ASYMPTOTIC SOLUTIONS OF A DEGENERATE GARNIER SYSTEM OF THE FIRST PAINLEVÉ TYPE

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Abstract. We consider a degenerate Garnier system of the first Painlevé type on $P^1(\mathbb{C}) \times P^1(\mathbb{C})$. Around a singular locus of irregular type, we present a three-parameter family of solutions of it. Restriction of them to a certain hyperplane yields asymptotic solutions of the fourth order version of the first Painlevé equation.

1. Introduction

Suppose that the linear differential equation

$$\frac{d^2y}{dx^2} - \Big(\sum_{k=1,2} \frac{1}{x-\lambda_k}\Big)\frac{dy}{dx} - \Big(9x^5 + 9t_1x^3 + 3t_2x^2 + 3K_2x + 3K_1 - \sum_{k=1,2}\frac{\mu_k}{x-\lambda_k}\Big)y = 0 \tag{1.1}$$

has non-logarithmic singular points at $x = \lambda_1, \lambda_2$. Then K_1 and K_2 are given by

$$3K_1 = \sum_{k=1,2} \frac{\Pi_1(\lambda_k)}{\Pi_0'(\lambda_k)}\Big(\mu_k^2 - \frac{\mu_k}{\Pi_1(\lambda_k)} - 9\lambda_k^5 - 9t_1\lambda_k^3 - 3t_2\lambda_k^2\Big),$$

$$3K_2 = \sum_{k=1,2} \frac{1}{\Pi_0'(\lambda_k)}\Big(\mu_k^2 - 9\lambda_k^5 - 9t_1\lambda_k^3 - 3t_2\lambda_k^2\Big)$$

with

$$\Pi_0(\xi) = (\xi-\lambda_1)(\xi-\lambda_2), \quad \Pi_1(\xi) = \xi - \lambda_1 - \lambda_2.$$

The isomonodromic deformation with respect to the parameters t_1, t_2 yields the completely integrable Hamiltonian system

$$\frac{\partial\lambda_j}{\partial t_k} = \frac{\partial K_k}{\partial \mu_j}, \quad \frac{\partial \mu_j}{\partial t_k} = -\frac{\partial K_k}{\partial\lambda_j} \quad (j, k = 1, 2);$$

which is equivalent to

$$
\text{(G)} \qquad \begin{aligned}
&\frac{\partial q_j}{\partial s} = \frac{\partial H_1}{\partial p_j}, \quad \frac{\partial p_j}{\partial s} = -\frac{\partial H_1}{\partial q_j}, \\
&\frac{\partial q_j}{\partial t} = \frac{\partial H_2}{\partial p_j}, \quad \frac{\partial p_j}{\partial t} = -\frac{\partial H_2}{\partial q_j} \qquad (j = 1, 2)
\end{aligned}
$$

with the Hamiltonians

$$
\begin{aligned}
3H_1 :=& \Big(q_2^2 - q_1 - \frac{s}{3}\Big)p_1^2 + 2q_2p_1p_2 + p_2^2 \\
&\qquad\qquad + 9\Big(q_1 + \frac{s}{3}\Big)q_2\Big(q_2^2 - 2q_1 + \frac{s}{3}\Big) - 3tq_1, \\
3H_2 :=& q_2p_1^2 + 2p_1p_2 + 9\Big(q_2^4 - 3q_1q_2^2 + q_1^2 - \frac{s}{3}q_1 - \frac{t}{3}q_2\Big).
\end{aligned}
$$

Here the new unknowns and variables are given by

$$
\begin{aligned}
&(q_1, q_2) = (\lambda_1\lambda_2 - t_1/3, \lambda_1 + \lambda_2), \\
&(p_1, p_2) = \Big(\frac{\mu_1 - \mu_2}{\lambda_2 - \lambda_1}, \frac{\lambda_1\mu_1 - \lambda_2\mu_2}{\lambda_1 - \lambda_2}\Big), \\
&(s, t) = (t_1, -t_2)
\end{aligned}
$$

(see [1]). This system may be regarded as a two-variable version of the first Painlevé equation PI. Let us consider (G) on $P^1(\mathbb{C}) \times P^1(\mathbb{C})$ $(\ni (s, t))$. Then, (G) admits singular loci along $s = \infty$ and $t = \infty$. Restricting (G) to the complex line $s = s_0$ $(\in \mathbb{C})$, we obtain the fourth order nonlinear differential equation

$$
(\mathrm{PI}_4) \qquad q^{(4)} = 20qq'' + 10(q')^2 - 40q^3 - 8s_0q - \frac{8}{3}t \qquad ({}' = d/dt,\ q := q_2),
$$

which belongs to the PI-hierarchy. For the PI-hierarchy written in the Hamiltonian form containing a large parameter, Y. Takei ([6]) constructed instanton-type formal solutions containing many free parameters, by reducing to the Birkhoff normal form. For Painlevé equations PI, ..., PV, two-parameter families of solutions were obtained near irregular singularities ([3], [4], [5], [7]). Furthermore, H. Kimura et al. ([2]) gave a reduction theorem for a class of Hamiltonian systems containing a Garnier system of PVI type around a regular singular locus.

In this paper, we give a three-parameter family of solutions of (G) near the singular locus $t = \infty$, by constructing a canonical transformation which reduces (G) to

$$
(\mathrm{G}_0) \qquad \begin{aligned}
&\frac{\partial Q_j}{\partial s} = \frac{\partial L_1}{\partial P_j}, \quad \frac{\partial P_j}{\partial s} = -\frac{\partial L_1}{\partial Q_j}, \\
&\frac{\partial Q_j}{\partial t} = \frac{\partial L_2}{\partial P_j}, \quad \frac{\partial P_j}{\partial t} = -\frac{\partial L_2}{\partial Q_j} \qquad (j = 1, 2)
\end{aligned}
$$

with the Hamiltonians

$$\begin{aligned} L_1 &:= \Lambda_1^{(1)}(t)Q_1P_1 + \Lambda_2^{(1)}(t)Q_2P_2, \\ L_2 &:= \Lambda_1^{(2)}(s,t)Q_1P_1 + \Lambda_2^{(2)}(s,t)Q_2P_2 \\ &\qquad + t^{-1}\big(\kappa_{20}(Q_1P_1)^2 + \kappa_{11}Q_1P_1Q_2P_2 + \kappa_{02}(Q_2P_2)^2\big). \end{aligned}$$

Here

$$\begin{aligned} &\Lambda_1^{(1)}(t) := -(4\sqrt{5})^{-1}i\rho^3t^{1/2}, \quad \Lambda_2^{(1)}(t) := (4\sqrt{5})^{-1}i\overline{\rho}^3t^{1/2}, \\ &\Lambda_1^{(2)}(s,t) := -\rho t^{1/6} - (8\sqrt{5})^{-1}i\rho^3st^{-1/2}, \\ &\Lambda_2^{(2)}(s,t) := -\overline{\rho}t^{1/6} + (8\sqrt{5})^{-1}i\overline{\rho}^3st^{-1/2} \end{aligned}$$

with

$$\begin{aligned} &\rho := ir_0e^{-i\omega}, \quad r_0 := 2^{3/4}15^{1/12}, \quad \omega := \frac{1}{2}\tan^{-1}(1/\sqrt{5}) \in (0,\pi/4), \\ &\kappa_{20} = (-7+2\sqrt{5}\,i)/24, \quad \kappa_{11} = 2\sqrt{30}/5, \quad \kappa_{02} = \overline{\kappa_{20}}\,. \end{aligned} \tag{1.2}$$

Furthermore, restricting solutions of (G_0) to the hyperplane $s = s_0$, we obtain asymptotic solutions of (PI_4) near $t = \infty$. In the final section, we sketch the process of construction of the formal canonical transformation. We employ the standard method for obtaining such transformations ([2], [5], [6]).

2. Results

To state our results, we explain the following notation:
(a) Consider the matrix

$$J_0 := \begin{pmatrix} 0 & 2\beta/3 & 0 & 2/3 \\ -6 & 0 & 18\beta & 0 \\ 0 & 2/3 & 0 & 0 \\ 18\beta & 0 & -9\beta^2 & 0 \end{pmatrix} \tag{2.1}$$

with

$$\beta := -15^{-1/3}. \tag{2.2}$$

The eigenvalues of J_0 are $\pm\rho$, $\pm\overline{\rho}$, and we have

$$T_0^{-1}J_0T_0 = \mathrm{diag}[-\rho,\ \rho,\ -\overline{\rho},\ \overline{\rho}],$$

where

$$(2.3)\qquad T_0 := (2\sqrt{5})^{-1/2} D_0 \Omega,$$

$$D_0 := \mathrm{diag}\Big[r_0^{-1/2}\beta^{1/2},\ \frac{\sqrt{6}}{2} r_0^{1/2}\beta^{-1/2},\ \frac{\sqrt{6}}{3} r_0^{-1/2}\beta^{-1/2},\ \frac{3}{2} r_0^{1/2}\beta^{1/2} \Big],$$

$$\Omega := \begin{pmatrix} e^{-3i\omega/2} & e^{-3i\omega/2} & e^{3i\omega/2} & e^{3i\omega/2} \\ -e^{-i\omega/2} & e^{-i\omega/2} & -e^{i\omega/2} & e^{i\omega/2} \\ -ie^{i\omega/2} & -ie^{i\omega/2} & ie^{-i\omega/2} & ie^{-i\omega/2} \\ -ie^{3i\omega/2} & ie^{3i\omega/2} & ie^{-3i\omega/2} & -ie^{-3i\omega/2} \end{pmatrix}.$$

(b) We fix the arguments of the eigenvalues of J_0 in such a way that

$$-\pi < \arg(-\rho) < -\frac{\pi}{2} < \arg\overline{\rho} < 0 < \arg\rho < \frac{\pi}{2} < \arg(-\overline{\rho}) < \pi,$$

where $\arg\overline{\rho} = -\arg\rho$, $\arg(-\overline{\rho}) = \pi - \arg\rho$, $\arg(-\rho) = -\pi + \arg\rho$. Let Σ_0 be the sector in the t-plane defined by

$$(2.4)\qquad \Sigma_0 :\quad -\arg\rho < \frac{7}{6}\arg t < \pi - \arg\rho.$$

(c) For an arbitrary sector Σ in the t-plane, and for a function $f(s,t)$ holomorphic for $(s,t) \in \mathbb{C}\times\Sigma$, we write

$$f(s,t) \in \mathcal{A}(\Sigma),$$

if, for any positive number R, the function $f(s,t)$ admits the asymptotic representation

$$f(s,t) \sim \sum_{\nu\geq 0} f_\nu(s) t^{-\nu/6}$$

uniformly for $|s| < R$ as $t\to\infty$ through Σ, where $f_\nu(s)$ is an entire function of s.

(d) For a vector $\mathbf{v} = (v_1, ..., v_m)$, we denote by ${}^T\mathbf{v}$ the transpose of it, and for a multi-index $\mathbf{k} = (k_1, ..., k_m) \in (\mathbb{N}\cup\{0\})^m$, we write

$$|\mathbf{k}| := k_1 + \cdots + k_m, \quad \mathbf{v}^{\mathbf{k}} := v_1^{k_1}\cdots v_m^{k_m}.$$

The formal canonical transformation is given by the following:

Theorem 2.1. *There exists a formal canonical transformation*

$$(2.5)\qquad {}^T(\mathbf{q},\mathbf{p}) = U(s,t,\mathbf{Q},\mathbf{P}),$$

$$\mathbf{q} = (q_1,q_2),\quad \mathbf{p} = (p_1,p_2),\quad \mathbf{Q} = (Q_1,Q_2),\quad \mathbf{P} = (P_1,P_2),$$

which reduces (G) *into* (G_0). *The right-hand side of* (2.5) *is a formal series given by*

$$U(s,t,\mathbf{Q},\mathbf{P}) = {}^T(\mathbf{u}(s,t),\mathbf{v}(s,t)) + t^{\Delta_0} T_0 \Big(\Gamma_0(s,t) + \sum_{|\mathbf{j}|+|\mathbf{k}|\geq 1} \Gamma_{\mathbf{jk}}(s,t)\mathbf{Q}^{\mathbf{j}}\mathbf{P}^{\mathbf{k}} \Big)\, {}^T(\mathbf{Q},\mathbf{P}).$$

Here

(i) $\mathbf{u} = (u_1, u_2)$ *and* $\mathbf{v} = (v_1, v_2)$ *satisfy*

$$t^{-2/3}u_1,\ t^{-1/3}u_2,\ t^{2/3}v_1,\ t^{1/3}v_2 \in \mathcal{A}(\Sigma_0);$$

(ii) Γ_0 *and* $\Gamma_{\mathbf{jk}}$ *are* 4 *by* 4 *matrices such that each entry of*

$$\Gamma_0 = I(s) + O(t^{-1/6}), \quad t^{(7/12)(|\mathbf{j}|+|\mathbf{k}|-2)}\Gamma_{\mathbf{jk}}$$

belongs to $\mathcal{A}(\Sigma_0)$, *where* $I(s)$ *is a diagonal matrix satisfying* $\det I(s) \neq 0$ *with entries entire in* s;

(iii) T_0 *is given by* (2.3) *and*

$$\Delta_0 := \mathrm{diag}[1/12,\ -1/4,\ -1/12,\ 1/4].$$

It is easy to see that system (G_0) admits a general solution

$$Q_1 = \Phi_1(C_1, C_2, s, t) := C_1 t^{2\kappa_{20}C_1C_2 + \kappa_{11}C_3C_4} \exp\Big(-\frac{6}{7}\rho t^{7/6} - \frac{i\rho^3 s t^{1/2}}{4\sqrt{5}}\Big),$$

$$P_1 = \Psi_1(C_1, C_2, s, t) := C_2 t^{-2\kappa_{20}C_1C_2 - \kappa_{11}C_3C_4} \exp\Big(\frac{6}{7}\rho t^{7/6} + \frac{i\rho^3 s t^{1/2}}{4\sqrt{5}}\Big),$$

$$Q_2 = \Phi_2(C_3, C_4, s, t) := C_3 t^{2\kappa_{02}C_3C_4 + \kappa_{11}C_1C_2} \exp\Big(-\frac{6}{7}\overline{\rho} t^{7/6} + \frac{i\overline{\rho}^3 s t^{1/2}}{4\sqrt{5}}\Big),$$

$$P_2 = \Psi_2(C_3, C_4, s, t) := C_4 t^{-2\kappa_{02}C_3C_4 - \kappa_{11}C_1C_2} \exp\Big(\frac{6}{7}\overline{\rho} t^{7/6} - \frac{i\overline{\rho}^3 s t^{1/2}}{4\sqrt{5}}\Big),$$

where C_j $(1 \le j \le 4)$ are integration constants. For generic values of C_j, there is no direction in t-plane along which Φ_1, Ψ_1, Φ_2, Ψ_2 are simultaneously bounded. In the sector Σ_0, however, there exists a curve tending to ∞ such that $\Phi_1(C_1, C_2, s, t)$, $\Psi_1(C_1, C_2, s, t)$, $\Phi_2(C_3, 0, s, t)$, and $\Psi_2(C_3, 0, s, t) \equiv 0$ are bounded. Indeed, along the ray $(7/6)\arg t = \pi/2 - \arg \rho$, we have $\arg(\overline{\rho} t^{7/6}) = \pi/2 - 2\arg \rho$, implying that $\mathrm{Re}\,(-\overline{\rho} t^{7/6}) < 0$. Substituting these into the formal transformation of Theorem 2.1, and rearranging the terms, we get a three-parameter family of solutions of (G_0). The asymptotic property is justified by using the method of successive approximation together with Borel-Ritt type reasoning (see [8]).

Theorem 2.2. *Let* R_0 *be an arbitrary positive number, and let* δ_0 *be an arbitrary small positive number. Then system* (G) *admits a family of solutions*:

$$\begin{aligned} &{}^{T}(\mathbf{q}, \mathbf{p}) = {}^{T}(\mathbf{u}(s,t), \mathbf{v}(s,t)) \\ &+ t^{\Delta_0} T_0\, \Xi(s, t, t^{-1/12}\Phi_1(C_1, C_2, s, t), t^{-1/12}\Psi_1(C_1, C_2, s, t), t^{-1/12}\Phi_2(C_3, 0, s, t)), \end{aligned}$$

$$(C_1, C_2, C_3) \in \mathbb{C}^3, \quad |C_1 C_2| < \delta_1$$

for (s,t) *in the domain given by*

(2.6) $$\big|(7/6)\arg t-(\pi/2-\arg\rho)\big|<\delta_0,$$

(2.7)
$$|t^{-1/12}\Phi_1(C_1,C_2,s,t)|+|t^{-1/12}\Psi_1(C_1,C_2,s,t)|+|t^{-1/12}\Phi_2(C_3,0,s,t)|<R_0,$$
$$|s|<R_0,$$

where δ_1 *is a sufficiently small positive number. The vector function* Ξ *is written in the form*

$$\Xi(s,t,X_1,Y_1,X_2)=t^{1/12}\big[\Gamma_0(s,t)+F(s,t,X_1,Y_1,X_2)\big]\,{}^T(X_1,Y_1,X_2,0),$$

where the vector function F *admits the asymptotic representation*

$$F(s,t,X_1,Y_1,X_2)\sim\sum_{\nu\ge1}\Gamma_\nu(s,X_1,Y_1,X_2)t^{-\nu/6}$$

as $t\to\infty$ *through the sector* (2.6) *uniformly for* (s,X_1,Y_1,X_2) *satisfying*

$$|s|<R_0,\quad |t|^{1/6}|X_1Y_1|<\delta_1,\quad |X_1|+|Y_1|+|X_2|<R_0.$$

The coefficients Γ_ν *are vector functions whose entries are polynomials in* X_1, Y_1, X_2 *with coefficients entire in* s.

Let us denote by $q_2=\chi(C_1,C_2,C_3,s,t)$ the second entry of each solution given above. For (PI_4) with $s_0\in\mathbb{C}$, we immediately obtain the following:

Corollary 2.3. *Equation* (PI_4) *admits a family of solutions*:

$$q=\chi(C_1,C_2,C_3,s_0,t),\quad (C_1,C_2,C_3)\in\mathbb{C}^3,\quad |C_1C_2|<\tilde{\delta}_1$$

for t *in the domain given by* (2.6) *and* (2.7) *with* $s=s_0$, *where* $\tilde{\delta}_1$ *is a sufficiently small positive number depending on* δ_0 *and* s_0.

Remark. In the sectors

$$\Sigma_1:\quad -\pi-\arg\rho<\frac{7}{6}\arg t<-\arg\rho,$$
$$\Sigma_2:\quad -\pi+\arg\rho<\frac{7}{6}\arg t<\arg\rho,$$
$$\Sigma_3:\quad \arg\rho<\frac{7}{6}\arg t<\pi+\arg\rho,$$

we can construct analogous formal canonical transformations, which yield asymptotic solutions corresponding to the triples

(i) $\Phi_1(C_1,C_2,s,t),\quad \Psi_1(C_1,C_2,s,t),\quad \Psi_2(0,C_4,s,t),$

(ii) $\Phi_1(C_1,0,s,t),\quad \Phi_2(C_3,C_4,s,t),\quad \Psi_2(C_3,C_4,s,t),$

(iii) $\Psi_1(0,C_2,s,t),\quad \Phi_2(C_3,C_4,s,t),\quad \Psi_2(C_3,C_4,s,t),$

respectively.

3. Construction of the formal transformation

The construction of (2.5) is divided into several steps, and it is obtained by composing the transformations given in these steps. For the simplicity of description, in every step, we use the following common notation: we denote initial Hamiltonians and variables by H_k and q_j, p_j $(j, k = 1, 2)$, namely an initial Hamiltonian system by

$$\frac{\partial q_j}{\partial s} = \frac{\partial H_1}{\partial p_j}, \quad \frac{\partial p_j}{\partial s} = -\frac{\partial H_1}{\partial q_j},$$
$$\frac{\partial q_j}{\partial t} = \frac{\partial H_2}{\partial p_j}, \quad \frac{\partial p_j}{\partial t} = -\frac{\partial H_2}{\partial q_j} \quad (j = 1, 2);$$

a canonical transformation by

$$(q_j, p_j) \mapsto (Q_j, P_j); \tag{3.1}$$

and the resultant Hamiltonians by K_k $(k = 1, 2)$ with variables Q_j, P_j. Note that transformation (3.1) is canonical, if

$$\sum_j dp_j \wedge dq_j = \sum_j dP_j \wedge dQ_j. \tag{3.2}$$

Then, K_k are computed by using the identity

$$\begin{aligned} \sum_j dp_j \wedge dq_j - dH_1 \wedge ds - dH_2 \wedge dt \\ = \sum_j dP_j \wedge dQ_j - dK_1 \wedge ds - dK_2 \wedge dt. \end{aligned} \tag{3.3}$$

In each step, our computation is concentrated on H_2. The corresponding expression of H_1 is derived by using the completely integrable condition.

3.1. **Step 1.** To eliminate the term $-3tq_2$ of $3H_2$, we put

$$q_1 = Q_1 + \alpha t^{2/3}, \qquad q_2 = Q_2 + \beta t^{1/3} \qquad (\alpha,\, \beta \in \mathbb{C}).$$

It is easy to see that this is a canonical transformation. Substitution into H_2 for (G) yields

$$\begin{aligned} 3\widetilde{H}_2 := {} & Q_2 P_1^2 + \beta t^{1/3} P_1^2 + 2P_1P_2 \\ & + 9\Big(Q_2^4 + 4\beta t^{1/3} Q_2^3 - 3Q_1Q_2^2 + (6\beta^2 - 3\alpha)t^{2/3}Q_2^2 - 6\beta t^{1/3} Q_1 Q_2 + Q_1^2 \\ & \qquad + (-3\beta^2 + 2\alpha)t^{2/3} Q_1 - \frac{s}{3} Q_1 + (4\beta^3 - 6\alpha\beta - \frac{1}{3}) t Q_2\Big). \end{aligned}$$

Choosing $\beta = -15^{-1/3}$, $\alpha = 3\beta^2/2$ (cf. (2.2)), and using (3.3), we have the Hamiltonian of the resultant system:

$$
\begin{aligned}
(3.4)\quad 3K_2 &= 3\widetilde{H}_2 - 2\alpha t^{-1/3}P_1 - \beta t^{-2/3}P_2 \\
&= Q_2P_1^2 + \beta t^{1/3}P_1^2 + 2P_1P_2 \\
&\quad + 9\Big(Q_2^4 + 4\beta t^{1/3}Q_2^3 - 3Q_1Q_2^2 + \alpha t^{2/3}Q_2^2 - 6\beta t^{1/3}Q_1Q_2 + Q_1^2\Big) \\
&\qquad - 3sQ_1 - 2\alpha t^{-1/3}P_1 - \beta t^{-2/3}P_2.
\end{aligned}
$$

3.2. **Step 2.** To make the quadratic part non-degenerate, we apply the sharing transformation

$$q_1 = t^{1/12}Q_1, \quad p_1 = t^{-1/12}P_1, \quad q_2 = t^{-1/4}Q_2, \quad p_2 = t^{1/4}P_2,$$

which is canonical. Then by (3.4) and (3.3), we have

$$
\begin{aligned}
(3.5)\quad 3K_2 &= 9t^{-1}Q_2^4 + t^{-5/12}\big(Q_2P_1^2 + 36\beta Q_2^3 - 27Q_1Q_2^2\big) \\
&\quad + t^{1/6}\Big(\beta P_1^2 + 2P_1P_2 + 9Q_1^2 - 54\beta Q_1Q_2 + \frac{27}{2}\beta^2Q_2^2\Big) \\
&\quad + t^{-1}\Big(-\frac{1}{4}Q_1P_1 + \frac{3}{4}Q_2P_2\Big) - 3st^{1/12}Q_1 - t^{-5/12}(3\beta^2P_1 + \beta P_2).
\end{aligned}
$$

3.3. **Step 3.** Observe that the quadratic part of (3.5)

$$t^{1/6}\big(\beta p_1^2 + 2p_1p_2 + 9q_1^2 - 54\beta q_1q_2 + \frac{27}{2}\beta^2q_2^2\big)$$

corresponds to the matrix J_0 (cf. (2.1)), which is a coefficient of the linear part of the Hamiltonian system for H_2. The matrix T_0 satisfying $T_0^{-1}J_0T_0 = \mathrm{diag}[-\rho, \rho, -\overline{\rho}, \overline{\rho}]$ is chosen so that the transformation

$${}^T(q_1, p_1, q_2, p_2) = T_0\,{}^T(Q_1, P_1, Q_2, P_2)$$

is canonical, namely that it satisfies (3.2). Then we have

$$
\begin{aligned}
3K_2 &= \frac{6}{5}r_0^{-2}\beta^{-2}t^{-1}K_2^{(4)} + \frac{\sqrt{6}r_0^{-3/2}\beta^{-1/2}}{(2\sqrt{5})^{1/2}}t^{-5/12}K_2^{(3)} + 3t^{1/6}K_2^{(2,1)} \\
&\quad + \frac{\sqrt{3}}{8\sqrt{10}}t^{-1}K_2^{(2,2)} - \frac{3r_0^{-1/2}\beta^{1/2}}{(2\sqrt{5})^{1/2}}st^{1/12}K_2^{(1,1)} - r_0^{1/2}\beta^{3/2}t^{-5/12}K_2^{(1,2)},
\end{aligned}
$$

where $K_2^{(4)}, \ldots$ are homogeneous polynomials in $Q_1,\ P_1,\ Q_2,\ P_2$ given by

$$\begin{aligned}
K_2^{(4)} &:= e^{2i\omega}Q_1^2P_1^2 + e^{-2i\omega}Q_2^2P_2^2 + 4Q_1Q_2P_1P_2 + \cdots,\\
K_2^{(3)} &:= (4+\sqrt{5}\,i)e^{3i\omega/2}(Q_1+P_1)Q_1P_1 + (4-\sqrt{5}\,i)e^{-3i\omega/2}(Q_2+P_2)Q_2P_2\\
&\qquad + 2\sqrt{6}\,ie^{3i\omega/2}(Q_1+P_1)Q_2P_2 - 2\sqrt{6}\,ie^{-3i\omega/2}(Q_2+P_2)Q_1P_1 + \cdots,\\
K_2^{(2,1)} &:= -\rho Q_1P_1 - \overline{\rho}Q_2P_2,\\
K_2^{(2,2)} &:= -e^{-2i\omega}(Q_1^2 - P_1^2) - \cdots,\\
K_2^{(1,1)} &:= e^{-3i\omega/2}(Q_1+P_1) + e^{3i\omega/2}(Q_2+P_2),\\
K_2^{(1,2)} &:= a_1(Q_1-P_1) + a_2(Q_2-P_2) \qquad (a_1, a_2 \in \mathbb{C}).
\end{aligned}$$

3.4. **Step 4.** We would like to eliminate the linear parts of the Hamiltonians H_j $(j=1,2)$. Using a classical result for nonlinear equations (see e.g. [8]), we can choose the canonical transformation

$$q_1 = Q_1 + u_1, \quad q_2 = Q_2 + u_2, \quad p_1 = P_1 + v_1, \quad p_2 = P_2 + v_2$$

(for $u_j,\ v_j$ cf. Theorem 2.1) such that $t^{1/12}u_j,\ t^{1/12}v_j \in \mathcal{A}(\Sigma)$, where Σ is some sector satisfying $\Sigma \supset \overline{\Sigma_0}$ and $|\Sigma| > \pi$ ($|\Sigma|$ denotes the opening of Σ). By this transformation, we have

$$K_2 = \sum_{2\le|\mathbf{j}|+|\mathbf{k}|\le 4} t^{1/6-(7/12)(|\mathbf{j}|+|\mathbf{k}|-2)} h_{\mathbf{jk}}(s,t)\mathbf{Q}^{\mathbf{j}}\mathbf{P}^{\mathbf{k}}$$

with $h_{\mathbf{jk}}(s,t) \in \mathcal{A}(\Sigma)$; in particular, for $|\mathbf{j}|+|\mathbf{k}|=2$,

$$\begin{aligned}
t^{1/6}h_{(1,0,1,0)}(s,t) &= \Lambda_1^{(2)}(s,t) + O(t^{-7/6}),\\
t^{1/6}h_{(0,1,0,1)}(s,t) &= \Lambda_2^{(2)}(s,t) + O(t^{-7/6}),\\
t^{1/6}h_{\mathbf{jk}}(s,t) &= O(t^{-1/2}) \qquad \text{(otherwise)}.
\end{aligned}$$

By a further linear canonical transformation, we have

$$h_{\mathbf{jk}}(s,t) \equiv 0 \quad \text{for } |\mathbf{j}|+|\mathbf{k}|=2,\ (\mathbf{j},\mathbf{k}) \ne (1,0,1,0),\ (0,1,0,1).$$

Using the completely integrable condition and the fact $|\Sigma| > \pi$, we can check that the linear terms of H_1 are simultaneously eliminated by the transformation above. For example,

$$\begin{aligned}
K_1 = \big(\Lambda_1^{(1)}(t) + f_1(s) + O(t^{-1/6})\big)Q_1P_1&\\
+ \big(\Lambda_2^{(1)}(t) + f_2(s) + O(t^{-1/6})\big)Q_2P_2 + \cdots&
\end{aligned}$$

with some polynomials $f_j(s)$ $(j=1,2)$.

3.5. **Step 5.** We eliminate higher order terms in H_2 for $\mathbf{j} \neq \mathbf{k}$. Suppose that $h_{\mathbf{jk}}(s,t) \equiv 0$ for $\mathbf{j}$, $\mathbf{k}$ satisfying $\mathbf{j} \neq \mathbf{k}$ and $|\mathbf{j}| + |\mathbf{k}| \le \iota_0 - 1$ ($\iota_0 \ge 3$). Put

$$W = Q_1 p_1 + Q_2 p_2 + \sum_{\substack{|\mathbf{j}|+|\mathbf{k}|=\iota_0 \\ \mathbf{j}\neq\mathbf{k}}} f_{\mathbf{jk}}(s,t)\mathbf{Q}^{\mathbf{j}}\mathbf{p}^{\mathbf{k}}.$$

Then

$$q_1 = W_{p_1}, \quad q_2 = W_{p_2}, \quad P_1 = W_{Q_1}, \quad P_2 = W_{Q_2}$$

is a canonical transformation, and

$$K_2 = H_2 - W_t = \cdots + \sum_{\iota_0} \big((k_1 - j_1)\Lambda_1^{(2)} + (k_2 - j_2)\Lambda_2^{(2)} \big) f_{\mathbf{jk}} \mathbf{Q}^{\mathbf{j}}\mathbf{p}^{\mathbf{k}}$$

$$+ \sum_{\iota_0} h_{\mathbf{jk}} \mathbf{Q}^{\mathbf{j}}\mathbf{p}^{\mathbf{k}} - \sum_{\iota_0} (\partial f_{\mathbf{jk}}/\partial t) \mathbf{Q}^{\mathbf{j}}\mathbf{p}^{\mathbf{k}} + \cdots .$$

Choose $f_{\mathbf{jk}}(s,t)$ so that

$$\partial f_{\mathbf{jk}}/\partial t = h_{\mathbf{jk}} + \big((k_1 - j_1)\Lambda_1^{(2)} + (k_2 - j_2)\Lambda_2^{(2)} \big) f_{\mathbf{jk}}.$$

Since $\mathbf{j} \neq \mathbf{k}$, there exists $f_{\mathbf{jk}}$ such that

$$t^{(7/12)(\iota_0 - 2)} f_{\mathbf{jk}}, \ t^{-1/6 + (7/12)(\iota_0 - 2)} (\partial f_{\mathbf{jk}}/\partial t) \in \mathcal{A}(\Sigma_{\mathbf{jk}}),$$

for some sector $\Sigma_{\mathbf{jk}}$, $|\Sigma_{\mathbf{jk}}| > \pi$. Thus we get the canonical transformation

$${}^T(\mathbf{q}, \mathbf{p}) = {}^T(\mathbf{Q}, \mathbf{P}) + \sum_{|\mathbf{j}|+|\mathbf{k}| \ge \iota_0 - 1} \varphi_{\mathbf{jk}}(s,t) \mathbf{Q}^{\mathbf{j}}\mathbf{P}^{\mathbf{k}},$$

$$\varphi_{\mathbf{jk}} = O(t^{-(7/12)(|\mathbf{j}|+|\mathbf{k}|-1)})$$

such that the coefficients of the terms in K_2 for $|\mathbf{j}| + |\mathbf{k}| = \iota_0$, $\mathbf{j} \neq \mathbf{k}$ vanish. Applying the procedure above, we inductively obtain the required transformation.

3.6. **Step 6.** By a transformation of the form

$${}^T(\mathbf{q}, \mathbf{p}) = \begin{pmatrix} Q_1 \exp\left(S_x(Q_1 P_1,\, Q_2 P_2)\right) \\ Q_2 \exp\left(S_y(Q_1 P_1,\, Q_2 P_2)\right) \\ P_1 \exp\left(-S_x(Q_1 P_1,\, Q_2 P_2)\right) \\ P_2 \exp\left(-S_y(Q_1 P_1,\, Q_2 P_2)\right) \end{pmatrix}$$

($S_x = \partial S/\partial x,\ S_y = \partial S/\partial y$) with

$$S(x,y) = \sum_{|\mathbf{j}| \ge 1} \psi_{\mathbf{j}}(s,t) x^{j_1} y^{j_2}, \quad \mathbf{j} = (j_1, j_2),$$

$$t^{d(\mathbf{j})} \psi_{\mathbf{j}} \in \mathcal{A}(\Sigma_0),$$

$$d(\mathbf{j}) := \begin{cases} 1/6 & |\mathbf{j}| = 1, \\ 2/3 & |\mathbf{j}| = 2, \\ (7/6)(|\mathbf{j}| - 2) & |\mathbf{j}| \ge 3, \end{cases}$$

we get the reduced system (G_0).

Composing the transformations given above, we obtain the formal power series in $\mathbf{Q}$, $\mathbf{P}$ given in Theorem 2.1, whose coefficients are functions expressible by asymptotic series in t uniformly valid for s.

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