Kazhdan-Lusztig Conjecture for Symmetrizable Kac-Moody Lie Algebra. II —Intersection Cohomologies of Schubert Varieties—

Masaki KASHIWARA*
Toshiyuki TANISAKI**

dedicated to Professor Jacques Dixmier on his sixty-fifth birthday

0. Introduction

0.0. This article is a continuation of Kashiwara [K3]. We shall complete the proof of a generalization of the Kazhdan-Lusztig conjecture to the case of symmetrizable Kac-Moody Lie algebras.

0.1. The original Kazhdan-Lusztig conjecture [KL1] describes the characters of irreducible highest weight modules of finite-dimensional semisimple Lie algebras in terms of certain combinatorially defined polynomials, called Kazhdan-Lusztig polynomials. It was simultaneously solved by two parties, Beilinson-Bernstein and Brylinski-Kashiwara, by similar methods ([BB], [BK]). The proof consists of the following two parts.

(i) The algebraic part — the correspondence between $\mathcal{D}$-modules on the flag variety and representations of the semisimple Lie algebra.
(ii) The topological part — the description of geometry of Schubert varieties in terms of the Kazhdan-Lusztig polynomials.

Note that the topological part had been already established by Kazhdan and Lusztig themselves ([KL2]).

0.2. Our proof of the generalization of the Kazhdan-Lusztig conjecture in the symmetrizable Kac-Moody Lie algebra case is similar to that in the finite-dimensional case mentioned above. The algebraic part has

*R.I.M.S., Kyoto University, Kyoto 606, Japan.
**College of General Education, Osaka University, Toyonaka 560, Japan.

already appeared in [K3] and this paper is devoted to the topological part. The proof is again similar to the finite-dimensional case except two points.

The first point is that we use the theory of mixed Hodge modules of M. Saito [S] instead of the Weil sheaves. Note that mixed Hodge modules and Weil sheaves are already employed by several authors in order to relate the Hecke-Iwahori algebra of the Weyl group with the geometry of Schubert varieties ([LV], [Sp], [T]).

The second point is that we interpret the inverse Kazhdan-Lusztig polynomials as the coefficients of certain elements of the dual of the Hecke-Iwahori algebra. The appearance of the dual of the Hecke-Iwahori algebra is natural because the open Schubert cell corresponds to the identity element of the Weyl group, contrary to the finite-dimensional case in which the open Schubert cell corresponds to the longest element.

0.3. We shall state our results more precisely. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra, $\mathfrak{h}$ the Cartan subalgebra and W the Weyl group (see [K']). For $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$ (resp. $L(\lambda)$) be the Verma module (resp. irreducible module) with highest weight λ. For $w \in W$ we define a new action of W on $\mathfrak{h}^*$ by $w \circ \lambda = w(\lambda + \rho) - \rho$, where ρ is an element of $\mathfrak{h}^*$ such that $\langle \rho, h_i \rangle = 1$ for any simple coroot $h_i \in \mathfrak{h}$. For $w, z \in W$ let $P_{w,z}(q)$ be the Kazhdan-Lusztig polynomial and $Q_{w,z}(q)$ the inverse Kazhdan-Lusztig polynomial ([KL1], [KL2]). They are defined through a combinatorics in the Hecke-Iwahori algebra of the Weyl group, and are related by

$$\sum_{w \in W} (-1)^{\ell(w)-\ell(y)} Q_{y,w} P_{w,z} = \delta_{y,z}. \tag{0.3.1}$$

Our main result is the following.

Theorem. *For a dominant integral weight $\lambda \in \mathfrak{h}^*$ we have*

$$\operatorname{ch} L(w \circ \lambda) = \sum_{z \in W} (-1)^{\ell(z)-\ell(w)} Q_{w,z}(1) \operatorname{ch} M(z \circ \lambda),$$

or equivalently

$$\operatorname{ch} M(w \circ \lambda) = \sum_{z \in W} P_{w,z}(1) \operatorname{ch} L(z \circ \lambda).$$

Here ch *denotes the character and $\ell(w)$ is the length of w.*

0.4. Let X be the flag variety of $\mathfrak{g}$ constructed in [K2] and let X_w be the Scubert cell corresponding to $w \in W$. Note that X_w is a finitecodimensional locally closed subvariety of the infinite-dimensional variety X.

By the algebraic part [K3] $\mathfrak{g}$-modules correspond to holonomic $\mathcal{D}_X$-modules. Hence by taking the solutions of holonomic $\mathcal{D}_X$-modules, we obtain a correspondence between $\mathfrak{g}$-modules and perverse sheaves on X. Since $M(w\circ\lambda)$ and its dual $M^*(w\circ\lambda)$ have the same characters and since the perverse sheaf corresponding to the highest weight module $L(w\circ\lambda)$ (resp. $M^*(w\circ\lambda)$) is ${}^{\pi}\mathbf{C}_{X_w}[-\ell(w)]$ (resp. $\mathbf{C}_{X_w}[-\ell(w)]$), the proof of the theorem is reduced to

$$(0.4.1) \qquad [{}^{\pi}\mathbf{C}_{X_w}[-\ell(w)]] = \sum_{z\in W}(-1)^{\ell(z)-\ell(w)}Q_{w,z}(1)[\mathbf{C}_{X_z}[-\ell(z)]]$$

(in the Grothendieck group of perverse sheaves). We shall prove it for any (not necessarily symmetrizable) Kac-Moody Lie algebra in §6 by using Hodge modules.

0.5. We finally remark that the Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebras is explicitly stated in Deodhar-Gabber-Kac [DGK]. We also note that we have received the following short note announcing the similar result: L. Cassian, Formule de multiplicité de Kazhdan-Lusztig dans le case de Kac-Moody, preprint.

1. Infinite-dimensional schemes

1.0. In this section we shall briefly discuss infinite-dimensional schemes.

1.1. A scheme X is called *coherent* if the structure ring $\mathcal{O}_X$ is coherent. A scheme X over $\mathbf{C}$ is said to be *of countable type* if the $\mathbf{C}$-algebra $\mathcal{O}_X(U)$ is generated by a countable number of elements for any affine open subset U of X (cf. [K2]). A morphism $f: X \to Y$ of schemes is called *weakly smooth* if $\Omega^1_{X/Y}$ is a flat $\mathcal{O}_X$-module, where $\Omega^1_{X/Y}$ is the sheaf of relative differentials.

1.2. We say that a $\mathbf{C}$-scheme X satisfies (S) if $X \simeq \varprojlim_{n\in\mathbf{N}} S_n$ for some projective system $\{S_n\}_{n\in\mathbf{N}}$ of $\mathbf{C}$-schemes satisfying the following conditions:

(1.2.1) S_n is quasi-compact and smooth over $\mathbf{C}$ for any n.
(1.2.2) The morphism $p_{nm}: S_m \to S_n$ is smooth and affine for $m \geqq n$.

In particular, X is quasi-compact.

Remark that by [EGA IV, Proposition (8.13.1)], the pro-object "$\varprojlim$"S_n is uniquely determined in the category of $\mathbf{C}$-schemes of finite type. More precisely, we have

$$\varinjlim_n \mathrm{Hom}(S_n, Y) \xrightarrow{\sim} \mathrm{Hom}(X, Y) \tag{1.2.3}$$

for any **C**-scheme Y locally of finite type.

Note that the projection $p_n : X \to S_n$ is flat and and we have

$$\Omega^1_X \simeq \varinjlim_n (p_n)^* \Omega^1_{S_n}, \tag{1.2.4}$$

where $\Omega^1_X = \Omega^1_{X/\mathbf{C}}$. Thus we obtain

(1.2.5) Ω^1_X is locally a direct sum of locally free $\mathcal{O}_X$-modules of finite rank.

We see from the following lemma that, if X is separated, we may assume that S_n is also separated for any n.

Lemma 1.2.1. *Let X be an affine (resp. separated) scheme such that $X \simeq \varprojlim_n S_n$, where $\{S_n\}_{n\in\mathbf{N}}$ is a projective system of schemes satisfying the following conditions:*

(1.2.6) *S_n is quasi-compact and quasi-separated for any n.*
(1.2.7) *$p_{nm} : S_m \to S_n$ is affine for $m \geqq n$.*

Then S_n is also affine (resp. separated) for $n \gg 0$.

Proof. Let $p_n : X \to S_n$ be the projection.

(1) Assume that X is affine. We see from the assumptions that there exist an affine open covering $S_0 = \cup_{i\in I} U_i$ and $f_i \in \Gamma(X; \mathcal{O}_X)$ $(i \in I)$ such that $p_0^{-1}(U_i) \supset X_{f_i}$ and $X = \cup_{i\in I} X_{f_i}$, where I is a finite index set and $X_{f_i} = X \setminus \mathrm{Supp}(\mathcal{O}_X/\mathcal{O}_X f_i)$. Setting $A = \Gamma(X; \mathcal{O}_X)$ and $A_n = \Gamma(S_n; \mathcal{O}_{S_n})$, we have $A = \varinjlim_n A_n$ by [EGA IV, Theorem(8.5.2)], and hence there exists some n satisfying $f_i \in A_n$ $(i \in I)$. Thus we may assume that $f_i \in A_0$ from the beginning. It is easily seen from the assumptions that $(S_n)_{f_i} \subset p_{0n}^{-1} U_i$ and $A_n = \sum_{i\in I} A_n f_i$ for $n \gg 0$. Then $(S_n)_{f_i}$ is affine, and hence $S_n \to \mathrm{Spec}(A_n)$ is an affine morphism.

(2) Assume that X is separated. In order to prove that S_n is separated for $n \gg 0$, it is enough to show that, for any affine open subsets U and V of S_0, $p_{0n}^{-1}(U \cap V) \to p_{0n}^{-1}(U) \times p_{0n}^{-1}(V)$ is a closed embedding for $n \gg 0$. Since $p_0^{-1}(U \cap V)$ is affine, $p_{0n}^{-1}(U \cap V)$ is affine for $n \gg 0$ by (1), and hence we may assume from the beginning that $U \cap V$ is affine.

Since $\mathcal{O}_{S_0}(U \cap V)$ is of finite type over $\mathcal{O}_{S_0}(U)$, $\mathcal{O}_{S_0}(U \cap V)$ is generated by finitely many elements a_i over $\mathcal{O}_{S_0}(U)$. Since $p_0^{-1}(U \cap V) \to p_0^{-1}(U) \times p_0^{-1}(V)$ is a closed embedding, $(p_0)^* a_i$ is contained in the image of $\mathcal{O}_X(p_0^{-1}(U)) \otimes \mathcal{O}_X(p_0^{-1}(V)) \to \mathcal{O}_X(p_0^{-1}(U \cap V))$. Thus $(p_{0n})^* a_i$ is contained in the image of $\mathcal{O}_{S_n}(p_{0n}^{-1}(U)) \otimes \mathcal{O}_{S_n}(p_{0n}^{-1}(V)) \to \mathcal{O}_{S_n}(p_{0n}^{-1}(U \cap V))$ for $n \gg 0$. Therefore $\mathcal{O}_{S_n}(p_{0n}^{-1}(U)) \otimes \mathcal{O}_{S_n}(p_{0n}^{-1}(V)) \to \mathcal{O}_{S_n}(p_{0n}^{-1}(U \cap V))$ is surjective. □

1.3. Let (L) (resp. (LA)) denote the category of quasi-compact smooth **C**-schemes and smooth (resp. smooth affine) morphisms.

Proposition 1.3.1. *Let X be a* **C**-*scheme satisfying* (S). *Then* "$\varprojlim$"S_n *as a pro-object in* (LA) *does not depend on the choice of the projective system* $\{S_n\}_{n \in \mathbf{N}}$ *as in* §1.2.

Proof. It is enough to show that, for any quasi-compact smooth **C**-scheme Y, the natural map

(1.3.1)
$$\varinjlim_n \mathrm{Hom}_{(\mathrm{L})}(S_n, Y) \to \{ f \in \mathrm{Hom}(X,Y) \,;\, (f^*\Omega_Y^1)(x) \to \Omega_X^1(x) \text{ is injective for any } x \in X \}$$

is bijective. Here, for an $\mathcal{O}_X$-module $\mathcal{F}$ and $x \in X$, $\mathcal{F}(x)$ denotes $\mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$. In fact, by Lemma 1.2.1, we then have

(1.3.2)
$$\varinjlim_n \mathrm{Hom}_{(\mathrm{LA})}(S_n, Y) \xrightarrow{\sim} \{ f \in \mathrm{Hom}(X,Y) \,;\, f \text{ is affine and } (f^*\Omega_Y^1)(x) \to \Omega_X^1(x) \text{ is injective for any } x \in X \}.$$

The injectivity of (1.3.1) follows from (1.2.3). Let $f : X \to Y$ be a **C**-morphism such that $(f^*\Omega_Y^1)(x) \to \Omega_X^1(x)$ is injective for any $x \in X$. Then f splits into the composition of $p_n : X \to S_n$ and $\tilde{f} : S_n \to Y$ for some n. Since $(\tilde{f}^*\Omega_Y^1)(p_n(x)) \to (f^*\Omega_Y^1)(x)$ is injective for any $x \in X$, $(\tilde{f}^*\Omega_Y^1)(s) \to \Omega_{S_n}^1(s)$ is also injective for any $s \in p_n(X)$. Hence there exists an open neighborhood Ω of $p_n(X)$ such that $(\tilde{f}^*\Omega_Y^1)(s) \to \Omega_{S_n}^1(s)$ is injective for any $s \in \Omega$. Now [EGA IV, Proposition (1.9.2)] guarantees that there exists $m \geqq n$ such that $p_{nm}^{-1}(\Omega) = S_m$, and hence

$((\tilde{f} \circ p_{nm})^* \Omega^1_Y)(s) \to \Omega^1_{S_m}(s)$ is injective for any $s \in S_m$. This means that $\tilde{f} \circ p_{nm}$ is smooth. □

Lemma 1.3.2. *Let $f: X \to Y$ be a morphism of* **C***-schemes satisfying* (S). *Then the following conditions are equivalent.*

(i) *f is weakly smooth (i.e. $\Omega^1_{X/Y}$ is flat).*

(ii) *For any $x \in X$, $(f^*\Omega^1_Y)(x) \to \Omega^1_X(x)$ is injective.*

(iii) *There exist projective systems $\{X_n\}$, $\{Y_n\}$ satisfying* (1.2.1), (1.2.2) *and a morphism $\{f_n\}: \{X_n\} \to \{Y_n\}$ of projective systems such that $X \simeq \varprojlim_n X_n$, $Y \simeq \varprojlim_n Y_n$, $f = \varprojlim_n f_n$, and f_n is smooth for any n.*

Proof. (i)⇒(ii) is evident. (iii)⇒(i) follows from the fact that $\Omega^1_{X/Y}$ is the inductive limit of the flat $\mathcal{O}_X$-modules $(p_n)^*\Omega^1_{X_n/Y_n}$, where $p_n: X \to X_n$ is the projection. Assume (ii). By (1.2.3), there exist $\{X_n\}$, $\{Y_n\}$ and $\{f_n\}$ such that $X \simeq \varprojlim_n X_n$, $Y \simeq \varprojlim_n Y_n$, $f = \varprojlim_n f_n$. Then we see from the bijectivity of (1.3.1) that, for any n, there exists some $m \geqq n$ such that the composition $X_m \to X_n \to Y_n$ is smooth. This implies (iii). □

1.4. A **C**-scheme X is called *pro-smooth* if it is covered by open subsets satisfying (S).

Lemma 1.4.1 (cf. [K2]). *A pro-smooth* **C***-scheme is coherent and of countable type.*

Lemma 1.4.2. *Let $f: X \to Y$ be a smooth morphism of* **C***-schemes. If Y satisfies* (S), *so does X.*

Proof. Let $Y \simeq \varprojlim_n S_n$, where $\{S_n\}$ is as in §1.2. By [EGA IV, Theorem (8.8.2)] there exist some n and a morphism $f_n: X_n \to S_n$ satisfying $f \simeq f_n \times_{S_n} Y$. Then, by [EGA IV, Proposition (17.7.8)], $f_n \times_{S_n} S_m$ is smooth for $m \gg 0$. □

Corollary 1.4.3. *A* **C***-scheme smooth over a pro-smooth* **C***-scheme is also pro-smooth.*

1.5. We give several examples of pro-smooth **C**-schemes.

(a) $\mathbf{A}^\infty = \mathrm{Spec}(\mathbf{C}[X_n\,;\, n = 1, 2, \ldots])$ (cf. [K2]). Let $p_n: \mathbf{A}^\infty \to \mathbf{A}^n$ be the projection given by $(X_1, \ldots, X_n)$. Then we have $\mathbf{A}^\infty \simeq \varprojlim_n \mathbf{A}^n$.

(b) $\mathbf{P}^\infty$ (cf. [K2]).
(c) Let E be a countable subset of $\mathbf{C}$, and let A be the $\mathbf{C}$-subalgebra of the rational function field $\mathbf{C}(x)$ generated by x and $\{(x-a)^{-1}\,;\, a \in E\}$. Then $X = \mathrm{Spec}(A)$ is a pro-smooth $\mathbf{C}$-scheme and we have $X(\mathbf{C}) \simeq \mathbf{C} - E$.
(d) Let A be the $\mathbf{C}$-algebra which is generated by the elements e_n ($n \in \mathbf{Z}$) satisfying the fundamental relations $e_n e_m = \delta_{n,m} e_n$. Let x_n ($n \in \mathbf{Z}$) and ξ be the points of $X = \mathrm{Spec}(A)$ given by the prime ideals $A(1-e_n)$ ($n \in \mathbf{Z}$) and $\sum_{n\in\mathbf{Z}} Ae_n$ respectively. Then X is a pro-smooth $\mathbf{C}$-scheme consisting of x_n ($n \in \mathbf{Z}$) and ξ. The underlying topological space is homeomorphic to the one-point compactification of $\mathbf{Z}$ with discrete topology, and the structure sheaf $\mathcal{O}_X$ is isomorphic to the sheaf of locally constant $\mathbf{C}$-valued functions.

1.6. A $\mathbf{C}$-scheme X is called *essentially smooth* if it is covered by open subsets U, each of which is either smooth over $\mathbf{C}$ or isomorphic to $W \times \mathbf{A}^\infty$ for a smooth $\mathbf{C}$-scheme W. An essentially smooth $\mathbf{C}$-scheme is obviously pro-smooth.

Proposition 1.6.1. *If W is a $\mathbf{C}$-scheme of finite type such that $W \times \mathbf{A}^\infty$ is pro-smooth, then W is smooth.*

Proof. We may assume that $W \times \mathbf{A}^\infty$ satisfies (S). Hence we have $W \times \mathbf{A}^\infty \simeq \varprojlim S_n$ for some $\{S_n\}$ satisfying (1.2.1) and (1.2.2). Then there exist n and m such that the morphism $p_{0m}: S_m \to S_0$ splits into $S_m \to W \times \mathbf{A}^n \to S_0$. Hence $W \times \mathbf{A}^n \to S_0$ is smooth at the image of S_m. Therefore $W \times \mathbf{A}^n$ is smooth and hence so is W. □

Proposition 1.6.2. *Let X and Y be $\mathbf{C}$-schemes and let $f: Y \to X$ be a morphism of finite presentation. Assume $X \simeq W \times \mathbf{A}^\infty$ for a $\mathbf{C}$-scheme W of finite type. Then there exist some n and a $\mathbf{C}$-morphism $f': U \to W \times \mathbf{A}^n$ of finite type satisfying $f = f' \times \mathbf{A}^\infty$ (Note that we have $\mathbf{A}^\infty \simeq \mathbf{A}^n \times \mathbf{A}^\infty$).*

Proof. Since $X \simeq \varprojlim_n W \times \mathbf{A}^n$, there exist some n and a $\mathbf{C}$-scheme U of finite presentation over $W \times \mathbf{A}^n$ such that $Y \simeq X \times_{W\times\mathbf{A}^n} U$ by [EGA IV, Theorem (8.8.2)]. Then $f': U \to W \times \mathbf{A}^n$ satisfies the desired condition. □

Corollary 1.6.3. *A $\mathbf{C}$-scheme smooth over an essentially smooth $\mathbf{C}$-scheme is also essentially smooth.*

Lemma 1.6.4. *Any essentially smooth $\mathbf{C}$-scheme is a disjoint union of open irreducible subsets.*

Proof. Let X be an essentially smooth $\mathbf{C}$-scheme. Since X is covered by open irreducible subsets, it is enough to show that, if U is an open irreducible subset of X, then $\overline{U}$ is also an open subset of X. Let $x \in \overline{U}$ and let W be an irreducible open subset of X containing x. Since $W \cap U \neq \emptyset$, we have $\overline{W} = \overline{W \cap U} = \overline{U}$. This shows that $\overline{U}$ is a neighborhood of x. □

1.7. We shall recall the definition of $\mathcal{D}_X$ and admissible $\mathcal{D}_X$-modules for a pro-smooth $\mathbf{C}$-scheme X.

For a morphism $f\colon X \to Y$ of pro-smooth $\mathbf{C}$-schemes we set

(1.7.1)
$$F_n(\mathcal{D}_{X\to Y}) = 0 \quad (n < 0),$$
(1.7.2)
$$F_n(\mathcal{D}_{X\to Y}) = \{ P \in \mathcal{H}om_{\mathbf{C}}(f^{-1}\mathcal{O}_Y, \mathcal{O}_X);\ [P, a] \in F_{n-1}(\mathcal{D}_{X\to Y}) \text{ for any } a \in \mathcal{O}_Y \} \quad (n \geqq 0),$$
(1.7.3)
$$F(\mathcal{D}_{X\to Y}) = \cup_n F_n(\mathcal{D}_{X\to Y}) \subset \mathcal{H}om_{\mathbf{C}}(f^{-1}\mathcal{O}_Y, \mathcal{O}_X).$$

We have

$$F_0(\mathcal{D}_{X\to Y}) \simeq \mathcal{O}_X, \tag{1.7.4}$$
$$F_1(\mathcal{D}_{X\to Y}) \simeq \mathcal{O}_X \oplus \Theta_{X\to Y}, \tag{1.7.5}$$

where $\Theta_{X\to Y} := \mathcal{H}om_{f^{-1}\mathcal{O}_Y}(f^{-1}\Omega^1_Y, \mathcal{O}_X)$ is the sheaf of derivations.

Let I_{Δ_Y} denote the defining ideal of the diagonal Δ_Y in $Y \times_{\mathbf{C}} Y$ and set $\mathcal{O}_{\Delta_{Y}(n)} = \mathcal{O}_{Y\times_{\mathbf{C}} Y}/(I_{\Delta_Y})^{n+1}$. Then $\mathcal{O}_{\Delta_Y(n)}$ is locally a direct sum of locally free $\mathcal{O}_Y$-modules of finite rank with respect to the $\mathcal{O}_Y$-module structure induced by the first projection. Then we have

$$F_n(\mathcal{D}_{X\to Y}) = \mathcal{H}om_{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{O}_{\Delta_Y(n)}, \mathcal{O}_X),$$

and hence $F_n(\mathcal{D}_{X\to Y})$ has a structure of a sheaf of linear topological spaces induced from the pseudo-discrete topology of $\mathcal{O}_{\Delta_Y(n)}$ (cf. [EGA 0, 3.8]). More concretely, for an affine open subset U of X and an affine open subset V of Y such that $U \subset f^{-1}(V)$,

$$\{ P \in F_n(\mathcal{D}_{X\to Y})(U);\ P(f_i) = 0 \quad (i \in I) \}$$

form a neighborhood system of 0 in $\Gamma(U; F_n(\mathcal{D}_{X\to Y}))$, where $\{f_i\}_{i\in I}$ ranges over finite subsets of $\mathcal{O}_Y(V)$.

If $g: Y \to Z$ is also a morphism of pro-smooth $\mathbf{C}$-schemes, we can define the composition

(1.7.6) $$\mathcal{D}_{X\to Y} \otimes f^{-1}\mathcal{D}_{Y\to Z} \to \mathcal{D}_{X\to Z}.$$

In particular, $D_X := D_{X \xrightarrow{\mathrm{id}} X}$ is a sheaf of rings and $\mathcal{D}_{X\to Y}$ is a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule.

If Y is a smooth $\mathbf{C}$-scheme, we have

(1.7.7) $$\mathcal{D}_{X\to Y} \simeq \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y.$$

Definition 1.7.1. Let X be a pro-smooth $\mathbf{C}$-scheme. A $\mathcal{D}_X$-module $\mathfrak{M}$ is called *admissible* if it satisfies the following conditions:

(1.7.8) For any affine open subset U of X and any $s \in \Gamma(U; \mathfrak{M})$, there exists a finitely generated subalgebra A of $\Gamma(U; \mathcal{O}_X)$ such that $Ps = 0$ for any $P \in \Gamma(U; \mathcal{D}_X)$ satisfying $P(A) = 0$.
(1.7.9) $\mathfrak{M}$ is quasi-coherent as an $\mathcal{O}_X$-module.

The condition (1.7.8) is equivalent to saying that $\mathcal{D}_X$ acts continuously on $\mathfrak{M}$ with the pseudo-discrete topology.

1.8. Let $f: X \to Y$ be a morphism of pro-smooth $\mathbf{C}$-schemes. Then for any admissible $\mathcal{D}_Y$-module $\mathfrak{N}$, $f^*\mathfrak{N} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathfrak{N}$ has a structure of $\mathcal{D}_X$-module. Moreover $f^*\mathfrak{N}$ is admissible (cf. §1.9).

1.9. Let X be a $\mathbf{C}$-scheme satisfying (S). Let $\{S_n\}_{n\in\mathbf{N}}$ be a projective system as in §1.2 and let $p_n: X \to S_n$ be the projection. Then we have

(1.9.1) $$F_k(\mathcal{D}_{X\to S_n}) \simeq \varinjlim_m p_m^{-1} F_k(\mathcal{D}_{S_m\to S_n}) = \mathcal{O}_X \otimes_{p_n^{-1}\mathcal{O}_{S_n}} p_n^{-1} F_k(\mathcal{D}_{S_n}),$$

(1.9.2) $$F_k(\mathcal{D}_X) \simeq \varprojlim_n F_k(\mathcal{D}_{X\to S_n}).$$

If $\mathfrak{M}$ is an admissible $\mathcal{D}_X$-module locally of finite type, then there exist some n and a coherent $\mathcal{D}_{S_n}$-module $\mathfrak{N}$ such that

(1.9.3) $$\mathfrak{M} \simeq (p_n)^*\mathfrak{N} = \mathcal{O}_X \otimes_{p_n^{-1}\mathcal{O}_{S_n}} p_n^{-1}\mathfrak{N} \simeq \mathcal{D}_{X\to S_n} \otimes_{p_n^{-1}\mathcal{D}_{S_n}} p_n^{-1}\mathfrak{N}.$$

Conversely, for a quasi-coherent $\mathcal{D}_{S_n}$-module $\mathfrak{N}$, the $\mathcal{D}_X$-module $(p_n)^*\mathfrak{N}$ is an admissible $\mathcal{D}_X$-module.

1.10. Let X be a pro-smooth $\mathbf{C}$-scheme. A $\mathcal{D}_X$-module $\mathfrak{M}$ is called *holonomic* (resp. *regular holonomic*) if it satisfies the following condition:

(1.10.1) For any point $x \in X$, there exist a morphism $f: U \to Y$ from an open neighborhood U of x to a smooth C-scheme Y and a holonomic (resp. regular holonomic) $\mathcal{D}_Y$-module $\mathfrak{N}$ such that $\mathfrak{M}|U$ is isomorphic to $f^*\mathfrak{N}$.

Let $X \simeq \varprojlim_n S_n$, where $\{S_n\}_{n\in\mathbf{N}}$ is a projective system as in §1.2 and let $\mathfrak{N}$ be a coherent $\mathcal{D}_{S_0}$-module. It is seen that, if $(p_0)^*\mathfrak{N}$ is a holonomic (resp. regular holonomic) $\mathcal{D}_X$-module, then $(p_{0n})^*\mathfrak{N}$ is a holonomic (resp. regular holonomic) $\mathcal{D}_{S_n}$-module for any n.

2. The analytic structure on C-schemes

2.0. In this section, ringed spaces, schemes and their morphisms are all over **C**.

2.1. Let X be an affine **C**-scheme. We define a local ringed space X_{an} as follows. The underlying set of X_{an} is the set $X(\mathbf{C})$ of the **C**-valued points of X. The topology on X_{an} is the weakest one such that, for any $f \in \mathcal{O}_X(X)$, $f(\mathbf{C})\colon X(\mathbf{C}) \to \mathbf{C}$ is continuous with respect to the Euclidian topology on **C**. We define the sheaf of rings $\mathcal{O}_{X_{\text{an}}}$ on $X(\mathbf{C})$ by

$$\mathcal{O}_{X_{\text{an}}} = \varinjlim_f f(\mathbf{C})^{-1}\mathcal{O}_{S_{\text{an}}}, \tag{2.1.1}$$

where $f\colon X \to S$ ranges over morphisms with schemes S of finite type as targets. Here S_{an} denotes the complex analytic space associated to S. .

2.2. More generally, let X be a **C**-scheme. We endow with $X(\mathbf{C})$ the weakest topology such that, for any affine open subset U of X, any open subset of U_{an} is open in $X(\mathbf{C})$. Let X_{an} denote this topological space. We define the sheaf of rings $\mathcal{O}_{X_{\text{an}}}$ by $\mathcal{O}_{X_{\text{an}}}|U_{\text{an}} \simeq \mathcal{O}_{U_{\text{an}}}$ for any affine open subset U of X.

Then we can check easily the following.

Lemma 2.2.1. (i) *The ringed space $(X_{\text{an}}, \mathcal{O}_{X_{\text{an}}})$ is well-defined.*
(ii) *The correspondence $X \mapsto (X_{\text{an}}, \mathcal{O}_{X_{\text{an}}})$ is functorial.*
(iii) $(X \times Y)_{\text{an}} = X_{\text{an}} \times Y_{\text{an}}$ *as a topological space.*
(iv) *If $X \to Y$ is an open (resp. closed) embedding, so is $X_{\text{an}} \to Y_{\text{an}}$.*
(v) *Let $\{X_n\}_{n\in\mathbf{N}}$ be a projective system of* **C**-*schemes such that $X_m \to X_n$ is affine for $m \geqq n$. Then we have $(\varprojlim X_n)_{\text{an}} \simeq \varprojlim (X_n)_{\text{an}}$ as a ringed space.*
(vi) *If X is separated, then X_{an} is Hausdorff.*

(vii) *If X is quasi-compact and of countable type, then X_{an} has a countable base of open subsets.*

2.3. There exists a natural morphism of ringed spaces

$$\iota = \iota_X : X_{\mathrm{an}} \to X \tag{2.3.1}$$

functorial in X, and we have a natural $\iota_X^{-1}\mathcal{D}_X$-module structure on $\mathcal{O}_{X_{\mathrm{an}}}$.

2.4. A *quasi-compact stratification* of a $\mathbf{C}$-scheme X is a locally finite family $\{X_\alpha\}$ of locally closed subsets of X such that

(2.4.1) $X = \sqcup X_\alpha$ as a set,
(2.4.2) $\overline{X}_\alpha \cap X_\beta \neq \emptyset$ implies $\overline{X}_\alpha \supset X_\beta$,
(2.4.3) The inclusion $X_\alpha \hookrightarrow X$ is a quasi-compact morphism.

2.5. Let X be a coherent $\mathbf{C}$-scheme and let k be a field.

Definition 2.5.1 A sheaf F of k-vector spaces on X_{an} is called *weakly constructible* if there exists a quasi-compact stratification $X = \sqcup X_\alpha$ such that $F|(X_\alpha)_{\mathrm{an}}$ is locally constant. If moreover F_x is finite-dimensional for any $x \in X_{\mathrm{an}}$, we call F *constructible*.

Let $D(X_{\mathrm{an}};k)$ be the derived category of the category of sheaves of k-vector spaces on X_{an}. An object K of $D(X_{\mathrm{an}};k)$ is called *constructible* (resp. *weakly constructible*) if it satisfies the following conditions.

(2.5.1) $H^n(K)$ is constructible (resp. weakly constructible) for any n.
(2.5.2) For any quasi-compact open subset Ω of X, $H^n(K)|\Omega_{\mathrm{an}} = 0$ except for finitely many n.

The full subcategory of $D(X_{\mathrm{an}};k)$ consisting of constructible (resp. weakly constructible) objects will be denoted by $D_c(X;k)$ (resp. $D_{w.c.}(X;k)$).

Proposition 2.5.2. *Let W be a smooth $\mathbf{C}$-scheme. Set $X = W \times \mathbf{A}^\infty$, $X_n = W \times \mathbf{A}^n$ and let $p_n : X \to X_n$ be the projection. Then for any $K \in \mathrm{Ob}(D_c(X;k))$, there exist some n and $K_n \in \mathrm{Ob}(D_c(X_n;k))$ satisfying $(p_n)_{\mathrm{an}}^{-1} K_n \simeq K$.*

Proof. We can take a finite coherent stratification $X = \sqcup X_\alpha$ such that $H^j(K)|X_\alpha$ is locally constant for any j and α. Then there exist some n and a stratification $X_n = \sqcup \tilde{X}_\alpha$ such that $X_\alpha = p_n^{-1}\tilde{X}_\alpha$. Let $i: X_n \to X_n \times \mathbf{A}^\infty$ be the embedding by the origin $\in \mathbf{A}^\infty$. Since

$X \simeq X_n \times \mathbf{A}^\infty$ and $(\mathbf{A}^\infty)_{\mathrm{an}}$ is contractible to the origin, we have $K \simeq (p_n)_{\mathrm{an}}^{-1} K_n$ with $K_n = (i_{\mathrm{an}})^{-1} K$. □

Proposition 2.5.3. *Let $X = \mathbf{A}^\infty \times W$, where W is a smooth $\mathbf{C}$-scheme, and let $p\colon X \to W$ be the projection. Then for a cohomologically bounded object K of $D(W_{\mathrm{an}}; k)$, we have*

$$\mathbf{R}Hom((p_{\mathrm{an}})^{-1}K, k_{X_{\mathrm{an}}}) \simeq (p_{\mathrm{an}})^{-1}\mathbf{R}Hom(K, k_{W_{\mathrm{an}}}).$$

Proof. We shall show that the functorial morphism

$$(p_{\mathrm{an}})^{-1}\mathbf{R}Hom(K, k_{W_{\mathrm{an}}}) \to \mathbf{R}Hom((p_{\mathrm{an}})^{-1}K, k_{X_{\mathrm{an}}})$$

is an isomorphism. In order to see this, it suffices to show that

$$\begin{aligned}&\mathbf{R}\Gamma((p_{\mathrm{an}})^{-1}V; (p_{\mathrm{an}})^{-1}\mathbf{R}Hom(K, k_{W_{\mathrm{an}}}))\\ \to &\mathbf{R}\Gamma((p_{\mathrm{an}})^{-1}V; \mathbf{R}Hom((p_{\mathrm{an}})^{-1}K, k_{X_{\mathrm{an}}}))\end{aligned}$$

is an isomorphism for any open subset V of W_{an} (Observe that $U \times (\mathbf{A}^\infty)_{\mathrm{an}} \times V$ form a base of open subsets of $(\mathbf{A}^\infty \times W)_{\mathrm{an}}$, where $\mathbf{A}^\infty \simeq \mathbf{A}^n \times \mathbf{A}^\infty$, U is an open subset of $(\mathbf{A}^n)_{\mathrm{an}}$ and V is an open subset of W_{an}). This follows from the following lemma.

Lemma 2.5.4. *Let X, Y and S be topological spaces and let $p_X : X \to S$ and $p_Y : Y \to S$ be continuous maps. Let $p\colon X \times [0,1] \to X$ be the projection, and let $h\colon X \times [0,1] \to Y$ be a continuous map satisfying $p_Y \circ h = p_X \circ p$. Define $i_\nu : X \to X \times [0,1]$ ($\nu = 0,1$) by $i_\nu(x) = (x, \nu)$, and set $f_\nu = h \circ i_\nu$. Let K (resp. F) be a cohomologically bounded (resp. lower bounded) object in the derived category of the category of sheaves of k-vector spaces on S and let $f_\nu^\sharp$ be the composition of*

$$\begin{aligned}&\mathbf{R}(p_Y)_*\mathbf{R}Hom(p_Y^{-1}K, p_Y^{-1}F)\\ \to &\mathbf{R}(p_Y)_*\mathbf{R}(f_\nu)_*\mathbf{R}Hom(f_\nu^{-1}p_Y^{-1}K, f_\nu^{-1}p_Y^{-1}F)\\ \xrightarrow{\sim} &\mathbf{R}(p_X)_*\mathbf{R}Hom(p_X^{-1}K, p_X^{-1}F).\end{aligned}$$

Then we have $f_0^\sharp = f_1^\sharp$.

Proof. Set $Z = X \times [0,1]$ and $p_Z = p_X \circ p$. Then $f_\nu^\sharp$ is obtained by

$$\begin{aligned}&\mathbf{R}(p_Y)_*\mathbf{R}Hom(p_Y^{-1}K, p_Y^{-1}F)\\ \to &\mathbf{R}(p_Y)_*\mathbf{R}h_*\mathbf{R}Hom(h^{-1}p_Y^{-1}K, h^{-1}p_Y^{-1}F)\\ \simeq &\mathbf{R}(p_Z)_*\mathbf{R}Hom(p_Z^{-1}K, p_Z^{-1}F)\\ \to &\mathbf{R}(p_Z)_*\mathbf{R}(i_\nu)_*\mathbf{R}Hom(i_\nu^{-1}p_Z^{-1}K, i_\nu^{-1}p_Z^{-1}F)\\ \simeq &\mathbf{R}(p_X)_*\mathbf{R}Hom(p_X^{-1}K, p_X^{-1}F).\end{aligned}$$

Set $\tilde{K} = p_X^{-1}K$ and $\tilde{F} = p_X^{-1}F$. Since $\mathbf{R}(p_Z)_* = \mathbf{R}(p_X)_*\mathbf{R}p_*$, it is enough to show that the morphism

$$i_\nu^\sharp : \mathbf{R}p_*\mathbf{R}Hom(p^{-1}\tilde{K}, p^{-1}\tilde{F}) \to \mathbf{R}p_*\mathbf{R}(i_\nu)_*\mathbf{R}Hom(i_\nu^{-1}p^{-1}\tilde{K}, i_\nu^{-1}p^{-1}\tilde{F}) \simeq \mathbf{R}Hom(\tilde{K}, \tilde{F})$$

does not depend on ν. Since

$$p^* : \mathbf{R}Hom(\tilde{K}, \tilde{F}) \to \mathbf{R}p_*\mathbf{R}Hom(p^{-1}\tilde{K}, p^{-1}\tilde{F}) \simeq \mathbf{R}Hom(\tilde{K}, \mathbf{R}p_*p^{-1}\tilde{F})$$

is an isomorphism and $i_\nu^\sharp \circ p^* = \mathrm{id}$, we obtain the desired result. □

For a quasi-compact separated essentially smooth $\mathbf{C}$-scheme X we set

$$\mathbf{D}_X(K) = \mathbf{R}Hom(K, k_{X_{an}}) \tag{2.5.3}$$

for $K \in D_c(X; k)$.

Corollary 2.5.5. *Let X be a quasi-compact separated essentially smooth $\mathbf{C}$-scheme. Then*

(i) $\mathbf{D}_X$ *preserves* $D_c(X; k)$.

(ii) $\mathbf{D}_X \circ \mathbf{D}_X \simeq \mathrm{id}$.

2.6. Let X be a quasi-compact separated essentially smooth $\mathbf{C}$-scheme. Define full subcategories ${}^pD_c^{\leqq 0}(X; k)$ and ${}^pD_c^{\geqq 0}(X; k)$ of $D_c(X; k)$ by

(2.6.1) K belongs to ${}^pD_c^{\leqq 0}(X; k)$ if and only if $\operatorname{codim}\operatorname{Supp} H^n(K) \geqq n$ for any n.

(2.6.2) K belongs to ${}^pD_c^{\geqq 0}(X; k)$ if and only if $\mathbf{D}_X(K)$ belongs to ${}^pD_c^{\leqq 0}(X; k)$.

The following theorem is similarly proven as in the finite-dimensional case (see [BBD], [KS]), and we omit the proof.

Theorem 2.6.1. (i) *$({}^pD_c^{\leqq 0}(X; k), {}^pD_c^{\geqq 0}(X; k))$ is a t-structure of $D_c(X; k)$.*

(ii) *For $K_1 \in \mathrm{Ob}({}^pD_c^{\leqq 0}(X; k))$ and $K_2 \in \mathrm{Ob}({}^pD_c^{\geqq 0}(X; k))$, we have*

$$H^n(\mathbf{R}Hom(K_1, K_2)) = 0 \ (n < 0).$$

(iii) $\mathrm{Perv}(X; k) = {}^pD_c^{\leqq 0}(X; k) \cap {}^pD_c^{\geqq 0}(X; k)$ *is a stack, i.e.*

(a) *For* $K_1, K_2 \in \mathrm{Ob}(\mathrm{Perv}(X;k))$, $U \mapsto \mathrm{Hom}(K_1|U_{\mathrm{an}}, K_2|U_{\mathrm{an}})$ *is a sheaf on* X.

(b) *Let* $X = \cup_j U_j$ *be an open covering. Assume that we are given objects* K_j *of* $\mathrm{Perv}(U_j;k)$ *and isomorphisms* $f_{ij}: K_j|(U_i)_{\mathrm{an}} \cap (U_j)_{\mathrm{an}} \to K_i|(U_i)_{\mathrm{an}} \cap (U_j)_{\mathrm{an}}$ *such that* $f_{ij} \circ f_{jk} = f_{ik}$ *on* $(U_i)_{\mathrm{an}} \cap (U_j)_{\mathrm{an}} \cap (U_k)_{\mathrm{an}}$. *Then there exist* $K \in \mathrm{Ob}(\mathrm{Perv}(X;k))$ *and isomorphisms* $f_i: K|(U_i)_{\mathrm{an}} \to K_i$ *such that* $f_{ij} \circ f_j = f_i$ *on* $(U_i)_{\mathrm{an}} \cap (U_j)_{\mathrm{an}}$.

We call an object of $\mathrm{Perv}(X;k)$ a *perverse sheaf*. When X is smooth, this definition coincides with the one in [BBD] up to shift.

Proposition 2.6.2. *Let* X *be a separated essentially smooth* **C**-*scheme such that* $X \simeq \varprojlim_n X_n$ *for some projective system* $\{X_n\}$ *satisfying* (1.2.1) *and* (1.2.2). *Then we have* $\mathrm{Perv}(X;k) \simeq \varinjlim_n \mathrm{Perv}(X_n;k)$; *i.e. the following two properties hold.*

(2.6.3) *For* $M_1, M_2 \in \mathrm{Ob}(\mathrm{Perv}(X_n;k))$ *we have*

$$\varinjlim_m \mathrm{Hom}((p_{nm})^* M_1, (p_{nm})^* M_2) \simeq \mathrm{Hom}((p_n)^* M_1, (p_n)^* M_2).$$

(2.6.4) *For any* $M \in \mathrm{Ob}(\mathrm{Perv}(X;k))$ *there exist some* n *and* $M_n \in \mathrm{Ob}(\mathrm{Perv}(X_n;k))$ *such that* $M \simeq (p_n)^* M_n$.

Here, $p_{nm}: X_m \to X_n$ *and* $p_n: X \to X_n$ *are the projections.*

2.7. Let X be a **C**-scheme satisfying (S) and let $\{S_n\}_{n \in \mathbf{N}}$ be a projective system as in §1.2. We denote by $p_{nm}: S_m \to S_n$ $(m \geqq n)$ and $p_n: X \to S_n$ the projections. Let $\mathfrak{B}^{(p,q)}_{(S_n)_{\mathrm{an}}}$ be the sheaf of (p,q)-forms on $(S_n)_{\mathrm{an}}$ with hyperfunction coefficients. Then we have natural homomorphisms

$$(2.7.1) \qquad (p_{nm})^{-1}_{\mathrm{an}} \mathfrak{B}^{(p,q)}_{(S_n)_{\mathrm{an}}} \to \mathfrak{B}^{(p,q)}_{(S_m)_{\mathrm{an}}},$$

$$(2.7.2) \qquad \iota_X^{-1} p_m^{-1} \mathcal{D}_{S_m \to S_n} \times (p_n)^{-1}_{\mathrm{an}} \mathfrak{B}^{(0,p)}_{(S_n)_{\mathrm{an}}} \to (p_m)^{-1}_{\mathrm{an}} \mathfrak{B}^{(0,p)}_{(S_m)_{\mathrm{an}}}.$$

By (2.7.1) we obtain a sheaf $\mathfrak{B}^{(p,q)}_{X_{\mathrm{an}}} = \varinjlim (p_n)^{-1}_{\mathrm{an}} \mathfrak{B}^{(p,q)}_{(S_n)_{\mathrm{an}}}$ on X_{an}, and this does not depend on the choice of $\{S_n\}_{n \in \mathbf{N}}$ by Proposition 1.3.1. Taking the inductive limit in (2.7.2) with respect to m, we obtain

$$(2.7.3) \qquad \iota_X^{-1} \mathcal{D}_{X \to S_n} \times (p_n)^{-1}_{\mathrm{an}} \mathfrak{B}^{(0,p)}_{(S_n)_{\mathrm{an}}} \to \mathfrak{B}^{(0,p)}_{X_{\mathrm{an}}}.$$

Taking again the limit in (2.7.3) with respect to n, we obtain

$$\iota_X^{-1}\mathcal{D}_X \times \mathfrak{B}_{X_{\mathrm{an}}}^{(0,p)} \to \mathfrak{B}_{X_{\mathrm{an}}}^{(0,p)}, \tag{2.7.4}$$

and this defines a structure of an $\iota_X^{-1}\mathcal{D}_X$-module on $\mathfrak{B}_{X_{\mathrm{an}}}^{(0,p)}$. We have also

$$\begin{aligned}\varinjlim_m (p_m)_{\mathrm{an}}^{-1} Hom_{\iota_{S_m}^{-1}\mathcal{D}_{S_m}}(\iota_{S_m}^{-1}\mathcal{D}_{S_m\to S_n}, \mathfrak{B}_{(S_m)_{\mathrm{an}}}^{(0,p)}) \\ \simeq Hom_{\iota_X^{-1}\mathcal{D}_X}(\iota_X^{-1}\mathcal{D}_{X\to S_n}, \mathfrak{B}_{X_{\mathrm{an}}}^{(0,p)}).\end{aligned} \tag{2.7.5}$$

2.8. More generally, let X be a pro-smooth $\mathbf{C}$-scheme. We can patch the sheaves $\mathfrak{B}_{U_{\mathrm{an}}}^{(p,q)}$ for affine open subschemes U of X satisfying (S), and obtain a sheaf $\mathfrak{B}_{X_{\mathrm{an}}}^{(p,q)}$ on X_{an} such that

$$\mathfrak{B}_{X_{\mathrm{an}}}^{(p,q)}|U_{\mathrm{an}} \simeq \mathfrak{B}_{U_{\mathrm{an}}}^{(p,q)}. \tag{2.8.1}$$

We can define the derivatives

$$\partial\colon \mathfrak{B}_{X_{\mathrm{an}}}^{(p,q)} \to \mathfrak{B}_{X_{\mathrm{an}}}^{(p+1,q)}, \tag{2.8.2}$$

$$\bar\partial\colon \mathfrak{B}_{X_{\mathrm{an}}}^{(p,q)} \to \mathfrak{B}_{X_{\mathrm{an}}}^{(p,q+1)}, \tag{2.8.3}$$

and we have the exact sequence:

$$0 \to \mathcal{O}_{X_{\mathrm{an}}} \to \mathfrak{B}_{X_{\mathrm{an}}}^{(0,0)} \xrightarrow{\bar\partial} \mathfrak{B}_{X_{\mathrm{an}}}^{(0,1)} \xrightarrow{\bar\partial} \cdots \tag{2.8.4}$$

of $\iota_X^{-1}\mathcal{D}_X$-modules. The Dolbeault complex:

$$\mathfrak{B}_{X_{\mathrm{an}}}^{(0,0)} \xrightarrow{\bar\partial} \mathfrak{B}_{X_{\mathrm{an}}}^{(0,1)} \xrightarrow{\bar\partial} \cdots$$

is denoted by $\mathfrak{B}^{\cdot}_{X_{\mathrm{an}}}$.

2.9. Let X be a pro-smooth $\mathbf{C}$-scheme. For a holonomic $\mathcal{D}_X$-module $\mathfrak{M}$, we set

$$\mathrm{Sol}(\mathfrak{M}) = Hom_{\mathcal{D}_X}(\mathfrak{M}, \mathfrak{B}^{\cdot}_{X_{\mathrm{an}}})\,(= Hom_{\iota_X^{-1}\mathcal{D}_X}(\iota_X^{-1}\mathfrak{M}, \mathfrak{B}^{\cdot}_{X_{\mathrm{an}}})), \tag{2.9.1}$$

and regard this as an object of $D(X_{\mathrm{an}};\mathbf{C})$. When X is smooth, we have

$$\mathrm{Sol}(\mathfrak{M}) = \mathbf{R}Hom_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{O}_{X_{\mathrm{an}}})$$

by [K1].

Let U be an open subset of X satisfying (S) and let $\{S_n\}$ be a projective system of $\mathbf{C}$-schemes satisfying (1.2.1), (1.2.2) such that $U \simeq \varprojlim S_n$. Then we have $\mathfrak{M}|U \simeq (p_n)^* \mathfrak{N}$ for some n and some holonomic $\mathcal{D}_{S_n}$-module $\mathfrak{N}$, where $p_n : U \to S_n$ is the projection. By (2.7.5) we have

(2.9.2)

$$\begin{aligned}
&\mathcal{H}om_{\mathcal{D}_U}(\mathfrak{M}, \mathfrak{B}^{(0,p)}_{U_{\mathrm{an}}}) \\
\simeq &\mathcal{H}om_{\mathcal{D}_U}(\mathcal{D}_{U\to S_n} \otimes_{\mathcal{D}_{S_n}} \mathfrak{N}, \mathfrak{B}^{(0,p)}_{U_{\mathrm{an}}}) \\
\simeq &\mathcal{H}om_{\mathcal{D}_{S_n}}(\mathfrak{N}, \mathcal{H}om_{\mathcal{D}_U}(\mathcal{D}_{U\to S_n}, \mathfrak{B}^{(0,p)}_{U_{\mathrm{an}}})) \\
\simeq &\varinjlim_m \mathcal{H}om_{\mathcal{D}_{S_n}}(\mathfrak{N}, (p_m)^{-1}_{\mathrm{an}} \mathcal{H}om_{\mathcal{D}_{S_m}}(\mathcal{D}_{S_m\to S_n}, \mathfrak{B}^{(0,p)}_{(S_m)_{\mathrm{an}}})) \\
\simeq &\varinjlim_m (p_m)^{-1}_{\mathrm{an}} \mathcal{H}om_{\mathcal{D}_{S_m}}((p_{nm})^* \mathfrak{N}, \mathfrak{B}^{(0,p)}_{(S_m)_{\mathrm{an}}}).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
(2.9.3) \qquad &H^q(\mathcal{H}om_{\mathcal{D}_{S_m}}((p_{nm})^* \mathfrak{N}, \mathfrak{B}_{(S_m)_{\mathrm{an}}})) \\
\simeq &\mathcal{E}xt^q_{\mathcal{D}_{S_m}}((p_{nm})^* \mathfrak{N}, \mathcal{O}_{(S_m)_{\mathrm{an}}}) \\
\simeq &(p_{nm})^{-1}_{\mathrm{an}} \mathcal{E}xt^q_{\mathcal{D}_{S_n}}(\mathfrak{N}, \mathcal{O}_{(S_n)_{\mathrm{an}}}).
\end{aligned}$$

Thus

$$(2.9.4) \qquad H^q(\mathrm{Sol}(\mathfrak{M})|U_{\mathrm{an}}) \simeq H^q((p_n)^{-1}_{\mathrm{an}} \mathrm{Sol}(\mathfrak{N})),$$

and we finally obtain

$$(2.9.5) \qquad \mathrm{Sol}(\mathfrak{M})|U_{\mathrm{an}} \simeq (p_n)^{-1}_{\mathrm{an}} \mathrm{Sol}(\mathfrak{N}) \quad \text{in} \quad D^b(U_{\mathrm{an}}; \mathbf{C}).$$

This shows in particular

Lemma 2.9.1. *Let X be a quasi-compact separated essentially smooth $\mathbf{C}$-scheme. If $\mathfrak{M}$ is a holonomic $\mathcal{D}_X$-module, then* Sol($\mathfrak{M}$) *is a perverse sheaf, and* Sol *is a contravariant exact functor from the category of holonomic $\mathcal{D}_X$-modules to* Perv(X).

3. Mixed Hodge modules on essentially smooth C-schemes

3.0. We shall study mixed Hodge modules on essentially smooth **C**-schemes. In this section all schemes are over **C** and assumed to be *quasi-compact and separated.*

3.1. In [S], M. Saito constructed mixed Hodge modules on finite-dimensional manifolds. In his formulation, the weights behave well under direct images, but not under inverse images. Since we treat infinite-dimensional manifolds, we have to modify his definition so that the weights behave well under inverse images.

3.2. Let X be a quasi-compact essentially smooth **C**-scheme. Let $\tilde{M}FW(X)$ be the category consisting of $M = (\mathfrak{M}, F, K, W, \iota)$, where

(3.2.1) $\mathfrak{M}$ is a regular holonomic $\mathcal{D}_X$-module,
(3.2.2) F is a filtration of $\mathfrak{M}$ by coherent $\mathcal{O}_X$-submodules which is compatible with $(\mathcal{D}_X, F)$,
(3.2.3) $W(\mathfrak{M})$ is a finite filtration of $\mathfrak{M}$ by regular holonomic $\mathcal{D}_X$-modules,
(3.2.4) K is an object of $\mathrm{Perv}(X; \mathbf{Q})$,
(3.2.5) $W(K)$ is a finite filtration of K in $\mathrm{Perv}(X; \mathbf{Q})$,
(3.2.6) ι is an isomorphism $\mathbf{C}_X \otimes_{\mathbf{Q}_X} K \xrightarrow{\sim} \mathrm{Sol}(\mathfrak{M})$ in $D_c(X; \mathbf{C})$, compatible with W; i.e. ι induces a commutative diagram

$$\begin{array}{ccc} \mathbf{C}_X \otimes_{\mathbf{Q}_X} W_k(K) & \xrightarrow{\sim} & \mathrm{Sol}(\mathfrak{M}/W_{-k-1}(\mathfrak{M})) \\ \downarrow & & \downarrow \\ \mathbf{C}_X \otimes_{\mathbf{Q}_X} K & \xrightarrow{\sim} & \mathrm{Sol}(\mathfrak{M}). \end{array}$$

We define morphisms of $\tilde{M}FW(X)$ so that $M \mapsto K \in \mathrm{Perv}(X; \mathbf{Q})$ is a covariant functor and $M \mapsto \mathfrak{M}$ is a contravariant functor.

Sometimes, ι in $(\mathfrak{M}, F, K, W, \iota)$ will be omitted.

3.3. Let X be a smooth **C**-scheme and let $MHM(X)$ be the category of mixed Hodge modules on X defined in Saito [S]. We define a contravariant functor

$$\varphi_X : MHM(X) \to \tilde{M}FW(X) \tag{3.3.1}$$

as follows. Let $M = (\mathfrak{M}, F, K, W)$ be an object of $MHM(X)$ and let $\mathbf{D}_X(M) = (\mathfrak{M}^*, F, K^*, W)$ be the dual of M (cf. [S]). Then we define

$\varphi_X(M) = (\mathfrak{N}, F, \tilde{K}, W)$ by

$$\mathfrak{N} = \mathfrak{M}^* \otimes_{\mathcal{O}_X} (\Omega_X^{\dim X})^{\otimes -1}, \tag{3.3.2}$$

$$F_p(\mathfrak{N}) = F_p(\mathfrak{M}^*) \otimes_{\mathcal{O}_X} (\Omega_X^{\dim X})^{\otimes -1}, \tag{3.3.3}$$

$$W_k(\mathfrak{N}) = W_{k+\dim X}(\mathfrak{M}^*) \otimes_{\mathcal{O}_X} (\Omega_X^{\dim X})^{\otimes -1}, \tag{3.3.4}$$

$$\tilde{K} = K[-\dim X], \tag{3.3.5}$$

$$W_k(\tilde{K}) = W_{k+\dim X}(K)[-\dim X]. \tag{3.3.6}$$

Note that $\mathfrak{N}$ is a left $\mathcal{D}_X$-module since $\mathfrak{M}$ and $\mathfrak{M}^*$ are right $\mathcal{D}_X$-modules.

Let $\tilde{M}HM(X)$ be the image of φ_X. It is a full subcategory of $\tilde{M}FW(X)$, isomorphic to $MHM(X)$.

We define

$$\varphi_X : D^b(MHM(X)) \xrightarrow{\sim} D^b(\tilde{M}HM(X)) \tag{3.3.7}$$

by $M^\cdot \mapsto \varphi_X(M^\cdot)[\dim X]$. Hence φ_X is compatible with $i_X : D^b(MHM(X)) \to D_c(X; \mathbf{Q})$ and $i_X : D^b(\tilde{M}HM(X)) \to D_c(X; \mathbf{Q})$.

The duality functor $\mathbf{D}_X$ on $MHM(X)$ defines the duality functor

$$\mathbf{D}_X : \tilde{M}HM(X) \to \tilde{M}HM(X)^{\mathrm{op}} \tag{3.3.8}$$

by $\mathbf{D}_X \circ \varphi_X = \varphi_X \circ \mathbf{D}_X$. Then we have $i_X \circ \mathbf{D}_X = \mathbf{D}_X \circ i_X$, where $\mathbf{D}_X(K) = \mathbf{R}Hom(K, \mathbf{Q}_{X_{\mathrm{an}}})$ for $K \in \mathrm{Perv}(X; \mathbf{Q})$.

3.4. For a morphism $f : X \to Y$ of smooth $\mathbf{C}$-schemes, we define functors

$$f^*, f^! : D^b(\tilde{M}HM(Y)) \to D^b(\tilde{M}HM(X)) \tag{3.4.1}$$

$$f_*, f_! : D^b(\tilde{M}HM(X)) \to D^b(\tilde{M}HM(Y)) \tag{3.4.2}$$

using those defined in [S] and the isomorphism (3.3.7). In particular, if $f : X \to Y$ is smooth and $M = (\mathfrak{M}, F, K, W)$ is an object of $\tilde{M}HM(Y)$, then we have

$$f^*(M) = (f^*\mathfrak{M}, F, f^*K, W) \in \mathrm{Ob}(\tilde{M}HM(X)), \tag{3.4.3}$$

where

$$F_p(f^*\mathfrak{M}) = f^* F_p(\mathfrak{M}), \tag{3.4.4}$$

$$W_k(f^*\mathfrak{M}) = f^*(W_k(\mathfrak{M})), \quad W_k(f^*K) = f^*(W_k(K)). \tag{3.4.5}$$

We extend this definition when X is essentially smooth, Y is smooth and f is weakly smooth. Hence in this case, f^* is a functor from $\tilde{M}HM(Y)$ into $\tilde{M}FW(X)$ defined by (3.4.3), (3.4.4) and (3.4.5).

3.5. For an essentially smooth **C**-scheme X, we define a full subcategory $\tilde{M}HM(X)$ of $\tilde{M}FW(X)$ as follows. An object M of $\tilde{M}FW(X)$ belongs to $\tilde{M}HM(X)$ if and only if X is covered by open subsets U such that there are a weakly smooth morphism $f: U \to Y$ to a smooth **C**-scheme Y and an object M' of $\tilde{M}HM(Y)$ satisfying $M|U \simeq f^*M'$. We can easily see that $\tilde{M}HM(X)$ is a stack. In this paper we call objects of $\tilde{M}HM(X)$ *mixed Hodge modules* on X. Note that $\tilde{M}HM(X)$ is an abelian category.

We can define the duality functor

$$\mathbf{D}_X : \tilde{M}HM(X) \to \tilde{M}HM(X)^{\mathrm{op}} \tag{3.5.1}$$

by $\mathbf{D}_X M|U \simeq f^*\mathbf{D}_Y M'$. It is an exact functor satisfying $\mathbf{D}_X \circ \mathbf{D}_X \simeq \mathrm{id}$. Hence this extends to

$$\mathbf{D}_X : D^b(\tilde{M}HM(X)) \to D^b(\tilde{M}HM(X))^{\mathrm{op}}. \tag{3.5.2}$$

3.6. Let X be an essentially smooth **C**-scheme satisfying (S) and let $\{S_n\}_{n\in\mathbf{N}}$ be a projective system as in §1.2. Then we have

$$\tilde{M}HM(X) \simeq \varinjlim_n \tilde{M}HM(S_n), \tag{3.6.1}$$

$$D^b(\tilde{M}HM(X)) \simeq \varinjlim_n D^b(\tilde{M}HM(S_n)) \tag{3.6.2}$$

(cf. Proposition 2.6.2).

3.7. For a morphism $f: X \to Y$ of essentially smooth **C**-schemes satisfying (S), we define

$$f^* : D^b(\tilde{M}HM(Y)) \to D^b(\tilde{M}HM(X)) \tag{3.7.1}$$

as follows. Let $X \simeq \varprojlim_n X_n$ and $Y \simeq \varprojlim_n Y_n$, where $\{X_n\}$ and $\{Y_n\}$ satisfy (1.2.1) and (1.2.2). We may assume that there are morphisms $f_n : X_n \to Y_n$ $(n \in \mathbf{N})$ such that $f = \varprojlim_n f_n$. Let $p_{X,n} : X \to X_n$ and $p_{Y,n} : Y \to Y_n$ be the projections. For a bounded complex $M^\cdot$ of mixed Hodge modules on Y, there exist some n and a bounded complex $M_n^\cdot$

of mixed Hodge modules on Y_n such that $M^\cdot \simeq (p_{Y,n})^* M_n^\cdot$. Then we define (3.7.1) by

$$(3.7.2) \qquad f^* M^\cdot = (p_{X,n})^*((f_n)^* M_n^\cdot).$$

It is easy to check that this is well-defined.

3.8. Let $f: X \to Y$ be a morphism of essentially smooth **C**-schemes. Then, for each $i \in \mathbf{Z}$, we can define a functor

$$(3.8.1) \qquad H^i f^*: \tilde{M}HM(Y) \to \tilde{M}HM(X).$$

In fact, locally on X, $(H^i f^*)(M)$ is defined as $H^i(f^*(M))$, and they can be patched together. It satisfies the following properties:

(3.8.2) If f is weakly smooth, then we have $(H^i f^*)(M) = 0$ for $i \neq 0$, and $(H^0 f^*)(M)$ is given by (3.4.4) and (3.4.5).
(3.8.3) If $(H^i f^*)(M) = 0$ for $i \neq p$ and if $g: W \to X$ is another morphism, then we have $(H^i g^*)(H^p f^*)(M) \simeq (H^{i+p}(f \circ g)^*)(M)$.

3.9. Let $f: X \to Y$ be a morphism of finite presentation. Assume that Y satisfies (S), so that X also satisfies (S). Then we define

$$(3.9.1) \qquad f_*: D^b(\tilde{M}HM(X)) \to D^b(\tilde{M}HM(Y)),$$

$$(3.9.2) \qquad f_!: D^b(\tilde{M}HM(X)) \to D^b(\tilde{M}HM(Y)),$$

$$(3.9.3) \qquad f^!: D^b(\tilde{M}HM(Y)) \to D^b(\tilde{M}HM(X))$$

as follows. Let $X \simeq \varprojlim_n X_n$ and $Y \simeq \varprojlim_n Y_n$, where $\{X_n\}$ and $\{Y_n\}$ satisfy (1.2.1) and (1.2.2), and let $M^\cdot$ be a bounded complex of mixed Hodge modules on X (resp. Y). We may assume that there exists a morphism $f_0: X'_0 \to Y_0$ such that $X \simeq X'_0 \times_{Y_0} Y$ and $f = f_0 \times_{Y_0} Y$. Set $f_n = f_0 \times_{Y_0} Y_n$. We may further assume that there exists $\{g_n\}: \{X'_0 \times_{Y_0} Y_n\} \to \{X_n\}$ (resp. $\{h_n\}: \{X_n\} \to \{X'_0 \times_{Y_0} Y_n\}$) such that $\varprojlim g_n = \mathrm{id}_X$ (resp. $\varprojlim h_n = \mathrm{id}_X$). Let $p_{X,n}: X \to X_n$ and $p_{Y,n}: Y \to Y_n$ be the projections. There exist some n and a bounded complex $M_n^\cdot$ of mixed Hodge modules on X_n (resp. Y_n) such that $M^\cdot \simeq (p_{X,n})^* M_n^\cdot$ (resp. $M^\cdot \simeq (p_{Y,n})^* M_n^\cdot$) . Then we define (3.9.1), (3.9.2) (resp. (3.9.3)) by

$$(3.9.4) \qquad f_* M^\cdot = (p_{Y,n})^* \varphi_{Y_n} (f_n)_* (g_n)^* \varphi_{X_n}^{-1}(M_n^\cdot),$$

$$(3.9.5) \qquad f_! M^\cdot = (p_{Y,n})^* \varphi_{Y_n} (f_n)_! (g_n)^* \varphi_{X_n}^{-1}(M_n^\cdot)$$

$$(3.9.6) \qquad (\text{ resp. } f^! M^\cdot = (p_{X,n})^* \varphi_{X_n} (h_n)^* (f_n)^! \varphi_{Y_n}^{-1}(M_n^\cdot)\,).$$

It is easy to check that they are well-defined. We have the following properties concerning them.

(3.9.7) The functor f_* (resp. $f^!$) is a right adjoint functor of f^* (resp. $f_!$).
(3.9.8) $f_* \circ \mathbf{D}_X \simeq \mathbf{D}_Y \circ f_!$.
(3.9.9) If f is proper, then we have $f_* = f_!$.

Note that (3.9.9) follows from the fact that f_n is proper for $n \gg 0$ if f is proper.

3.10. For an essentially smooth **C**-scheme X, we define

$$\mathbf{Q}_X^H = (\mathcal{O}_X, F, \mathbf{Q}_X, W, \iota) \in \mathrm{Ob}(\tilde{M}HM(X)) \tag{3.10.1}$$

by

(3.10.2) $F_p(\mathcal{O}_X) = \begin{cases} \mathcal{O}_X & (p \geqq 0) \\ 0 & (p < 0), \end{cases}$

(3.10.3) $W_k(\mathcal{O}_X) = \begin{cases} \mathcal{O}_X & (k \geqq 0) \\ 0 & (k < 0), \end{cases}$

(3.10.4) $W_k(\mathbf{Q}_X) = \begin{cases} \mathbf{Q}_X & (k \geqq 0) \\ 0 & (k < 0), \end{cases}$

(3.10.5) $\iota\colon \mathbf{C}_X \otimes_{\mathbf{Q}_X} \mathbf{Q}_X \xrightarrow{\sim} \mathrm{Sol}(\mathcal{O}_X)$ is induced by

$$1 \mapsto \mathrm{id} \in Hom_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X) \subset Hom_{\mathcal{D}_X}(\mathcal{O}_X, \mathfrak{B}^{(0,0)}_{X_{an}}).$$

Set (pt) = Spec(**C**) and let $a_X : X \to$ (pt) be the projection. We shall identify $\tilde{M}HM((\mathrm{pt}))$ with the category MHS of mixed Hodge structures. Then we have $\mathbf{Q}_X^H = (a_X)^*\mathbf{Q}^H$, where $\mathbf{Q}^H$ is the trivial mixed Hodge structure on **Q**.

3.11. Let X and Y be essentially smooth **C**-schemes and let $j : Y \hookrightarrow X$ be an emmbeding of finite presentation. Then for any $M \in \tilde{M}HM(Y)$, there exists an object ${}^\pi M$ of $D^b(\tilde{M}HM(X))$ satisfying the following properties:

(3.11.1) ${}^\pi M[-\operatorname{codim} Y] \in \mathrm{Ob}(\tilde{M}HM(X))$,
(3.11.2) $\operatorname{Supp} {}^\pi M[-\operatorname{codim} Y] \subset \overline{Y}$,
(3.11.3) $j^*({}^\pi M) \simeq M$,
(3.11.4) ${}^\pi M[-\operatorname{codim} Y]$ has neither non-zero quotient nor non-zero subobject whose support is contained in $\overline{Y} - Y$.

Such ${}^{\pi}M$ is unique up to isomorphism, and we call it the *minimal extension* of M.

3.12. The following descent theorem being proven in a canonical way, we leave its proof to the readers.

Proposition 3.12.1. *Let $f: X \to Y$ be a weakly smooth morphism of essentially smooth* **C***-schemes. Assume that f admits a section locally on Y. Let $p_i: X \times_Y X \to X$ $(i = 1, 2)$ and $p_{ij}: X \times_Y X \times_Y X \to X \times_Y X$ $(i, j = 1, 2, 3)$ be the obvious projections.*

(i) *For any $M, M' \in \mathrm{Ob}(\tilde{M}HM(Y))$ we have an exact sequence:*

$$0 \to Hom(M, M') \to Hom(f^*M, f^*M')$$
$$\xrightarrow{(p_1)^* - (p_2)^*} Hom((p_1)^* f^* M, (p_1)^* f^* M').$$

(ii) *Let $M \in \mathrm{Ob}(\tilde{M}HM(X))$ and let $\alpha: (p_1)^*M \xrightarrow{\sim} (p_2)^*M$ be an isomorphism satisfying $(p_{23})^*\alpha \circ (p_{12})^*\alpha = (p_{13})^*\alpha$ (Note that we have $(p_{13})^*(p_1)^*M = (p_{12})^*(p_1)^*M$, $(p_{12})^*(p_2)^*M = (p_{23})^*(p_1)^*M$ and $(p_{13})^*(p_2)^*M = (p_{23})^*(p_2)^*M$). Then there exist some $N \in \mathrm{Ob}(\tilde{M}HM(Y))$ and an isomorphism $\beta: M \xrightarrow{\sim} f^*N$ satisfying $(p_1)^*\beta = (p_2)^*\beta \circ \alpha$.*

3.13. Let G be an essentially smooth affine group scheme acting on an essentially smooth **C**-scheme X. Let $\mu: G \times X \to X$ be the composition morphism, $pr: G \times X \to X$ the projection and $i: X \to G \times X$ the embedding by the identity element $e \in G$. We define morphisms $p_i: G \times G \times X \to G \times X$ $(i = 1, 2, 3)$ by $p_1(g_1, g_2, x) = (g_1, g_2 x)$, $p_2(g_1, g_2, x) = (g_1 g_2, x)$ and $p_3(g_1, g_2, x) = (g_2, x)$. For a mixed Hodge module M on X we have mixed Hodge modules μ^*M and pr^*M since μ and pr are weakly smooth.

We define an abelian category $\tilde{M}HM^G(X)$ as follows. An object is a mixed Hodge module M on X, together with an isomorphism $\alpha_M: \mu^*M \xrightarrow{\sim} pr^*M$, satisfying the following conditions:

(3.13.1) $i^*\alpha_M: i^*\mu^*M \to i^*pr^*M$ coincides with $\mathrm{id}: M \to M$ under the identificaions $i^*\mu^*M = M$ amd $i^*pr^*M = M$.
(3.13.2) We have $(p_2)^*\alpha_M = (p_3)^*\alpha_M \circ (p_1)^*\alpha_M$ under the identifications $(p_2)^*\mu^*M = (p_1)^*\mu^*M$, $(p_2)^*pr^*M = (p_3)^*pr^*M$ and $(p_1)^*pr^*M = (p_3)^*\mu^*M$.

A morphism $\varphi: M \to N$ in $\tilde{M}HM^G(X)$ is a morphism of mixed Hodge modules satisfying $pr^*\varphi \circ \alpha_M = \alpha_N \circ \mu^*\varphi$. An object of $\tilde{M}HM^G(X)$ is called a *G-equivariant mixed Hodge module* on X.

Note that (3.13.1) is a consequence of (3.13.2) since α_M is an isomorphism.

If M is a mixed Hodge structue, then $(a_X)^*M$ is naturally endowed with a structure of G-equivariant mixed Hodge module, and we call it a *constant* G-equivariant mixed Hodge module. Here a_X is the morphism $X \to (\mathrm{pt})$.

Lemma 3.13.1. *Any G-equivariant mixed Hodge module on G (with respect to the left multiplication) is constant.*

Proof. Let $M \in \mathrm{Ob}(\tilde{M}HM^G(G))$. Let $\iota: (\mathrm{pt}) \to G$ be the embedding by e and let $\iota': G \to G \times G$ be the morphism given by $g \mapsto (g, e)$. Since $\mu \circ \iota' = \mathrm{id}$ and $pr \circ \iota' = \iota \circ a_G$, we have $M = \iota'^* \mu^* M \simeq \iota'^* pr^* M = (a_G)^* \iota^* M$. We can easily check that the action of G on M coincides with the one on the constant mixed Hodge module $(a_G)^* \iota^* M$. □

Set $MHS^G = \tilde{M}HM^G((\mathrm{pt}))$.

Lemma 3.13.2. *The abelian category MHS^G is naturally equivalent to MHS^{G/G^0}, where G^0 is the connected component of G containing the identity element e (Note that G/G^0 is a finite group by Lemma 1.6.4).*

Proof. This follows from the fact that, for any $M \in \mathrm{Ob}(MHS^G)$, the restriction of $\alpha_M: (a_G)^*M \to (a_G)^*M$ to each connected component of G comes from an automorphism of M in MHS. □

The following theorem can be easily proven by Lemma 3.13.2 and Proposition 3.12.1.

Theorem 3.13.3. *Let G be an affine group scheme and H an essentially smooth closed subgroup scheme of G. Assume that H acts locally freely on G and that G/H is separated and essentially smooth. Let i: $(\mathrm{pt}) \to G/H$ be the embedding by $e \in G$. Then $M \mapsto i^*M$ gives an equivalence: $\tilde{M}HM^G(G/H) \xrightarrow{\sim} MHS^{H/H^0}$.*

4. Kac-Moody Lie algebras and flag varieties

4.1. Let $A = (a_{ij})_{1 \leqq i,j \leqq \ell}$ be a matrix of integers satisfying $a_{ii} = 2$, $a_{ij} \leqq 0\,(i \neq j)$, $a_{ij} \neq 0 \Leftrightarrow a_{ji} \neq 0$. Assume that we are given a finite-dimensional $\mathbf{C}$-vector space $\mathfrak{h}$, and elements $h_1, \ldots, h_\ell \in \mathfrak{h}$, $\alpha_1, \ldots, \alpha_\ell \in \mathfrak{h}^*$ satisfying the following conditions:

$$\text{(4.1.1)} \qquad \langle h_i, \alpha_j \rangle = a_{ij} \quad (i, j = 1, \ldots, \ell),$$

$$\text{(4.1.2)} \qquad \{\alpha_1, \ldots, \alpha_\ell\} \text{ is linearly independent},$$

$$\text{(4.1.3)} \qquad \{h_1, \ldots, h_\ell\} \text{ is linearly independent}.$$

A Kac-Moody Lie algebra associated to these data is a Lie algebra $\mathfrak{g}$ over $\mathbf{C}$ which contains $\mathfrak{h}$ as an abelian subalgebra and which is generated by $\mathfrak{h}$ and elements $e_1, \ldots, e_\ell$, $f_1, \ldots, f_\ell$ satisfying the following relations:

$$\text{(4.1.4)} \qquad [h, e_i] = \alpha_i(h) e_i, \quad (h \in \mathfrak{h}, i = 1, \ldots, \ell),$$

$$\text{(4.1.5)} \qquad [h, f_i] = -\alpha_i(h) f_i \quad (h \in \mathfrak{h}, i = 1, \ldots, \ell),$$

$$\text{(4.1.6)} \qquad [e_i, f_j] = \delta_{ij} h_i \quad (i, j = 1, \ldots, \ell),$$

$$\text{(4.1.7)} \qquad (\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0 \quad (i \neq j),$$

$$\text{(4.1.8)} \qquad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0 \quad (i \neq j).$$

4.2. For $i = 1, \ldots, \ell$, let s_i be the linear automorphism of $\mathfrak{h}^*$ given by

$$\text{(4.2.1)} \qquad s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i.$$

The Weyl group W of $(\mathfrak{g}, \mathfrak{h})$ is the subgroup of $\operatorname{Aut}(\mathfrak{h}^*)$ generated by $S = \{ s_1, \ldots, s_\ell \}$. It is well known that (W, S) is a Coxeter group with

$a_{ij}a_{ji}$	0	1	2	3	$\geqq 4$
$\operatorname{ord}(s_i s_j)$	2	3	4	6	∞

for $i \neq j$. We denote the length function and the Bruhat order on W by ℓ and $\geqq$, respectively.

Set

$$\text{(4.2.2)} \qquad \mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \,;\, [h, x] = \alpha(h) x \,(h \in \mathfrak{h}) \} \quad (\alpha \in \mathfrak{h}^*),$$

$$\text{(4.2.3)} \qquad \Delta = \{ \alpha \in \mathfrak{h}^* \setminus \{0\} \,;\, \mathfrak{g}_\alpha \neq 0 \},$$

$$\text{(4.2.4)} \qquad \Delta^+ = \Delta \cap \sum_{i=1}^{\ell} \mathbf{Z}_{\geqq 0} \alpha_i, \quad \Delta^- = \Delta \cap \sum_{i=1}^{\ell} \mathbf{Z}_{\leqq 0} \alpha_i.$$

Let $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) be the subalgebra of $\mathfrak{g}$ generated by $e_1, \ldots, e_\ell$ (resp. $f_1, \ldots, f_\ell$). Then we have

$$\mathfrak{n} = \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \oplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha, \tag{4.2.5}$$

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}. \tag{4.2.6}$$

Set

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}, \quad \mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-, \tag{4.2.7}$$

$$\mathfrak{g}_i = \mathfrak{h} \oplus \mathbf{C}e_i \oplus \mathbf{C}f_i \quad (i = 1, \ldots, \ell), \tag{4.2.8}$$

(4.2.9)

$$\mathfrak{n}_i = \oplus_{\alpha \in \Delta^+ \setminus \{\alpha_i\}} \mathfrak{g}_\alpha, \quad \mathfrak{n}_i^- = \oplus_{\alpha \in \Delta^- \setminus \{-\alpha_i\}} \mathfrak{g}_\alpha \quad (i = 1, \ldots, \ell),$$

$$\mathfrak{p}_i = \mathfrak{g}_i \oplus \mathfrak{n}_i \quad \mathfrak{p}_i^- = \mathfrak{g}_i \oplus \mathfrak{n}_i^- \quad (i = 1, \ldots, \ell). \tag{4.2.10}$$

They are subalgebras of $\mathfrak{g}$.

4.3. We shall define groups corresponding to certain subalgebras of $\mathfrak{g}$ (see [M], [K2]). Fix a $\mathbf{Z}$-lattice P in $\mathfrak{h}^*$ satisfying

$$\alpha_i \in P, \quad \langle h_i, P \rangle \subset \mathbf{Z} \quad (i = 1, \ldots, \ell). \tag{4.3.1}$$

Let

(4.3.2) $T = \mathrm{Spec}(\mathbf{C}[P])$,
(4.3.3) $U = \varprojlim_k \exp(\mathfrak{n}/(\mathrm{ad}\,\mathfrak{n})^k \mathfrak{n})$, $U^- = \varprojlim_k \exp(\mathfrak{n}^-/(\mathrm{ad}\,\mathfrak{n}^-)^k \mathfrak{n}^-)$,
(4.3.4) B (resp. B^-) is the semi-direct product of T and U (resp. U^-),
(4.3.5) G_i is the algebraic group with $G_i \supset T, \mathrm{Lie}(G_i) = \mathfrak{g}_i, \mathrm{Lie}(T) = \mathfrak{h}$,
(4.3.6) $U_i = \varprojlim_k \exp(\mathfrak{n}_i/(\mathrm{ad}\,\mathfrak{n})^k \mathfrak{n}_i)$, $U_i^- = \varprojlim_k \exp(\mathfrak{n}_i^-/(\mathrm{ad}\,\mathfrak{n}^-)^k \mathfrak{n}_i^-)$,
(4.3.7) P_i (resp. P_i^-) is the semi-direct product of G_i and U_i (resp. U_i^-).

Here we denote by $\exp(\mathfrak{a})$ the unipotent algebraic group corresponding to a finite-dimensional nilpotent Lie algebra $\mathfrak{a}$. The groups defined above are naturally endowed with group scheme structures (see [M], [K2]).

4.4. In [K2] the first-named author has given a scheme theoretic construction of the flag variety of $(\mathfrak{g}, \mathfrak{h}, P)$. It is the quotient $X = G/B^-$, where G is the scheme defined in [K2] which has a locally free action of B^-. Let $x_0 = (1 \mod B^-) \in X$ and set $X_w = Bwx_0 \subset X$ for $w \in W$. As in the finite-dimensional case we have the following.

Proposition 4.4.1 ([K2]).
(i) *X_w is an affine scheme with codimension $\ell(w)$ in X.*

(ii) $X = \sqcup_{w \in W} X_w$.

(iii) $\overline{X}_w = \sqcup_{z \geqq w} X_z$.

For $i = 1, \dots, \ell$ set $X^i = G/P_i^-$, $x_i = (1 \mod P_i^-) \in X^i$ (P_i^- acts on G locally freely). Let $q_i : X \to X^i$ be the natural morphism.

Proposition 4.4.2 ([K3]).

(i) q_i *is a* $\mathbf{P}^1$*-bundle.*

(ii) $X^i = \sqcup_{\ell(ws_i) > \ell(w)} Bwx_i$.

(iii) $q_i^{-1}(Bwx_i) = X_w \sqcup X_{ws_i}$.

(iv) q_i *induces an isomorphism* $Bwx_0 \simeq Bwx_i$ *for* $\ell(ws_i) < \ell(w)$.

(v) q_i *induces an* $\mathbf{A}^1$*-bundle* $Bwx_0 \to Bwx_i$ *for* $\ell(ws_i) > \ell(w)$.

Lemma 4.4.3. *Any* B*-invariant quasi-compact open subset of* X *or* X^i *satisfies* (S).

Proof. The proof being similar, we shall prove the theorem only for X. Let Ω be a B-invariant quasi-compact open subset of X. Then there exists a finite subset J of W such that $\Omega = \cup_{w \in J} Bwx_0 = \cup_{w \in J} wBx_0$. Let Θ be a subset of Δ^+ such that $\Delta^+ \setminus \Theta$ is a finite set, $(\Theta + \Theta) \cap \Delta^+ \subset \Theta$ and $w^{-1}\Theta \subset \Delta^+$ for any $w \in J$. We denote by U_Θ the closed subgroup of U corresponding to $\mathfrak{n}_\Theta = \sum_{\alpha \in \Theta} \mathfrak{g}_\alpha$; i.e. $U_\Theta = \varprojlim_k \exp(\mathfrak{n}_\Theta / (\mathrm{ad}\, \mathfrak{n})^k \mathfrak{n}_\Theta)$. For $w \in J$ the action of U_Θ on wBx_0 is equivalent to the action of $w^{-1} U_\Theta w$ ($\subset U$) on Bx_0, and hence U_Θ acts on wBx_0 freely. Thus Ω / U_Θ exists and it is a quasi-compact smooth $\mathbf{C}$-scheme. Since $\Omega \simeq \varprojlim_\Theta \Omega / U_\Theta$, the assertion follows from Lemma 1.2.1. □

4.5. Let us recall the results of [K3]. Assume that $\mathfrak{g}$ is *symmetrizable* until the end of §4. For $\lambda \in P$, let $\mathcal{O}_X(\lambda)$ be the corresponding invertible $\mathcal{O}_X$-module. Set $\mathcal{D}_\lambda = \mathcal{O}_X(\lambda) \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\lambda)$ and $\mathcal{F}(\lambda) = \mathcal{O}_X(\lambda) \otimes_{\mathcal{O}_X} \mathcal{F}$ for an $\mathcal{O}_X$-module $\mathcal{F}$. Note that, if $\mathfrak{M}$ is a $\mathcal{D}_X$-module, then $\mathfrak{M}(\lambda)$ is a $\mathcal{D}_\lambda$-module. For $w \in W$ set $\mathfrak{B}_w = \mathcal{H}^{\ell(w)}_{X_w}(\mathcal{O}_X)$, where $\ell(w)$ is the length of w. Let $\mathfrak{M}_w$ be the dual of the $\mathcal{D}_X$-module $\mathfrak{B}_w$ and let $\mathfrak{L}_w$ be the image of the unique non-zero homomorphism $\mathfrak{M}_w \to \mathfrak{B}_w$. Then $\mathfrak{L}_w$ is the minimal extension of $\mathfrak{B}_w | wBx_0$.

4.6. For $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$ be the Verma module with highest weight λ, $M^*(\lambda)$ the $\mathfrak{h}$-finite part of the dual of the Verma module with lowest weight $-\lambda$ and $L(\lambda)$ the image of the unique non-zero homomorphism $M(\lambda) \to M^*(\lambda)$. Then $L(\lambda)$ is the irreducible module with highest weight λ.

Set $w \circ \lambda = w(\lambda + \rho) - \rho$ for $w \in W$ and $\lambda \in \mathfrak{h}^*$, where ρ is an element of $\mathfrak{h}^*$ such that $\langle h_i, \rho \rangle = 1$ for any i.

4.7. Let $\lambda \in P_+ = \{\lambda \in P; \langle h_i, \lambda \rangle \geqq 0 \text{ for any } i\}$. For a B-equivariant $\mathcal{D}_\lambda$-module $\mathfrak{M}$ we set

$$\tilde{H}^n(X; \mathfrak{M}) = \oplus_{\mu \in P} \varprojlim_{\Omega} (H^n(\Omega; \mathfrak{M}))_\mu, \tag{4.7.1}$$

$$\tilde{\Gamma}(X; \mathfrak{M}) = \tilde{H}^0(X; \mathfrak{M}), \tag{4.7.2}$$

where Ω ranges over B-invariant quasi-compact open subsets of X, and for a semisimple $\mathfrak{h}$-module M the weight space with weight μ is denoted by M_μ. By [K3, Theorem 5.2.1] we have

$$\tilde{H}^n(X; \mathfrak{M}) = 0 \text{ for any } n \neq 0, \tag{4.7.3}$$

$$\tilde{\Gamma}(X; \mathfrak{M}_w(\lambda)) = M(w \circ \lambda), \tag{4.7.4}$$

$$\tilde{\Gamma}(X; \mathfrak{B}_w(\lambda)) = M^*(w \circ \lambda), \tag{4.7.5}$$

$$\tilde{\Gamma}(X; \mathfrak{L}_w(\lambda)) = L(w \circ \lambda). \tag{4.7.6}$$

4.8. Our main theorem is the following.

Theorem 4.8.1. *Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra. Then, for $\lambda \in P_+$ and $w \in W$, we have:*

$$\operatorname{ch} L(w \circ \lambda) = \sum_{z \geqq w} (-1)^{\ell(z) - \ell(w)} Q_{w,z}(1) \operatorname{ch} M(z \circ \lambda),$$

where $Q_{w,z}$ is the inverse Kazhdan-Lusztig polynomial (see [KL2] *and* §5.3 *below).*

In order to prove this theorem, it is sufficient to show that, for any B-invariant quasi-compact open subset Ω, we have:

$$[\mathfrak{L}_w | \Omega] = \sum_{z \geqq w} (-1)^{\ell(z) - \ell(w)} Q_{w,z}(1) [\mathfrak{B}_z | \Omega] \tag{4.8.1}$$

in the Grothendieck group of the abelian category of B-equivariant holonomic $\mathcal{D}_\Omega$-modules. Note that $M(w \circ \lambda)$ and $M^*(w \circ \lambda)$ have the same characters. Since we have

$$\operatorname{Sol}(\mathfrak{L}_w) = {}^\pi \mathbf{C}_{X_w}[-\ell(w)], \tag{4.8.2}$$

$$\operatorname{Sol}(\mathfrak{B}_w) = \mathbf{C}_{X_w}[-\ell(w)], \tag{4.8.3}$$

this is again reduced to:

$$(4.8.4) \quad [{}^{\pi}\mathbf{C}_{X_w}[-\ell(w)]|\Omega] = \sum_{z \geqq w} (-1)^{\ell(z)-\ell(w)} Q_{w,z}(1)[\mathbf{C}_{X_z}[-\ell(z)]|\Omega]$$

in the Grothendieck group of the abelian category of B-equivariant perverse sheaves on Ω.

The last statement will be proven for any (not necessarily symmetrizable) Kac-Moody Lie algebras in §6 by the aid of mixed Hodge modules.

5. Hecke-Iwahori Algebras

5.0. In this section W denotes a Coxeter group with canonical generator system S. The length function and the Bruhat order on W are denoted by ℓ and $\geqq$, respectively.

5.1. The Hecke-Iwahori algebra $H(W)$ is the associative algebra over the Laurent polynomial ring $\mathbf{Z}[q, q^{-1}]$ which has a free $\mathbf{Z}[q, q^{-1}]$-basis $\{T_w\}_{w \in W}$ satisfying the following relations:

$$(5.1.1) \qquad (T_s + 1)(T_s - q) = 0 \quad \text{for} \quad s \in S,$$

$$(5.1.2) \qquad T_{w_1} T_{w_2} = T_{w_1 w_2} \quad \text{if} \quad \ell(w_1) + \ell(w_2) = \ell(w_1 w_2).$$

Let $h \mapsto \overline{h}$ be the automorphism of the ring $H(W)$ given by

$$(5.1.3) \qquad \overline{q} = q^{-1}, \qquad \overline{T_w} = T_{w^{-1}}^{-1},$$

and define $R_{y,w} \in \mathbf{Z}[q, q^{-1}]$ for $y, w \in W$ by

$$(5.1.4) \qquad \overline{T_w} = \sum_{y \in W} \overline{R}_{y,w} q^{-\ell(y)} T_y.$$

The following is easily checked by direct calculations (see [KL1]).

(5.1.5) $R_{y,w} \neq 0$ if and only if $y \leqq w$.
(5.1.6) $R_{y,w}$ is a ploynomial in q with degree $\ell(w) - \ell(y)$ for $y \leqq w$.
(5.1.7) $R_{w,w} = 1$.

Following [KL1] we introduce a free $\mathbf{Z}[q, q^{-1}]$-basis $\{C_w\}_{w \in W}$ of $H(W)$.

Proposition 5.1.1 ([KL1]). *For $w \in W$ there exists a unique element*
$$C_w = \sum_{y \leqq w} (-q)^{\ell(w)-\ell(y)} \overline{P}_{y,w} T_y \in H(W)$$
satisfying the following conditions:

(a) $P_{w,w} = 1$.
(b) *If $y < w$, then $P_{y,w}$ is a polynomial in q with degree $\leqq (\ell(w) - \ell(y) - 1)/2$.*
(c) $\overline{C}_w = q^{-\ell(w)} C_w$.

We set $P_{y,w} = 0$ if $y \nleqq w$.

5.2. Set $H^*(W) = \mathrm{Hom}_{\mathbf{Z}[q,q^{-1}]}(H(W), \mathbf{Z}[q, q^{-1}])$. For $w \in W$ let S_w be the element of $H^*(W)$ determined by

$$\langle S_w, \overline{T}_y \rangle = \delta_{w,y^{-1}} q^{-\ell(w)}, \tag{5.2.1}$$

where $\langle \ , \ \rangle$ denotes the natural paring of $H^*(W)$ and $H(W)$. Any element of $H^*(W)$ is uniquely written as a formal infinite sum $\sum_{w \in W} a_w S_w$ ($a_w \in \mathbf{Z}[q, q^{-1}]$).

Define an endomorphism $u \mapsto \overline{u}$ of the abelian group $H^*(W)$ by

$$\langle \overline{u}, h \rangle = \overline{\langle u, \overline{h} \rangle} \qquad (u \in H^*(W), h \in H(W)). \tag{5.2.2}$$

We also define a right $H(W)$-module structure on $H^*(W)$ by

$$\langle u \cdot h_1, h_2 \rangle = \langle u, h_1 h_2 \rangle \qquad (u \in H^*(W), h_1, h_2 \in H(W)). \tag{5.2.3}$$

We can check the following lemma by direct calculations.

Lemma 5.2.1.
(i) $\overline{\sum_{w \in W} a_w S_w} = \sum_{w \in W} q^{\ell(w)} (\overline{\sum_{y \leqq w} a_y R_{y^{-1}, w^{-1}}}) S_w$.
(ii) *For $s \in S$ we have*

$$\left(\sum_{w \in W} a_w S_w\right) \cdot T_s = \sum_{ws > w} ((q-1) a_w + a_{ws}) S_w + \sum_{ws < w} q a_{ws} S_w.$$

(iii) $\langle u, h \rangle = \epsilon(u \cdot h)$ $(u \in H^*(W), h \in H(W))$, *where $\epsilon \colon H^*(W) \to R$ is given by $\epsilon(\sum_{w \in W} a_w S_w) = a_e$.*

5.3. For $w \in W$ we define an element D_w of $H^*(W)$ by

$$\langle D_w, \overline{C}_y \rangle = \delta_{w,y^{-1}} q^{-\ell(w)}. \tag{5.3.1}$$

Set $D_w = \sum_{z \in W} Q_{w,z} S_z$ ($Q_{w,z} \in \mathbf{Z}[q,q^{-1}]$). It is easily seen that $Q_{w,z} = 0$ unless $z \geqq w$, and $Q_{y,w}$ for $y \leqq w$ are uniquely determined by

$$(5.3.2) \qquad \sum_{y \leqq w \leqq z} (-1)^{\ell(w)-\ell(y)} Q_{y,w} P_{w,z} = \delta_{y,z} \quad (y \leqq z).$$

By definition we have the following properties:

(5.3.3) $Q_{w,w} = 1$.
(5.3.4) If $z > w$, then $Q_{w,z}$ is a polynomial in q with degree $\leqq (\ell(z) - \ell(w) - 1)/2$.
(5.3.5) $\overline{D_w} = q^{\ell(w)} D_w$.

Moreover these properties characterize the element D_w. We shall formulate this uniquenes in a more general setting.

Let R be a commutative ring with 1 containing $\mathbf{Z}[q,q^{-1}]$. Assume that we are given a grading $R = \oplus_{i \in \mathbf{Z}} R_i$ and an involutive automorphism $r \mapsto \overline{r}$ of the ring R satisfying

$$(5.3.6) \qquad R_i R_j \subset R_{i+j}, \quad q \in R_2, \quad \overline{R_i} = R_{-i}, \quad \overline{q} = q^{-1}.$$

Set $H_R(W) = R \otimes_{\mathbf{Z}[q,q^{-1}]} H(W)$ and $H_R^*(W) = \mathrm{Hom}_R(H_R(W), R)$. Similarly to (5.2.2) and (5.2.3), we have an involution $u \mapsto \overline{u}$ of $H_R^*(W)$ and a right $H_R(W)$-module structure on $H_R^*(W)$.

Proposition 5.3.1. *Let $w \in W$. If $D'_w = \sum_{z \geqq w} Q'_{w,z} S_z$ ($Q'_{w,z} \in R$) is an element of $H_R^*(W)$ satisfying the following conditions* (a), (b), (c), *then we have $D'_w = D_w$.*

(a) $Q'_{w,w} = 1$.
(b) $Q'_{w,z} \in \oplus_{i \leqq \ell(z)-\ell(w)-1} R_i$ *for* $z > w$.
(c) $\overline{D'_w} = q^{\ell(w)} D'_w$.

Proof. We shall show $Q'_{w,z} = Q_{w,z}$ for $z \geqq w$ by induction on $\ell(z) - \ell(w)$. If $\ell(z) - \ell(w) = 0$, we have $w = z$, and the assertion is trivial. Assume that $z > w$. By Lemma 5.2.1 (i) we have

$$\overline{D'_w} = \sum_{v \geqq w} q^{\ell(v)} \Big(\sum_{v \geqq y \geqq w} \overline{Q'_{w,y} R_{y^{-1},v^{-1}}} \Big) S_v,$$

and hence (c) implies :

$$Q'_{w,z} = q^{\ell(z)-\ell(w)} \Big(\overline{Q'_{w,z}} + \sum_{z > y \geqq w} \overline{Q'_{w,y} R_{y^{-1},z^{-1}}} \Big).$$

By the inductive hypothesis we have

$$Q'_{w,z} - q^{\ell(z)-\ell(w)}\overline{Q'_{w,z}} = q^{\ell(z)-\ell(w)} \sum_{z>y\geqq w} \overline{Q_{w,y}R_{y^{-1},z^{-1}}}. \tag{5.3.7}$$

On the other hand (b) implies

$$Q'_{w,z} \in \oplus_{i\leqq \ell(z)-\ell(w)-1} R_i, \tag{5.3.8}$$

$$q^{\ell(z)-\ell(w)}\overline{Q'_{w,z}} \in \oplus_{i\geqq \ell(z)-\ell(w)+1} R_i, \tag{5.3.9}$$

and hence the equation (5.3.7) uniquely determines $Q'_{w,z}$. Since $Q_{w,z}$ also satisfies the same equation, we have $Q'_{w,z} = Q_{w,z}$. □

6. Hodge modules on flag varieties

6.0. In this section we shall give a proof of (4.8.4) for any (not necessarily symmetrizable) Kac-Moody Lie algebra $\mathfrak{g}$. For an abelian category $\mathcal{A}$ we denote its Grothendieck group by $K(\mathcal{A})$.

6.1. Set $R = K(MHS)$. The abelian group R is naturally endowed with a structure of commutative ring with 1 via the tensor product. Since MHS is an Artinian category, R has a free $\mathbf{Z}$-basis consisting of simple objects. For $i \in \mathbf{Z}$ we denote by R_i the $\mathbf{Z}$-submodule of R generated by the elements corresponding to pure Hodge structures of weight i. Since any simple object of MHS is a pure Hodge structure, we have

$$R = \oplus_{i\in\mathbf{Z}} R_i, \quad R_iR_j \subset R_{i+j}. \tag{6.1.1}$$

In the following we regard $\mathbf{Z}[q,q^{-1}]$ as a subring of R via $q^i = [\mathbf{Q}^H(-i)] \in R_{2i}$, where $\mathbf{Q}^H(-i)$ is the pure Hodge structure of weight $2i$ obtained by twisting the trivial Hodge structure $\mathbf{Q}^H$. Let $r \mapsto \overline{r}$ be the involutive automorphism of the ring R induced by the duality operation in MHS. Then we have

$$\overline{R_i} = R_{-i}, \quad \overline{q} = q^{-1}, \tag{6.1.2}$$

and hence the ring R satisfies the condition (5.3.6).

6.2. We have a natural R-module structure on $K(\tilde{M}HM^B(X_w))$ for $w \in W$.

Lemma 6.2.1. (i) *Any object M of $\tilde{M}HM^B(X_w)$ is isomorphic to a constant B-equivariant mixed Hodge module $\mathbf{Q}^H_{X_w} \otimes L\,(= (a_{X_w})^*(L))$ for some $L \in \mathrm{Ob}(MHS)$.*

(ii) *$K(\tilde{M}HM^B(X_w))$ is a rank one free R-module with basis $[\mathbf{Q}^H_{X_w}]$.*

Proof. This follows from Theorem 3.13.3 since the isotropy group with respect to the action of B on X_w is connected. □

6.3. We say that a subset J of W is ***admissible*** if J is a finite set satisfying the condition:

$$w \in J, \quad y \leqq w \Rightarrow y \in J. \tag{6.3.1}$$

We denote by $\mathcal{C}$ the set of admissible subsets of W. For a subset J of W set $\Omega_J = \sqcup_{w \in J} X_w$. By [K2] we see that Ω_J is a quasi-compact open subset of X if and only if J is admissible.

For admissible subsets J_1, J_2 satisfying $J_1 \subset J_2$, we have a natural functor and a natural homomorphism

$$\tilde{M}HM^B(\Omega_{J_2}) \to \tilde{M}HM^B(\Omega_{J_1}) \tag{6.3.2}$$

$$K(\tilde{M}HM^B(\Omega_{J_2})) \to K(\tilde{M}HM^B(\Omega_{J_1})) \tag{6.3.3}$$

by the restriction, and they give projective systems $\{\tilde{M}HM^B(\Omega_J)\}_{J \in \mathcal{C}}$ and
$\{K(\tilde{M}HM^B(\Omega_J))\}_{J \in \mathcal{C}}$. Set

$$\tilde{M}HM^B(X) = \varprojlim_{J \in \mathcal{C}} \tilde{M}HM^B(\Omega_J), \tag{6.3.4}$$

$$K^B(X) = \varprojlim_{J \in \mathcal{C}} K(\tilde{M}HM^B(\Omega_J)), \tag{6.3.5}$$

and let $p_J \colon K^B(X) \to K(\tilde{M}HM^B(\Omega_J))$ be the projection. The R-module $K^B(X)$ may be regarded as a completion of the Grothendieck group of $\tilde{M}HM^B(X)$.

Let $i_w \colon X_w \to X$ be the inclusion. Let $w \in W$ and $J \in \mathcal{C}$ such that $w \in J$, and let $i_{w,J} \colon X_w \to \Omega_J$ be the inclusion. We define objects $(i_w)_!\mathbf{Q}^H_{X_w}$ and ${}^\pi\mathbf{Q}^H_{X_w}$ of $D^b(\tilde{M}HM^B(X))$ by

$$(i_w)_!\mathbf{Q}^H_{X_w}|\Omega_J = (i_{w,J})_!(\mathbf{Q}^H_{X_w}), \tag{6.3.6}$$

(6.3.7)

$${}^\pi\mathbf{Q}^H_{X_w}|\Omega_J = (\text{ the minimal extension of } \mathbf{Q}^H_{X_w} \text{ with respect to } i_{w,J}).$$

Set $[M] = \sum_{k\in\mathbf{Z}}(-1)^k[H^k(M)]$ for $M \in \mathrm{Ob}(D^b(\tilde{M}HM(\Omega_J)))$. We have elements $[(i_w)_!\mathbf{Q}^H_{X_w}]$ and $[{}^\pi\mathbf{Q}^H_{X_w}]$ of $K^B(X)$ satisfying

$$p_J([(i_w)_!\mathbf{Q}^H_{X_w}]) = [(i_w)_!\mathbf{Q}^H_{X_w}|\Omega_J], \tag{6.3.8}$$

$$p_J([{}^\pi\mathbf{Q}^H_{X_w}]) = [{}^\pi\mathbf{Q}^H_{X_w}|\Omega_J]. \tag{6.3.9}$$

We nextly define an R-homomorhism

$$(i_w)^* : K^B(X) \to K(\tilde{M}HM^B(X_w)) \tag{6.3.10}$$

as follows. For an admissible subset J such that $w \in J$, we have an R-homomorphism

$$(i_{w,J})^* : K(\tilde{M}HM^B(\Omega_J)) \to K(\tilde{M}HM^B(X_w))$$

given by

$$(i_{w,J})^*([M]) = \sum_{k\in\mathbf{Z}}(-1)^k[(H^k(i_{w,J})^*)(M)]$$

for $M \in \mathrm{Ob}(K(\tilde{M}HM^B(\Omega_J)))$, and (6.3.10) is defined by

$$(i_w)^*(m) = (i_{w,J})^*(p_J(m)). \tag{6.3.11}$$

For $m \in K^B(X)$ and $w \in W$ we define $\varphi_w(m) \in R$ by

$$(i_w)^*(m) = \varphi_w(m)[\mathbf{Q}^H_{X_w}] \tag{6.3.12}$$

(see Lemma 6.2.1). We also define an R-homomorphism $\varphi\colon K^B(X) \to H^*_R(W)$ by

$$\varphi(m) = \sum_{w\in W}\varphi_w(m)S_w. \tag{6.3.13}$$

Lemma 6.3.1. (i) $\varphi([(i_w)_!\mathbf{Q}^H_{X_w}]) = S_w$
(ii) *φ is an isomorphism of R-modules.*

Proof. (i) is clear. Let us show (ii). Let J be an admissible subset of W. Since $\tilde{M}HM^B(\Omega_J)$ is an Artinian category, its Grothendieck group has a free $\mathbf{Z}$-basis consisting of the simple objects. Since any simple object of $\tilde{M}HM^B(\Omega_J)$ is isomorphic to $({}^\pi\mathbf{Q}^H_{X_w}|\Omega_J)[-\ell(w)]\otimes L$ for some $w \in J$ and some simple object L of MHS, we see that $K(\tilde{M}HM^B(\Omega_J))$ is a free R-module with basis $\{[{}^\pi\mathbf{Q}^H_{X_w}|\Omega_J]\,;\, w \in J\}$. Since we have

$$[{}^\pi\mathbf{Q}^H_{X_w}|\Omega_J] \in [(i_w)_!\mathbf{Q}^H_{X_w}|\Omega_J] + \sum_{\substack{y\in J\\ y>w}} R[(i_y)_!\mathbf{Q}^H_{X_y}|\Omega_J]$$

for $w \in J$, $\{[(i_w)_!\mathbf{Q}^H_{X_w}|\Omega_J]\,;\, w \in J\}$ is also a free basis of the R-module $K(\tilde{M}HM^B(\Omega_J))$. Therefore the assertion follows from (i). □

6.4. We shall define an R-homomorphism

$$\tau_i\colon K^B(X) \to K^B(X) \tag{6.4.1}$$

for each $i = 1, \dots, \ell$ as follows. Let $\mathcal{C}_i$ be the set of admissible subsets J of W such that $ws_i \in J$ if $w \in J$. For $J \in \mathcal{C}_i$ let $q_{i,J}\colon \Omega_J \to q_i(\Omega_J)$ be the restriction of $q_i\colon X \to X^i$ and define an endomorphism $\tau_{i,J}$ of the R-module $K(\tilde{M}HM^B(\Omega_J))$ by

$$\tau_{i,J}([M]) = [(q_{i,J})^*(q_{i,J})_!M] \quad \text{for} \quad M \in \mathrm{Ob}(\tilde{M}HM^B(\Omega_J)).$$

Since $q_{i,J}$ is a B-equivariant $\mathbf{P}^1$-bundle, $\tau_{i,J}$ is well-defined. Then we define an endomorphism τ_i of $K^B(X) = \varprojlim_{J\in\mathcal{C}_i} K(\tilde{M}HM^B(\Omega_J))$ by $\tau_i = \varprojlim_J \tau_{i,J}$.

Lemma 6.4.1. *$\varphi(\tau_i(m)) = \varphi(m)\cdot(T_{s_i}+1)$ for $m \in K^B(X)$.*

Proof. By Lemma 5.2.1 (ii) and Lemma 6.3.1 it is sufficient to show

$$\tau_i([(i_w)_!\mathbf{Q}^H_{X_w}]) = \begin{cases} [(i_{ws_i})_!\mathbf{Q}^H_{X_{ws_i}}] + [(i_w)_!\mathbf{Q}^H_{X_w}] & (ws_i < w) \\ q([(i_{ws_i})_!\mathbf{Q}^H_{X_{ws_i}}] + [(i_w)_!\mathbf{Q}^H_{X_w}]) & (ws_i > w). \end{cases}$$

Let $J \in \mathcal{C}_i$ such that $w \in J$. Set $\tilde{X}_w = q_i^{-1}q_i(X_w) = X_w \sqcup X_{ws_i}$ and let $j_{w,J}\colon \tilde{X}_w \to \Omega_J$ be the inclusion. Since $\tilde{X}_w \to q_i(\tilde{X}_w)$ is a $\mathbf{P}^1$-bundle and since $X_w \to q_i(\tilde{X}_w)$ is an isomorphism (resp. $\mathbf{A}^1$-bundle) for $ws_i < w$ (resp. $ws_i > w$), we have

$$(q_{i,J})^*(q_{i,J})_!((i_{w,J})_!\mathbf{Q}^H_{X_w}) = \begin{cases} (j_{w,J})_!\mathbf{Q}^H_{\tilde{X}_w} & (ws_i < w) \\ (j_{w,J})_!\mathbf{Q}^H_{\tilde{X}_w}[-2](-1) & (ws_i > w). \end{cases}$$

On the other hand, if $ws_i > w$, we have an exact sequence:

$$\begin{aligned} 0 \to (i_{ws_i,J})_!\mathbf{Q}^H_{X_{ws_i}}[-\ell(w)-1] &\to (i_{w,J})_!\mathbf{Q}^H_{X_w}[-\ell(w)] \\ &\to (j_{w,J})_!\mathbf{Q}^H_{\tilde{X}_w}[-\ell(w)] \to 0 \end{aligned}$$

in $\tilde{M}HM^B(\Omega_J)$. Hence the assertion is clear. □

By Lemma 6.3.1 and Lemma 6.4.1 we can define a right $H(W)$-module structure on $K^B(X)$ by

$$m \cdot (T_{s_i} + 1) = \tau_i(m) \quad (m \in K^B(X)). \tag{6.4.2}$$

6.5. We denote by $m \mapsto m^*$ the endomorphisms of the abelian groups $K^B(X)$ and $K(\tilde{M}HM^B(X_e))$ induced by the duality operation of mixed Hodge modules.

Lemma 6.5.1. *$\varphi(m^*) = \overline{\varphi(m)}$ for $m \in K^B(X)$.*

Proof. We have to show $\langle \varphi(m^*), h \rangle = \overline{\langle \varphi(m), \overline{h} \rangle}$ for $m \in K^B(X)$, $h \in H(W)$. By Lemma 5.2.1 (iii) and §§6.3, 6.4 this is equivalent to

$$(i_e)^*(m^* \cdot T_z) = ((i_e)^*(m \cdot \overline{T}_z))^* \quad (m \in K^B(X), z \in W), \tag{6.5.1}$$

where the right action of $H(W)$ on $K^B(X)$ is given by (6.4.2). Let us prove (6.5.1) by induction on $\ell(z)$. The case $z = e$ being trivial, we take $w \in W$ satisfying $s_i w > w$ and prove (6.5.1) for $z = s_i w$ assuming (6.5.1) for $z = w$.

Let $J \in \mathcal{C}_i$. Since $q_{i,J}$ is a $\mathbf{P}^1$-bundle, we have

$$(q_{i,J})^*(q_{i,J})_! \mathbf{D}_{\Omega_J}(M) = (\mathbf{D}_{\Omega_J}(q_{i,J})^*(q_{i,J})_!(M))[-2](-1) \tag{6.5.2}$$

for $M \in \mathrm{Ob}(\tilde{M}HM^B(\Omega_J))$ and hence we have

$$\tau_i(m^*) = (q^{-1}\tau_i(m))^* \quad (m \in K^B(X)). \tag{6.5.3}$$

Therefore we have

$$\begin{aligned}(i_e)^*(m^* \cdot T_{s_i w}) &= (i_e)^*((\tau_i(m^*) - m^*) \cdot T_w) \\ &= (i_e)^*((q^{-1}\tau_i(m) - m)^* \cdot T_w) \\ &= ((i_e)^*((q^{-1}\tau_i(m) - m) \cdot \overline{T}_w))^* \\ &= ((i_e)^*(m \cdot \overline{T}_{s_i w}))^*. \quad \square\end{aligned}$$

6.6. We shall determine $H^i((i_{z,J})^*({}^\pi\mathbf{Q}^H_{X_w}|\Omega_J))$ for any admissible subset J and $z, w \in J$. Since this does not depend on J, we simply denote it by $H^i((i_z)^*({}^\pi\mathbf{Q}^H_{X_w}))$.

We first give a weaker result.

Proposition 6.6.1. $\varphi([{}^\pi\mathbf{Q}^H_{X_w}]) = D_w$ *for* $w \in W$.

Proof. Setting $Q'_{w,z} = \varphi_z([{}^\pi\mathbf{Q}^H_{X_w}]) \in R$ and $D'_w = \sum_{z \geqq w} Q'_{w,z} S_z \in H^*_R(W)$, we have $\varphi([{}^\pi\mathbf{Q}^H_{X_w}]) = D'_w$. Hence by Proposition 5.3.1, it is sufficient to show the following conditions:

(6.6.1) $Q'_{w,w} = 1$.
(6.6.2) $Q'_{w,z} \in \oplus_{i \leqq \ell(z)-\ell(w)-1} R_i$ for $z > w$.
(6.6.3) $\overline{D'_w} = q^{\ell(w)} D'_w$.

(6.6.1) is trivial, and (6.6.3) follows from Lemma 6.5.1 and

$$\mathbf{D}_{\Omega_J}({}^\pi\mathbf{Q}^H_{X_w}|\Omega_J) = ({}^\pi\mathbf{Q}^H_{X_w}|\Omega_J)[-2\ell(w)](-\ell(w)). \tag{6.6.4}$$

Let us show (6.6.2). Let $z > w$. Since ${}^\pi\mathbf{Q}^H_{X_w}|\Omega_J$ is pure of weight 0, $(i_{z,J})^*({}^\pi\mathbf{Q}^H_{X_w}|\Omega_J)$ is of weight $\leqq 0$, and hence $H^i((i_{z,J})^*({}^\pi\mathbf{Q}^H_{X_w}|\Omega_J))$ is of weight $\leqq i$. On the other hand we have $H^i((i_{z,J})^*({}^\pi\mathbf{Q}^H_{X_w}|\Omega_J)) = 0$ for $i \geqq \ell(z)-\ell(w)$ by the definition. Therefore $Q'_{w,z} \in \oplus_{i \leqq \ell(z)-\ell(w)-1} R_i$.
□

Lemma 6.6.2. *Let Y be an irreducible closed subvariety of $\mathbf{C}^n$ such that there exist integers $a_1, \dots, a_n > 0$ satisfying*

$$z \in \mathbf{C}^*, \quad (z_1, \dots, z_n) \in Y \Rightarrow (z^{a_1} z_1, \dots, z^{a_n} z_n) \in Y, \tag{6.6.5}$$

and let $i \colon \{0\} \to Y$ be the inclusion. Then $H^j(i^*({}^\pi\mathbf{Q}^H_Y))$ is a pure Hodge structure of weight j.

The proof is similar to [KL2, Lemma 4.5].

Lemma 6.6.3. $H^j((i_z)^*({}^\pi\mathbf{Q}^H_{X_w}))$ *is pure of weight j.*

Proof. We may assume that $z > w$. Let $x \in X_z$. By [K2, Remark 4.5.14] we can take an open neighborhood V of x in X such that there exists a commutative diagram

$$\begin{array}{ccccc} X_z \cap V & \longrightarrow & \overline{X}_w \cap V & \longrightarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} \times \mathbf{A}^\infty & \longrightarrow & Y \times \mathbf{A}^\infty & \longrightarrow & \mathbf{C}^n \times \mathbf{A}^\infty \end{array} \tag{6.6.6}$$

where Y is an irreducible closed subvariety of $\mathbf{C}^n$ satisfying the assumption of Lemma 6.6.2, the horizontal arrows are the natural inclusions and the vertical arrows are isomorphisms. Hence the assertion follows from Lemma 6.6.2. □

Set $Q_{w,z} = \sum_j c_{w,z,j} q^j$ ($c_{w,z,j} \in \mathbf{Z}$) for $z, w \in W$ with $z \geqq w$.

Theorem 6.6.4. *Let $z \geqq w$.*

(i) $H^{2j+1}((i_z)^*({}^\pi \mathbf{Q}^H_{X_w})) = 0$ *for any* $j \in \mathbf{Z}$.

(ii) *For any* $j \in \mathbf{Z}$ *we have* $c_{w,z,j} \geqq 0$, *and* $H^{2j}((i_z)^*({}^\pi \mathbf{Q}^H_{X_w}))$ *is isomorphic to* $(\mathbf{Q}^H_{X_z}(-j))^{\oplus c_{w,z,j}}$.

Proof. By Theorem 3.13.3 there exist some $N_k \in \mathrm{Ob}(MHS)$ ($k \in \mathbf{Z}$) such that $H^k((i_z)^*({}^\pi \mathbf{Q}^H_{X_w})) = (a_{X_z})^*(N_k)$. Then we see from Proposition 6.6.1 that

$$\sum_j c_{w,z,j} q^j = \sum_{k \in \mathbf{Z}} (-1)^k [N_k]. \tag{6.6.7}$$

Since $[N_k] \in R_k$ by Lemma 6.6.3, we have $[N_{2j+1}] = 0$ and $[N_{2j}] = c_{w,z,j} q^j$, and this implies that $N_{2j+1} = 0$ and $N_{2j} = (\mathbf{Q}^H)^{\oplus c_{w,z,j}}$. □

It is easily seen that (4.8.4) is a consequence of Theorem 6.6.4 (or even Lemma 6.6.1).

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