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On the fractional parts of powers of algebraic numbers

By

Hajime Kaneko*

Abstract

We study the fractional parts $\{\xi\alpha^n\}$ of the geometrical progressions $\xi\alpha^n$ $(n = 0, 1, \ldots)$, where α is an algebraic number greater than 1 and ξ is a positive real number. We also consider the distance $\|\xi\alpha^n\|$ from the number $\xi\alpha^n$ to the nearest integer. The main purpose of this paper is to estimate the maximal and minimal limit points of the sequences $\{\xi\alpha^n\}$ $(n = 0, 1, \ldots)$ and $\|\xi\alpha^n\|$ $(n = 0, 1, \ldots)$.

§ 1. Introduction

In this paper we study the fractional parts of geometric progressions. Let $\alpha > 1$ be a common ratio. Then Koksma [8] proved for almost all real numbers ξ that the sequence $\xi\alpha^n$ $(n = 0, 1, \ldots)$ is uniformly distributed modulo 1. Moreover, let ξ be any positive initial value. Then Koksma also showed that for almost all α greater than 1 the sequence $\xi\alpha^n$ is uniformly distributed modulo 1.

On the other hand, it is generally difficult to show that given geometric progressions are uniformly distributed. In fact, we know little on the fractional parts of given progressions. For instance, we can not disprove that

$$\lim\{e^n\} = 0,$$

where $\{x\}$ is the fractional part of a real number x. In the case where α is a transcendental number, it is generally difficult to prove that the sequence $\{\alpha^n\}$ $(n = 0, 1, \ldots)$ has two distinct limit points.

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*Department of Mathematics College of Science and Technology, Nihon University

In the case where α is an algebraic number, there is a criterion to decide whether the sequence $\{\xi\alpha^n\}$ $(n = 0, 1, \ldots)$ has infinitely many limit points. We recall the definition of Pisot and Salem numbers. Pisot numbers are algebraic integers greater than 1 whose conjugates except themselves have absolute values less than 1. Note that all integers greater than 1 are Pisot numbers. Salem numbers are also algebraic integers greater than 1 satisfying the following: the conjugates except themselves have absolute values less than or equal to 1; there is at least one conjugate with absolute value 1.

Let α be an algebraic number greater than 1 and let ξ be a positive real number. Pisot [9] proved that the sequence $\{\xi\alpha^n\}$ $(n = 0, 1, \ldots)$ has only finitely many limit points if and only if α is a Pisot number and $\xi \in \mathbb{Q}(\alpha)$. Dubickas [5] gave another proof of the result of Pisot. However, the limit points of the fractional parts of geometric progressions are mysterious. For instance, by the result of Pisot, the sequence $\{(3/2)^n\}$ $(n = 0, 1, \ldots)$ has infinitely many limit points. However, there is no real number proven to be the limit point of such a sequence. In what follows, we study ranges of limit points of the fractional parts of geometric progressions, estimating the maximal and minimal limit points. Namely, put

$$F(\xi, \alpha) := \limsup_{n\to\infty}\{\xi\alpha^n\},\ f(\xi, \alpha) := \liminf_{n\to\infty}\{\xi\alpha^n\}.$$

In Section 2, we consider the values $F(\xi, \alpha)$ and $f(\xi, \alpha)$ for fixed algebraic numbers α greater than 1.

Moreover, we also consider the distance $\|x\|$ from a real number to the nearest integer. Note that

$$\|x\| = \min\{\{x\}, \{1 - x\}\}.$$

Pisot and Salem numbers are characterized by using the function $\|\cdot\|$. Namely, let α be a Pisot number. Then we have

$$\lim_{n\to\infty} \|\alpha^n\| = 0.$$

In fact, let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ be the conjugates of α, where $|\alpha_i| < 1$ for any $2 \le i \le d$. Since, for any nonnegative integer n, the trace $\alpha_1^n + \cdots + \alpha_d^n$ is a rational integer, we get

$$\lim_{n\to\infty} \|\alpha\| = \|\alpha_2^n + \cdots + \alpha_d^n\| = 0.$$

Conversely, let α be an algebraic number greater than 1 such that there exists a positive ξ satisfying

$$\lim_{n\to\infty} \|\xi\alpha^n\| = 0.$$

Then Pisot's result above implies that α is a Pisot number and that $\xi \in \mathbb{Q}(\alpha)$.

Let α be a Pisot or Salem number. Then for an arbitrary positive real number ε, there exists a positive $\xi = \xi(\alpha, \varepsilon)$ depending on α and ε such that

$$\limsup_{n\to\infty} \|\xi\alpha^n\| \leq \varepsilon.$$

Namely, we have

$$\inf_{\xi>0} \limsup_{n\to\infty} \|\xi\alpha^n\| = 0. \tag{1.1}$$

Conversely, Dubickas showed for an algebraic number greater than 1 that if (1.1) holds, then α is a Pisot or Salem number. For more details on Pisot and Salem numbers, see [1].

If α is neither a Pisot nor Salem number, then the value

$$\inf_{\xi>0} \limsup_{n\to\infty} \|\xi\alpha^n\| (> 0)$$

is not known. Let

$$D(\xi, \alpha) := \limsup_{n\to\infty} \|\xi\alpha^n\|,\ d(\xi, \alpha) := \liminf_{n\to\infty} \|\xi\alpha^n\|.$$

In Sections 3 and 4, we study the values $D(\xi, \alpha)$ and $d(\xi, \alpha)$ for fixed algebraic numbers α.

§ 2. Limit points of the fractional parts of powers of algebraic numbers

Let α be an algebraic number greater than 1. Then the Koksma's results mentioned in Section 1 implies that

$$F(\xi, \alpha) = 1,\ f(\xi, \alpha) = 0$$

for almost all positive real numbers ξ. On the other hand, let α be a real number greater than 2. Tijdeman [10] proved the following: there exists a positive number $\xi_0 = \xi_0(\alpha)$ depending on α such that

$$F(\xi_0, \alpha) \leq \frac{1}{\alpha - 1}. \tag{2.1}$$

In this section we consider the length of the shortest interval including the limit points of the sequences $\{\xi\alpha^n\}$ $(n = 0, 1, \ldots)$. Namely, we estimate the value

$$F(\xi, \alpha) - f(\xi, \alpha)$$

in the case where α is an algebraic number greater than 1.

We give some notation for algebraic numbers. Let $P_\alpha(X) = a_d X^d + a_{d-1}X^{d-1} + \cdots + a_0 \in \mathbb{Z}[X]$ be the minimal polynomial of an algebraic number α. The length of $P_\alpha(X)$ is given by

$$L(P_\alpha(X)) = \sum_{i=0}^{d} |a_i|.$$

Moreover, let

$$L_+(P_\alpha(X)) = \sum_{i=0}^{d} \max\{0, a_i\},\ L_-(P_\alpha(X)) = \sum_{i=0}^{d} \max\{0, -a_i\}.$$

We introduce the reduced length defined in [3]. Put

$$l'(\alpha) := \inf\{L(B(X)P_\alpha(X)) \mid B(X) \in \mathbb{R}[X],\ B(X) \text{ is monic}\}.$$

Then the reduced length of α is

$$l(\alpha) = \min\{l'(\alpha), l'(\alpha^{-1})\}. \tag{2.2}$$

Let again α be an algebraic number greater than 1 and ξ a positive number. Suppose that $\xi \notin \mathbb{Q}(\alpha)$ in the case where α is a Pisot or Salem number. Then Dubickas [3] showed that

$$\left| \sum_{i=0}^{d} a_{d-i}\{\xi\alpha^{n-i}\} \right| \geq 1$$

for infinitely many positive integer n. In particular, we have

$$F(\xi, \alpha) \geq \min\left\{ \frac{1}{L_+(P_\alpha(X))}, \frac{1}{L_-(P_\alpha(X))} \right\}. \tag{2.3}$$

The author [7] improved (2.3) for certain classes of algebraic numbers α.

Dubickas [3] also proved that

$$F(\xi, \alpha) - f(\xi, \alpha) \geq \frac{1}{l(\alpha)}. \tag{2.4}$$

In the case where $\alpha = b$ is a rational integer, (2.4) implies that

$$F(\xi, b) - f(\xi, b) \geq \frac{1}{b}.$$

In the rest of this section we introduce the results by Bugeaud and Dubickas [2] on irrational numbers ξ satisfying

$$F(\xi, b) - f(\xi, b) = \frac{1}{b}.$$

For any bounded sequence of integers $\mathbf{w} = (w_n)_{n=1}^{\infty}$, put

$$(\mathbf{w})_b := \sum_{n=1}^{\infty} \frac{w_n}{b^n}.$$

Let ξ be an irrational number. Then the sequence $\{\xi b^n\}$ $(n = 0, 1, \ldots)$ lies in a closed interval of length $1/b$ if and only if

$$\xi = g + \frac{k}{b-1} + (\mathbf{w})_b,$$

where g, k are integers with $0 \le k \le b-2$ and $\mathbf{w}$ is a Sturmian word on the alphabet $\{0, 1\}$. In this case, ξ is a transcendental number.

§ 3. The distances between powers of algebraic numbers and those nearest integers

Let α be an algebraic number greater than 1 and ξ a positive real number. Let $P(X) = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_0 \in \mathbb{Z}[X]$ be the minimal polynomial of α. We use the same notation on the length and reduced length as that in Section 2. In this section we study lower bounds of $D(\xi, \alpha)$ for such α and ξ. If α is a Pisot or Salem number, then we assume that $\xi \notin \mathbb{Q}(\alpha)$. Then Dubickas [4] showed that

$$D(\xi, \alpha) \ge \max\left\{ \frac{1}{L(P_\alpha(X))}, \frac{1}{2l(\alpha)} \right\}. \tag{3.1}$$

In the same paper, he also improved the inequality above in the case where α is a rational number, using a fixed point a substitution. Let $\tau : \{1, 2\}^* \to \{1, 2\}$ be the substitution defined by

$$\tau(1) = 2,\ \tau(2) = 211.$$

Then τ has a unique fixed point

$$\lim_{m\to\infty} \tau^m(2) = 21122211211\ldots =: t_1 t_2 \ldots.$$

Let λ be an empty word. Define the sequence $\underline{\gamma} = (\gamma_m)_{m=1}^{\infty}$ by

$$\gamma_m = \begin{cases} \lambda & \text{if } t_m = 1, \\ 0 & \text{if } t_m = 2. \end{cases}$$

Moreover, the right infinite word $e_0 e_1 e_2 \ldots$ is given by

$$\begin{aligned} e_0 e_1 e_2 \ldots &= 1\gamma_1 \overline{1} \gamma_2 1 \gamma_3 \overline{1} \gamma_4 1 \gamma_5 \overline{1} \ldots \\ &= 10\overline{1}1\overline{1}010\overline{1}\ldots, \end{aligned}$$

where $\overline{1}$ denotes -1. Let m, r be integers with $m \geq 0, r \geq 1$. and $X_1, \ldots, X_r$ indeterminants. Put

$$\rho_m(X_1, \ldots, X_r) := \sum_{\substack{i_1, \ldots, i_r \geq 0 \\ i_1 + \cdots + i_r = m}} X_1^{i_1} \cdots X_r^{i_r}$$

and

$$E_r(X_1, \ldots, X_r) = \sum_{i=0}^{\infty} \rho_i(X_1, \ldots, X_r) e_i.$$

In the case of $r = 1$, Dubickas [4] showed that

$$E_1(X) = \frac{1 - (1 - X) \prod_{i=0}^{\infty} (1 - X^{2^i})}{2X}.$$

The infinite product

$$\Psi(X) = \prod_{i=0}^{\infty} (1 - X^{2^i})$$

is called a Mahler function because it satisfies the functional equation

$$\Psi(X^2) = \frac{\Psi(X)}{1 - X}.$$

In the case of $r \geq 2$, the author [6] proved that

$$E_r(X_1, \ldots, X_r) = \sum_{i=1}^{r} \left(\prod_{\substack{1 \leq j \leq r \\ j \neq i}} \frac{1}{X_i - X_j} \right) X_i^{r-1} E_1(X_i).$$

In particular, the power series $E_d(X_1, \ldots, X_d)$ is represented by the Mahler function $\Psi(X)$.

Now we assume that $\alpha = p/q$ is a rational number, where p and q are relatively coprime integers with $p > q > 0$. In the case where α is an integer, then suppose that ξ is a irrational number. Then Dubickas improved (3.1) as follows:

$$D\left(\xi, \frac{p}{q}\right) \geq \frac{1}{p} E_1\left(\frac{q}{p}\right). \tag{3.2}$$

(3.2) is stronger than (3.1). In fact, (3.1) implies that

$$D\left(\xi, \frac{p}{q}\right) \geq \frac{1}{p+q}.$$

We have

$$\frac{1}{p}E_1\left(\frac{q}{p}\right)-\frac{1}{p+q} = \frac{p-q}{2q(p+q)}\left(1-\left(1-\frac{q^2}{p^2}\prod_{i=1}^{\infty}\left(1-\frac{q^{2^i}}{p^{2^i}}\right)\right)\right)>0.$$

In the rest of this section we assume that α is quadratic irrational number for simplicity. Let α_2 be the conjugate of α and $a_2X^2+a_1X+a_0\in\mathbb{Z}[X]$ be the minimal polynomial of α, where $a_2>0$ and $\gcd(a_2,a_1,a_0)=1$. In the case of $|\alpha_2|>1$, the author [6] improved (3.1), using the function $E_2(X_1,X_2)$.

First we consider the case of $\alpha_2>1$. If

$$\alpha^{-1}+\alpha_2^{-1}\le\frac{\sqrt{5}-1}{2}, \tag{3.3}$$

then we have

$$D(\xi,\alpha)\ge\frac{1}{a_0}E_2(\alpha^{-1},\alpha_2^{-1}). \tag{3.4}$$

Under the conditions (3.3) and $\alpha_2>1$, we have

$$\frac{1}{a_0}E_2(\alpha^{-1},\alpha_2^{-1})>\max\left\{\frac{1}{L(P_\alpha(X))},\frac{1}{2l(\alpha)}\right\}.$$

So (3.4) is stronger than (3.1).

Next, assume that $\alpha_2<-1$. Define the number ζ by

$$\zeta=\begin{cases}1 & \text{if } \alpha<|\alpha_2|,\\ -1 & \text{if } \alpha\ge|\alpha_2|.\end{cases}$$

Suppose for $i=0,1$ that

$$0<\rho_{m+1}(\zeta\alpha^{-1},\zeta\alpha_2^{-1})\le\frac{1}{2}\rho_m(\zeta\alpha^{-1},\zeta\alpha_2^{-1}). \tag{3.5}$$

Then

$$D(\xi,\alpha)\ge\frac{1}{|a_0|}E_2(\zeta\alpha^{-1},\zeta\alpha_2^{-1}). \tag{3.6}$$

Under the conditions (3.5) and $\alpha_2<-1$, we get

$$\frac{1}{|a_0|}E_2(\zeta\alpha^{-1},\zeta\alpha_2^{-1})>\max\left\{\frac{1}{L(P_\alpha(X))},\frac{1}{2l(\alpha)}\right\}.$$

Thus, (3.6) is stronger than (3.1). In the case where α is an algebraic number with an arbitrary degree, see [6].

We give a numerical example. Let $\alpha = 4 + \sqrt{2}$. Then we have $\alpha_2 = 4 - \sqrt{2}$. (3.1) implies that

$$D(\xi, 4 + \sqrt{2}) \geq 0.0434\ldots$$

for any positive number ξ. On the other hand, since α and α_2 satisfies (3.3), the lower bound (3.4) means that

$$D(\xi, 4 + \sqrt{2}) \geq 0.0581\ldots$$

for all positive numbers ξ.

In the same way as the proof of (2.1), we can prove following: Let α be any real number α greater than 2. Then there exists a positive real number $\xi_1 = \xi_1(\alpha)$ depending on α such that

$$D(\xi_1, \alpha) \leq \frac{1}{2\alpha - 2}.$$

Thus, there exists a positive ξ_1 such that

$$0.0581\ldots \leq D(\xi_1, 4 + \sqrt{2}) \leq 0.113\ldots.$$

§ 4. Main results

In this section we show that if b is an even positive integer, then the value

$$\inf_{\xi > 0, \xi \notin \mathbb{Q}} (D(\xi, b) - d(\xi, b))$$

is described by using certain kinds of substitutions. We introduce some notation. Recall that, for any bounded sequence of integers $\mathbf{w} = (w_n)_{n=1}^{\infty}$,

$$(\mathbf{w})_b = (w_1 w_2 \ldots)_b = \sum_{n=1}^{\infty} \frac{w_n}{b^n}.$$

and that the substitution $\tau : \{1, 2\}^* \to \{1, 2\}$ is defined by $\tau(1) = 2$ and $\tau(2) = 211$. Moreover, let $\kappa : \{1, 2\}^* \to \{1, 2\}$ be the substitution defined by

$$\kappa(1) = 1,\ \kappa(2) = 21.$$

For any (finite or infinite) word $\mathbf{x} = (x_n)_{n=1}^{R}$ on the alphabet $\{1, 2\}$, where $1 \leq R \leq \infty$, we define the word $\underline{\mu}(\mathbf{x}) = (\mu_n(\mathbf{x}))_{n=1}^{R+1}$ as follows: Put $\mu_1(\mathbf{x}) := 1$ and

$$\mu_{n+1}(\mathbf{x}) := \begin{cases} -\mu_n(\mathbf{x}) \text{ (if } x_n = 1), \\ \ \ \mu_n(\mathbf{x}) \ \text{ (if } x_n = 2), \end{cases}$$

for $1 \leq n \leq R$. Recall that $\tau^m(2)$ converges to the right-infinity sequence $\mathbf{t} = (t_n)_{n=1}^{\infty}$. Put

$$\mathbf{y}_1 := \kappa(\mathbf{t}),\, \mathbf{y}_2 := 1\mathbf{y}_1$$

and

$$\mathbf{w}_1 := \underline{\mu}(\mathbf{y}_1),\, \mathbf{w}_2 := \underline{\mu}(\mathbf{y}_2)$$

Moreover, let

$$L_b := \frac{1}{2}(\mathbf{w}_1)_b - \frac{1}{2}(\mathbf{w}_2)_b.$$

Now we state the main results.

Theorem 4.1. *Let b be an even integer greater than 2.*
(1) *For any irrational number ξ,*

$$D(\xi, b) - d(\xi, b) \geq L_b.$$

(2) *There exists a irrational number ξ_0 satisfying*

$$D(\xi_0, b) - d(\xi_0, b) = L_b. \tag{4.1}$$

(3) *Suppose that an irrational number ξ_0 satisfies (4.1). Then*

$$D(\xi_0, b) = \frac{1}{2}(\mathbf{w}_1)_b,\, d(\xi_0, b) = \frac{1}{2}(\mathbf{w}_2)_b.$$

Consequently, we obtain the following:

Corollary 4.2. *Let b be an integer greater than 2. Then*

$$\inf_{\xi > 0, \xi \notin \mathbb{Q}} (D(\xi, b) - d(\xi, b)) = L_b.$$

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