p + \frac{a}{q}, q, t; a, b) =: F(p, q, t; a, b). \tag{31}$$

Changing the variables $p \mapsto p - \frac{a}{q}$, we have

$$H(p, q, t; a, b) = F(p - \frac{a}{q}, q, t; a, b). \tag{32}$$

Applying the transformation r on this equation, we get

$$r(H) = H(\frac{1}{p}, -p(pq - a), t; a, b) = F(\frac{q}{pq - a}, -p(pq - a), t; a, b). \tag{33}$$

Then this expression should be polynomial since $H(p, q, \cdots)$ and $F(p, q, \cdots)$ are polynomials. ∎

This proposition says that the holomorphy is a necessary condition for the symmetry. This is a reason why the holomorphy approach is effective to find the differential system with desired symmetry.

6. Garnier Systems

Here we give a characterization of Garnier system via holomorphy conditions. We give an example in case of 3-variables. n-variable case is similar (the 2-variable case was first obtained in [7]). Let us define birational symplectic transformations:

$$\begin{aligned}
& r_1(q_1) = \frac{1}{q_1}, \quad r_1(q_j) = \frac{q_j}{q_1}, \quad (j = 2,3) \\
& r_1(p_1) = -q_1(\sum_{j=1}^{3} q_j p_j + \rho), \quad r_1(p_j) = p_j q_1, \quad (j = 2,3) \\
& r_2 = r_1|_{\rho \mapsto \rho + \vartheta_6 + 1} \\
& r_3(p_1) = \frac{1}{p_1}, \quad r_3(q_1) = -p_1(p_1 q_1 - \vartheta_1), \\
& r_4(p_2) = \frac{1}{p_2}, \quad r_4(q_2) = -p_2(p_2 q_2 - \vartheta_2), \\
& r_5(p_3) = \frac{1}{p_3}, \quad r_5(q_3) = -p_3(p_3 q_3 - \vartheta_3), \\
& r_6(p_1) = \frac{1}{p_1}, \quad r_6(p_j) = p_j + \frac{1}{p_1} - p_1, \quad (j = 2,3) \\
& r_6(q_1) = 1 - q_2 - q_3 - p_1\big(p_1(q_1 + q_2 + q_3 - 1) - \vartheta_5\big), \\
& r_7(p_1) = \frac{1}{p_1}, \quad r_7(p_j) = p_j + \frac{t_1}{t_j}(\frac{1}{p_1} - p_1), \quad (j = 2,3) \\
& r_7(q_1) = t_1(1 - \frac{q_2}{t_2} - \frac{q_3}{t_3}) - p_1\big(p_1 t_1(\frac{q_1}{t_1} + \frac{q_2}{t_2} + \frac{q_3}{t_3} - 1) - \vartheta_4\big).
\end{aligned} \tag{34}$$

The parameters satisfy the Fuchs relation:

$$2\rho + \sum_{i=1}^{6} \vartheta_i = 0. \tag{35}$$

Theorem 6.1. *There exist unique polynomials K_i for $i = 1,2,3$, such that the Hamiltonian system*

$$\frac{\partial q_i}{\partial t_j} = \frac{\partial K_j}{\partial p_i}, \quad \frac{\partial p_i}{\partial t_j} = -\frac{\partial K_j}{\partial q_i}, \quad (i,j = 1,2,3) \tag{36}$$

is transformed into a polynomial Hamiltonian system under the action of each r_k. This system coincides with the Garnier system in 3 variables (equation (1.9) [9]*).*

Remark 6.2. We have checked similar characterization of the n-variable Garnier systems for $n \leq 4$. General cases are, however, not proved yet.

7. System with $W(E_6^{(1)})$-symmetry

Canonical variables (q_i, p_i), $(i = 1,2,3)$, parameters α_i, $(i = 0,\dots,6)$ and dependent variable t. We put $\alpha_0 + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 = 1$.

The invariant divisors are

$$\begin{aligned}
& f_0 = q_3 - t, \quad f_1 = q_1, \quad f_2 = p_1, \\
& f_3 = q_1 q_2 - q_1 - q_2 + q_3, \\
& f_4 = p_2, \quad f_5 = q_2, \quad f_6 = p_3.
\end{aligned} \tag{37}$$

Define the actions s_i ($i = 0, \dots, 6$) by the same formulas as (19) with the Cartan matrix for $E_6^{(1)}$:

$$\begin{pmatrix} 2 & & & & & & -1 \\ & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & -1 \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & \\ -1 & & & -1 & & & 2 \end{pmatrix}. \tag{38}$$

These actions define the representation of affine Weyl group $W(E_6^{(1)})$.

Define birational symplectic transformations r_i ($0 \le i \le 6$) as follows:

$$\begin{aligned}
&r_0(p_3) = \frac{1}{p_3}, \quad r_0(q_3) = t - p_3(p_3 f_0 - \alpha_0), \\
&r_1(p_1) = \frac{1}{p_1}, \quad r_1(q_1) = -p_1(p_1 f_1 - \alpha_1), \\
&r_2(q_1) = \frac{1}{q_1}, \quad r_2(p_1) = -q_1(q_1 f_2 + \alpha_2), \\
&r_3(p_3) = \frac{1}{p_3}, \quad r_3(q_3) = -q_1 q_2 + q_1 + q_2 - p_3(p_3 f_3 - \alpha_3), \\
&r_3(p_2) = p_2 + (1 - q_1)(p_3 - \frac{1}{p_3}), \quad r_3(p_1) = p_1 + (1 - q_2)(p_3 - \frac{1}{p_3}), \\
&r_4(q_2) = \frac{1}{q_2}, \quad r_4(p_2) = -q_2(q_2 f_4 + \alpha_4), \\
&r_5(p_2) = \frac{1}{p_2}, \quad r_5(q_2) = -p_2(p_2 f_5 - \alpha_5), \\
&r_6(q_3) = \frac{1}{q_3}, \quad r_6(p_3) = -q_3(q_3 f_6 + \alpha_6).
\end{aligned} \tag{39}$$

Theorem 7.1. *There exists a unique polynomial Hamiltonian system of degree 5, which is also polynomial in the transformed coordinates given by r_i ($0 \le i \le 6$). This system has the affine Weyl group symmetry $W(E_6^{(1)})$ given above, and the Hamiltonian is explicitly given by*

$$\begin{aligned}
&t(t-1)H \\
&= H_{\rm VI}(q_1, p_1; \alpha_{3560}, \alpha_{34}, \alpha_1, \alpha_2\alpha_{2346}) \\
&+ H_{\rm VI}(q_2, p_2; \alpha_{1360}, \alpha_{23}, \alpha_5, \alpha_4\alpha_{2346}) \\
&+ H_{\rm VI}(q_3, p_3; \alpha_0, \alpha_{234}, \alpha_{135}, \alpha_6\alpha_{2346}) \\
&+ (t-1)q_1 p_1 q_2 p_2 - q_1(p_1(q_1 - 1) + \alpha_2)q_2(p_2(q_2 - 1) + \alpha_4) \\
&+ q_1 p_1 (q_3 - 1)(p_3(q_3 - t) + \alpha_6) + q_1(p_1(q_1 - 1) + \alpha_2)(q_3 - t)p_3 \\
&+ q_2 p_2 (q_3 - 1)(p_3(q_3 - t) + \alpha_6) + q_2(p_2(q_2 - 1) + \alpha_4)(q_3 - t)p_3,
\end{aligned} \tag{40}$$

where $H_{\rm VI}(q, p : a, b, c, d)$ is given in (26) and $\alpha_{ij\cdots k} = \alpha_i + \alpha_j + \cdots + \alpha_k$.

Remark 7.2. The Hamiltonian system with Hamiltonian (40) was discovered by T.Suzuki [10] through a similarity reduction of Drinfeld-Sokolov hierachy of type $E_6^{(1)}$.

8. Democratic form of the coupled $P_{\rm VI}$ system

For $n = 2k + 2$, the system (25) with $W(D_n^{(1)})$ symmetry can be described more symmetric way by introducing independent variables t_μ ($\mu = 1, \dots, 4$). Then the original system (25) is obtained under the specialization $(t_1, t_2, t_3, t_4) = (t, \infty, 1, 0)$. In case of $k = 1$ (original $P_{\rm VI}$) this result corresponds to the work by Kawamuko[1].

Let us put

$$\begin{aligned} &f_0 = q_1 - t_1, \quad f_1 = q_1 - t_2, \\ &f_{2i} = p_i, \quad f_{2i+1} = q_i - q_{i+1}, \quad (1 \le i \le k-1) \\ &f_{n-1} = q_k - t_3, \quad f_n = q_k - t_4. \end{aligned} \tag{41}$$

The Weyl group representation $s_i \in W(D_n^{(1)})$ is defined by the same formulas (19).

Define birational symplectic transformations r_i $(i = 0, \dots, n)$ as follows.

$$\begin{aligned} &r_0(p_1) = \frac{1}{p_1}, \quad r_0(q_1) = t_1 - p_1(p_1(q_1 - t_1) - \alpha_0), \\ &r_1(p_1) = \frac{1}{p_1}, \quad r_1(q_1) = t_2 - p_1(p_1(q_1 - t_2) - \alpha_1), \\ &r_{2i}(q_i) = \frac{1}{q_i}, \quad r_{2i}(p_i) = -q_i(q_i p_i + \alpha_{2i}), \quad (2i = 2, 4, \dots, n-2) \\ &r_{2i+1}(p_i) = \frac{1}{p_i}, \quad r_{2i+1}(q_i) = q_{i+1} - p_i(p_i(q_i - q_{i+1}) - \alpha_{2i+1}), \\ &\qquad r_{2i+1}(p_{i+1}) = p_{i+1} + p_i - \frac{1}{p_i}, \quad (2i+1 = 3, 5, \dots, n-3) \\ &r_{n-1}(p_k) = \frac{1}{p_k}, \quad r_{n-1}(q_k) = t_3 - p_k(p_k(q_k - t_3) - \alpha_{n-1}), \\ &r_n(p_k) = \frac{1}{p_k}, \quad r_n(q_k) = t_4 - p_k(p_k(q_k - t_4) - \alpha_n). \end{aligned} \tag{42}$$

We have confirmed the following up to $k \le 4$.

Conjecture 8.1. *There exists a unique polynomial Hamiltonian system of degree 6, which is also holomorphic for r_i (42) $(0 \le i \le n)$. This system has $W(D_n^{(1)})$ symmetry and explicitly given as follows:*

$$\frac{\partial q_i}{\partial t_\mu} = \frac{\partial H_\mu}{\partial p_i}, \quad \frac{\partial p_i}{\partial t_\mu} = -\frac{\partial H_\mu}{\partial q_i}, \quad (i = 1, \dots, k, \ \mu = 0, \dots, 4) \tag{43}$$

where

$$\begin{aligned} H_\mu &= \prod_{\nu(\neq\mu)=1}^{4} \frac{1}{t_\mu - t_\nu} H|_{c_{i\mu} \mapsto c_{i\mu} - 1}, \\ H &= \sum_{i=1}^{k} H_{(i)} + 2 \sum_{1 \le i < j \le k} W_{(ij)}, \\ H_{(i)} &= \prod_{\mu=1}^{4} (q_i - t_\mu) \left[p_i^2 - p_i \sum_{\mu=1}^{4} \frac{c_{i\mu}}{q_i - t_\mu} \right] \\ &\quad + \alpha_{2i} q_i \left[\sum_{\mu=1}^{2} (c_{i5} + c_{i\mu})(q_i - t_\mu) + \sum_{\mu=3}^{4} (c_{i6} + c_{i\mu})(q_i - t_\mu) - \alpha_{2i} q_i \right], \\ W_{(ij)} &= \Big[p_i(q_i - t_1)(q_i - t_2) + \alpha_{2i}(q_i - q_j) \Big] \\ &\quad \times \Big[p_j(q_j - t_3)(q_j - t_4) + \alpha_{2j}(q_j - q_i) \Big] - \alpha_{2i}\alpha_{2j} q_i q_j, \end{aligned} \tag{44}$$

and

$$\left\{ \begin{array}{c} c_{i1} \\ c_{i2} \end{array} \right\} = \left\{ \begin{array}{c} \alpha_0 \\ \alpha_1 \end{array} \right\} + \alpha_3 + \alpha_5 + \cdots + \alpha_{2i-1},$$
$$\left\{ \begin{array}{c} c_{i3} \\ c_{i4} \end{array} \right\} = \alpha_{2i+1} + \alpha_{2i+3} + \cdots + \alpha_{2k-1} + \left\{ \begin{array}{c} \alpha_{n-1} \\ \alpha_n \end{array} \right\}, \tag{45}$$
$$c_{i5} = 2(\alpha_2 + \alpha_4 + \cdots + \alpha_{2i-2}) + \alpha_{2i},$$
$$c_{i6} = \alpha_{2i} + 2(\alpha_{2i+2} + \alpha_{2i+4} + \cdots + \alpha_{2k}).$$

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