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Perfect Crystals and q-deformed Fock Spaces

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Abstract. In [S], [KMS] the semi-infinite wedge construction of level 1 $U_q(A_n^{(1)})$ Fock spaces and their decomposition into the tensor product of an irreducible $U_q(A_n^{(1)})$-module and a bosonic Fock space were given. Here a general scheme for the wedge construction of q-deformed Fock spaces using the theory of perfect crystals is presented.

Let $U_q(\mathfrak{g})$ be a quantum affine algebra. Let V be a finite-dimensional $U_q'(\mathfrak{g})$-module with a perfect crystal base of level l. Let $V_{\rm aff} \simeq V \otimes \mathbb{C}[z, z^{-1}]$ be the affinization of V, with crystal base $(L_{\rm aff}, B_{\rm aff})$. The wedge space $V_{\rm aff} \wedge V_{\rm aff}$ is defined as the quotient of $V_{\rm aff} \otimes V_{\rm aff}$ by the subspace generated by the action of $U_q(\mathfrak{g})[z^a \otimes z^b + z^b \otimes z^a]_{a,b\in\mathbb{Z}}$ on $v \otimes v$ (v an extremal vector). The wedge space $\bigwedge^r V_{\rm aff}$ ($r \in \mathbb{N}$) is defined similarly. Normally ordered wedges are defined by using the energy function $H : B_{\rm aff} \otimes B_{\rm aff} \to \mathbb{Z}$. Under certain assumptions, it is proved that normally ordered wedges form a base of $\bigwedge^r V_{\rm aff}$.

A q-deformed Fock space is defined as the inductive limit of $\bigwedge^r V_{\rm aff}$ as $r \to \infty$, taken along the semi-infinite wedge associated to a ground state sequence. It is proved that normally ordered wedges form a base of the Fock space and that the Fock space has the structure of an integrable $U_q(\mathfrak{g})$-module. An action of the bosons, which commute with the $U_q'(\mathfrak{g})$-action, is given on the Fock space. It induces the decomposition of the q-deformed Fock space into the tensor product of an irreducible $U_q(\mathfrak{g})$-module and a bosonic Fock space.

As examples, Fock spaces for types $A_{2n}^{(2)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$ at level 1 and $A_1^{(1)}$ at level k are constructed. The commutation relations of the bosons in each of these cases are calculated, using two point functions of vertex operators.

Contents

1. Introduction

Let $\mathfrak{g}$ be an affine Lie algebra. The construction of integrable highest weight modules for $\mathfrak{g}$ has been studied extensively for more than 15 years, with applications to problems in mathematical physics like soliton equations and conformal field theories. More recently, a further item was added to the list of interactions between representation theory and integrable systems: the link between quantum affine algebras, $U_q(\mathfrak{g})$, and solvable lattice models (see [JM] and references therein).

The link is twofold: (a) the R-matrices, which appear as the Boltzmann weights of solvable lattice models, are intertwiners of level 0 $U_q(\mathfrak{g})$-modules, and (b) the irreducible integrable highest weight modules for $U_q(\mathfrak{g})$ appear as the spaces of the eigenvectors of the corner transfer matrices. This suggests a construction of integrable highest weight modules by means of semi-infinite tensor products of level 0 modules. In fact, in the crystal limit, such a construction was given for a large class of representations known as the representations with perfect crystals [KMN1].

The idea of using Fock spaces of bosons or fermions goes back to earlier works before the above link was found. In fact, the literature is vast. Let us mention some of the works that are closely related to the present work. In [LW], [KKLW], bosonic Fock spaces were used to construct some level 1 highest weight modules of affine Lie algebras using the fact that the actions of the principal Heisenberg subalgebras are irreducible. In [DJKM] the level 1 highest weight modules of $\mathfrak{gl}_\infty$ were constructed in the fermionic Fock space. By the boson-fermion correspondence one has the action of bosons on the Fock space. The action of affine Lie algebras such as $\widehat{\mathfrak{sl}}_n$, as subalgebras of $\mathfrak{gl}_\infty$, was then realized as the commutant of bosons of degree divisible by n. Likewise, level 1 highest weight modules of other affine Lie algebras $\mathfrak{g}$ were constructed by realizing $\mathfrak{g}$ as a subalgebra of $\mathfrak{go}_\infty$ (see also [JY]) or $\mathfrak{go}_{2\infty}$.

Under the influence of quantum groups, several further developments were made in this direction. A q-deformed construction of the fermion Fock space was achieved in [H]. In [MM], this was connected to the crystal base theory of Kashiwara [K1].

These works and the developments in solvable lattice models led to the semi-infinite construction of affine crystals mentioned above.

Very recently, in [S], Stern gave a semi-infinite construction of the level 1 Fock spaces for $U_q(\mathfrak{g})$ when $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$. Subsequently, in [KMS], the decomposition of the Fock spaces into the level 1 irreducible highest weight modules and the bosonic Fock space was given. In the present paper, we give a similar construction of Fock spaces and their decomposition, for various cases in the class of representations with perfect crystals. The case in [S], [KMS] corresponds to the perfect crystal of level 1 for $A_n^{(1)}$. Here we treat

$$\text{level 1 } A_{2n}^{(2)},\ B_n^{(1)},\ A_{2n-1}^{(2)},\ D_n^{(1)},\ D_{n+1}^{(2)} \quad \text{and} \quad \text{level } k\ A_1^{(1)}.$$

In order to handle these cases, we not only follow the basic strategy in [S], [KMS], but also develop some new machinery, where the R-matrix and crystal bases play an important role.

In the following we recall the basic construction in [KMS] and compare it with the newer version developed in this paper, by taking the examples of level l $A_1^{(1)}$, $l = 1, 2$.

1.1 The kernel of $R - 1$

Let V be a finite-dimensional $U_q'(\mathfrak{g})$-module, and $V_{\mathrm{aff}} = V \otimes \mathbb{C}[z, z^{-1}]$ its affinization. The r-th q-wedge space is given by

$$\bigwedge^r V_{\mathrm{aff}} = V_{\mathrm{aff}}^{\otimes r} / N_r,$$

where

$$N_r = \sum_{i=0}^{r-2} V_{\mathrm{aff}}^{\otimes i} \otimes N \otimes V_{\mathrm{aff}}^{\otimes (r-2-i)}$$

and the space N is a certain subspace of $V_{\mathrm{aff}} \otimes V_{\mathrm{aff}}$. Namely, the q-wedge space is defined as a quotient of the tensor product of V_{aff} modulo certain relations of nearest neighbour type.

For the level 1 $A_1^{(1)}$ case, the space V is the 2-dimensional representation of $U_q'(\widehat{\mathfrak{sl}}_2)$, $V = \mathbb{Q}v_0 \oplus \mathbb{Q}v_1$. In [S], [KMS], the action of the Hecke algebra generator T was given on $V_{\mathrm{aff}} \otimes V_{\mathrm{aff}}$, and the space N was defined by

$$N = \mathrm{Ker}(T + 1).$$

It was also noted that $N = U_q'(\widehat{\mathfrak{sl}}_2) \cdot v_0 \otimes v_0$. In this paper, we define, in general,

$$N = U_q'(\mathfrak{g})[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot v \otimes v, \tag{1.1.1}$$

where v is an extremal vector in V_{aff} (see §3.1 for the definition). For $l = 1$, any $z^n v_i$ ($n \in \mathbb{Z}, i = 0, 1$) is extremal. For $l = 2$, we take

$$V = \mathbb{Q}v_0 \oplus \mathbb{Q}v_1 \oplus \mathbb{Q}v_2 .$$

The extremal vectors are $z^n v_0$ and $z^n v_2$ ($n \in \mathbb{Z}$). For $l = 1$, in the $q = 1$ limit, the construction gives rise to ordinary wedges with anti-commutation relations

$$z^m v_i \wedge z^n v_j + z^n v_j \wedge z^m v_i = 0.$$

For $l = 2$, this is not the case, e.g. $v_1 \wedge v_1 \neq 0$, even in the $q = 1$ limit.

The definition (1.1.1) is appropriate for computational use. For theoretical use, we have the equivalent definition

$$N = \mathrm{Ker}(R - 1).$$

Here R is the R-matrix acting on $V_{\mathrm{aff}} \otimes V_{\mathrm{aff}}$ (strictly speaking, the image of R belongs to a certain completion of $V_{\mathrm{aff}} \otimes V_{\mathrm{aff}}$).

The R-matrix satisfies the Yang-Baxter equation

$$R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23},$$

commutes with the $U_q(\mathfrak{g})$-action on $V_{\mathrm{aff}} \otimes V_{\mathrm{aff}}$, satisfies

$$R(z \otimes 1) = (1 \otimes z) R, \quad R(1 \otimes z) = (z \otimes 1) R,$$

and is normalized as

$$R(v \otimes v) = v \otimes v,$$

where v is an extremal vector.

1.2. Energy function and the normal ordering rules

In [KMS], it was shown that the q-wedge relations give a normal ordering rule of products of vectors. Define u_m ($m \in \mathbb{Z}$) by

$$z^n v_i = u_{2n-i}. \tag{1.2.1}$$

It was shown that the vectors

$$u_{m_1} \wedge \cdots \wedge u_{m_r} \quad (m_1 < \cdots < m_r)$$

form a base of $\bigwedge^r V_{\mathrm{aff}}$.

To describe the normal ordering rules in the general case, we use the energy function

$$H : B_{\mathrm{aff}} \otimes B_{\mathrm{aff}} \longrightarrow \mathbb{Z}.$$

The set B_{aff} is the crystal of V_{aff}. For each element b in B_{aff}, we have a corresponding vector $G(b)$ in V_{aff}. In this section we use the same symbol for b and $G(b)$: e.g. a general element of B_{aff} for the level 1 $A_1^{(1)}$ case and that of V_{aff} are denoted by $z^n v_i$. The energy function H is such that

$$R\big(G(b_1) \otimes G(b_2)\big) = z^{H(b_1 \otimes b_2)} G(b_1) \otimes z^{-H(b_1 \otimes b_2)} G(b_2) \bmod qL(V_{\mathrm{aff}}) \otimes L(V_{\mathrm{aff}}),$$

where $L(V_{\mathrm{aff}})$ is the free module generated by $G(b)$ ($b \in B_{\mathrm{aff}}$) over $A \stackrel{\mathrm{def}}{=} \{f \in \mathbb{Q}(q);$ f is regular at $q = 0\}$.

For the level 2 $A_1^{(1)}$ case,

$$B_{\mathrm{aff}} = \{z^m v_i; m \in \mathbb{Z}, i = 0, 1, 2\}$$

and

$$H(z^m v_i \otimes z^n v_j) = -m + n + h_{ij}$$

where the $(h_{ij})_{i,j=0,1,2}$ are given by

$$\begin{array}{c} \\ i=0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} j=0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} \end{array}.$$

We show that the set of vectors

$$G(b_1) \wedge \cdots \wedge G(b_r)$$

such that

$$H(b_i \otimes b_{i+1}) > 0 \quad (i = 1, \ldots, r-1) \tag{1.2.2}$$

is a base of $\bigwedge^r V_{\mathrm{aff}}$.

The vectors satisfying (1.2.2) are called normally ordered wedges. To show that the normally ordered wedges span the q-wedge space, we need to write down the basic q-wedge relations explicitly. This part of the work is technically much involved. We do it case by case. The generality in handling examples in this paper is narrower than that of [KMN2] because of this limitation.

In [KMS] the linear independence of the normally ordered wedges is proved by reduction to the $q = 1$ limit. Since the $q = 1$ result is not known for the general case, we prove the linear independence directly by using the Yang-Baxter equation for R and the crystal base theory.

1.3. Fock representations

In [KMS] the Fock spaces are constructed by means of an inductive limit of $\bigwedge^r V_{\mathrm{aff}}$. In the case of level 1 $A_1^{(1)}$, we take the sequence $(u_m)_{m\in\mathbb{Z}}$ as in (1.2.1). The Fock space $\mathcal{F}_m$ is defined as the space spanned by the semi-infinite wedges

$$u_{j_1} \wedge u_{j_2} \wedge u_{j_3} \wedge \cdots$$

such that $j_k = m + k - 1$ for sufficiently large k. The action of $U_q(\widehat{\mathfrak{sl}}_2)$ on $\mathcal{F}_m$ is defined by using the semi-infinite coproduct. It was shown that $\mathcal{F}_m$ is the tensor product

$$V(\lambda_m) \otimes \mathbb{C}[H_-].$$

Here $V(\lambda_m)$ is the irreducible highest weight representation with the highest weight λ_m, where

$$\lambda_m = \begin{cases} \Lambda_1 & \text{if } m \equiv 0 \bmod 2; \\ \Lambda_0 & \text{if } m \equiv 1 \bmod 2, \end{cases}$$

and $\mathbb{C}[H_-]$ is the Fock space of the Heisenberg algebra generated by B_n $(n \in \mathbb{Z}\backslash\{0\})$ that acts on $\mathcal{F}_m$ by

$$B_n = \sum_{k=1}^{\infty} 1 \otimes \cdots \otimes 1 \otimes \overset{\overset{k}{\vee}}{z^n} \otimes 1 \otimes \cdots .$$

To construct Fock spaces in the general case, we use the construction of affine crystals developed in [KMN1]. We assume that V has a perfect crystal B of level l. Then we can choose a sequence b_m° in B_{aff} such that

$$\langle c, \varepsilon(b_m^\circ)\rangle = l,$$
$$\varepsilon(b_m^\circ) = \varphi(b_{m+1}^\circ),$$
$$H(b_m^\circ \otimes b_{m+1}^\circ) = 1$$

(see subsection 3.1 for the definition of $\varepsilon(b)$ and $\varphi(b)$). In the case of level 2 $A_1^{(1)}$, we have

$$b_m^\circ = \begin{cases} z^k v_j & \text{if } m \text{ is odd;} \\ z^{k+1-j} v_{2-j} & \text{if } m \text{ is even,} \end{cases} \tag{1.3.1}$$

for some $k \in \mathbb{Z}$ and $j \in \{0, 1, 2\}$ independent of m. Then we shall define the Fock space $\mathcal{F}_m$ as a certain quotient of the space spanned by the semi-infinite wedges

$$G(b_1) \wedge G(b_2) \wedge G(b_3) \wedge \cdots$$

such that $b_n = b^\circ_{m+n-1}$ for sufficiently large n. In particular, the Fock space contains the highest weight vector

$$|m\rangle = G(b^\circ_m) \wedge G(b^\circ_{m+1}) \wedge G(b^\circ_{m+2}) \wedge \cdots$$

with the highest weight

$$\lambda_m = \begin{cases} j\Lambda_1 + (2-j)\Lambda_0 & \text{if } m \text{ is odd;} \\ (2-j)\Lambda_1 + j\Lambda_0 & \text{if } m \text{ is even.} \end{cases}$$

The quotient is such that if

$$H(b \otimes b^\circ_m) \leq 0$$

we require that

$$G(b) \wedge |m\rangle = 0.$$

Here is a significant difference between level 1 $A^{(1)}_n$ and other cases. For the former if $H(b \otimes b^\circ_m) \leq 0$ then

$$G(b) \wedge G(b^\circ_m) \wedge \cdots \wedge G(b^\circ_{m'}) = 0$$

for sufficiently large m'. But, this is not true in general. The correct statement is that for any n we can find m' such that the q-wedge $G(b) \wedge G(b^\circ_m) \wedge \cdots \wedge G(b^\circ_{m'})$ is a linear combination of normally ordered wedges whose coefficients are $O(q^n)$ at $q = 0$. Therefore we need to impose the separability of the q-adic topology, taking the quotient by the closure of $\{0\}$.

It is necessary to check that the action of $U_q(\mathfrak{g})$ given by the semi-infinite coproduct, is well-defined. A careful study of the q-wedges shows that

$$\Delta^{(\infty/2)}(f_i)|m\rangle = G(\tilde{f}_i b^\circ_m) \wedge |m+1\rangle, \tag{1.3.2}$$

where

$$\Delta^{(\infty/2)}(f_i) = \sum_{n=1}^{\infty} 1 \otimes \cdots \otimes 1 \otimes \overset{\overset{n}{\vee}}{f_i} \otimes t_i \otimes t_i \otimes \cdots .$$

In the case in [KMS], the action of $\Delta^{(\infty/2)}(f_i)$ on each vector in $\mathcal{F}_m$ is such that only finitely many terms in the sum are different from 0. This is not true in general. For example, consider the case $k = 1$ and $j = 1$ in (1.3.1). We have $f_1 v_1 = [2]v_2$ ($[2] = q + q^{-1}$) and $t_1|m\rangle = q|m\rangle$. Therefore we have

$$\begin{aligned}\Delta^{(\infty/2)}(f_1)(v_1 \wedge v_1 \wedge v_1 \wedge \cdots) = q[2](v_2 \wedge v_1 \wedge v_1 \wedge \cdots) + q[2](v_1 \wedge v_2 \wedge v_1 \wedge \cdots) \\ + q[2](v_1 \wedge v_1 \wedge v_2 \wedge \cdots) + \cdots\cdots .\end{aligned}$$

On the other hand, we have

$$v_1 \wedge v_2 + q^2 v_2 \wedge v_1 = 0,$$

and hence

$$\Delta^{(\infty/2)}(f_1)(v_1 \wedge v_1 \wedge \cdots) = v_2 \wedge v_1 \wedge v_1 \wedge \cdots,$$

by summing up

$$1 + (-q^2) + (-q^2)^2 + \cdots = \frac{1}{1+q^2}$$

in the q-adic topology.

In general, based on (1.3.2) we can show the well-definedness of the $U_q(\mathfrak{g})$-action.

The decomposition of the q-Fock spaces into the irreducible $U_q(\mathfrak{g})$-modules and the bosonic Fock space goes the same as the level 1 $A_n^{(1)}$ case. We carry out the computation of the exact commutation relations of the bosons in each case by reducing it to the commutation relations of vertex operators.

The plan of this paper is as follows. We list the notations in Section 2. We define the finite q-wedges in Section 3 and prove that the normally ordered wedges form a base. In Section 4, we define the q-Fock space and the actions of $U_q(\mathfrak{g})$ and the Heisenberg algebra. We give level 1 examples in Section 5 for which we check the conditions assumed in Section 3. We compute the level 1 two point functions in Section 6 in order to find the commutation relations of the bosons. Section 7 is devoted to a higher level example. We add four appendices. In Appendix A we prove a proposition on crystal base which is necessary in this paper but was not proved in [KMN1]. Appendix B is a proof that the Serre relations follow from the integrability of representations. Appendix C is the computation of the two-point correlation functions of the q-vertex operators in the $D_{n+1}^{(2)}$ case. In Appendix D we consider the $q \to 1$ limit for the $A_{2n}^{(2)}$ case and compare it to the result in [JY].

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2. Preliminary

2.1. Notations

In this paper we use the following notations.

$$\delta(P) = \begin{cases} 1 & \text{if a statement } P \text{ is true} \\ 0 & \text{if } P \text{ is false.} \end{cases}$$

- $\mathfrak{g}$: an affine Lie algebra.
- $\mathfrak{h}$: its Cartan subalgebra with dimension $\mathrm{rank}(\mathfrak{g}) + 1$.
- I : the index set for simple roots.
- α_i : a simple root $\in \mathfrak{h}^*$ corresponding to $i \in I$.
- h_i : a simple coroot $\in \mathfrak{h}$ corresponding to $i \in I$. We assume that the simple roots and the simple coroots are linearly independent.
- W : the Weyl group of $\mathfrak{g}$.
- $(\ ,\ )$: a W-invariant non-degenerate bilinear symmetric form on $\mathfrak{h}^*$ such that $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$.
- $\langle\ ,\ \rangle$: the coupling $\mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$.
- P : a weight lattice $\subset \mathfrak{h}^*$.
- $Q = \sum_i \mathbb{Z}\alpha_i$ the root lattice.
- $Q_\pm = \pm \sum_i \mathbb{Z}_{\geq 0}\alpha_i$.
- δ : an element of Q_+ such that $\mathbb{Z}\delta = \{\lambda \in Q; \langle h_i, \lambda\rangle = 0\}$.
- c : an element of $\sum_i \mathbb{Z}_{>0}h_i$ such that $\mathbb{Z}c = \{h \in \sum_i \mathbb{Z}h_i; \langle h, \alpha_i\rangle = 0\}$. We write
  $\delta = \sum_i a_i\alpha_i$ and
  $c = \sum_i a_i^\vee h_i$.
- $P_{\mathrm{cl}} = P/\mathbb{Z}\delta$.
- cl : $P \to P_{\mathrm{cl}}$. We assume for the sake of simplicity
  $P_{\mathrm{cl}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\oplus_{i\in I}\mathbb{Z}h_i, \mathbb{Z})$.
  This implies $\{\lambda \in P\,; \langle h_i, \lambda\rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\delta$.
- Λ_i : a fundamental weight in P, i.e. an element of P such that $\langle h_j, \Lambda_i\rangle = \delta_{ij}$.
- $\Lambda_i^{\mathrm{cl}} = \mathrm{cl}(\Lambda_i)$, the fundamental weight in P_{cl}. Note that Λ_i is determined modulo $\mathbb{Z}\delta$.
- P^0 : the level 0 part of P, i.e. $\{\lambda \in P : \langle c, \lambda\rangle = 0\}$.
- P^0_{cl} : the level 0 part of P_{cl}, i.e. $\mathrm{cl}(P^0)$.
- $U_q(\mathfrak{g})$: the quantized universal enveloping algebra with $\{q^h; h \in P^*\}$ as its Cartan part.
- $U'_q(\mathfrak{g})$: the quantized universal enveloping algebra with $\{q^h; h \in P_{\mathrm{cl}}{}^*\}$ as its Cartan part. Hence $U'_q(\mathfrak{g})$ is a subalgebra of $U_q(\mathfrak{g})$.

$K = \mathbb{Q}(q)$.
We consider $U_q(\mathfrak{g})$ and $U'_q(\mathfrak{g})$ over K.
$A = \{f \in K;\ f$ has no pole at $q = 0\}$.
$U'_q(\mathfrak{g})_{\mathbb{Z}}$: the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U'_q(\mathfrak{g})$ generated by the divided powers $e_i^{(n)}$, $f_i^{(n)}$, t_i and $\left\{ {t_i \atop n} \right\}$.
$U_q(\mathfrak{g})_{\mathbb{Z}}$: the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by $U'_q(\mathfrak{g})_{\mathbb{Z}}$ and $\left\{ {q^h \atop n} \right\}$ $(h \in P^*)$.

The quantized affine algebra $U_q(\mathfrak{g})$ is a K-algebra generated by e_i, f_i $(i \in I)$ and q^h $(h \in P^*)$ with the commutation relations

$$
\begin{aligned}
&q^h = 1 \text{ for } h = 0,\\
&q^{h+h'} = q^h q^{h'} \text{ for } h, h' \in P^*,\\
&q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i \quad \text{and} \quad q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i,\\
&[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}},\\
&\text{for } i \neq j \in I\\
&\quad \sum_k (-1)^k e_i^{(k)} e_j e_i^{(-\langle h_i, \alpha_j \rangle - k)} = 0,\\
&\quad \sum_k (-1)^k f_i^{(k)} f_j f_i^{(-\langle h_i, \alpha_j \rangle - k)} = 0.
\end{aligned}
$$

Here

$$q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}} \text{ and } t_i = q^{\frac{(\alpha_i, \alpha_i)}{2} h_i}.$$

2.2. Coproducts

There are several coproducts of $U_q(\mathfrak{g})$ used in the literature. In this paper, we use a coproduct different from the ones used in [DJO], [JM], [K1], [K2], [KMN1]. In this subsection, we shall explain the relations among four coproducts:

$$
\Delta_+ : \begin{cases} q^h \mapsto q^h \otimes q^h \\ e_i \mapsto e_i \otimes 1 + t_i \otimes e_i \\ f_i \mapsto f_i \otimes t_i^{-1} + 1 \otimes f_i \end{cases} \tag{2.2.1}
$$

$$
\Delta_- : \begin{cases} q^h \mapsto q^h \otimes q^h \\ e_i \mapsto e_i \otimes t_i^{-1} + 1 \otimes e_i \\ f_i \mapsto f_i \otimes 1 + t_i \otimes f_i \end{cases} \tag{2.2.2}
$$

$$\bar{\Delta}_+ : \begin{cases} q^h \mapsto q^h \otimes q^h \\ e_i \mapsto e_i \otimes 1 + t_i^{-1} \otimes e_i \\ f_i \mapsto f_i \otimes t_i + 1 \otimes f_i \end{cases} \tag{2.2.3}$$

$$\bar{\Delta}_- : \begin{cases} q^h \mapsto q^h \otimes q^h \\ e_i \mapsto e_i \otimes t_i + 1 \otimes e_i \\ f_i \mapsto f_i \otimes 1 + t_i^{-1} \otimes f_i \end{cases} . \tag{2.2.4}$$

Their antipodes are given by

$$a_+ : \begin{cases} q^h \mapsto q^{-h} \\ e_i \mapsto -t_i^{-1} e_i \\ f_i \mapsto -f_i t_i \end{cases} \tag{2.2.5}$$

$$a_- : \begin{cases} q^h \mapsto q^{-h} \\ e_i \mapsto -e_i t_i \\ f_i \mapsto -t_i^{-1} f_i \end{cases} \tag{2.2.6}$$

$$\bar{a}_+ : \begin{cases} q^h \mapsto q^{-h} \\ e_i \mapsto -t_i e_i \\ f_i \mapsto -f_i t_i^{-1} \end{cases} \tag{2.2.7}$$

$$\bar{a}_- : \begin{cases} q^h \mapsto q^{-h} \\ e_i \mapsto -e_i t_i^{-1} \\ f_i \mapsto -t_i f_i \end{cases} \tag{2.2.8}$$

For two $U_q(\mathfrak{g})$-modules M_1 and M_2, let us denote by $M_1 \otimes_+ M_2$, $M_1 \otimes_- M_2$, $M_1 \overline{\otimes}_+ M_2$ and $M_1 \overline{\otimes}_- M_2$ the vector space $M_1 \otimes_K M_2$ endowed with the $U_q(\mathfrak{g})$-module structure via the coproduct Δ_+, Δ_-, $\bar{\Delta}_+$ and $\bar{\Delta}_-$, respectively.

We have functorial isomorphisms of $U_q(\mathfrak{g})$-modules

$$M_1 \otimes_+ M_2 \xrightarrow{\sim} M_2 \overline{\otimes}_- M_1 \tag{2.2.9}$$

$$M_1 \otimes_- M_2 \xrightarrow{\sim} M_2 \overline{\otimes}_+ M_1 \tag{2.2.10}$$

by $u_1 \otimes u_2 \mapsto u_2 \otimes u_1$.

We have functorial isomorphisms of $U_q(\mathfrak{g})$-modules

$$q^{-(\cdot,\cdot)} : M_1 \otimes_+ M_2 \xrightarrow{\sim} M_1 \otimes_- M_2 \tag{2.2.11}$$

$$q^{(\cdot,\cdot)} : M_1 \overline{\otimes}_+ M_2 \xrightarrow{\sim} M_1 \overline{\otimes}_- M_2 \tag{2.2.12}$$

Here $q^{-(\cdot,\cdot)}$ sends $u_1 \otimes_+ u_2$ to $q^{-(\mathrm{wt}\,(u_1),\mathrm{wt}\,(u_2))} u_1 \otimes_- u_2$ and $q^{(\cdot,\cdot)}$ sends $u_1 \overline{\otimes}_+ u_2$ to $q^{(\mathrm{wt}\,(u_1),\mathrm{wt}\,(u_2))} u_1 \overline{\otimes}_- u_2$.

The tensor products $\otimes_+$ and $\overline{\otimes}_-$ behave well under upper crystal bases and $\otimes_-$ and $\overline{\otimes}_+$ behave well under lower crystal bases. Namely, if (L_j, B_j) is an upper crystal base of an integrable $U_q(\mathfrak{g})$-module M_j $(j = 1,2)$, then $(L_1 \otimes_A L_2, B_1 \otimes B_2)$ is an upper crystal base of $M_1 \otimes_+ M_2$ and $M_1 \overline{\otimes}_- M_2$. Similarly, if (L_j, B_j) is a lower crystal base of M_j, then $(L_1 \otimes_A L_2, B_1 \otimes B_2)$ is a lower crystal base of $M_1 \otimes_- M_2$ and $M_1 \overline{\otimes}_+ M_2$. If we use $\otimes_+$ or $\otimes_-$, the tensor product of crystal base is described as follows. For two crystals B_1, B_2 and $b_1 \in B_1$, $b_2 \in B_2$,

$$
\begin{aligned}
&\mathrm{wt}\,(b_1 \otimes b_2) = \mathrm{wt}\,(b_1) + \mathrm{wt}\,(b_2),\\
&\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}\,(b_1)\rangle),\\
&\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_1) + \langle h_i, \mathrm{wt}\,(b_2)\rangle, \varphi_i(b_2)),\\
&\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \ge \varepsilon_i(b_2),\\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2),\end{cases}\\
&\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2),\\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \le \varepsilon_i(b_2).\end{cases}
\end{aligned}
$$

If we use the other tensor products $\overline{\otimes}_+$ or $\overline{\otimes}_-$, we have to exchange the first and the second factors in the formulas above. Namely the tensor product of crystals is given as

$$
\begin{aligned}
&\mathrm{wt}\,(b_1 \otimes b_2) = \mathrm{wt}\,(b_1) + \mathrm{wt}\,(b_2),\\
&\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1) - \langle h_i, \mathrm{wt}\,(b_2)\rangle, \varepsilon_i(b_2)),\\
&\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_1), \varphi_i(b_2) + \langle h_i, \mathrm{wt}\,(b_1)\rangle),\\
&\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) > \varphi_i(b_2),\\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varepsilon_i(b_1) \le \varphi_i(b_2),\end{cases}\\
&\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) \ge \varphi_i(b_2),\\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varepsilon_i(b_1) < \varphi_i(b_2).\end{cases}
\end{aligned}
\tag{2.2.13}
$$

In this article, we mainly use the tensor product $\overline{\otimes}_+$ and lower crystal bases. The rule of the tensor product of crystals is therefore by (2.2.13). Note that $\otimes_+$ is used in [DJO], [JM] and $\otimes_-$ in [K2], [KMN1].

3. Wedge products

3.1 Perfect crystal

Let us take an integrable finite-dimensional representation V of $U'_q(\mathfrak{g})$. Let $V =$

$\oplus_{\lambda\in P^0_{\rm cl}} V_\lambda$ be its weight space decomposition. Its affinization is defined by

$$V_{\rm aff} = \bigoplus_{\lambda\in P} (V_{\rm aff})_\lambda$$

where $(V_{\rm aff})_\lambda = V_{{\rm cl}(\lambda)}$ for $\lambda \in P$. Let $\mathrm{cl} : (V_{\rm aff})_\lambda \to V_{{\rm cl}(\lambda)}$ denote the canonical isomorphism. Then $V_{\rm aff}$ has a natural structure of a $U_q(\mathfrak{g})$-module such that $\mathrm{cl} : V_{\rm aff} \to V$ is $U'_q(\mathfrak{g})$-linear (see [KMN1]).

Let $z : V_{\rm aff} \to V_{\rm aff}$ be the endomorphism of weight δ given by

$$\begin{tikzcd} (V_{\rm aff})_\lambda \arrow[r, "z"] \arrow[d, "\wr"'] & (V_{\rm aff})_{\lambda+\delta} \arrow[d, "\wr"'] \\ V_{{\rm cl}(\lambda)} \arrow[r, equal] & V_{{\rm cl}(\lambda+\delta)} \end{tikzcd}$$

The endomorphism z is $U'_q(\mathfrak{g})$-linear.

Taking a section of $\mathrm{cl} : P \to P_{\rm cl}$, $V_{\rm aff}$ may be identified with $V \otimes \mathbb{C}[z, z^{-1}]$ (see section 5.1).

We assume that

(P) *V has a perfect crystal base (L, B).*

Let us recall its definition in [KMN1]. A crystal base (L, B) is called perfect of level $l \in \mathbb{Z}_{>0}$ if it satisfies the following axioms (P1)–(P3).

(P1) There is a weight $\lambda^\circ \in P^0_{\rm cl}$ such that the weights of V are contained in the convex hull of $W\lambda^\circ$ and that $\dim V_{w\lambda^\circ} = 1$ for any w in the Weyl group W. We call a vector in $V_{w\lambda^\circ}$ an *extremal* vector with extremal weight $w\lambda^\circ$.

(P2) $B \otimes B$ is connected.

(P3) There is a positive integer l satisfying the following conditions.

(i) For every $b \in B$, $\langle c, \varepsilon(b)\rangle = \langle c, \varphi(b)\rangle \geq l$. Here we set

$$\begin{aligned} \varepsilon(b) &= \sum_{i\in I} \varepsilon_i(b)\Lambda_i^{\rm cl} \in P_{\rm cl} \\ \varphi(b) &= \sum_{i\in I} \varphi_i(b)\Lambda_i^{\rm cl} \in P_{\rm cl} \end{aligned} \tag{3.1.1}$$

with the fundamental weights $\Lambda_i^{\rm cl} \in P_{\rm cl}$.

(ii) Set $B_{\min} = \{b \in B; \langle c, \varepsilon(b)\rangle = l\}$ and $(P^+_{\rm cl})_l = \{\lambda \in P_{\rm cl}\,; \langle c, \lambda\rangle = l$ and $\langle h_i, \lambda\rangle \geq 0$ for every $i \in I\}$. Then

$$\varepsilon, \varphi : B_{\min} \to (P^+_{\rm cl})_l \quad \text{are bijective.}$$

Note that (P1) is equivalent to the irreducibility of V (see [CP]).

Note that the equality $\langle c, \varepsilon(b)\rangle = \langle c, \varphi(b)\rangle$ in (P3) (i) follows from

$$\varphi(b) = \mathrm{wt}\,(b) + \varepsilon(b)$$

and the fact that V is a $U_q'(\mathfrak{g})$-module of level 0.

Remark. The map $\varepsilon(b) \mapsto \varphi(b)$ ($b \in B_{\min}$) defines an automorphism of $(P_{\mathrm{cl}}^+)_l$. In all the examples of perfect crystals that we know, this automorphism is induced by a Dynkin diagram automorphism.

We have constructed V_{aff} out of V. Similarly we construct the crystal base $(L_{\mathrm{aff}}, B_{\mathrm{aff}})$ of V_{aff} out of (L, B). We define similarly $\mathrm{cl} : B_{\mathrm{aff}} \to B$ and $z : B_{\mathrm{aff}} \to B_{\mathrm{aff}}$.

We assume further that V has a good base $\{G(b)\}_{b\in B}$:

(G) *V has a lower global base $\{G(b)\}_{b\in B}$.*

This means that the base $\{G(b)\}_{b\in B}$ satisfies the following conditions (cf. [K2]).

(i) $\bigoplus_{b\in B} \mathbb{Z}[q,q^{-1}]G(b)$ is a $U_q'(\mathfrak{g})_{\mathbb{Z}}$-submodule of V.
(ii) $b \equiv G(b) \mod L/qL$.
(iii) $e_iG(b) = [\varphi_i(b)+1]_iG(\tilde{e}_ib) + \sum E^i_{b,b'}G(b')$,
(iv) $f_iG(b) = [\varepsilon_i(b)+1]_iG(\tilde{f}_ib) + \sum F^i_{b,b'}G(b')$.

In both cases, the sum ranges over b' that belongs to an i-string strictly longer than that of b ($\Leftrightarrow \varepsilon_i(b') \geq \varepsilon_i(b)$ or $\varphi_i(b') \geq \varphi_i(b)$ respectively for (iii) or (iv)). Moreover the coefficients satisfy

$$E^i_{b,b'} \in qq_i^{-\varphi_i(b')}\mathbb{Z}[q] \cup q^{-1}q_i^{\varphi_i(b')}\mathbb{Z}[q^{-1}] \tag{3.1.2}$$

$$F^i_{b,b'} \in qq_i^{-\varepsilon_i(b')}\mathbb{Z}[q] \cup q^{-1}q_i^{\varepsilon_i(b')}\mathbb{Z}[q^{-1}]. \tag{3.1.3}$$

Remark. The reason why we choose a lower global base is explained in Theorem 4.2.5 and the remark after Proposition 4.2.8.

We define the base $\{G(b)\}_{b\in B_{\mathrm{aff}}}$ of V_{aff} by $\mathrm{cl}(G(b)) = G(\mathrm{cl}(b))$. We have $G(z^nb) = z^nG(b)$ for $n \in \mathbb{Z}$ and $b \in B_{\mathrm{aff}}$.

3.2. Energy function

Let H be an energy function (see [KMN1]). Namely $H : B_{\mathrm{aff}} \otimes B_{\mathrm{aff}} \to \mathbb{Z}$ satisfies

(E1) $H(zb_1 \otimes b_2) = H(b_1 \otimes b_2) - 1$.
(E2) $H(b_1 \otimes zb_2) = H(b_1 \otimes b_2) + 1$.
(E3) H is constant on every connected component of the crystal graph $B_{\mathrm{aff}} \otimes B_{\mathrm{aff}}$.

By (E1–3), H is uniquely determined up to a constant. We normalize H by

(E4) $H(b\otimes b) = 0$ for any (or equivalently some) extremal $b \in B_{\mathrm{aff}}$ (i.e. $\mathrm{cl}\bigl(\mathrm{wt}\,(b)\bigr) \in W\lambda^\circ$).

We know already its existence and uniqueness ([KMN1]). The existence is in fact proved by using the R-matrix. Let us explain their relation. There is a $U_q(\mathfrak{g})$-linear endomorphism (R-matrix) R of $V_{\text{aff}} \otimes V_{\text{aff}}$ such that

$$R \circ (z \otimes 1) = (1 \otimes z) \circ R \tag{3.2.1}$$

$$R \circ (1 \otimes z) = (z \otimes 1) \circ R \tag{3.2.2}$$

and normalized by

$$R(u \otimes u) = u \otimes u \qquad \text{for every extremal } u \in V_{\text{aff}}. \tag{3.2.3}$$

Strictly speaking, R is a homomorphism from $V_{\text{aff}} \otimes V_{\text{aff}}$ to its completion $V_{\text{aff}} \widehat{\otimes} V_{\text{aff}}$. It is proved in [KMN1] that R sends $L_{\text{aff}} \otimes L_{\text{aff}}$ to $L_{\text{aff}} \widehat{\otimes} L_{\text{aff}}$ and

$$\begin{aligned} &R(G(b_1) \otimes G(b_2)) \\ &\quad \equiv G(z^{H(b_1 \otimes b_2)} b_1) \otimes G(z^{-H(b_1 \otimes b_2)} b_2) \mod qL_{\text{aff}} \widehat{\otimes} L_{\text{aff}} \\ &\quad \text{for every } b_1,\, b_2 \in B_{\text{aff}}. \end{aligned} \tag{3.2.4}$$

We know that R has finitely many poles. It means that there is a non-zero $\psi \in K[z \otimes z^{-1}, z^{-1} \otimes z]$ such that ψR sends $V_{\text{aff}} \otimes V_{\text{aff}}$ into itself. We assume that the denominator ψ of R satisfies the following property:

(D) *$\psi \in A[z \otimes z^{-1}]$ and $\psi = 1$ at $q = 0$.*

We take a linear form $s : P \to \mathbb{Q}$ such that $s(\alpha_i) = 1$ for every $i \in I$, and define

$$l : B_{\text{aff}} \to \mathbb{Z}$$

by $l(b) = s(\text{wt}\,(b)) + c$ for some constant c. With a suitable choice of c, l is $\mathbb{Z}$-valued. It satisfies

(i) $l(zb) = l(b) + a$ for any $b \in B_{\text{aff}}$. Here a is a positive integer independent of b.

(ii) $l(\tilde{e}_i b) = l(b) + 1$ if $i \in I$ and $b \in B_{\text{aff}}$ satisfy $\tilde{e}_i b \neq 0$.

We assume that it satisfies

(L) *If $H(b_1 \otimes b_2) \leq 0$, then $l(b_1) \geq l(b_2)$.*

3.3. Wedge products

We define $L(V_{\text{aff}}^{\otimes 2})$ by $L_{\text{aff}} \otimes_A L_{\text{aff}}$. Let us set $\tilde{R} = \psi(z \otimes 1, 1 \otimes z)R = R\psi(1 \otimes z, z \otimes 1)$. Then it is an endomorphism of $V_{\text{aff}}^{\otimes 2}$ and $L(V_{\text{aff}}^{\otimes 2})$ is stable by $\tilde{R}$. We shall denote by the same letter $\tilde{R}$ the endomorphism of $L(V_{\text{aff}}^{\otimes 2})/qL(V_{\text{aff}}^{\otimes 2})$ induced by $\tilde{R}$. Then by (D) and (3.2.4) we have the equality in $L(V_{\text{aff}}^{\otimes 2})/qL(V_{\text{aff}}^{\otimes 2})$

$$\tilde{R}(b_1 \otimes b_2) = z^{H(b_1 \otimes b_2)} b_1 \otimes z^{-H(b_1 \otimes b_2)} b_2 \quad \text{for every } b_1, b_2 \in B_{\text{aff}}. \tag{3.3.1}$$

Since $R^2 = 1$, we have

$$\big(\tilde{R} - \psi(z \otimes 1, 1 \otimes z)\big) \circ \big(\tilde{R} + \psi(1 \otimes z, z \otimes 1)\big) = 0. \tag{3.3.2}$$

Let us choose an extremal vector $u \in V_{\mathrm{aff}}$. Then we define

$$N = U_q(\mathfrak{g})[z \otimes z,\ z^{-1} \otimes z^{-1},\ z \otimes 1 + 1 \otimes z]\,(u \otimes u)\,.$$

This definition does not depend on the choice of u, because an extremal vector u of weight λ satisfies

$$\begin{aligned} (f_i^{(n)}u) \otimes (f_i^{(n)}u) &= f_i^{(2n)}(u \otimes u) \quad \text{if } \langle h_i, \lambda\rangle = n \geq 0, \\ (e_i^{(n)}u) \otimes (e_i^{(n)}u) &= e_i^{(2n)}(u \otimes u) \quad \text{if } \langle h_i, \lambda\rangle = -n \leq 0. \end{aligned}$$

By definition, we have

$$\begin{aligned} &f(z \otimes 1, 1 \otimes z)N \subset N \\ &\qquad \text{for any symmetric Laurent polynomial } f(z_1, z_2). \end{aligned} \tag{3.3.3}$$

We make the following postulate.

(R) *For every pair (b_1, b_2) in B_{aff} with $H(b_1 \otimes b_2) = 0$, there exists $C_{b_1,b_2} \in N$ which has the form*

$$C_{b_1,b_2} = G(b_1) \otimes G(b_2) - \sum_{b'_1, b'_2} a_{b'_1,b'_2}\, G(b'_1) \otimes G(b'_2)\,.$$

Here the sum ranges over (b'_1, b'_2) such that

$$\begin{aligned} H(b'_1 \otimes b'_2) &> 0\,, \\ l(b_2) \leq l(b'_1) &< l(b_1)\,, \\ l(b_2) < l(b'_2) &\leq l(b_1)\,, \end{aligned}$$

and the coefficients $a_{b'_1,b'_2}$ belong to $\mathbb{Z}[q, q^{-1}]$.

Later in Lemma 3.3.2, we see that $a_{b'_1,b'_2}$ belong to $q\mathbb{Z}[q]$.
Since we have normalized the R-matrix by $R(u \otimes u) = u \otimes u$, we have

$$\tilde{R}(v) = \psi(z \otimes 1, 1 \otimes z)v \quad \text{for every } v \in N. \tag{3.3.4}$$

Hence $\tilde{R}$ sends N to itself.

We set

$$L(N) = N \cap L(V_{\mathrm{aff}}^{\otimes 2}).$$

Then by (D) and (3.3.4), we have the equality in $L(V_{\mathrm{aff}}^{\otimes 2})/qL(V_{\mathrm{aff}}^{\otimes 2})$

$$\tilde{R}(b) = b \quad \text{for every } b \in L(N)/qL(N). \tag{3.3.5}$$

We define the wedge product by

$$\bigwedge^2 V_{\mathrm{aff}} = V_{\mathrm{aff}}^{\otimes 2}/N\,.$$

For $v_1, v_2 \in V$, let us denote by $v_1 \wedge v_2$ the element of $\bigwedge^2 V_{\mathrm{aff}}$ corresponding to $v_1 \otimes v_2$. We set

$$L(\bigwedge^2 V_{\mathrm{aff}}) = L(V_{\mathrm{aff}}^{\otimes 2})/L(N) \subset \bigwedge^2 V_{\mathrm{aff}}.$$

Now we shall study the properties of $\bigwedge^2 V_{\mathrm{aff}}$ under conditions (P), (G), (D), (L) and (R). We conjecture that (P) and (G) imply the other conditions (D), (L) and (R).

Lemma 3.3.1. *If $\sum_{H(b_1\otimes b_2)>0} a_{b_1,b_2} G(b_1)\otimes G(b_2)$ belongs to* Ker $\big(\tilde{R}-\psi(z\otimes 1, 1\otimes z)\big)$, *then all a_{b_1,b_2} vanish.*

Proof. It is enough to show that for $n \in \mathbb{Z}$

$$\text{if } a_{b_1,b_2} \in q^n A \text{ for all } b_1, b_2, \text{ then } a_{b_1,b_2} \in q^{n+1}A. \tag{3.3.6}$$

By (D), (3.3.4) and (3.3.1), we obtain the identity in $L(V_{\mathrm{aff}}^{\otimes 2})/qL(V_{\mathrm{aff}}^{\otimes 2})$,

$$\sum_{H(b_1\otimes b_2)>0} (q^{-n}a_{b_1,b_2}) b_1 \otimes b_2 = \sum_{H(b_1\otimes b_2)>0} (q^{-n}a_{b_1,b_2}) z^{H(b_1\otimes b_2)} b_1 \otimes z^{-H(b_1\otimes b_2)} b_2.$$

Since $H\big(z^{H(b_1\otimes b_2)} b_1 \otimes z^{-H(b_1\otimes b_2)} b_2\big) = -H(b_1\otimes b_2) < 0$, we obtain the desired assertion (3.3.6). □

A similar argument leads to the following result.

Lemma 3.3.2. *If $H(b_1 \otimes b_2) = 0$ and $G(b_1)\otimes G(b_2) - \sum_{H(b_1'\otimes b_2')>0} a_{b_1',b_2'} G(b_1')\otimes G(b_2')$ belongs to N, then $a_{b_1',b_2'} \in qA$.*

We shall call a pair (b_1, b_2) of elements in B_{aff} *normally ordered* and $G(b_1)\wedge G(b_2)$ a *normally ordered wedge* if $H(b_1\otimes b_2) > 0$. The axiom (R) may be considered as a rule to write $G(b_1)\wedge G(b_2)$ as a linear combination of normally ordered wedges when $H(b_1\otimes b_2) = 0$. In order to treat the case $H(b_1\otimes b_2) = -c < 0$, we introduce an element of N (see (3.3.3))

$$\begin{aligned} C'_{b_1,b_2} &= (1\otimes z^{-c} + z^{-c}\otimes 1) C_{b_1, z^c b_2} \\ &= (1\otimes z^{c} + z^{c}\otimes 1) C_{z^{-c}b_1, b_2}. \end{aligned} \tag{3.3.7}$$

Note that $H(b_1\otimes z^c b_2) = H(z^{-c} b_1 \otimes b_2) = 0$.

Lemma 3.3.3. *If $H(b_1 \otimes b_2) \leq 0$, then C'_{b_1,b_2} has the form*

$$G(b_1) \otimes G(b_2) - \sum_{b'_1,b'_2} a_{b'_1,b'_2} G(b'_1) \otimes G(b'_2).$$

Here the sum ranges over (b'_1, b'_2) such that

$$\begin{aligned} &H(b'_1 \otimes b'_2) > H(b_1 \otimes b_2)\,, \\ &l(b_2) \leq l(b'_1) < l(b_1)\,, \\ &l(b_2) < l(b'_2) \leq l(b_1)\,. \end{aligned}$$

Moreover $a_{b'_1,b'_2}$ belongs to $\mathbb{Z}[q]$.

Proof. Assume $H(b_1 \otimes b_2) = -c < 0$. Set

$$C'_{z^{-c}b_1,b_2} = G(z^{-c}b_1) \otimes G(b_2) - \sum_{H(b'_1 \otimes b'_2)>0} a_{b'_1,b'_2} G(b'_1) \otimes G(b'_2).$$

Here the sum ranges over

$$\begin{aligned} &l(b_2) \leq l(b'_1) < l(z^{-c}b_1)\,, \\ &l(b_2) < l(b'_2) \leq l(z^{-c}b_1)\,. \end{aligned}$$

Then

$$\begin{aligned} C'_{b_1,b_2} &= G(b_1) \otimes G(b_2) + G(z^{-c}b_1) \otimes G(z^{c}b_2) \\ &\quad - \sum_{H(b'_1 \otimes b'_2)>0} a_{b'_1,b'_2} \big(G(b'_1) \otimes G(z^{c}b'_2) + G(z^{c}b'_1) \otimes G(b'_2)\big). \end{aligned}$$

The desired properties can be easily checked. □

By the repeated use of the proposition above, we obtain the following result.

Corollary 3.3.4. *If $H(b_1 \otimes b_2) \leq 0$ then N contains an element C_{b_1,b_2}, which has the form*

$$G(b_1) \otimes G(b_2) - \sum_{b'_1,b'_2} a_{b'_1,b'_2} G(b'_1) \otimes G(b'_2).$$

Here the sum ranges over (b'_1, b'_2) such that

$$\begin{aligned} &H(b'_1 \otimes b'_2) > 0\,, \\ &l(b_2) \leq l(b'_1) < l(b_1)\,, \\ &l(b_2) < l(b'_2) \leq l(b_1)\,. \end{aligned}$$

and $a_{b'_1,b'_2} \in \mathbb{Z}[q]$.

By Lemma 3.3.1, C_{b_1,b_2} is uniquely determined. Note that we shall see

$$a_{b'_1,b'_2}(0) = -\delta(b'_1 \otimes b'_2 = z^{H(b_1\otimes b_2)}b_1 \otimes z^{-H(b_1\otimes b_2)}b_2)$$

(see Lemma 3.3.8).

The following corollary is a consequence of the corollary above and Lemma 3.3.1.

Lemma 3.3.5. *$L(N)$ is a free A-module with $\{C_{b_1,b_2}\}_{H(b_1\otimes b_2)\leq 0}$ as its basis.*

Proposition 3.3.6.

(i) *The normally ordered wedges form a base of $\bigwedge^2 V_{\mathrm{aff}}$.*

(ii) *$L(\bigwedge^2 V_{\mathrm{aff}})$ is a free A-module with the normally ordered wedges as a base.*

Proof. Lemma 3.3.1 implies the linear independence of the normally ordered wedges and Corollary 3.3.4 implies that they generate $\bigwedge^2 V_{\mathrm{aff}}$.

(ii) follows from (i) and Corollary 3.3.4. □

Corollary 3.3.7. $N = \mathrm{Ker}\left(\tilde{R} - \psi(z\otimes 1, 1\otimes z)\right)$.

Proof. We know already that N is contained in $\mathrm{Ker}\left(\tilde{R} - \psi(z\otimes 1, 1\otimes z)\right)$. Since the normally ordered wedges are linearly independent in $V_{\mathrm{aff}}^{\otimes 2}/\,\mathrm{Ker}\left(\tilde{R} - \psi(z\otimes 1, 1\otimes z)\right)$ by Lemma 3.3.1, $\bigwedge^2 V_{\mathrm{aff}} \to V_{\mathrm{aff}}^{\otimes 2}/\,\mathrm{Ker}\left(\tilde{R} - \psi(z\otimes 1, 1\otimes z)\right)$ is injective. □

We define for $n > 0$

$$N_n = \sum_{k=0}^{n-2} \left(V_{\mathrm{aff}}^{\otimes k} \otimes N \otimes V_{\mathrm{aff}}^{\otimes(n-k-2)}\right) \subset V_{\mathrm{aff}}^{\otimes n}$$

and then

$$\bigwedge^n V_{\mathrm{aff}} = V_{\mathrm{aff}}^{\otimes n}/N_n\,.$$

For $u_1, u_2, \ldots, u_n \in V_{\mathrm{aff}}$, we denote by $u_1\wedge u_2\wedge\cdots\wedge u_n$ the image of $u_1\otimes u_2\otimes\cdots\otimes u_n$ in $\bigwedge^n V_{\mathrm{aff}}$.

There is a $U_q(\mathfrak{g})$-linear homomorphism

$$\wedge : \bigwedge^n V_{\mathrm{aff}} \otimes \bigwedge^m V_{\mathrm{aff}} \to \bigwedge^{n+m} V_{\mathrm{aff}}.$$

Let us set $L(V_{\mathrm{aff}}^{\otimes n}) = L_{\mathrm{aff}}^{\otimes n}$ and let $L(\bigwedge^n V_{\mathrm{aff}})$ be the image of $L(V_{\mathrm{aff}}^{\otimes n})$ in $\bigwedge^n V_{\mathrm{aff}}$. We call a sequence $(b_1, b_2, \ldots, b_n)$ normally ordered if its every consecutive pair is normally ordered, i.e. if $H(b_j\otimes b_{j+1}) > 0$ for $j = 1, \ldots, n-1$. In this case we call $G(b_1)\wedge\cdots\wedge G(b_n)$ a *normally ordered wedge.* Set

$$L(N_n) = \sum_{k=0}^{n-2} L(V_{\mathrm{aff}})^{\otimes k} \otimes_A L(N) \otimes_A L(V_{\mathrm{aff}})^{\otimes(n-2-k)} \subset L(V_{\mathrm{aff}}^{\otimes n}).$$

Note that we have not yet seen $L(N_n) \supset N_n \cap L(V_{\mathrm{aff}}^{\otimes n})$, which will follow from Lemma 3.3.11. In the formulae below, we have to pay attention to a difference between modulo $qL(N_n)$ and modulo $qL(V_{\mathrm{aff}}^{\otimes n})$.

Lemma 3.3.8.

(i) *If* $H(b_1 \otimes b_2) = 0$ *then*

$$G(z^a b_1) \wedge G(z^b b_2) \equiv -G(z^b b_1) \wedge G(z^a b_2) \mod qL(\bigwedge^2 V_{\mathrm{aff}}).$$

(ii) *If* $H(b_1 \otimes b_2) \le 0$ *then*

$$C_{b_1,b_2} \equiv b_1 \otimes b_2 + \delta(H(b_1 \otimes b_2) < 0) z^{H(b_1 \otimes b_2)} b_1 \otimes z^{-H(b_1 \otimes b_2)} b_2 \mod qL(V_{\mathrm{aff}}^{\otimes 2}).$$

(iii) *If* $H(b_j \otimes b_{j+1}) = 0$ *for* $j = 1, \dots, n-1$, *then for any* $\sigma \in S_n$,

$$G(z^{a_1} b_1) \wedge G(z^{a_2} b_2) \wedge \cdots \wedge G(z^{a_n} b_n) \equiv \operatorname{sgn}(\sigma) G(z^{a_{\sigma(1)}} b_1) \wedge G(z^{a_{\sigma(2)}} b_2) \wedge \cdots \wedge G(z^{a_{\sigma(n)}} b_n) \mod qL(\bigwedge^n V_{\mathrm{aff}}).$$

Proof. By Lemma 3.3.3, (i) holds for $a = b = 0$. The general case is obtained by operating $z^a \otimes z^b + z^b \otimes z^a$ on $G(b_1) \otimes G(b_2) \equiv 0$. The other assertions follow from (i). □

Proposition 3.3.9. *Let* $a, c \in \mathbb{Z}$ *and* $n \in \mathbb{Z}_{>0}$. *Then for* $b_1, \dots, b_n \in B_{\mathrm{aff}}$ *with* $a \le l(b_j) \le c$, *we have*

$$G(b_1) \otimes \cdots \otimes G(b_n) \in \sum \mathbb{Z}[q] G(b'_1) \otimes \cdots \otimes G(b'_n) + L(N_n)$$

where the sum ranges over normally ordered sequences $(b'_1, \dots, b'_n)$ *with* $a \le l(b'_j) \le c$ *and* $l(b'_1) \le l(b_1)$.

Proof. We shall prove this by induction on n and $l(b_1)$. By the induction hypothesis on n, we may assume that $(b_2, \dots, b_n)$ is normally ordered. If $H(b_1 \otimes b_2) > 0$, then we are done. Assume that $H(b_1 \otimes b_2) \le 0$. Then by Corollary 3.3.4, we can write

$$G(b_1) \otimes G(b_2) \equiv \sum_{b'_1, b'_2} a_{b'_1, b'_2} G(b'_1) \otimes G(b'_2) \mod L(N)$$

with $H(b'_1 \otimes b'_2) > 0$ and $l(b_2) \le l(b'_1) < l(b_1)$ and $l(b_2) < l(b'_2) \le l(b_1)$. Then we have

$$G(b_1) \otimes G(b_2) \otimes \cdots \otimes G(b_n) \equiv \sum a_{b'_1, b'_2} G(b'_1) \otimes G(b'_2) \otimes G(b_3) \otimes \cdots \otimes G(b_n) \mod L(N_n).$$

Since $a \le l(b_2) \le l(b'_1) < l(b_1)$, the induction proceeds. □

This proposition says in particular that $\bigwedge^n V_{\mathrm{aff}}$ is generated by the normally ordered wedges. In order to see their linear independence, we need the compatibility of the relations, which follow from the Yang–Baxter equation for R.

Lemma 3.3.10. *Assume $H(b_1 \otimes b_2) = H(b_2 \otimes b_3) = 0$. Then for $a \geq b \geq c$, we have*

$$\begin{aligned}
&(1+\delta_{a,b})C_{z^a b_1, z^b b_2} \otimes G(z^c b_3) \\
&\qquad + (1+\delta_{b,c})C_{z^b b_1, z^c b_2} \otimes G(z^a b_3) + (1+\delta_{a,c})C_{z^a b_1, z^c b_2} \otimes G(z^b b_3) \\
\equiv\ &(1+\delta_{b,c})G(z^a b_1) \otimes C_{z^b b_2, z^c b_3} \\
&\qquad + (1+\delta_{a,c})G(z^b b_1) \otimes C_{z^a b_2, z^c b_3} + (1+\delta_{a,b})G(z^c b_1) \otimes C_{z^a b_2, z^b b_3} \\
&\qquad \bmod\ qL(N_3).
\end{aligned}$$

Proof. We have the Yang-Baxter equation

$$\tilde{R}_{12} \circ \tilde{R}_{23} \circ \tilde{R}_{12} = \tilde{R}_{23} \circ \tilde{R}_{12} \circ \tilde{R}_{23}.$$

Here $\tilde{R}_{ij}$ is the action of $\tilde{R}$ on the i, j-th components on $V_{\mathrm{aff}}^{\otimes 3}$. Set $\psi_{21} = \psi(1 \otimes z \otimes 1, z \otimes 1 \otimes 1)$, etc. Since $\tilde{R} + \psi(1 \otimes z, z \otimes 1)$ sends $L(V_{\mathrm{aff}}^{\otimes 2})$ to $L(N)$, $R_{ij} + \psi_{ji}$ sends $L(V_{\mathrm{aff}}^{\otimes 3})$ to $L(N_3)$. Also we have

$$\begin{aligned}
&\big(\tilde{R} + \psi(1 \otimes z, z \otimes 1)\big)(G(z^a b_1) \otimes G(z^b b_2)) \\
&\qquad \equiv G(z^b b_1) \otimes G(z^a b_2) + G(z^a b_1) \otimes G(z^b b_2) \\
&\qquad \equiv (1+\delta_{a,b})C_{z^a b_1, z^b b_2} \qquad \bmod\ q\,L(V_{\mathrm{aff}}^{\otimes 2}).
\end{aligned}$$

Since $L(N) = N \cap L(V_{\mathrm{aff}}^{\otimes 2})$, the above congruence is also true modulo $qL(N)$. Since we have

$$\begin{aligned}
&\tilde{R}_{23} \circ \tilde{R}_{12}\big(G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3)\big) \\
&\qquad \equiv G(z^b b_1) \otimes G(z^c b_2) \otimes G(z^a b_3) \qquad \bmod\ qL(V_{\mathrm{aff}}^{\otimes 3}),
\end{aligned}$$

etc., we have

$$\begin{aligned}
&(\tilde{R}_{12} + \psi_{21}) \circ \tilde{R}_{23} \circ \tilde{R}_{12}\big(G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3)\big) \\
&\qquad \equiv (\tilde{R}_{12} + \psi_{21})\big(G(z^b b_1) \otimes G(z^c b_2) \otimes G(z^a b_3)\big) \\
&\qquad \equiv (1+\delta_{b,c})C_{z^b b_1, z^c b_2} \otimes G(z^a b_3) \qquad \bmod\ qL(N_3),
\end{aligned}$$

and similarly

$$\begin{aligned}
&(\tilde{R}_{23} + \psi_{23}) \circ \tilde{R}_{12}\big(G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3)\big) \\
&\qquad \equiv (1+\delta_{a,c})G(z^b b_1) \otimes C_{z^a b_2, z^c b_3} \qquad \bmod\ qL(N_3).
\end{aligned}$$

They imply

$$
\begin{aligned}
&\tilde{R}_{12} \circ \tilde{R}_{23} \circ \tilde{R}_{12}\big(G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3)\big) \\
&\equiv (1+\delta_{b,c}) C_{z^b b_1, z^c b_2} \otimes G(z^a b_3) \\
&\qquad - \psi_{21} \tilde{R}_{23} \circ \tilde{R}_{12}\big(G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3)\big) \\
&\equiv (1+\delta_{b,c}) C_{z^b b_1, z^c b_2} \otimes G(z^a b_3) - (1+\delta_{a,c}) \psi_{21} G(z^b b_1) \otimes C_{z^a b_2, z^c b_3} \\
&\qquad + \psi_{21}\psi_{32} \tilde{R}_{12}\big(G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3)\big) \\
&\equiv (1+\delta_{b,c}) C_{z^b b_1, z^c b_2} \otimes G(z^a b_3) - (1+\delta_{a,c}) G(z^b b_1) \otimes C_{z^a b_2, z^c b_3} \\
&\qquad + (1+\delta_{a,b}) C_{z^a b_1, z^b b_2} \otimes G(z^c b_3) \\
&\qquad\qquad - \psi_{21}\psi_{32}\psi_{31} G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3).
\end{aligned}
$$

Here $\equiv$ is taken modulo $qL(N_3)$. Similarly we have

$$
\begin{aligned}
&\tilde{R}_{23} \circ \tilde{R}_{12} \circ \tilde{R}_{23}\big(G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3)\big) \\
&\equiv (1+\delta_{a,b}) G(z^c b_1) \otimes C_{z^a b_2, z^b b_3} - (1+\delta_{a,c}) C_{z^a b_1, z^c b_2} \otimes G(z^b b_3) \\
&\quad + (1+\delta_{b,c}) G(z^a b_1) \otimes C_{z^b b_2, z^c b_3} - \psi_{32}\psi_{31}\psi_{21} G(z^a b_1) \otimes G(z^b b_2) \otimes G(z^c b_3).
\end{aligned}
$$

Comparing these two identities, we obtain the desired result. □

Lemma 3.3.11. *The $\mathbb{Q}$-vector space $L(N_n)/qL(N_n)$ is generated by $G(b_1) \otimes \cdots \otimes G(b_{i-1}) \otimes C_{b_i, b_{i+1}} \otimes G(b_{i+2}) \otimes \cdots \otimes G(b_n)$ where $(b_1, \ldots, b_n)$ ranges over the elements in B^n_{aff} such that $(b_{i+1}, \ldots, b_n)$ is normally ordered and $H(b_i \otimes b_{i+1}) \le 0$.*

Proof. $L(N_n)$ is generated by $G(b_1) \otimes \cdots \otimes G(b_{i-1}) \otimes C_{b_i, b_{i+1}} \otimes G(b_{i+2}) \otimes \cdots \otimes G(b_n)$. Here $H(b_i \otimes b_{i+1}) \le 0$ but $(b_{i+1}, \ldots, b_n)$ is not necessarily normally ordered. We shall prove that such a vector can be written as a $\mathbb{Q}$-linear combination of vectors satisfying the conditions as in the lemma, by induction on n and descending induction on i. Arguing by induction on n, we may assume $i = 1$. Write $b_k = z^{a_k} \tilde{b}_k$ with $H(\tilde{b}_k \otimes \tilde{b}_{k+1}) = 0$. Then $a_1 \ge a_2$. By Lemma 3.3.8 (iii), we may assume that $a_3 < a_4 < \cdots < a_n$. If $a_2 < a_3$, there is nothing to prove. Assume $a_2 \ge a_3$. Then the preceding lemma implies

$$
\begin{aligned}
&(1+\delta_{a_1,a_2}) C_{z^{a_1}\tilde{b}_1, z^{a_2}\tilde{b}_2} \otimes G(z^{a_3}\tilde{b}_3) \\
&\equiv\ -(1+\delta_{a_2,a_3}) C_{z^{a_2}\tilde{b}_1, z^{a_3}\tilde{b}_2} \otimes G(z^{a_1}\tilde{b}_3) - (1+\delta_{a_1,a_3}) C_{z^{a_1}\tilde{b}_1, z^{a_3}\tilde{b}_2} \otimes G(z^{a_2}\tilde{b}_3) \\
&\quad + (1+\delta_{a_2,a_3}) G(z^{a_1}\tilde{b}_1) \otimes C_{z^{a_2}\tilde{b}_2, z^{a_3}\tilde{b}_3} + (1+\delta_{a_1,a_3}) G(z^{a_2}\tilde{b}_1) \otimes C_{z^{a_1}\tilde{b}_2, z^{a_3}\tilde{b}_3} \\
&\quad + (1+\delta_{a_1,a_2}) G(z^{a_3}\tilde{b}_1) \otimes C_{z^{a_1}\tilde{b}_2, z^{a_2}\tilde{b}_3} \qquad\qquad \bmod\ qL(N_3).
\end{aligned}
$$

Note that a_3 is the smallest among $(a_1, \ldots, a_n)$. After tensoring $G(z^{a_4}\tilde{b}_4) \otimes \cdots \otimes G(z^{a_n}\tilde{b}_n)$, the first two terms can be written in the desired form by Lemma 3.3.8 (iii), and the last three terms can be written in the desired form by the hypothesis of induction on i. □

Theorem 3.3.12. *The normally ordered wedges form a base of* $\bigwedge^n V_{\mathrm{aff}}$.

Proof. The normally ordered wedges generate $\bigwedge^n V_{\mathrm{aff}}$ by Proposition 3.3.9. We shall show that any linear combination of normally ordered tensors in N_n vanishes. Let C be such a linear combination. Since $\bigcap_k q^k L(N_n) \subset \bigcap_k q^k L(\bigwedge^n V_{\mathrm{aff}}) = 0$, it is enough to show that $C \in L(N_n)$ implies $C \in qL(N_n)$. By the preceding lemma, we can write

$$C \equiv \sum_{i=1}^{n-1} \sum_{(b_1,\dots,b_n)\in K_i} a_i(b_1,\dots,b_n)\, G(b_1)\otimes\cdots\otimes G(b_{i-1})\otimes C_{b_i,b_{i+1}} \otimes G(b_{i+2})\otimes\cdots\otimes G(b_n) \mod qL(N_n).$$

Here the coefficients $a_i(b_1,\dots,b_n)$ belong to $\mathbb{Q}$ and $(b_{i+1},\dots,b_n)$ is normally ordered for $(b_1,\dots,b_n)\in K_i$. In order to show the vanishing of $a_i(b_1,\dots,b_n)$, let us calculate C modulo $qL(V_{\mathrm{aff}}^{\otimes n})$.

$$C \equiv \sum_{i=1}^{n-1} \sum_{(b_1,\dots,b_n)\in K_i} a_i(b_1,\dots,b_n)\, b_1\otimes\cdots\otimes b_{i-1}\otimes C_{b_i,b_{i+1}}\otimes b_{i+2}\otimes\cdots\otimes b_n \mod qL(V_{\mathrm{aff}}^{\otimes n}).$$

Since Lemma 3.3.8 (ii) implies

$$C_{b_i,b_{i+1}} \equiv b_i\otimes b_{i+1} + \delta(H(b_i\otimes b_{i+1})<0)z^{H(b_i\otimes b_{i+1})}b_i\otimes z^{-H(b_i\otimes b_{i+1})}b_{i+1},$$

we have

$$\begin{aligned} C \equiv \sum_{i=1}^{n-1} \sum_{(b_1,\dots,b_n)\in K_i} & a_i(b_1,\dots,b_n)\, b_1\otimes\cdots\otimes b_{i-1} \\ &\otimes \left(b_i\otimes b_{i+1} + \delta(H(b_i\otimes b_{i+1})<0)z^{H(b_i\otimes b_{i+1})}b_i\otimes z^{-H(b_i\otimes b_{i+1})}b_{i+1}\right) \\ &\otimes b_{i+2}\otimes\cdots\otimes b_n \qquad \mod qL(V_{\mathrm{aff}}^{\otimes n}). \end{aligned} \tag{3.3.8}$$

We shall show $a_i(b_1,\dots,b_n)=0$ by the descending induction on i. Assume that $a_k(b_1,\dots,b_n)=0$ for $k>i$. Note that $H(b_i\otimes b_{i+1})\le 0$, and $H\big(z^{H(b_i\otimes b_{i+1})}b_i\otimes z^{-H(b_i\otimes b_{i+1})}b_{i+1}\big)>0$ when $H(b_i\otimes b_{i+1})<0$. We also note that $(b_i,\dots,b_n)$ is not normally ordered for $(b_1,\dots,b_n)\in K_i$ but it is normally ordered for $(b_1,\dots,b_n)\in K_k$ with $k<i$. By these observations, for $(b_1,\dots,b_n)\in K_i$, the coefficient of $b_1\otimes b_2\otimes\cdots\otimes b_n$ on the right hand side of (3.3.8) is $a_i(b_1,\dots,b_n)$ and $b_1\otimes b_2\otimes\cdots\otimes b_n$ does not appear in C. Hence $a_i(b_1,\dots,b_n)$ must vanish. □

Corollary 3.3.13. *$L(\bigwedge^n V_{\mathrm{aff}})$ is a free A-module with the normally ordered wedges as a base.*

In fact, the normally ordered wedges generate $L(\bigwedge^n V_{\mathrm{aff}})$ by Proposition 3.3.9 and are linearly independent by the theorem above.

Let $B(\bigwedge^n V_{\mathrm{aff}})$ be the set of normally ordered sequences. Let us regard $B(\bigwedge^n V_{\mathrm{aff}})$ as a subset of $B_{\mathrm{aff}}^{\otimes n}$. Since it is invariant by $\tilde{e}_i$ and $\tilde{f}_i$, we can endow $B(\bigwedge^n V_{\mathrm{aff}})$ with the structure of crystal induced by $B_{\mathrm{aff}}^{\otimes n}$. We regard $B(\bigwedge^n V_{\mathrm{aff}})$ as a basis of $L(\bigwedge^n V_{\mathrm{aff}})/qL(\bigwedge^n V_{\mathrm{aff}})$. Then we have

Proposition 3.3.14. $\left(L(\bigwedge^n V_{\mathrm{aff}}), B(\bigwedge^n V_{\mathrm{aff}})\right)$ *is a crystal base of* $\bigwedge^n V_{\mathrm{aff}}$.

The following lemma follows immediately from (3.3.3).

Lemma 3.3.15. *Let $f(z_1, \ldots, z_n)$ be a symmetric Laurent polynomial. Then $f(z \otimes 1 \otimes \cdots \otimes 1, 1 \otimes z \otimes 1 \otimes \cdots \otimes 1, \ldots, 1 \otimes \cdots \otimes 1 \otimes z)$ induces an endomorphism of $\bigwedge^n V_{\mathrm{aff}}$.*

4. Fock space

4.1. Ground state sequence

In this section we shall introduce a q-deformed Fock space in a similar way to the $A_n^{(1)}$–case ([KMS]).

We continue the discussion on the perfect crystal B of level l. Let us take a sequence $\{b_m^\circ\}_{m\in\mathbb{Z}}$ in B_{aff} such that

$$\begin{aligned} \langle c, \varepsilon(b_m^\circ)\rangle &= l, \\ \varepsilon(b_m^\circ) &= \varphi(b_{m+1}^\circ) \\ \text{and} \qquad H(b_m^\circ \otimes b_{m+1}^\circ) &= 1\,. \end{aligned}$$

We call $(\cdots, b_{-1}^\circ, b_0^\circ, b_1^\circ, \ldots)$ a ground state sequence. If we give one of b_m°, then the other members of a ground state sequence are uniquely determined. Since B is a finite set, there exists a positive integer N and an integer c such that

$$b_{k+N}^\circ = z^c b_k^\circ \quad \text{for every } k. \tag{4.1.1}$$

Take weights $\lambda_m \in P$ of level l satisfying

$$\begin{aligned} \lambda_m &= \mathrm{wt}\,(b_m^\circ) + \lambda_{m+1} \\ \text{and} \qquad \mathrm{cl}(\lambda_m) &= \varphi(b_m^\circ) = \varepsilon(b_{m-1}^\circ). \end{aligned}$$

Set $v_m^\circ = G(b_m^\circ) \in V_{\mathrm{aff}}$.

4.2. Definition of Fock space

For $m \in \mathbb{Z}$, let us define first a (fake) q-deformed Fock space $\overline{\mathcal{F}}_m$ as the inductive limit ($k \to \infty$) of $\bigwedge^{k-m} V_{\text{aff}}$, where $\bigwedge^{k-m} V_{\text{aff}} \to \bigwedge^{k+1-m} V_{\text{aff}}$ is given by $u \mapsto u \wedge v_k^\circ$. Intuitively $\overline{\mathcal{F}}_m$ is the subspace of $\bigwedge^\infty V_{\text{aff}}$ generated by the vectors of the form $u_m \wedge u_{m+1} \wedge \cdots$ with $u_k = v_k^\circ$ for $k \gg m$. Similarly we define $L(\overline{\mathcal{F}}_m)$ as the inductive limit of $L(\bigwedge^{k-m} V_{\text{aff}})$. We define the vacuum vector $\overline{|m\rangle} = v_m^\circ \wedge v_{m+1}^\circ \wedge \cdots \in \overline{\mathcal{F}}_m$. Then any vector can be written as $v \wedge \overline{|m+r\rangle}$ for some positive integer r and $v \in \bigwedge^r V_{\text{aff}}$. Note that $v \wedge \overline{|m+r\rangle} = 0$ if and only if $v \wedge v_{m+r}^\circ \wedge \cdots \wedge v_{m+s}^\circ = 0$ for some $s > r$. Then we introduce the true (q-deformed) Fock space by

$$\mathcal{F}_m = \overline{\mathcal{F}}_m / \Big(\bigcap_{n>0} q^n L(\overline{\mathcal{F}}_m) \Big) .$$

Let $L(\mathcal{F}_m) \subset \mathcal{F}_m$ be the image of $L(\overline{\mathcal{F}}_m)$, and $|m\rangle$ the image of $\overline{|m\rangle}$. We have the homomorphism

$$\wedge : \bigwedge^r V_{\text{aff}} \otimes \mathcal{F}_{m+r} \to \mathcal{F}_m .$$

For a normally ordered sequence $(b_m, b_{m+1}, \dots)$ in B_{aff} such that $b_k = b_k^\circ$ for $k \gg m$, we call $G(b_m) \wedge G(b_{m+1}) \wedge \cdots \in \mathcal{F}_m$ a *normally ordered wedge.*

Theorem 4.2.1. *The normally ordered wedges form a base of $\mathcal{F}_m$.*

In order to prove this theorem, we need some preparations.

Lemma 4.2.2. *If $l(b) > l(b_m^\circ)$, then $H(b \otimes b_{m+1}^\circ) \le 0$.*

Proof. If $l(b) \gg 0$, then the assertion holds. Let us prove it by descending induction on $l(b)$. Assume that there is $i \in I$ such that $\tilde{e}_i(b \otimes b_{m+1}^\circ) = (\tilde{e}_i b) \otimes b_{m+1}^\circ \neq 0$. Then $l(b) < l(\tilde{e}_i b)$ and hence $H(b \otimes b_{m+1}^\circ) = H(\tilde{e}_i b \otimes b_{m+1}^\circ) \le 0$ by the hypothesis of induction. Hence we may assume that there is no such i. Then $\varepsilon_i(b) \le \varphi_i(b_{m+1}^\circ)$ for any i, and hence $b = z^a b_m^\circ$ for some $a \in \mathbb{Z}$. Since $l(b) > l(b_m^\circ)$, we have $a > 0$. Therefore $H(b \otimes b_{m+1}^\circ) = 1 - a \le 0$. □

Proposition 4.2.3. *Assume $H(b \otimes b_m^\circ) \le 0$. Then for every n we can find $m_1 \ge m$ such that*

$$G(b) \wedge v_m^\circ \wedge \cdots \wedge v_{m_1}^\circ \in q^n L(\bigwedge^{m_1 - m + 2} V_{\text{aff}}).$$

Proof. We shall prove this by induction on n and $H(b \otimes b_m^\circ)$. Set $H(b \otimes b_m^\circ) = -c$ and

$$G(b) \wedge v_m^\circ = \sum a(b_1, b_2) G(b_1) \wedge G(b_2).$$

Here the sum ranges over normally ordered pairs (b_1, b_2) such that

$$\begin{aligned} l(b_m^\circ) &\le l(b_1) < l(b), \\ l(b_m^\circ) &< l(b_2) \le l(b). \end{aligned} \tag{4.2.1}$$

By the preceding lemma $H(b_2 \otimes b^\circ_{m+1}) \leq 0$. Lemma 3.3.8 (i) implies

$$a(b_1, b_2) \equiv -\delta\big(c < 0 \text{ and } (b_1, b_2) = (z^{-c}b, z^c b^\circ_m)\big) \quad \bmod \ qA.$$

We have

$$G(b) \wedge v^\circ_m \wedge \cdots \wedge v^\circ_{m_1} = \sum a(b_1, b_2) G(b_1) \wedge G(b_2) \wedge v^\circ_{m+1} \wedge \cdots \wedge v^\circ_{m_1}.$$

Since $l(b_2) > l(b^\circ_m)$, we have $G(b_2) \wedge v^\circ_{m+1} \wedge \cdots \wedge v^\circ_{m_1} \in q^{n-1}L(\bigwedge V_{\mathrm{aff}})$. Hence $a(b_1, b_2)G(b_1) \wedge G(b_2) \wedge v^\circ_{m+1} \wedge \cdots \wedge v^\circ_{m_1}$ belongs to $q^n L(\bigwedge V_{\mathrm{aff}})$ except $c < 0$ and $(b_1, b_2) = (z^{-c}b, z^c b^\circ_m)$.

Assume that $c < 0$ and $(b_1, b_2) = (z^{-c}b, z^c b^\circ_m)$. Then we have $0 \geq H(z^c b^\circ_m \otimes b^\circ_{m+1}) = 1 - c > H(b \otimes b^\circ_m)$. Hence $a(b_1, b_2)G(b_1) \wedge G(b_2) \wedge v^\circ_{m+1} \wedge \cdots \wedge v^\circ_{m_1}$ belongs to $q^n L(\bigwedge^{m_1 - m + 2} V_{\mathrm{aff}})$ by the hypothesis of induction on $H(b \otimes b^\circ_m)$. □

Remark. Assume that c in (4.1.1) is positive (or equivalently, $l(b^\circ_m)$ tends to infinity as m tends to infinity). Then $H(b \otimes b_m) \leq 0$ implies $G(b) \wedge v^\circ_m \wedge \cdots \wedge v^\circ_{m_1} = 0$ for $m_1 \gg m$. In fact by the same argument as above we have $G(b) \wedge v^\circ_m \wedge \cdots \wedge v^\circ_{m_1} \in \sum_{b'} \bigwedge^{m_1 - m + 1} V_{\mathrm{aff}} \wedge G(b')$ where b' satisfies $l(b^\circ_{m_1}) < l(b') \leq l(b)$.

Note that, under the condition of the proposition, $G(b) \wedge v^\circ_m \wedge v^\circ_{m+1} \wedge \cdots \wedge v^\circ_k = 0$ for $k \gg m$ is false in general.

A similar argument shows the following dual statement.

Proposition 4.2.4. *Assume $H(b^\circ_m \otimes b) \leq 0$. Then for every n we can find $m_1 \leq m$ such that*

$$v^\circ_{m_1} \wedge \cdots \wedge v^\circ_m \wedge G(b) \in q^n L(\textstyle\bigwedge^{m - m_1 + 2} V_{\mathrm{aff}}).$$

As an immediate consequence of Proposition 4.2.3, we obtain the following result.

Theorem 4.2.5. *For any vector $b \in B_{\mathrm{aff}}$ such that $H(b \otimes b^\circ_m) \leq 0$, we have the equality in $\mathcal{F}_m$*

$$G(b) \wedge |m\rangle = 0\,.$$

Proof of Theorem 4.2.1. Any vector in $\mathcal{F}_m$ can be written in the form $v \wedge |m + r\rangle$ with $v \in \bigwedge^r V_{\mathrm{aff}}$. We may assume that v is a normally ordered wedge

$$G(b_m) \wedge \cdots \wedge G(b_{m+r-1}).$$

If $H(b_{m+r-1} \otimes b^\circ_{m+r}) > 0$, then $v \wedge |m + r\rangle$ is a normally ordered wedge and otherwise $v \wedge |m + r\rangle = 0$ by Proposition 4.2.3.

The linear independence follows immediately from the corresponding statement for the wedge space (Corollary 3.3.13). □

By a similar argument, we have

Proposition 4.2.6. *$L(\mathcal{F}_m)$ is a free A-submodule of $\mathcal{F}_m$ generated by the normally ordered wedges.*

Proposition 4.2.7.

$$\begin{aligned}\bigcap_{n>0} q^n L(\overline{\mathcal{F}}_m) &= \sum_{H(b\otimes b^\circ_{m+r})\leq 0} \bigwedge^{r-1} V_{\mathrm{aff}} \wedge G(b) \wedge \overline{|m+r\rangle} \\ &= \sum_{l(b)>l(b^\circ_{m+r-1})} \bigwedge^{r-1} V_{\mathrm{aff}} \wedge G(b) \wedge \overline{|m+r\rangle}.\end{aligned}$$

Proof. The first equality follows from Theorems 4.2.1 and 4.2.5 and the last follows from Lemma 4.2.2 and (4.2.1). □

As a corollary of Theorem 4.2.5 we have the following result concerning vertex operators.

Proposition 4.2.8. *Let $V(\lambda_m)$ be the irreducible $U_q(\mathfrak{g})$-module with highest weight λ_m and u_{λ_m} its highest weight vector. Let $\Phi : V_{\mathrm{aff}} \otimes V(\lambda_m) \to V(\lambda_{m-1})$ be an intertwiner. Then for any vector $b \in B_{\mathrm{aff}}$ such that $H(b \otimes b^\circ_m) \leq 0$, $\Phi(G(b) \otimes u_{\lambda_m}) = 0$.*

Proof. As proved in [DJO], the intertwiner is unique up to a constant. As seen in the next two subsections, $\mathcal{F}_m$ has a $U_q(\mathfrak{g})$-module structure and contains $V(\lambda_m)$ as a direct summand. By this embedding, the highest vector u_{λ_m} of $V(\lambda_m)$ corresponds to $|m\rangle$. Therefore Φ is given as the composition:

$$V_{\mathrm{aff}} \otimes V(\lambda_m) \to V_{\mathrm{aff}} \otimes \mathcal{F}_m \to \mathcal{F}_{m-1} \to V(\lambda_{m-1}).$$

Now the result follows from Theorem 4.2.5.

Remark. It is known (see e.g. [DJO]) that $\Phi(v \otimes u_{\lambda_m}) = 0$ for $v \in (V_{\mathrm{aff}})_{\lambda_{m-1}-\lambda_m}$ such that $v \in \sum_i e_i^{1+\langle h_i,\lambda_{m-1}\rangle} V_{\mathrm{aff}}$. On the other hand, by the property of the lower global base ([K2]), $G(b)$ belongs to $\sum_i e_i^{1+\langle h_i,\lambda_{m-1}\rangle} V_{\mathrm{aff}}$ if and only if $\varphi_i(b) > \langle h_i, \lambda_{m-1}\rangle$ for some i. Therefore, $\Phi(G(b) \otimes u_{\lambda_m}) = 0$ for $b \in (B_{\mathrm{aff}})_{\lambda_{m-1}-\lambda_m}$ other than b°_{m-1}.

This observation shows that we have to take a lower global base in order to have Theorem 4.2.5. Theorem 4.2.5, as well as Proposition 4.2.8, does not hold for an arbitrary choice of base other than the lower global base. In the course of our construction of the Fock space, we have not used explicitly the property of the lower global base. This is hidden in postulate (R). This postulate fails for an arbitrary choice of base.

4.3. $U_q(\mathfrak{g})$-module structure on the Fock space

Let us define the action of $U_q(\mathfrak{g})$ on $\mathcal{F}_m$. We define first the action of the Cartan part of $U_q(\mathfrak{g})$ by assigning weights. We set $\mathrm{wt}\,(|m\rangle) = \lambda_m$ and $\mathrm{wt}\,(v \wedge |m+r\rangle) = \mathrm{wt}\,(v) + \mathrm{wt}\,(|m+r\rangle)$ for $v \in \bigwedge^r V_{\mathrm{aff}}$. This defines the weight decomposition of the Fock space.

Let $B(\mathcal{F}_m)$ denote the set of normally ordered sequences $(b_m, b_{m+1}, \dots)$ in B_{aff} such that $b_k = b_k^\circ$ for $k \gg m$. Then it has a crystal structure as in [KMN1]. Moreover $B(\mathcal{F}_m)$ may be considered as a base of $L(\mathcal{F}_m)/qL(\mathcal{F}_m)$ by Proposition 4.2.6. We write $b_m \wedge b_{m+1} \wedge \cdots$ for $(b_m, b_{m+1}, \dots)$.

Proposition 4.3.1.

(i) $\mathrm{ch}\,(\mathcal{F}_m) = \mathrm{ch}\,(V(\lambda_m)) \prod_{k>0}(1 - \mathrm{e}^{-k\delta})^{-1}$.

(ii) *The weights of $\mathcal{F}_m$ appear as weights of $V(\lambda_m)$. In particular, any weight μ of $\mathcal{F}_m$ satisfies $s(\mu) \le s(\lambda_m)$ (see the end of §3.2 for $s : P \to \mathbb{Q}$). Moreover, $s(\mu) = s(\lambda_m)$ implies $\mu = \lambda_m$.*

(iii) *For any $\mu \in P$, $\dim(\mathcal{F}_m)_\mu < \infty$.*

(iv) $(\mathcal{F}_m)_{\lambda_m - n\alpha_i} = \begin{cases} KG(\tilde{f}_i^n b_m^\circ) \wedge |m+1\rangle & \text{if } 0 \le n \le \langle h_i, \lambda_m \rangle, \\ 0 & \text{otherwise.} \end{cases}$

(v) *If $b \in B_{\mathrm{aff}}$ satisfies $\mathrm{wt}\,(b) = \mathrm{wt}\,(b_m^\circ) - n\alpha_i$, then $G(b) \wedge |m+1\rangle = 0$ unless $0 \le n \le \langle h_i, \lambda_m \rangle$ and $b = \tilde{f}_i^n b_m^\circ$.*

(vi) *Any highest weight element of $B(\mathcal{F}_m)$ has the form $z^{a_m} b_m^\circ \wedge z^{a_{m+1}} b_{m+1}^\circ \wedge \cdots$ with $a_m \le a_{m+1} \le \cdots$ and $a_k = 0$ for $k \gg m$.*

(vii) *For $b_m \wedge b_{m+1} \wedge \cdots \in B(\mathcal{F}_m)$, $b_m = b_m^\circ$ implies $b_k = b_k^\circ$ for any $k \ge m$.*

Proof. By Proposition 4.6.4 in [KMN1] (see also Appendix A), we have

$$\mathrm{ch}\,(V(\lambda_m)) = \mathrm{e}^{\lambda_m} \sum \mathrm{e}^{\sum_{n \ge m}(\mathrm{wt}\,(b_n) - \mathrm{wt}\,(b_n^\circ))}$$

where the sum ranges over the family $\mathcal{B}_0$ of sequences $b_m, b_{m+1}, \dots$ in B_{aff} such that $b_n = b_n^\circ$ for $n \gg m$ and $H(b_n \otimes b_{n+1}) = 1$ for any $n \ge m$. On the other hand, we have

$$\mathrm{ch}\,(\mathcal{F}_m) = \mathrm{e}^{\lambda_m} \sum \mathrm{e}^{\sum_{n \ge m}(\mathrm{wt}\,(b_n) - \mathrm{wt}\,(b_n^\circ))}$$

where the sum ranges over the family $\mathcal{B}$ of normally ordered $b_m, b_{m+1}, \dots$ such that $b_n = b_n^\circ$ for $n \gg m$. We have

$$\begin{aligned}\mathcal{B} = \{&(z^{-a_m} b_m, z^{-a_{m+1}} b_{m+1}, \dots)\,;\\ &(b_m, b_{m+1}, \dots) \in \mathcal{B}_0,\ a_m \ge a_{m+1} \ge \cdots \text{ and } a_n = 0 \text{ for } n \gg m\}.\end{aligned}$$

To obtain (i), it is enough to remark that z has weight δ.

The assertions (ii)–(vi) follow from (i) and Theorem 4.2.5. The assertion (vii) follows from (vi) and

$$\tilde{f}_i(z^{a_m} b_m^\circ \wedge z^{a_{m+1}} b_{m+1}^\circ \wedge \cdots) = z^{a_m} \tilde{f}_i b_m^\circ \wedge z^{a_{m+1}} b_{m+1}^\circ \wedge \cdots .$$

□

Now we shall define the action of e_i and f_i on $\mathcal{F}_m$.

Taking $\{q^n L(\mathcal{F}_m)\}_n$ as a neighborhood system of 0, $\mathcal{F}_m$ is endowed with a so-called q-adic topology. Since $\bigcap_n q^n L(\mathcal{F}_m) = 0$ by construction, the q-adic topology is separated. Since we use $K = \mathbb{Q}(q)$ as a base field, $\mathcal{F}_m$ is not complete with respect to this topology. For any $\mu \in P$, the completion of $(\mathcal{F}_m)_\mu$ is $\mathbb{Q}((q)) \otimes_K (\mathcal{F}_m)_\mu$.

Proposition 4.3.2. *For any vectors $u_m, u_{m+1}, \cdots \in V_{\mathrm{aff}}$ such that $u_k = v_k^\circ$ for $k \gg m$,*

$$\sum_{k \geq m} t_i^{-1}(u_m \wedge \cdots \wedge u_{k-1}) \wedge e_i u_k \wedge u_{k+1} \wedge \cdots \tag{4.3.1}$$

and

$$\sum_{k \geq m} u_m \wedge \cdots \wedge u_{k-1} \wedge f_i u_k \wedge t_i(u_{k+1} \wedge \cdots) \tag{4.3.2}$$

converge in the q-adic topology to elements of $\mathbb{Q}((q)) \otimes_K \mathcal{F}_m$.

Proof. First note that $(e_i v_k^\circ) \wedge |k+1\rangle = 0$ because $\lambda_k + \alpha_i$ is not a weight of $\mathcal{F}_k$. Hence, only finitely many terms survive in (4.3.1).

In order to prove the convergence of (4.3.2), we may assume that $u_k = v_k^\circ$ for every $k \geq m$. Then

$$v_m^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge f_i v_k^\circ \wedge t_i(v_{k+1}^\circ \wedge \cdots) = q_i^{\langle h_i, \lambda_{k+1} \rangle} v_m^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge f_i v_k^\circ \wedge |k+1\rangle.$$

Since $\langle h_i, \lambda_{k+1} \rangle$ takes only finitely many values, it is enough to show that $v_m^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge f_i v_k^\circ \wedge |k+1\rangle$ converges in the q-adic topology. This follows from the following lemma. □

Lemma 4.3.3. *Let C be an endomorphism of the K-vector space V_{aff} of weight $\mu \neq 0$. Assume that $Cz = zC$. Then for any m, $v_m^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge C v_k^\circ \wedge |k+1\rangle$ converges to 0 in the q-adic topology when k tends to infinity.*

Proof. Write

$$C v_k^\circ = \sum_\nu c_{k,\nu} G(b_{k,\nu}).$$

Take N and c as in (4.1.1). Then we have also the periodicity $b_{k+N,\nu} = z^c b_{k,\nu}$ and $c_{k+N,\nu} = c_{k,\nu}$. Hence $c_{k,\nu}$ is bounded with respect to the q-adic topology. Therefore it is enough to show that $v_m^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge G(b_{k,\nu}) \wedge |k+1\rangle$ converges to 0.

By Proposition 4.3.1 (vi), $(b^\circ_m, \dots, b^\circ_{k-1}, b_{k,\nu}, b^\circ_{k+1}, \dots)$ is not normally ordered. It means that either $H(b_{k,\nu} \otimes b^\circ_{k+1}) \le 0$ or $H(b^\circ_{k-1} \otimes b_{k,\nu}) \le 0$. If $H(b_{k,\nu} \otimes b^\circ_{k+1}) \le 0$ then $G(b_{k,\nu}) \wedge |k+1\rangle$ vanishes. If $H(b^\circ_{k-1} \otimes b_{k,\nu}) \le 0$, then $v^\circ_{m-s} \wedge \cdots v^\circ_{k-1} \wedge G(b_{k,\nu}) \wedge |k+1\rangle$ converges to 0 when s tends to infinity by Proposition 4.2.4. By shifting the indices, $v^\circ_m \wedge \cdots \wedge v^\circ_{k-1} \wedge G(b_{k,\nu}) \wedge |k+1\rangle$ converges to 0. □

Let us set

$$f_i\overline{|m\rangle} = f_i v^\circ_m \wedge t_i|m+1\rangle + v^\circ_m \wedge f_i v^\circ_{m+1} \wedge t_i|m+2\rangle + \cdots \tag{4.3.3}$$

Then it is an element of $\mathbb{Q}((q)) \otimes_K \mathcal{F}_m$.

Lemma 4.3.4. *$f_i\overline{|m\rangle}$ belongs to $\mathcal{F}_m$.*

Proof. Let us take c and N as in (4.1.1). We define the isomorphism $\psi_m : \mathcal{F}_m \to \mathcal{F}_{m+N}$ by $u_m \wedge u_{m+1} \wedge \cdots \mapsto z^c u_m \wedge z^c u_{m+1} \wedge \cdots$. Then $f_i\overline{|m\rangle}$ satisfies the recurrence relation

$$\begin{aligned} &f_i\overline{|m\rangle} - v^\circ_m \wedge v^\circ_{m+1} \wedge \cdots \wedge v^\circ_{m+N-1} \wedge \psi_m(f_i\overline{|m\rangle}) \\ &= f_i(v^\circ_m \wedge v^\circ_{m+1} \wedge \cdots \wedge v^\circ_{m+N-1}) \wedge t_i|m+N\rangle \in \mathcal{F}_m. \end{aligned}$$

Hence the result follows from the following lemma. □

Lemma 4.3.5. *For $\mu \in P\backslash\{\lambda_m\}$, the endomorphism of $(\mathcal{F}_m)_\mu$ given by $w \mapsto w - v^\circ_m \wedge v^\circ_{m+1} \wedge \cdots \wedge v^\circ_{m+N-1} \wedge \psi_m(w)$ is an isomorphism.*

Proof. It is enough to show its injectivity. We show that $w = v^\circ_m \wedge v^\circ_{m+1} \wedge \cdots \wedge v^\circ_{m+N-1} \wedge \psi_m(w)$ implies $w = 0$.

For $b_m \wedge b_{m+1} \wedge \cdots \in B(\mathcal{F}_m)_\mu$, $(b^\circ_{m-1}, b_m, b_{m+1}, \dots)$ is not normally ordered by Proposition 4.3.1 (vii), and hence $H(b^\circ_{m-1} \otimes b_m) \le 0$. Proposition 4.2.4 implies that $v^\circ_{m-kN} \wedge \cdots \wedge v^\circ_{m-1} \wedge G(b_m) \wedge G(b_m+1) \wedge \cdots$ belongs to $qL(\mathcal{F}_{m-kN})$ for $k \gg 0$. Shifting the indices, we conclude that

$$v^\circ_m \wedge \cdots \wedge v^\circ_{m+kN-1} \wedge \psi_{m+(k-1)N} \cdots \psi_{m+N}\psi_m\big(G(b_m) \wedge G(b_{m+1}) \wedge \cdots\big)$$

belongs to $qL(\mathcal{F}_m)$ for $k \gg 0$. Therefore the homomorphism

$$C : w \mapsto v^\circ_m \wedge \cdots \wedge v^\circ_{m+kN-1} \wedge \psi_{m+(k-1)N} \cdots \psi_{m+N}\psi_m(w)$$

sends $L(\mathcal{F}_m)_\mu$ to $qL(\mathcal{F}_m)_\mu$ for $k \gg 0$. This shows the injectivity of the endomorphism $\mathrm{id}_{(\mathcal{F}_m)_\mu} - C$. □

Now we define

$$\begin{aligned} &e_i(v \wedge \overline{|m+r\rangle}) = e_i v \wedge |m+r\rangle, \\ &f_i(v \wedge \overline{|m+r\rangle}) = f_i v \wedge t_i|m+r\rangle + v \wedge f_i\overline{|m+r\rangle} \end{aligned} \tag{4.3.4}$$

for $v \in \bigwedge^r V_{\mathrm{aff}}$. Then e_i and f_i are well-defined homomorphisms from $\overline{\mathcal{F}}_m$ to $\mathcal{F}_m$. They satisfy

$$\begin{aligned} e_i(v \wedge u) &= e_i v \wedge u + t_i^{-1} v \wedge e_i u \\ f_i(v \wedge u) &= f_i v \wedge t_i u + v \wedge f_i u \end{aligned} \tag{4.3.5}$$

for $v \in \bigwedge^r V_{\mathrm{aff}}$ and $u \in \overline{\mathcal{F}}_{m+r}$. In order to see that they define endomorphisms of $\mathcal{F}_m$, we need to show the following proposition (see Proposition 4.2.7).

Proposition 4.3.6. *Assume that $b \in B_{\mathrm{aff}}$ satisfies $l(b) > l(b^\circ_m)$. Then we have the equalities in $\mathcal{F}_m$.*

$$e_i\big(G(b) \wedge \overline{|m+1\rangle}\big) = 0, \tag{4.3.6}$$

$$f_i\big(G(b) \wedge \overline{|m+1\rangle}\big) = 0. \tag{4.3.7}$$

The first equality (4.3.6) follows from the fact that $\mathrm{wt}\,(b) + \alpha_i + \lambda_{m+1}$ is not a weight of $\mathcal{F}_m$ (following Proposition 4.3.1 (ii)).

Let us prove (4.3.7). First note that the same consideration on the weight implies that

$$\text{if } l(b) > l(b^\circ_m) \text{ and } \mathrm{wt}\,(b) \neq \mathrm{wt}\,(b^\circ_m) + \alpha_i, \text{ then (4.3.7) holds.} \tag{4.3.8}$$

Hence in order to prove (4.3.7), we may assume that

$$\mathrm{wt}\,(b) = \mathrm{wt}\,(b^\circ_m) + \alpha_i. \tag{4.3.9}$$

Sublemma 4.3.7. *Under the condition (4.3.9), we can write*

$$G(b) \wedge v^\circ_{m+1} \wedge \cdots \wedge v^\circ_{m+r} = u + \sum a(b_0, \ldots, b_r) G(b_0) \wedge \cdots \wedge G(b_r). \tag{4.3.10}$$

Here u satisfies $f_i(u \wedge \overline{|m+r+1\rangle}) = 0$, the coefficients $a(b_0, \ldots, b_r)$ belong to $q^r A$, and the sum ranges over $(b_0, \ldots, b_r)$ such that

$$\mathrm{wt}\,(b_j) = \begin{cases} \mathrm{wt}\,(b^\circ_{m+j}) & \text{for } 0 \le j < r \\ \mathrm{wt}\,(b^\circ_{m+r}) + \alpha_i & \text{for } j = r \end{cases}. \tag{4.3.11}$$

Proof. We shall prove this by induction on r. Assuming (4.3.10) for r, let us prove (4.3.10) for $r+1$. Since $H(b_r \otimes b^\circ_{m+r+1}) \le 0$ by Lemma 4.2.2, we can write

$$G(b_r) \wedge v^\circ_{m+r+1} = \sum a_{b',b''} G(b') \wedge G(b''). \tag{4.3.12}$$

Here (b', b'') ranges over normally ordered pairs such that

$$\begin{aligned} &l(b^\circ_{m+r+1}) \le l(b') < l(b_r), \\ &l(b^\circ_{m+r+1}) < l(b'') \le l(b_r). \end{aligned} \tag{4.3.13}$$

If $\mathrm{wt}\,(b'') \neq \mathrm{wt}\,(b^\circ_{m+r+1}) + \alpha_i$, then we have $G(b'') \wedge |m+r+2\rangle = 0$ and $f_i(G(b'') \wedge \overline{|m+r+2\rangle}) = 0$ by (4.3.13) and (4.3.8). Therefore $f_i(G(b_0) \wedge \cdots \wedge G(b_{r-1}) \wedge G(b') \wedge G(b'') \wedge \overline{|m+r+2\rangle}) = 0$. If $\mathrm{wt}\,(b'') = \mathrm{wt}\,(b^\circ_{m+r+1}) + \alpha_i$, then $\mathrm{wt}\,(b') = \mathrm{wt}\,(b^\circ_{m+r})$. Moreover Lemma 3.3.8 (i) implies $a_{b',b''} \in qA$. Thus the induction proceeds. □

We resume the proof of Proposition 4.3.6. We have

$$\begin{aligned} f_i(G(b) \wedge \overline{|m+1\rangle}) = \sum a(b_0, \ldots, b_r) \\ \big(G(b_0) \wedge \cdots \wedge G(b_r) \wedge f_i \overline{|m+r+1\rangle} \\ + \sum_{0 \le j \le r} G(b_0) \wedge \cdots \wedge G(b_{j-1}) \wedge f_i G(b_j) \wedge \\ t_i G(b_{j+1}) \wedge \cdots \wedge t_i G(b_r) \wedge t_i |m+r+1\rangle\big). \end{aligned}$$

There is a constant s such that $f_i L_{\mathrm{aff}} \in q^s L_{\mathrm{aff}}$, and $f_i \overline{|m+r+1\rangle}$ is bounded with respect to the q-adic topology. Moreover, $t_i G(b_{j+1}) \wedge \cdots \wedge t_i G(b_r) \wedge t_i |m+r+1\rangle = q_i^{\langle h_i, \lambda_{m+j+1}\rangle + 2\delta(j<r)} G(b_{j+1}) \wedge \cdots \wedge G(b_r) \wedge |m+r+1\rangle$ and $\langle h_i, \lambda_{m+j}\rangle$ is bounded from below. Hence there is a constant d independent of r such that

$$f_i\big(G(b) \wedge \overline{|m+1\rangle}\big) \in q^{r+d} L(\mathcal{F}_m) \quad \text{for every } r.$$

This implies the equality (4.3.7) in $\mathcal{F}_m$. This completes the proof of Proposition 4.3.6.

Thus we have defined the action of e_i and f_i on $\mathcal{F}_m$. Now we shall show the commutation relations between them.

Proposition 4.3.8. *On $\mathcal{F}_m$ we have*

$$[e_i, f_j] = \delta_{ij}(t_i - t_i^{-1})/(q_i - q_i^{-1}). \tag{4.3.14}$$

Proof. First note that (4.3.5) implies

$$[e_i, f_j](v \wedge u) = [e_i, f_j]v \wedge t_j u + t_i^{-1} v \wedge [e_i, f_j]u. \tag{4.3.15}$$

for $v \in \bigwedge^r V_{\mathrm{aff}}$ and $u \in \mathcal{F}_{m+r}$. Hence, it is enough to prove that the equality (4.3.14) holds when it is applied to the vacuum vector. If $i \neq j$ then $[e_i, f_j]|m\rangle = 0$ because $\lambda_m + \alpha_i - \alpha_j$ is not a weight of $\mathcal{F}_m$.

Now we shall show $[e_i, f_i] = \{t_i\}_i$. Here $\{x\}_i = (x - x^{-1})/(q_i - q_i^{-1})$.

Since $[e_i, f_i]|m\rangle$ has weight λ_m, there is $c_m \in K$ such that $[e_i, f_i]|m\rangle = c_m|m\rangle$. Then by (4.3.15), we have

$$\begin{aligned}[e_i, f_i]|m\rangle &= [e_i, f_i]v_m^\circ \wedge t_i|m+1\rangle + t_i^{-1}v_m^\circ \wedge [e_i, f_i]|m+1\rangle \\ &= \left(q_i^{\langle h_i,\lambda_{m+1}\rangle}[\langle h_i, \lambda_m - \lambda_{m+1}\rangle]_i + q_i^{\langle h_i,\lambda_{m+1}-\lambda_m\rangle}c_{m+1}\right)|m\rangle.\end{aligned}$$

Hence we have a recurrence relation

$$c_m = q_i^{\langle h_i,\lambda_{m+1}\rangle}[\langle h_i, \lambda_m - \lambda_{m+1}\rangle]_i + q_i^{\langle h_i,\lambda_{m+1}-\lambda_m\rangle}c_{m+1}.$$

Solving this, there is a constant $a \in K$ such that

$$c_m = [\langle h_i, \lambda_m\rangle]_i + q_i^{-\langle h_i,\lambda_m\rangle}a \qquad \text{for every } m. \tag{4.3.16}$$

Namely we have $[e_i, f_i](|m\rangle) = (\{t_i\}_i + at_i^{-1})|m\rangle$. Hence for $v \in \bigwedge^r V_{\text{aff}}$

$$\begin{aligned}[e_i, f_i](v \wedge |m+r\rangle) &= [e_i, f_i]v \wedge t_i|m+r\rangle + t_i^{-1}v \wedge [e_i, f_i]|m+r\rangle \\ &= \{t_i\}_i v \wedge t_i|m+r\rangle + t_i^{-1}v \wedge (\{t_i\}_i + at_i^{-1})|m+r\rangle \\ &= (\{t_i\}_i + at_i^{-1})(v \wedge |m+r\rangle).\end{aligned}$$

Thus we obtain

$$[e_i, f_i] = \{t_i\}_i + at_i^{-1}. \tag{4.3.17}$$

Let us show the vanishing of a.

By induction on n we can see the following commutaion relation

$$e_i^{(n)}f_i^{(n)} = \sum_{k=0}^{n} f_i^{(n-k)}e_i^{(n-k)}\frac{\prod_{\nu=0}^{k-1}(\{q_i^{-\nu}t_i\}_i + aq_i^{\nu}t_i^{-1})}{[k]_i!}. \tag{4.3.18}$$

Setting $c = \langle h_i, \lambda_m\rangle$, we have $f_i^{(c+1)}|m\rangle = 0$ by Proposition 4.3.1 (iv). Hence

$$\begin{aligned}0 &= e_i^{(c+1)}f_i^{(c+1)}|m\rangle \\ &= \frac{\prod_{\nu=0}^{c+1}([c-\nu]_i + aq_i^{\nu-c})}{[c+1]_i!}|m\rangle.\end{aligned}$$

Therefore there is an integer s such that $a = -q_i^s[s]_i$. Then the commutation relation (4.3.17) can be rewritten as

$$[e_i, q_i^{-s}f_i] = \{q_i^{-s}t_i\}_i.$$

Hence e_i, $q_i^{-s}f_i$ and $q_i^{-s}t_i$ form $U_q(\mathfrak{sl}_2)$. Then the representation theory of $U_q(\mathfrak{sl}_2)$ and Proposition 4.3.1 (iv) implies $s = 0$. In fact, the string containing the weight of $|m\rangle$ (with respect to $q_i^{-s}t_i$) is $\{c - s - 2n; 0 \le n \le c\}$, and hence the symmetry of a string under the simple reflection implies $c - s = -(-c - s)$. □

Thus the actions of e_i and f_i satisfy the commutation relations. By Proposition 4.3.1 (ii), for any $i \in I$ and $\mu \in P$, $\mu + n\alpha_i$ is a weight of $\mathcal{F}_m$ only for a finitely many integers n. Therefore $\mathcal{F}_m$ is integrable over the $U_q(\mathfrak{sl}_2)_i = \langle e_i, f_i, t_i, t_i^{-1}\rangle$. This implies the Serre relations (see Appendix B).

Thus we obtain

Proposition 4.3.9. *$\mathcal{F}_m$ has the structure of an integrable $U_q(\mathfrak{g})$-module.*

By Proposition 4.3.1 (i), $\mathcal{F}_m$ is a direct sum of $V(\lambda_m - k\delta)$'s. This decomposition is studied in the next subsection through bosons.

Note that

$$\wedge : \bigwedge^r V_{\mathrm{aff}} \otimes \mathcal{F}_{m+r} \to \mathcal{F}_m$$

is $U_q(\mathfrak{g})$-linear.

Lemma 4.3.10.

$$f_i^{(k)}|m\rangle = G(\tilde{f}_i^k b_m^\circ) \wedge |m+1\rangle.$$

Proof. If $k > \langle h_i, \lambda_m\rangle$, then the both side vanish. Assume that $0 \le k \le \langle h_i, \lambda_m\rangle$. By Proposition 4.3.1 (iv), there is $c \in K$ such that

$$f_i^{(k)}|m\rangle = cG(\tilde{f}_i^k b_m^\circ) \wedge |m+1\rangle.$$

We have

$$e_i^{(k)} f_i^{(k)}|m\rangle = \begin{bmatrix} \langle h_i, \lambda_m\rangle \\ k \end{bmatrix}_i |m\rangle.$$

On the other hand, by the repeated use of (iii) in (G), we have

$$e_i^{(k)} G(\tilde{f}_i^k b_m^\circ) = \begin{bmatrix} \langle h_i, \lambda_m\rangle \\ k \end{bmatrix}_i v_m^\circ + \cdots .$$

Here, $\cdots$ is a linear combination of global bases other than v_m°, which is annihilated after being wedged with $|m+1\rangle$ by Proposition 4.3.1 (v). Hence we have

$$\begin{aligned} e_i^{(k)}\big(G(\tilde{f}_i^k b_m^\circ) \wedge |m+1\rangle\big) &= \big(e_i^{(k)} G(\tilde{f}_i^k b_m^\circ)\big) \wedge |m+1\rangle \\ &= \begin{bmatrix} \langle h_i, \lambda_m\rangle \\ k \end{bmatrix}_i v_m^\circ \wedge |m+1\rangle. \end{aligned}$$

Comparing these two identities, we obtain $c = 1$. □

Let $\mathcal{F}_m^{\mathbb{Z}}$ be the $\mathbb{Z}[q, q^{-1}]$-submodule of $\mathcal{F}_m$ generated by the normally ordered wedges. Then $\mathcal{F}_m^{\mathbb{Z}}$ is a module over $U_q(\mathfrak{g})_{\mathbb{Z}}$ by Lemma 4.3.10. Hence by specializing at $q = 1$, we obtain a Fock representation of $U(\mathfrak{g})$. However, the action of the bosons on $\mathcal{F}_m$ introduced in the next subsection may have a pole at $q = 1$ and it cannot be specialized at $q = 1$ in a naïve way.

4.4. The action of Bosons

We shall define the action of the bosons B_n ($n \neq 0$) on $\mathcal{F}_m$.

Proposition 4.4.1. *For $n \neq 0$ and any $u_m, u_{m+1}, \cdots \in V_{\text{aff}}$ such that $u_k = v_k^\circ$ for $k \gg m$,*

$$\begin{aligned}&(z^n u_m \wedge u_{m+1} \wedge u_{m+2} \wedge \cdots)\\ &+ (u_m \wedge z^n u_{m+1} \wedge u_{m+2} \wedge \cdots)\\ &+ (u_m \wedge u_{m+1} \wedge z^n u_{m+2} \wedge \cdots)\\ &+ \cdots\cdots\end{aligned} \tag{4.4.1}$$

converges in the q-adic topology.

Proof. Reducing to the case $u_k = v_k^\circ$ for every $k \geq m$, apply Lemma 4.3.3.

Lemma 4.4.2. $z^n v_m^\circ \wedge \overline{|m+1\rangle} + v_m^\circ \wedge z^n v_{m+1}^\circ \wedge \overline{|m+2\rangle} + \cdots$ *belongs to $\mathcal{F}_m$*

The proof is similar to the one for Lemma 4.3.4.

By these lemmas and Lemma 3.3.15, (4.4.1) defines a homomorphism from $\overline{\mathcal{F}}_m$ to $\mathcal{F}_m$. Since $L(\overline{\mathcal{F}}_m)$ is stable by the correspondence (4.4.1), it induces an endomorphism of $\mathcal{F}_m$. We denote it by B_n. It is clear that B_n is a $U_q'(\mathfrak{g})$-linear endomorphism of $\mathcal{F}_m$ with weight $n\delta$.

By the definition, we have

$$B_n(v \wedge u) = z^n v \wedge u + v \wedge B_n(u) \quad \text{for } v \in V_{\text{aff}} \text{ and } u \in \mathcal{F}_m. \tag{4.4.2}$$

Proposition 4.4.3. *There is $\gamma_n \in K$ (independent of m) such that*

$$[B_n, B_{n'}] = \delta_{n+n',0}\gamma_n.$$

Proof. (4.4.2) implies

$$[B_n, B_{n'}](v \wedge u) = v \wedge [B_n, B_{n'}]u.$$

Since $[B_n, B_{n'}]|m\rangle$ has weight $\lambda_m + (n+n')\delta$ and hence it must vanish when $n+n' > 0$. Therefore $[B_n, B_{n'}] = 0$ in this case.

Assume $n + n' < 0$. Write $[B_n, B_{n'}]|m\rangle$ as a linear combination of normally ordered wedges:

$$[B_n, B_{n'}]|m\rangle = \sum_\nu c_\nu G(b_{1,\nu}) \wedge \cdots .$$

Then $b_{1,\nu} \neq b_m^\circ$. Take N and c as in (4.1.1). Then we have

$$[B_n, B_{n'}]|m + jN\rangle = \sum_\nu c_\nu G(z^{jc} b_{1,\nu}) \wedge \cdots .$$

We have also $H(b_{m+jN-1}^\circ \otimes z^{jc} b_{1,\nu}) = H(b_{m-1}^\circ \otimes b_{1,\nu}) \leq 0$. Hence by Proposition 4.2.4, $v_m^\circ \wedge \cdots \wedge v_{m+jN-1}^\circ \wedge G(z^{jc} b_{1,\nu}) \wedge |m + jN + 1\rangle$ converges to 0 when j

tends to infinity. Hence

$$\begin{aligned}[B_n, B_{n'}]|m\rangle &= v_m^\circ \wedge \cdots \wedge v_{m+jN-1}^\circ \wedge [B_n, B_{n'}]|m+jN\rangle \\ &= \sum_\nu c_\nu v_m^\circ \wedge \cdots \wedge v_{m+jN-1}^\circ \wedge G(z^{jc} b_{1,\nu}) \wedge \cdots\end{aligned}$$

converges to 0. Therefore $[B_n, B_{n'}]|m\rangle$ must vanish.

Now assume that $n + n' = 0$. Since $[B_n, B_{-n}]|m\rangle$ has the same weight as $|m\rangle$, there is $\gamma_{m,n}$ such that $[B_n, B_{-n}]|m\rangle = \gamma_{m,n}|m\rangle$. Since

$$[B_n, B_{-n}]|m\rangle = v_m^\circ \wedge [B_n, B_{-n}]|m+1\rangle = \gamma_{m+1,n} v_m^\circ \wedge |m+1\rangle = \gamma_{m+1,n}|m\rangle,$$

$\gamma_{m,n}$ does not depend on m. Write γ_n for $\gamma_{m,n}$. Now we have $[B_n, B_{-n}](v \wedge |m\rangle) = v \wedge [B_n, B_{-n}]|m\rangle = \gamma_n v \wedge |m\rangle$. □

Now we shall show that γ_n does not vanish.

Lemma 4.4.4. *Let n be a positive integer.*

(i) $z^n v_k^\circ \wedge |k+1\rangle = 0$.

(ii) $v_m^\circ \wedge v_{m+1}^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge z^{-n} v_k^\circ \wedge |k+1\rangle \equiv 0$ *for* $k \geq m+n$.

(iii) $z^n v_m^\circ \wedge v_{m+1}^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge z^{-n} v_k^\circ \wedge |k+1\rangle \equiv 0$ *for* $m < k < m+n$.

Here $\equiv$ is modulo $qL(\mathcal{F}_m)$.

Proof. (i) follows from Theorem 4.2.5. In order to prove the other statements, write $b_k = z^{-k} b_k^\circ$. Then $H(b_k \otimes b_{k+1}) = 0$. We have

$$v_m^\circ \wedge v_{m+1}^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge z^{-n} v_k^\circ \equiv \\ z^m b_m \wedge z^{1+m} b_{m+1} \wedge \cdots \wedge z^{k-1} b_{k-1} \wedge z^{k-n} b_k.$$

Since $m \leq k-n \leq k-1$, it is zero modulo $qL(\bigwedge V_{\text{aff}})$ by Lemma 3.3.8 (iii). The proof of (iii) is similar to that of (ii). We have

$$z^n v_m^\circ \wedge v_{m+1}^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge z^{-n} v_k^\circ \wedge v_{k+1}^\circ \wedge \cdots \wedge v_{n+m}^\circ \equiv \\ z^{n+m} b_m \wedge z^{m+1} b_{m+1} \wedge z^{k-1} b_{k-1} \wedge z^{k-n} b_k \wedge z^{k+1} b_{k+1} \wedge \cdots \wedge z^{n+m} b_{n+m}.$$

Then it is zero modulo $qL(\bigwedge V_{\text{aff}})$ again by Lemma 3.3.8 (iii).

Proposition 4.4.5. *For $n \neq 0$, $\gamma_n \in K$ has no pole at $q = 0$ and $\gamma_n(0) = n$.*

Proof. We may assume $n > 0$. Noting that $B_{\pm n}$ sends $L(\mathcal{F}_m)$ to itself, let us calculate the commutator modulo $qL(\mathcal{F}_m)$. We have $[B_n, B_{-n}]|m\rangle = B_n B_{-n}|m\rangle$. By Lemma 4.4.4 (ii), we have

$$\begin{aligned}B_{-n}|m\rangle &= z^{-n} v_m^\circ \wedge |m+1\rangle + v_m^\circ \wedge z^{-n} v_{m+1}^\circ \wedge |m+2\rangle + \cdots \\ &\equiv \sum_{m \leq k < m+n} v_m^\circ \wedge v_{m+1}^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge z^{-n} v_k^\circ \wedge |k+1\rangle.\end{aligned}$$

Here $\equiv$ is taken modulo $qL(\mathcal{F}_m)$. Hence we have by Lemma 4.4.4 (i) and (iii)

$$
\begin{aligned}
&B_n B_{-n}|m\rangle \\
&\equiv \sum_{m\le k<m+n} \Big(\sum_{m\le j<k} v_m^\circ \wedge v_{m+1}^\circ \wedge \cdots \wedge z^n v_j^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge z^{-n} v_k^\circ \wedge |k+1\rangle \\
&\quad + |m\rangle \\
&\quad + \sum_{j>k} v_m^\circ \wedge v_{m+1}^\circ \wedge \cdots \wedge v_{k-1}^\circ \wedge z^{-n} v_k^\circ \wedge v_{k+1}^\circ \wedge \cdots \wedge z^n v_j^\circ \wedge |j+1\rangle \Big) \\
&\equiv n|m\rangle.
\end{aligned}
$$

□

Let H be the Heisenberg algebra generated by $\{B_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ with the defining relations $[B_n, B_{n'}] = \delta_{n+n',0}\gamma_n$. Then H acts on the Fock space $\mathcal{F}_m$ commuting with the action of $U_q'(\mathfrak{g})$. Let $\mathbb{Q}[H_-]$ be the Fock space for H. Namely, $\mathbb{Q}[H_-]$ is the H-module generated by the vacuum vector 1 with the defining relation $B_n 1 = 0$ for $n > 0$. Since $|m\rangle$ is annihilated by the e_i and the B_n with $n > 0$, we have an injective $U_q'(\mathfrak{g}) \otimes H$–linear homomorphism

$$\iota_m : V(\lambda_m) \otimes \mathbb{Q}[H_-] \to \mathcal{F}_m \tag{4.4.3}$$

sending $u_{\lambda_m} \otimes 1$ to $|m\rangle$. Comparing their characters (see Proposition 4.3.1 (i)), we obtain

Theorem 4.4.6. *$\iota_m : V(\lambda_m) \otimes \mathbb{Q}[H_-] \to \mathcal{F}_m$ is an isomorphism.*

4.5. Vertex operator

Similarly to the $A_n^{(1)}$ case in [KMS], the intertwiner

$$
\begin{aligned}
\Omega_m : V_{\mathrm{aff}} \otimes \mathcal{F}_{m+1} &\to \mathcal{F}_m, \\
v \otimes u &\mapsto v \wedge u \qquad (v \in V_{\mathrm{aff}}, u \in \mathcal{F}_{m+1}),
\end{aligned}
$$

induced by the wedge product is related with vertex operators. Let us describe it briefly. The proof is similar to [KMS].

Take an intertwiner

$$\Phi_m : V_{\mathrm{aff}} \otimes V(\lambda_{m+1}) \to V(\lambda_m) \tag{4.5.1}$$

and normalize it by

$$\Phi_m(v_m^\circ \otimes u_{\lambda_{m+1}}) = u_{\lambda_m}$$

(cf. Appendix A).

Let

$$\iota_m : V(\lambda_m) \otimes \mathbb{Q}[H_-] \xrightarrow{\sim} \mathcal{F}_m$$

be the isomorphism in (4.4.3). We define

$$\Omega'_m : V_{\mathrm{aff}} \otimes V(\lambda_{m+1}) \otimes \mathbb{Q}[H_-] \to V(\lambda_m) \otimes \mathbb{Q}[H_-]$$

by requiring the commutativity of the following diagram

$$\begin{tikzcd} V_{\mathrm{aff}} \otimes V(\lambda_{m+1}) \otimes \mathbb{Q}[H_-] \arrow[r, "\sim", "\mathrm{id} \otimes \iota_{m+1}"'] \arrow[d, "\Omega'_m"] & V_{\mathrm{aff}} \otimes \mathcal{F}_{m+1} \arrow[d, "\Omega_m"] \\ V(\lambda_m) \otimes \mathbb{Q}[H_-] \arrow[r, "\sim", "\iota_m"'] & \mathcal{F}_m \end{tikzcd} . \tag{4.5.2}$$

We shall write the intertwiners in the form of generating functions. Namely, introducing an indeterminate w (of weight δ), we set for $v \in V_{\mathrm{aff}}$

$$\begin{aligned} v(w) &= \sum_n z^n v \otimes w^{-n}, \\ \Phi_m(w)(v \otimes u) = \Phi_m(v(w) \otimes u) &= \sum_n \Phi_m(z^n v \otimes u) w^{-n}, \\ \Omega_m(w)(v \otimes u) = \Omega_m(v(w) \otimes u) &= \sum_n \Omega_m(z^n v \otimes u) w^{-n}, \\ \Omega'_m(w)(v \otimes u) = \Omega'_m(v(w) \otimes u) &= \sum_n \Omega'_m(z^n v \otimes u) w^{-n}. \end{aligned}$$

Here $u \in V(\lambda_{m+1})$, $\mathcal{F}_{m+1}$ or $V(\lambda_{m+1}) \otimes \mathbb{Q}[H_-]$.

We define the vertex operator for the bosons by

$$\Theta(w) = \exp\left(\sum_{n \geq 1} \frac{B_{-n} w^n}{\gamma_n}\right) \exp\left(-\sum_{n \geq 1} \frac{B_n w^{-n}}{\gamma_n}\right). \tag{4.5.3}$$

Theorem 4.5.1. $\Omega'_m(w) = \Phi_m(w) \otimes \Theta(w)$.

As a corollary of this theorem, we have the relations of the two-point functions of the vertex operators and γ_n as in [KMS].

Set

$$\Phi^v_m(w)(u) = \Phi_m(w)(v \otimes u)$$

for $u \in \mathcal{F}_{m+1}$. For $v, v' \in V_{\mathrm{aff}}$, we define $\langle \Phi^v_{m-1}(w_1) \Phi^{v'}_m(w_2) \rangle$ to be the coefficient of $u_{\lambda_{m-1}}$ in $\Phi^v_{m-1}(w_1) \Phi^{v'}_m(w_2) u_{\lambda_{m+1}} \in V(\lambda_{m-1})$. We introduce functions by

$$\omega_{v,v'}(w_2/w_1) = \langle m-1 | v(w_1) \wedge v'(w_2) \wedge | m+1 \rangle \tag{4.5.4}$$

$$\phi_{v,v'}(w_2/w_1) = \langle \Phi^v_{m-1}(w_1) \Phi^{v'}_m(w_2) \rangle \tag{4.5.5}$$

and

$$\theta(w_2/w_1) = \exp\left(-\sum_{n>0} \frac{(w_2/w_1)^n}{\gamma_n}\right). \tag{4.5.7}$$

Here for $u \in \mathcal{F}_{m-1}$, $\langle m-1|u$ means the coefficient of $|m-1\rangle$ in u. Then the theorem above implies

Proposition 4.5.2. *For $v, v' \in V_{\text{aff}}$, we have*

$$\omega_{v,v'}(w_2/w_1) = \phi_{v,v'}(w_2/w_1)\theta(w_2/w_1). \tag{4.5.8}$$

This formula will be used later to calculate γ_n.

5. Examples of level 1 Fock spaces

In this section we give some examples of the theory developed in the earlier sections. The case of level 1 type $A_n^{(1)}$ described in [S,KMS] is first reviewed in the perfect crystal language. Then we present results for types $A_{2n}^{(2)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$ at level 1, corresponding to the perfect crystals of [KMN1] Table 2.

5.1. Preliminaries

Define $[m,n] := \{i \in \mathbb{Z} \mid m \le i \le n\}$. We label the simple roots by $I = [0,n]$. We choose $0 \in I$ so that W_{cl} is generated by $\{s_i\}_{i\in I\setminus\{0\}}$ and $a_0 = 1$.

We take fundamental weights $\{\Lambda_i\}_{i\in I}$ such that

$$\alpha_i = \sum_{j\in I} \langle h_j, \alpha_i\rangle \Lambda_j + \delta_{i,0}\delta.$$

Let $s_0 : P_{\text{cl}}^0 \to P^0$ be a section of $\text{cl} : P^0 \to P_{\text{cl}}^0$ such that

$$s_0(P_{\text{cl}}^0) \subset \sum_{i\in I\setminus\{0\}} \mathbb{Q}\alpha_i = \sum_{i\in I\setminus\{0\}} \mathbb{Q}(a_0^\vee \Lambda_i - a_i^\vee \Lambda_0).$$

Then we have

$$s_0(\lambda + \text{cl}(\alpha_i)) = \begin{cases} s_0(\lambda) + \alpha_i & \text{for } i \in I\setminus\{0\}, \\ s_0(\lambda) + \alpha_0 - \delta & \text{for } i = 0. \end{cases}$$

We regard V as a subspace of V_{aff} by $V \supset V_\lambda \simeq (V_{\text{aff}})_{s_0(\lambda)} \subset V_{\text{aff}}$. Then V_{aff} is identified with $\mathbb{Q}[z,z^{-1}]\otimes V$. With this identification, the action of $U_q(\mathfrak{g})$ on $\mathbb{Q}[z,z^{-1}]\otimes V$ is given by

$$e_i(a\otimes v) = z^{\delta_{i,0}} a \otimes e_i v,$$
$$f_i(a\otimes v) = z^{-\delta_{i,0}} a \otimes f_i v.$$

Similarly we identify B as a subset of B_{aff}.

In the examples that we treat in this paper, the action of $U_q(\mathfrak{g})$ on the lower global base of V_{aff} (respectively V) is completely determined by its crystal structure as we have

$$\begin{aligned} e_i G(b) &= [1+\varphi_i(b)]_i G(\tilde{e}_i b), \\ f_i G(b) &= [1+\varepsilon_i(b)]_i G(\tilde{f}_i b), \\ q^h G(b) &= q^{\langle h, \mathrm{wt}\,(b)\rangle} G(b), \end{aligned} \tag{5.1.1}$$

for $b \in B_{\mathrm{aff}}$ $(b \in B)$, $i \in I$ and $h \in P^*$ $(h \in P_{\mathrm{cl}}^*)$.

5.2. Level 1 $A_n^{(1)}$

5.2.1. Cartan datum. The Dynkin diagram for $A_n^{(1)}$ $(n \geq 1)$ is

$$\begin{tikzcd} 0 \arrow[r, dash] \arrow[d, dash] & 1 \arrow[r, dash] & 2 \arrow[d, dash] \\ n \arrow[d, dash] & & 3 \arrow[d, dash] \\ n-1 \arrow[r, dash] & \cdots \arrow[r, dash] & 4 \end{tikzcd} .$$

For $A_n^{(1)}$ we have

$$\begin{aligned} \delta &= \sum_{i \in I} \alpha_i, \\ c &= \sum_{i \in I} h_i, \\ (\alpha_i, \alpha_i) &= 2 \qquad (i \in I). \end{aligned}$$

5.2.2. Perfect crystal. Let $J := [0, n]$. Let V be the $(n+1)$-dimensional $U_q'(A_n^{(1)})$-module with the level 1 perfect crystal $B := \{b_i\}_{i \in J}$ and crystal graph:

$$\begin{tikzcd} b_0 \arrow[r, "1"] & b_1 \arrow[r, "2"] & b_2 \arrow[d, "3"] \\ b_n \arrow[u, "0"] & & \vdots \arrow[d, "n-3"] \\ b_{n-1} \arrow[u, "n"] & b_{n-2} \arrow[l, "n-1"'] & b_{n-3} \arrow[l, "n-2"'] \end{tikzcd} .$$

The elements of B have the following weights

$$\mathrm{wt}\,(b_i) = \Lambda_{i+1} - \Lambda_i \qquad (i \in J).$$

Let $v_j := G(b_j)$ $(j \in J)$. The action of $U'_q(A_n^{(1)})$ on $v_j \in V$ obeys (5.1.1).

5.2.3. Energy function. The energy function H takes the following values on $B \otimes B$

$$H(b_i \otimes b_j) = \begin{cases} 1 & \text{for } i > j, \\ 0 & \text{for } i \leq j. \end{cases}$$

Write $H(i,j)$ for $H(b_i \otimes b_j)$ $(i, j \in J)$.

The Coxeter number of $A_n^{(1)}$ is $h = n + 1 = \dim V$. We take $l : B_{\mathrm{aff}} \to \mathbb{Z}$ to be

$$l(z^m b_j) = mh - j \qquad (m \in \mathbb{Z}, j \in J).$$

The functions H and l satisfy condition (L) (see end of §3.2). The map l gives a total ordering of B_{aff}.

5.2.4. Wedge relations. We have

$$N := U_q(A_n^{(1)})[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot v_0 \otimes v_0 \subset V_{\mathrm{aff}} \otimes V_{\mathrm{aff}}.$$

The following elements are contained in $U_q(A_n^{(1)}) \cdot v_0 \otimes v_0 \subset N$:

$$\begin{aligned} C_{i,i} &= v_i \otimes v_i \qquad\qquad (i \in J), \\ C_{i,j} &= v_i \otimes z^{-H(i,j)} v_j \\ &\quad + q z^{-H(i,j)} v_j \otimes v_i \left((i,j) \in J^2 \setminus \{(k,k)\}_{k \in J}\right). \end{aligned}$$

Proposition 5.2.1. *Identify $C_{i,j}$ with $C_{b_i, z^{-H(i,j)} b_j}$. Then $\{z^m \otimes z^m \cdot C_{i,j}\}_{m \in \mathbb{Z}; i,j \in J}$ with the function l satisfies condition (R) of subsection 3.3.*

5.2.5. Fock space. For $U_q(A_n^{(1)})$ we have

$$B_{\min} = B,$$
$$(P_{\mathrm{cl}}^+)_1 = \{\Lambda_i^{\mathrm{cl}}\}_{i \in I},$$

with

$$\varepsilon(b_j) = \Lambda_j^{\mathrm{cl}}, \qquad \varphi(b_j) = \Lambda_{j+1 \bmod h}^{\mathrm{cl}} \qquad (j \in J).$$

Since $H(b_j \otimes b_{j-1}) = 1$ $(j \in [1,n])$ and $H(b_0 \otimes z b_n) = 1$ there is a unique ground state sequence given as follows: every $m \in \mathbb{Z}$ fixes uniquely $a \in \mathbb{Z}$ and $j \in J$ such that $m = ah - j$, then

$$\begin{aligned} b_m^\circ &= z^a b_j \qquad && (m \in \mathbb{Z}), \\ \mathrm{cl}(\lambda_m) &= \Lambda_{j+1 \bmod h} && (m \in \mathbb{Z}). \end{aligned}$$

With $v_m^\circ = G(b_m^\circ)$, the vacuum vector of $\mathcal{F}_m$ is then given by

$$|m\rangle = v_m^\circ \wedge v_{m+1}^\circ \wedge v_{m+2}^\circ \wedge \cdots\cdots$$

with highest weight λ_m.

5.3. Level 1 $A_{2n}^{(2)}$

5.3.1. Cartan datum. The Dynkin diagram for $A_{2n}^{(2)}$ $(n \geq 1)$ is

$$0 \Longrightarrow 1 — 2 — \cdots\cdots — (n-2) — (n-1) \Longrightarrow n\,.$$

For $A_{2n}^{(2)}$ we have

$$\begin{aligned}
\delta &= \alpha_0 + \sum_{i=1}^{n} 2\alpha_i, \\
c &= \left(\sum_{i=0}^{n-1} 2h_i\right) + h_n, \\
(\alpha_i, \alpha_i) &= \begin{cases} 8 & \text{for } i = 0, \\ 4 & \text{for } i \in [1, n-1]\,, \\ 2 & \text{for } i = n. \end{cases}
\end{aligned}$$

5.3.2. Perfect crystal. Let $J := [-n, n]$ and let V be the $(2n+1)$-dimensional $U_q'(A_{2n}^{(2)})$-module with the level 1 perfect crystal $B := \{b_i\}_{i \in J}$ and crystal graph:

$$\begin{tikzcd}
b_1 \arrow[r, "1"] & b_2 \arrow[r, "2"] & \cdots\cdots \arrow[r, "n-2"] & b_{n-1} \arrow[r, "n-1"] & b_n \arrow[d, "n"] \\
 & & & & b_0 \arrow[d, "n"] \\
b_{-1} \arrow[uu, "0"] & b_{-2} \arrow[l, "1"'] & \cdots\cdots \arrow[l, "2"'] & b_{1-n} \arrow[l, "n-2"'] & b_{-n} \arrow[l, "n-1"']
\end{tikzcd}\,.$$

The elements of B have the following weights

$$\begin{aligned}
\mathrm{wt}\,(b_i) &= \sum_{k=i}^{n} \alpha_k = (1 + \delta_{i,n})\Lambda_i - \Lambda_{i-1} && (i \in [1, n]), \\
\mathrm{wt}\,(b_0) &= 0, \\
\mathrm{wt}\,(b_{-i}) &= -\,\mathrm{wt}\,(b_i) && (i \in [1, n]).
\end{aligned}$$

Let $v_j := G(b_j)$ $(j \in J)$. The action of $U_q'(A_{2n}^{(2)})$ on $v_j \in V$ obeys (5.1.1).

5.3.3. Energy function. Define the following ordering of J

$$1 \succ 2 \succ \cdots \succ n \succ 0 \succ -n \succ 1-n \succ \cdots \succ -1.$$

The energy function H takes the following values on $B \otimes B$

$$H(b_i \otimes b_j) = \begin{cases} 1 & \text{for } (i,j) \in \{(i',j') \in J^2 \mid i' \prec j'\} \cup \{(0,0)\}, \\ 0 & \text{for } (i,j) \in \{(i',j') \in J^2 \mid i' \succ j'\} \cup \{(k,k)\}_{k \in J \setminus \{0\}}. \end{cases}$$

Write $H(i,j)$ for $H(b_i \otimes b_j)$ $(i,j \in J)$.

The Coxeter number of $A_{2n}^{(2)}$ is $h = 2n+1 = \dim V$. We take $l : B_{\mathrm{aff}} \to \mathbb{Z}$ to be

$$l(z^m b_j) = \begin{cases} hm + n + 1 - j & \text{for } j \in [1,n], \\ hm & \text{for } j = 0, \\ hm - (n+1+j) & \text{for } j \in [-n,-1]. \end{cases}$$

The functions H and l satisfy condition (L) (see end of §3.2). The map l gives a total ordering of B_{aff}.

5.3.4. Wedge relations. In $V_{\mathrm{aff}} \otimes V_{\mathrm{aff}}$ we have

$$N := U_q(A_{2n}^{(2)})[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot v_1 \otimes v_1.$$

The following elements are contained in $U_q(A_{2n}^{(2)}) \cdot v_1 \otimes v_1 \subset N$:

$$\begin{aligned}
\tilde{C}_{i,i} &= v_i \otimes v_i && (i \in J \setminus \{0\}), \\
\tilde{C}_{i,-i} &= v_i \otimes z^{-H(i,-i)} v_{-i} + q^2 v_{i+1} \otimes z^{-H(i,-i)} v_{-i-1} \\
&\quad + q^2 z^{-H(i,-i)} v_{-i-1} \otimes v_{i+1} \\
&\quad + q^4 z^{-H(i,-i)} v_{-i} \otimes v_i && (i \in J \setminus \{-1,0,n\}), \\
\tilde{C}_{i,j} &= v_i \otimes z^{-H(i,j)} v_j \\
&\quad + q^2 z^{-H(i,j)} v_j \otimes v_i && \left((i,j) \in J^2 \setminus \{(k,k),(k,-k)\}_{k \in J}\right), \\
\tilde{C}_{0,0} &= v_0 \otimes z^{-1} v_0 + q^2 [2] v_{-n} \otimes z^{-1} v_n \\
&\quad + q^2 [2] z^{-1} v_n \otimes v_{-n} + q^2 z^{-1} v_0 \otimes v_0, \\
\tilde{C}_{n,-n} &= v_n \otimes v_{-n} + q v_0 \otimes v_0 + q^4 v_{-n} \otimes v_n, \\
\tilde{C}_{-1,1} &= v_{-1} \otimes z^{-1} v_1 + q^4 z^{-1} v_1 \otimes v_{-1}.
\end{aligned}$$

Notice that each $\tilde{C}_{i,j}$ has $v_i \otimes z^{-H(i,j)} v_j$ as its first term and a term in $z^{-H(i,j)} v_j \otimes v_i$.

Define the following elements in N.

$$C_{i,j} := \begin{cases} \tilde{C}_{i,j} & \text{for } (i,j) \in J^2 \setminus \{(k,-k)\}_{k\in J}, \\ \sum_{k=i}^{n}(-q^2)^{k-i}\tilde{C}_{k,-k} & \text{for } (i,j) \in \{(k,-k)\}_{k\in[1,n]}, \\ \sum_{k=1}^{j}(-q^2)^{j-k}\tilde{C}_{-k,k} & \text{for } (i,j) \in \{(-k,k)\}_{k\in[1,n]}, \\ \tilde{C}_{0,0} - q^2[2]C_{-n,n} & \text{for } (i,j) = (0,0). \end{cases}$$

Explicitly for $(i,j) \in \{(k,-k)\}_{k\in J}$, we have

$$\begin{aligned} C_{j,-j} &= v_j \otimes v_{-j} + q^4 v_{-j} \otimes v_j + q(-q^2)^{n-j} v_0 \otimes v_0 \\ &\quad - (1-q^4) \sum_{k=j+1}^{n} (-q^2)^{k-j} v_{-k} \otimes v_k \qquad (j \in [1,n]), \\ C_{-j,j} &= v_{-j} \otimes z^{-1}v_j + q^4 z^{-1} v_j \otimes v_{-j} \\ &\quad - (1-q^4) \sum_{k=1}^{j-1} (-q^2)^{j-k} z^{-1} v_k \otimes v_{-k} \qquad (j \in [1,n]), \\ C_{0,0} &= v_0 \otimes z^{-1} v_0 + q^2 z^{-1} v_0 \otimes v_0 \\ &\quad + q^2[2](1-q^4) \sum_{k=1}^{n} (-q^2)^{n-k} z^{-1} v_k \otimes v_{-k}. \end{aligned}$$

Proposition 5.3.1. *Identify $C_{i,j}$ with $C_{b_i, z^{-H(i,j)}b_j}$. Then $\{z^m \otimes z^m \cdot C_{i,j}\}_{m\in\mathbb{Z}; i,j\in J}$ with the function l satisfies condition (R) of subsection 3.3.*

5.3.5. Fock space. We have $B_{\min} = \{b_0\}$ and $(P_{\mathrm{cl}}^+)_1 = \{\Lambda_n^{\mathrm{cl}}\}$. Since $H(b_0, b_0) = 1$ we have a unique ground state sequence (up to an overall shift by z^k ($k \in \mathbb{Z}$)): $b_m^\circ = b_0$ and $\lambda_m = \Lambda_n$ ($m \in \mathbb{Z}$). Therefore the vacuum vector of the Fock space $\mathcal{F}_m$ is

$$|m\rangle := v_0 \wedge v_0 \wedge v_0 \wedge v_0 \wedge \cdots\cdots \qquad (m \in \mathbb{Z}).$$

We set $\mathrm{wt}(|m\rangle) = \Lambda_n$.

As an illustration of the use of the q-adic topology, let us check Proposition 4.3.8 on $|m\rangle$: i.e. that $[e_i, f_i] \cdot |m\rangle = \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \cdot |m\rangle$. The case $i \in I \setminus \{n\}$ is trivial. For $e_n|m\rangle$, consider first $v_n \wedge |m+1\rangle = (-q^2)^j (v_0)^{\wedge j} \wedge v_n \wedge |m+j+1\rangle$ ($j \in \mathbb{N}$). As $j \to \infty$, the vector vanishes by the q-adic topology on $\mathcal{F}_m$. Hence

$$\begin{aligned} e_n \cdot |m\rangle &= \sum_{j=0}^{\infty} (v_0)^{\wedge j} \wedge (e_n \cdot v_0) \wedge |m+j+1\rangle \\ &= [2] \sum_{j=0}^{\infty} (v_0)^{\wedge j} \wedge v_n \wedge |m+j+1\rangle \\ &= 0. \end{aligned}$$

For f_n we have

$$\begin{aligned} f_n \cdot |m\rangle &= \sum_{j=0}^{\infty} (v_0)^{\wedge j} \wedge (f_n \cdot v_0) \wedge t_n |m+j+1\rangle \\ &= q[2] \sum_{j=0}^{\infty} (v_0)^{\wedge j} \wedge v_{-n} \wedge |m+j+1\rangle \\ &= q[2] \sum_{j=0}^{\infty} (-q^2)^j v_{-n} \wedge |m+1\rangle \\ &= q[2](1+q^2)^{-1} v_{-n} \wedge |m+1\rangle \\ &= v_{-n} \wedge |m+1\rangle . \end{aligned}$$

Then since

$$\begin{aligned} e_n \cdot f_n \cdot |m\rangle &= e_n \cdot v_{-n} \wedge |m+1\rangle \\ &= v_0 \wedge |m+1\rangle \\ &= |m\rangle , \end{aligned}$$

and $[\langle h_n, \Lambda_n \rangle]_n = 1$, this completes the check.

5.4. Level 1 $B_n^{(1)}$

5.4.1. Cartan datum. The Dynkin diagram for $B_n^{(1)}$ $(n \geq 3)$ is

$$\begin{array}{l} 0 — 2 — 3 — \cdots\cdots — (n-2) — (n-1) \Longrightarrow n \quad . \\ \quad\;\; | \\ \quad\;\; 1 \end{array}$$

For $B_n^{(1)}$ we have

$$\begin{aligned} \delta &= \alpha_0 + \alpha_1 + \sum_{i=2}^{n} 2\alpha_i, \\ c &= h_0 + h_1 + \left(\sum_{i=2}^{n-1} 2h_i \right) + h_n, \\ (\alpha_i, \alpha_i) &= \begin{cases} 4 & \text{for } i \in [0, n-1], \\ 2 & \text{for } i = n. \end{cases} \end{aligned}$$

5.4.2 Perfect crystal. Let $J := [-n, n]$ and let V be the $(2n+1)$-dimensional

$U_q'(B_n^{(1)})$-module with the level 1 perfect crystal $B := \{b_i\}_{i\in J}$ and crystal graph:

$$\begin{tikzcd}
 & b_2 \arrow[r, "2"] & b_3 \arrow[r, "3"] & \cdots\cdots \arrow[r, "n-2"] & b_{n-1} \arrow[r, "n-1"] & b_n \arrow[d, "n"] \\
b_1 \arrow[ur, "1"] & & b_{-1} \arrow[ul, "0"'] & & & b_0 \arrow[d, "n"] \\
 & b_{-2} \arrow[ul, "0"] \arrow[ur, "1"'] & b_3 \arrow[l, "2"] & \cdots\cdots \arrow[l, "3"] & b_{n-1} \arrow[l, "n-2"] & b_{-n} \arrow[l, "n-1"]
\end{tikzcd}$$

The elements of B have the following weights

$$\begin{aligned}
\mathrm{wt}\,(b_i) &= \sum_{k=i}^{n} \alpha_k = (1+\delta_{i,n})\Lambda_i - \Lambda_{i-1} - \delta_{i,2}\Lambda_0 \quad (i \in [1,n]),\\
\mathrm{wt}\,(b_0) &= 0,\\
\mathrm{wt}\,(b_{-i}) &= -\,\mathrm{wt}\,(b_i) \qquad\qquad\qquad\qquad\qquad\qquad (i \in [1,n]).
\end{aligned}$$

Let $v_j := G(b_j)$ $(j \in J)$. The action of $U_q'(B_n^{(1)})$ on $v_j \in V$ obeys (5.1.1).

5.4.3. Energy function. Define the following ordering of J

$$1 \succ 2 \succ \cdots \succ n \succ 0 \succ -n \succ 1-n \succ \cdots \succ -1.$$

The energy function H takes the following values on $B \otimes B$

$$H(b_i \otimes b_j) = \begin{cases} 2 & \text{for } (i,j) = (-1,1),\\ 1 & \text{for } (i,j) \in \big\{(i',j') \in J^2 \setminus \{(-1,1)\} \mid i' \prec j'\big\} \cup \{(0,0)\},\\ 0 & \text{for } (i,j) \in \big\{(i',j') \in J^2 \mid i' \succ j'\big\} \cup \{(k,k)\}_{k\in J\setminus\{0\}}. \end{cases}$$

Write $H(i,j)$ for $H(b_i \otimes b_j)$ $(i,j \in J)$.

The Coxeter number of $B_n^{(1)}$ is $h = 2n = \dim V - 1$. We take l to be

$$l(z^m b_j) = \begin{cases} hm + n + 1 - j & \text{for } j \in [1,n],\\ hm & \text{for } j = 0,\\ hm - (n+1+j) & \text{for } j \in [-n,-1]. \end{cases}$$

The functions H and l satisfy condition (L). Note that $l(z^m b_1) = l(z^{m+1} b_{-1})$ ($m \in \mathbb{Z}$), so the map l gives a partial ordering of B_{aff}.

5.4.4. Wedge relations. We have

$$N := U_q(B_n^{(1)})[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot v_1 \otimes v_1 \subset V_{\text{aff}} \otimes V_{\text{aff}}.$$

The following elements are contained in $U_q(B_n^{(1)}) \cdot v_1 \otimes v_1 \subset N$:

$$\begin{aligned}
\tilde{C}_{i,i} &= v_i \otimes v_i && (i \in J \setminus \{0\}), \\
\tilde{C}_{i,-i} &= v_i \otimes z^{-H(i,-i)} v_{-i} + q^2 v_{i+1} \otimes z^{-H(i,-i)} v_{-i-1} \\
&\quad + q^2 z^{-H(i,-i)} v_{-i-1} \otimes v_{i+1} \\
&\quad + q^4 z^{-H(i,-i)} v_{-i} \otimes v_i && (i \in J \setminus \{-1, 0, n\}), \\
\tilde{C}_{i,j} &= v_i \otimes z^{-H(i,j)} v_j \\
&\quad + q^2 z^{-H(i,j)} v_j \otimes v_i && \left((i,j) \in J^2 \setminus \{(k,k),(k,-k)\}_{k \in J}\right), \\
\tilde{C}_{0,0} &= v_0 \otimes z^{-1} v_0 + q^2 [2] v_{-n} \otimes z^{-1} v_n \\
&\quad + q^2 [2] z^{-1} v_n \otimes v_{-n} + q^2 z^{-1} v_0 \otimes v_0, \\
\tilde{C}_{n,-n} &= v_n \otimes v_{-n} + q v_0 \otimes v_0 + q^4 v_{-n} \otimes v_n, \\
\tilde{C}_{-1,1} &= v_{-1} \otimes z^{-2} v_1 + q^2 z^{-1} v_{-2} \otimes z^{-1} v_2 \\
&\quad + q^2 z^{-1} v_2 \otimes z^{-1} v_{-2} + q^4 z^{-2} v_1 \otimes v_{-1}.
\end{aligned}$$

Each $\tilde{C}_{i,j}$ has $v_i \otimes z^{-H(i,j)} v_j$ as its first term and a term in $z^{-H(i,j)} v_j \otimes v_i$.

Define the following elements in N.

$$C_{i,j} := \begin{cases}
\tilde{C}_{i,j} & \text{for } (i,j) \in J^2 \setminus \{(k,-k)\}_{k \in J}, \\
\sum_{k=i}^{n} (-q^2)^{k-i} \tilde{C}_{k,-k} & \text{for } (i,j) \in \{(k,-k)\}_{k \in [1,n]}, \\
\sum_{k=2}^{j} (-q^2)^{j-k} \tilde{C}_{-k,k} & \text{for } (i,j) \in \{(-k,k)\}_{k \in [2,n]}, \\
\tilde{C}_{0,0} - q^2 [2] C_{-n,n} & \text{for } (i,j) = (0,0), \\
\tilde{C}_{-1,1} - q^2 (z^{-1} \otimes z^{-1}) C_{2,-2} & \text{for } (i,j) = (-1,1).
\end{cases}$$

Explicitly for $(i,j) \in \{(k,-k)\}_{k \in J}$, we have

$$
\begin{aligned}
C_{j,-j} &= v_j \otimes v_{-j} + q^4 v_{-j} \otimes v_j + q(-q^2)^{n-j} v_0 \otimes v_0 \\
&\quad - (1-q^4) \sum_{k=j+1}^{n} (-q^2)^{k-j} v_{-k} \otimes v_k && (j \in [1,n]), \\
C_{-j,j} &= v_{-j} \otimes z^{-1} v_j + q^4 z^{-1} v_j \otimes v_{-j} \\
&\quad - (1-q^4) \sum_{k=2}^{j-1} (-q^2)^{j-k} z^{-1} v_k \otimes v_{-k} \\
&\quad - (-q^2)^{j-1} (z^{-1} v_1 \otimes v_{-1} + v_{-1} \otimes z^{-1} v_1) && (j \in [2,n]), \\
C_{0,0} &= v_0 \otimes z^{-1} v_0 + q^2 z^{-1} v_0 \otimes v_0 \\
&\quad + q^2 [2] (1-q^4) \sum_{k=2}^{n} (-q^2)^{n-k} z^{-1} v_k \otimes v_{-k} \\
&\quad + q^2 [2] (-q^2)^{n-1} (z^{-1} v_1 \otimes v_{-1} + v_{-1} \otimes z^{-1} v_1), \\
C_{-1,1} &= v_{-1} \otimes z^{-2} v_1 + q^4 z^{-2} v_1 \otimes v_{-1} \\
&\quad + q(-q^2)^{n-1} z^{-1} v_0 \otimes v_0 - (1-q^4) \sum_{k=2}^{n} (-q^2)^{k-1} z^{-1} v_{-k} \otimes v_k .
\end{aligned}
$$

Proposition 5.4.1. *Identify $C_{i,j}$ with $C_{b_i, z^{-H(i,j)} b_j}$. Then $\{z^m \otimes z^m \cdot C_{i,j}\}_{m \in \mathbb{Z}; i,j \in J}$ with the function l satisfies condition (R) of subsection 3.3.*

5.4.5. Fock space. For $U_q(B_n^{(1)})$ we have

$$
\begin{aligned}
B_{\min} &= \{b_1, b_0, b_{-1}\}, \\
(P_{\mathrm{cl}}^+)_1 &= \{\Lambda_1^{\mathrm{cl}}, \Lambda_n^{\mathrm{cl}}, \Lambda_0^{\mathrm{cl}}\},
\end{aligned}
$$

with

$$
\begin{aligned}
\varepsilon(b_1) &= \Lambda_0^{\mathrm{cl}}, & \varepsilon(b_0) &= \Lambda_n^{\mathrm{cl}}, & \varepsilon(b_{-1}) &= \Lambda_1^{\mathrm{cl}}, \\
\varphi(b_1) &= \Lambda_1^{\mathrm{cl}}, & \varphi(b_0) &= \Lambda_n^{\mathrm{cl}}, & \varphi(b_{-1}) &= \Lambda_0^{\mathrm{cl}}.
\end{aligned}
$$

Since $H(b_0 \otimes b_0) = 1$, $H(b_1 \otimes z b_{-1}) = 1$ and $H(z b_{-1} \otimes b_1) = 1$, there are two ground state sequences (up to overall shifts by z^k $(k \in \mathbb{Z})$):

$$
\begin{aligned}
b_m^\circ &= b_0 \quad (m \in \mathbb{Z}) \\
\lambda_m &= \Lambda_n \quad (m \in \mathbb{Z})
\end{aligned}
$$

and

$$b_m^\circ = \begin{cases} b_1 & \text{for } m \in 2\mathbb{Z}, \\ zb_{-1} & \text{for } m \in 2\mathbb{Z}+1, \end{cases}$$

$$\lambda_m = \begin{cases} \Lambda_1 - \frac{m}{2}\delta & \text{for } m \in 2\mathbb{Z}, \\ \Lambda_0 - \frac{m-1}{2}\delta & \text{for } m \in 2\mathbb{Z}+1. \end{cases}$$

The vacuum vectors are respectively

$$|m\rangle := v_0 \wedge v_0 \wedge v_0 \wedge v_0 \wedge \cdots\cdots \qquad (m \in \mathbb{Z}),$$

with $\mathrm{wt}\,(|m\rangle) = \Lambda_n$, and

$$|m\rangle := \begin{cases} v_1 \wedge zv_{-1} \wedge v_1 \wedge \cdots\cdots & \text{for } m \in 2\mathbb{Z}, \\ zv_{-1} \wedge v_1 \wedge zv_{-1} \wedge \cdots\cdots & \text{for } m \in 2\mathbb{Z}+1, \end{cases}$$

with $\mathrm{wt}\,(|m\rangle) = \Lambda_1 - \frac{m}{2}\delta$ (m : even), $\Lambda_0 - \frac{m-1}{2}\delta$ (m : odd).

5.5. Level 1 $A^{(2)}_{2n-1}$

5.5.1 Cartan datum. The Dynkin diagram for $A^{(2)}_{2n-1}$ ($n \geq 3$) is

$$\begin{array}{ccccccccccccc} 0 & - & 2 & - & 3 & - & \cdots\cdots & - & (n-2) & - & (n-1) & \Longleftarrow & n \\ & & | & & & & & & & & & & \\ & & 1 & & & & & & & & & & \end{array} \quad .$$

For $A^{(2)}_{2n-1}$ we have

$$\delta = \alpha_0 + \alpha_1 + \left(\sum_{i=2}^{n-1} 2\alpha_i\right) + \alpha_n,$$

$$c = h_0 + h_1 + \left(\sum_{i=2}^{n} 2h_i\right),$$

$$(\alpha_i, \alpha_i) = \begin{cases} 2 & \text{for } i \in [0, n-1], \\ 4 & \text{for } i = n. \end{cases}$$

5.5.2. Perfect crystal. Let $J := [-n, -1] \cup [1, n]$. Let V be the $(2n)$-dimensional $U'_q(A^{(2)}_{2n-1})$-module with the level 1 perfect crystal $B := \{b_i\}_{i \in J}$ and crystal graph:

$$\begin{tikzcd}
 & b_2 \arrow[r, "2"] & b_3 \arrow[r, "3"] & \cdots\cdots \arrow[r, "n-2"] & b_{n-1} \arrow[r, "n-1"] & b_n \arrow[dd, "n"] \\
b_1 \arrow[ur, "1"] & & b_{-1} \arrow[ul, "0"'] & & & \\
 & b_{-2} \arrow[ul, "0"] \arrow[ur, "1"'] & b_{-3} \arrow[l, "2"] & \cdots\cdots \arrow[l, "3"] & b_{1-n} \arrow[l, "n-2"] & b_{-n} \arrow[l, "n-1"]
\end{tikzcd} .$$

The elements of B have the following weights

$$\begin{aligned}
\mathrm{wt}\,(b_i) &= \alpha_n/2 + \sum_{k\in[i,n-1]} \alpha_k = \Lambda_i - \Lambda_{i-1} - \delta_{i,2}\Lambda_0 \quad &(i \in [1,n]),\\
\mathrm{wt}\,(b_{-i}) &= -\,\mathrm{wt}\,(b_i) &(i \in [1,n]).
\end{aligned}$$

Let $v_j := G(b_j)$ $(j \in J)$. The action of $U_q'(A_{2n-1}^{(2)})$ on $v_j \in V$ obeys (5.1.1).

5.5.3. Energy function. Define the following ordering of J

$$1 \succ 2 \succ \cdots \succ n \succ -n \succ 1-n \succ \cdots \succ -1.$$

The energy function H takes the following values on $B \otimes B$

$$H(b_i \otimes b_j) = \begin{cases} 2 & \text{for } (i,j) = (-1,1),\\ 1 & \text{for } (i,j) \in \left\{(i',j') \in J^2 \setminus \{(-1,1)\} \mid i' \prec j'\right\},\\ 0 & \text{for } (i,j) \in \{(i',j') \in J^2 \mid i' \succ j'\} \cup \{(k,k)\}_{k\in J}. \end{cases}$$

Write $H(i,j)$ for $H(b_i \otimes b_j)$ $(i,j \in J)$.

The Coxeter number of $A_{2n-1}^{(2)}$ is $h = 2n-1 = \dim V - 1$. We take l to be

$$l(z^m b_j) = \begin{cases} hm + n - j & \text{for } j \in [1,n],\\ hm - (n+1+j) & \text{for } j \in [-n,-1]. \end{cases}$$

The functions H and l satisfy condition (L). Note that $l(z^m b_1) = l(z^{m+1} b_{-1})$ $(m \in \mathbb{Z})$, so l gives a partial ordering of B_{aff}.

5.5.4. Wedge relations. We have

$$N := U_q(A_{2n-1}^{(2)})[z\otimes z, z^{-1}\otimes z^{-1}, z\otimes 1 + 1\otimes z]\cdot v_1 \otimes v_1 \subset V_{\mathrm{aff}} \otimes V_{\mathrm{aff}}.$$

The following elements are contained in $U_q(A_{2n-1}^{(2)}) \cdot v_1 \otimes v_1 \subset N$:

$$\begin{aligned}
\tilde{C}_{i,i} &= v_i \otimes v_i & (i \in J),\\
\tilde{C}_{i,-i} &= v_i \otimes z^{-H(i,-i)} v_{-i} + q v_{i+1} \otimes z^{-H(i,-i)} v_{-i-1}\\
&\quad + q z^{-H(i,-i)} v_{-i-1} \otimes v_{i+1}\\
&\quad + q^2 z^{-H(i,-i)} v_{-i} \otimes v_i & (i \in J \setminus \{-1,n\}),\\
\tilde{C}_{i,j} &= v_i \otimes z^{-H(i,j)} v_j\\
&\quad + q z^{-H(i,j)} v_j \otimes v_i & \left((i,j) \in J^2 \setminus \{(k,k),(k,-k)\}_{k\in J}\right),\\
\tilde{C}_{n,-n} &= v_n \otimes v_{-n} + q^2 v_{-n} \otimes v_n,\\
\tilde{C}_{-1,1} &= v_{-1} \otimes z^{-2} v_1 + q z^{-1} v_{-2} \otimes z^{-1} v_2\\
&\quad + q z^{-1} v_2 \otimes z^{-1} v_{-2} + q^2 z^{-2} v_1 \otimes v_{-1}.
\end{aligned}$$

Each $\tilde{C}_{i,j}$ has $v_i \otimes z^{-H(i,j)} v_j$ as its first term and a term in $z^{-H(i,j)} v_j \otimes v_i$.

Define the following elements in N.

$$C_{i,j} := \begin{cases} \tilde{C}_{i,j} & \text{for } (i,j) \in J^2 \setminus \{(k,-k)\}_{k\in J}, \\ \sum_{k=i}^{n} (-q)^{k-i} \tilde{C}_{k,-k} & \text{for } (i,j) \in \{(k,-k)\}_{k\in[1,n]}, \\ \sum_{k=2}^{j} (-q)^{j-k} \tilde{C}_{-k,k} & \text{for } (i,j) \in \{(-k,k)\}_{k\in[2,n]}, \\ \tilde{C}_{-1,1} - q(z^{-1}\otimes z^{-1}) C_{2,-2} & \text{for } (i,j) = (-1,1). \end{cases}$$

Explicitly for $(i,j) \in \{(k,-k)\}_{k\in J}$, we have

$$\begin{aligned} C_{j,-j} &= v_j \otimes v_{-j} + q^2 v_{-j} \otimes v_j \\ &\quad - (1-q^2) \sum_{k=j+1}^{n} (-q)^{k-j} v_{-k} \otimes v_k \qquad (j \in [1,n]), \\ C_{-j,j} &= v_{-j} \otimes z^{-1} v_j + q^2 z^{-1} v_j \otimes v_{-j} \\ &\quad - (1-q^2) \sum_{k=2}^{j-1} (-q)^{j-k} z^{-1} v_k \otimes v_{-k} \\ &\quad - (-q)^{j-1} (z^{-1} v_1 \otimes v_{-1} + v_{-1} \otimes z^{-1} v_1) \quad (j \in [2,n]), \\ C_{-1,1} &= v_{-1} \otimes z^{-2} v_1 + q^2 z^{-2} v_1 \otimes v_{-1} \\ &\quad - (1-q^2) \sum_{k=2}^{n} (-q)^{k-1} z^{-1} v_{-k} \otimes z^{-1} v_k. \end{aligned}$$

Proposition 5.5.1. *Identify $C_{i,j}$ with $C_{b_i, z^{-H(i,j)} b_j}$. Then $\{z^m \otimes z^m \cdot C_{i,j}\}_{m\in\mathbb{Z}; i,j\in J}$ with the function l satisfies condition (R) of subsection 3.3.*

5.5.5. Fock space. For $U_q(A^{(2)}_{2n-1})$ we have

$$B_{\min} = \{b_1, b_{-1}\},$$
$$(P^+_{\mathrm{cl}})_1 = \{\Lambda^{\mathrm{cl}}_1, \Lambda^{\mathrm{cl}}_0\},$$

with

$$\varepsilon(b_1) = \Lambda^{\mathrm{cl}}_0, \qquad \varepsilon(b_{-1}) = \Lambda^{\mathrm{cl}}_1,$$
$$\varphi(b_1) = \Lambda^{\mathrm{cl}}_1, \qquad \varphi(b_{-1}) = \Lambda^{\mathrm{cl}}_0.$$

Since $H(b_1 \otimes z b_{-1}) = 1$ and $H(z b_{-1} \otimes b_1) = 1$ there is one ground state sequence:

$$b^\circ_m = \begin{cases} b_1 & \text{for } m \in 2\mathbb{Z}, \\ z b_{-1} & \text{for } m \in 2\mathbb{Z}+1, \end{cases}$$
$$\lambda_m = \begin{cases} \Lambda_1 - \frac{m}{2}\delta & \text{for } m \in 2\mathbb{Z}, \\ \Lambda_0 - \frac{m-1}{2}\delta & \text{for } m \in 2\mathbb{Z}+1. \end{cases}$$

The vacuum vector of $\mathcal{F}_m$ is

$$|m\rangle := \begin{cases} v_1 \wedge zv_{-1} \wedge v_1 \wedge \cdots\cdots & \text{for } m \in 2\mathbb{Z}, \\ zv_{-1} \wedge v_1 \wedge zv_{-1} \wedge \cdots\cdots & \text{for } m \in 2\mathbb{Z}+1, \end{cases}$$

with $\mathrm{wt}\,(|m\rangle) = \Lambda_1 - \frac{m}{2}\delta$ (m : even), $\Lambda_0 - \frac{m-1}{2}\delta$ (m : odd).

5.6. Level 1 $D_n^{(1)}$

5.6.1. Cartan datum. The Dynkin diagram for $D_n^{(1)}$ ($n \geq 4$) is

$$\begin{tikzcd}[arrows=-] 0 \ar[r] & 2 \ar[r] \ar[d] & 3 \ar[r] & \cdots\cdots \ar[r] & (n-3) \ar[r] & (n-2) \ar[r] \ar[d] & n \\ & 1 & & & & (n-1) & \end{tikzcd} .$$

For $D_n^{(1)}$ we have

$$\begin{aligned} \delta &= \alpha_0 + \alpha_1 + \left(\sum_{i=2}^{n-2} 2\alpha_i\right) + \alpha_{n-1} + \alpha_n, \\ c &= h_0 + h_1 + \left(\sum_{i=2}^{n-2} 2h_i\right) + h_{n-1} + h_n, \\ (\alpha_i, \alpha_i) &= 2 \qquad (i \in I). \end{aligned}$$

5.6.2. Perfect crystal. Let $J := [-n,-1] \cup [1,n]$. Let V be the $(2n)$-dimensional $U_q'(D_n^{(1)})$-module with the level 1 perfect crystal $B := \{b_i\}_{i\in J}$ and crystal graph:

$$\begin{tikzcd} & b_2 \ar[r, "2"] & b_3 \ar[r, "3"] & \cdots\cdots \ar[r, "n-3"] & b_{n-2} \ar[r, "n-2"] & b_{n-1} \ar[dl, "n-1"'] \ar[dr, "n"] & \\ b_1 \ar[ur, "1"] & & b_{-1} \ar[ul, "0"'] & & b_n \ar[dr, "n"'] & & b_{-n} \ar[dl, "n-1"] \\ & b_{-2} \ar[ul, "0"] \ar[ur, "1"'] & b_{-3} \ar[l, "2"] & \cdots\cdots \ar[l, "3"] & b_{2-n} \ar[l, "n-3"] & b_{1-n} \ar[l, "n-2"] & \end{tikzcd}$$

The elements of B have the following weights

$$\begin{aligned} \mathrm{wt}\,(b_i) &= \left(\sum_{k=i}^{n-2} \alpha_k\right) + (\alpha_{n-1}+\alpha_n)/2 \\ &= \Lambda_i - \Lambda_{i-1} + \delta_{i,n-1}\Lambda_n - \delta_{i,2}\Lambda_0 \qquad (i \in [1,n]), \\ \mathrm{wt}\,(b_{-i}) &= -\,\mathrm{wt}\,(b_i) \qquad (i \in [1,n]). \end{aligned}$$

Let $v_j := G(b_j)$ $(j \in J)$. The action of $U_q'(D_n^{(1)})$ on $v_j \in V$ obeys (5.1.1).

5.6.3. Energy function. Define the following ordering of J

$$1 \succ 2 \succ \cdots \succ n \succ -n \succ 1-n \succ \cdots \succ -1.$$

The energy function H takes the following values on $B \otimes B$

$$H(b_i \otimes b_j) = \begin{cases} 2 & \text{if } (i,j) = (-1,1), \\ 1 & \text{if } (i,j) \in \{(i',j') \in J^2 \setminus \{(-1,1)\} \mid i' \prec j'\} \cup \{(n,-n)\}, \\ 0 & \text{if } (i,j) \in \{(i',j') \in J^2 \setminus \{(n,-n)\} \mid i' \succ j'\} \cup \{(k,k)\}_{k \in J}. \end{cases}$$

Write $H(i,j)$ for $H(b_i \otimes b_j)$ $(i, j \in J)$.

The Coxeter number of $D_n^{(1)}$ is $h = n+1 = \dim V - 2$. We take l to be

$$l(z^m b_j) = \begin{cases} hm + n - j & \text{for } j \in [1,n], \\ hm - (n+j) & \text{for } j \in [-n,-1]. \end{cases}$$

The functions H and l satisfy condition (L). Note that $l(z^m b_1) = l(z^{m+1} b_{-1})$ and $l(z^m b_n) = l(z^m b_{-n})$ $(m \in \mathbb{Z})$, so l gives a partial ordering of B_{aff}.

5.6.4. Wedge relations. We have

$$N := U_q(D_n^{(1)})[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot v_1 \otimes v_1 \subset V_{\text{aff}} \otimes V_{\text{aff}}.$$

The following elements are contained in $U_q(D_n^{(1)}) \cdot v_1 \otimes v_1 \subset N$:

$$\begin{aligned}
\tilde{C}_{i,i} &= v_i \otimes v_i \qquad (i \in J), \\
\tilde{C}_{i,-i} &= v_i \otimes z^{-H(i,-i)} v_{-i} + q v_{i+1} \otimes z^{-H(i,-i)} v_{-i-1} \\
&\quad + q z^{-H(i,-i)} v_{-i-1} \otimes v_{i+1} \\
&\quad + q^2 z^{-H(i,-i)} v_{-i} \otimes v_i \qquad (i \in J \setminus \{-1, n\}), \\
\tilde{C}_{i,j} &= v_i \otimes z^{-H(i,j)} v_j \\
&\quad + q z^{-H(i,j)} v_j \otimes v_i \qquad \left((i,j) \in J^2 \setminus \{(k,k),(k,-k)\}_{k \in J}\right), \\
\tilde{C}_{n,-n} &= v_n \otimes z^{-1} v_{-n} + q v_{1-n} \otimes z^{-1} v_{n-1} \\
&\quad + q z^{-1} v_{n-1} \otimes v_{1-n} + q^2 z^{-1} v_{-n} \otimes v_n, \\
\tilde{C}_{-1,1} &= v_{-1} \otimes z^{-2} v_1 + q z^{-1} v_{-2} \otimes z^{-1} v_2 \\
&\quad + q z^{-1} v_2 \otimes z^{-1} v_{-2} + q^2 z^{-2} v_1 \otimes v_{-1}.
\end{aligned}$$

Each $\tilde{C}_{i,j}$ has $v_i \otimes z^{-H(i,j)} v_j$ as its first term and a term in $z^{-H(i,j)} v_j \otimes v_i$.

Define the following elements in N.

$$C_{i,j} := \begin{cases} \tilde{C}_{i,j} & \text{for } (i,j) \in J^2 \setminus \{(k,-k)\}_{k\in J}, \\ \sum_{k=i}^{n-1} (-q)^{k-i} \tilde{C}_{k,-k} & \text{for } (i,j) \in \{(k,-k)\}_{k\in[1,n-1]}, \\ \sum_{k=2}^{j} (-q)^{j-k} \tilde{C}_{-k,k} & \text{for } (i,j) \in \{(-k,k)\}_{k\in[2,n]}, \\ \tilde{C}_{n,-n} - qC_{1-n,n-1} & \text{for } (i,j) = (n,-n), \\ \tilde{C}_{-1,1} - q(z^{-1}\otimes z^{-1})C_{2,-2} & \text{for } (i,j) = (-1,1). \end{cases}$$

Explicitly for $(i,j) \in \{(k,-k)\}_{k\in J}$, we have

$$\begin{aligned} C_{j,-j} &= v_j \otimes v_{-j} + q^2 v_{-j} \otimes v_j \\ &\quad - (1-q^2) \sum_{k=j+1}^{n-1} (-q)^{k-j} v_{-k} \otimes v_k \\ &\quad - (-q)^{n-j} (v_n \otimes v_{-n} + v_{-n} \otimes v_n) && (j \in [1,n]), \\ C_{-j,j} &= v_{-j} \otimes z^{-1} v_j + q^2 z^{-1} v_j \otimes v_{-j} \\ &\quad - (1-q^2) \sum_{k=2}^{j-1} (-q)^{j-k} z^{-1} v_k \otimes v_{-k} \\ &\quad - (-q)^{j-1} (z^{-1} v_1 \otimes v_{-1} + v_{-1} \otimes z^{-1} v_1) && (j \in [2,n]), \\ C_{n,-n} &= v_n \otimes z^{-1} v_{-n} + q^2 z^{-1} v_{-n} \otimes v_n \\ &\quad - (1-q^2) \sum_{k=2}^{n-1} (-q)^{n-k} z^{-1} v_k \otimes v_{-k} \\ &\quad - (-q)^{n-1} (z^{-1} v_1 \otimes v_{-1} + v_{-1} \otimes z^{-1} v_1) \\ C_{-1,1} &= v_{-1} \otimes z^{-2} v_1 + q^2 z^{-2} v_1 \otimes v_{-1} \\ &\quad - (1-q^2) \sum_{k=2}^{n-1} (-q)^{k-1} z^{-1} v_{-k} \otimes z^{-1} v_k \\ &\quad - (-q)^{n-1} (z^{-1} v_n \otimes z^{-1} v_{-n} + z^{-1} v_{-n} \otimes z^{-1} v_n). \end{aligned}$$

Proposition 5.6.1. *Identify $C_{i,j}$ with $C_{b_i, z^{-H(i,j)} b_j}$. Then $\{z^m \otimes z^m \cdot C_{i,j}\}_{m\in\mathbb{Z}; i,j\in J}$ with the function l satisfies condition (R) of subsection 3.3.*

5.6.5. Fock space. For $U_q(D_n^{(1)})$ we have

$$B_{\min} = \{b_1, b_{-1}, b_n, b_{-n}\},$$
$$(P_{\mathrm{cl}}^+)_1 = \{\Lambda_1^{\mathrm{cl}}, \Lambda_0^{\mathrm{cl}}, \Lambda_{n-1}^{\mathrm{cl}}, \Lambda_n^{\mathrm{cl}}\},$$

with

$$\begin{aligned}
\varepsilon(b_1) &= \Lambda_0^{\mathrm{cl}}, & \varphi(b_1) &= \Lambda_1^{\mathrm{cl}}, \\
\varepsilon(b_{-1}) &= \Lambda_1^{\mathrm{cl}}, & \varphi(b_{-1}) &= \Lambda_0^{\mathrm{cl}}, \\
\varepsilon(b_n) &= \Lambda_n^{\mathrm{cl}}, & \varphi(b_n) &= \Lambda_{n-1}^{\mathrm{cl}}, \\
\varepsilon(b_{-n}) &= \Lambda_{n-1}^{\mathrm{cl}}, & \varphi(b_{-n}) &= \Lambda_n^{\mathrm{cl}}.
\end{aligned}$$

Since $H(b_1 \otimes zb_{-1}) = 1$, $H(zb_{-1} \otimes b_1) = 1$, $H(b_n \otimes b_{-n}) = 1$ and $H(b_{-n} \otimes b_n) = 1$, there are two ground state sequences:

$$b_m^\circ = \begin{cases} b_1 & \text{for } m \in 2\mathbb{Z}, \\ zb_{-1} & \text{for } m \in 2\mathbb{Z}+1, \end{cases}$$
$$\lambda_m = \begin{cases} \Lambda_1 - \frac{m}{2}\delta & \text{for } m \in 2\mathbb{Z}, \\ \Lambda_0 - \frac{m-1}{2}\delta & \text{for } m \in 2\mathbb{Z}+1, \end{cases}$$

and

$$b_m^\circ = \begin{cases} b_n & \text{for } m \in 2\mathbb{Z}, \\ b_{-n} & \text{for } m \in 2\mathbb{Z}+1, \end{cases}$$
$$\lambda_m = \begin{cases} \Lambda_{n-1} & \text{for } m \in 2\mathbb{Z}, \\ \Lambda_n & \text{for } m \in 2\mathbb{Z}+1. \end{cases}$$

The vacuum vector of $\mathcal{F}_m$ are respectively

$$|m\rangle := \begin{cases} v_1 \wedge zv_{-1} \wedge v_1 \wedge \cdots\cdots & \text{for } m \in 2\mathbb{Z}, \\ zv_{-1} \wedge v_1 \wedge zv_{-1} \wedge \cdots\cdots & \text{for } m \in 2\mathbb{Z}+1, \end{cases}$$

with $\mathrm{wt}\,(|m\rangle) = \Lambda_1 - \frac{m}{2}\delta$ (m : even), $\Lambda_0 - \frac{m-1}{2}\delta$ (m : odd), and

$$|m\rangle := \begin{cases} v_n \wedge v_{-n} \wedge v_n \wedge \cdots\cdots & \text{for } m \in 2\mathbb{Z}, \\ v_{-n} \wedge v_n \wedge v_{-n} \wedge \cdots\cdots & \text{for } m \in 2\mathbb{Z}+1, \end{cases}$$

with $\mathrm{wt}\,(|m\rangle) = \Lambda_{n-1}$ (m : even), Λ_n (m : odd).

5.7. Level 1 $D_{n+1}^{(2)}$

5.7.1. Cartan datum. The Dynkin diagram for $D_{n+1}^{(2)}$ ($n \geq 4$) is

$$0 \Longleftarrow 1 — 2 — \cdots\cdots — (n-2) — (n-1) \Longrightarrow n \quad .$$

For $D_{n+1}^{(2)}$ we have

$$\delta = \sum_{i \in I} \alpha_i,$$

$$c = h_0 + \left(\sum_{i=1}^{n-1} 2h_i \right) + h_n,$$

$$(\alpha_i, \alpha_i) = \begin{cases} 2 & \text{for } i \in \{0, n\}, \\ 4 & \text{for } i \in I \setminus \{0, n\}. \end{cases}$$

5.7.2. Perfect crystal. Let $J := [-n, n] \cup \{\phi\}$. Let V be the $(2n+2)$-dimensional $U_q'(D_{n+1}^{(2)})$-module with the level 1 perfect crystal $B := \{b_i\}_{i \in J}$ and crystal graph:

$$\begin{tikzcd}
b_1 \arrow[r, "1"] & b_2 \arrow[r, "2"] & \cdots \arrow[r, "n-2"] & b_{n-1} \arrow[r, "n-1"] & b_n \arrow[d, "n"] \\
b_\phi \arrow[u, "0"] & & & & b_0 \arrow[d, "n"] \\
b_{-1} \arrow[u, "0"] & b_{-2} \arrow[l, "1"'] & \cdots \arrow[l, "2"'] & b_{1-n} \arrow[l, "n-2"'] & b_{-n} \arrow[l, "n-1"']
\end{tikzcd}$$

The elements of B have the following weights

$$\begin{aligned}
\mathrm{wt}\,(b_i) &= \sum_{k=i}^{n} \alpha_k = (1 + \delta_{i,n})\Lambda_i - (1 + \delta_{i,1})\Lambda_{i-1} && (i \in [1, n]) \\
\mathrm{wt}\,(b_0) &= 0 \\
\mathrm{wt}\,(b_\phi) &= 0 \\
\mathrm{wt}\,(b_{-i}) &= -\,\mathrm{wt}\,(b_i) && (i \in [1, n]).
\end{aligned}$$

Let $v_j := G(b_j)$ $(j \in J)$. The action of $U_q'(D_{n+1}^{(2)})$ on $v_j \in V$ obeys (5.1.1).

Let $J_0 := J \setminus \{\phi\}$. Let V_0 denote that subspace of V spanned by $\{v_j\}_{j \in J_0}$. Then, V_{aff} decomposes into two $U_q(D_{n+1}^{(2)})$-modules:

$$\begin{aligned}
V_{\mathrm{aff}} = &\left(V_0 \otimes \mathbb{C}[z^2, z^{-2}] + v_\phi \otimes z\mathbb{C}[z^2, z^{-2}]\right) \\
&\oplus \left(V_0 \otimes z\mathbb{C}[z^2, z^{-2}] + v_\phi \otimes \mathbb{C}[z^2, z^{-2}]\right).
\end{aligned}$$

5.7.3. Energy function. Define the following ordering of J

$$1 \succ 2 \succ \cdots \succ n \succ 0 \succ -n \succ 1-n \succ \cdots \succ -1 \succ \phi. \tag{5.7.1}$$

The energy function H takes the following values on $B \otimes B$

$$H(b_i \otimes b_j) = \begin{cases} 2 & \text{for } (i,j) \in \{(i',j') \in J_0^2 \mid i' \prec j'\} \cup \{(0,0),(\phi,\phi)\}, \\ 1 & \text{for } (i,j) \in \{(k,\phi),(\phi,k) \in J^2 \mid k \in J \setminus \{\phi\}\}, \\ 0 & \text{for } (i,j) \in \{(i',j') \in J_0^2 \mid i' \succ j'\} \cup \{(k,k)\}_{k \in J \setminus \{0,\phi\}}. \end{cases}$$

Write $H(i,j)$ for $H(b_i \otimes b_j)$ $(i,j \in J)$.

The Coxeter number of $D_{n+1}^{(2)}$ is $h = n+1 = \frac{1}{2}\dim V$. We take l to be

$$l(z^m b_j) = \begin{cases} hm + n + 1 - j & \text{for } j \in [1,n], \\ hm & \text{for } j \in \{0,\phi\}, \\ hm - (n+1+j) & \text{for } j \in [-n,-1]. \end{cases}$$

The functions H and l satisfy condition (L). Note that $l(z^m b_0) = l(z^m b_\phi)$ and $l(z^m b_i) = l(z^{m+1} b_{i-h})$ ($m \in \mathbb{Z}$ and $i \in [1,n]$), so l gives a partial ordering of B_{aff}. (l gives a total ordering of each of the crystals of the two irreducible submodules.)

5.7.4. Wedge relations. In $V_{\mathrm{aff}} \otimes V_{\mathrm{aff}}$ we have

$$N := U_q(D_{n+1}^{(2)})[z \otimes z, z^{-1} \otimes z^{-1}, z \otimes 1 + 1 \otimes z] \cdot v_1 \otimes v_1.$$

The following elements are contained in $U_q(D_{n+1}^{(2)})[z \otimes z, z^{-1} \otimes z^{-1}] \cdot v_1 \otimes v_1 \subset N$:

$$\begin{aligned}
\tilde{C}_{i,i} &= v_i \otimes v_i \qquad (i \in J \setminus \{0,\phi\}), \\
\tilde{C}_{i,-i} &= v_i \otimes z^{-H(i,-i)} v_{-i} + q^2 v_{i+1} \otimes z^{-H(i,-i)} v_{-i-1} \\
&\quad + q^2 z^{-H(i,-i)} v_{-i-1} \otimes v_{i+1} \\
&\quad + q^4 z^{-H(i,-i)} v_{-i} \otimes v_i \qquad \big(i \in J \setminus \{-1,0,\phi,n\}\big), \\
\tilde{C}_{i,j} &= v_i \otimes z^{-H(i,j)} v_j + q^2 z^{-H(i,j)} v_j \otimes v_i \\
&\qquad \big((i,j) \in J^2 \setminus \{(k,k),(k,-k)\}_{k \in J}\big), \\
\tilde{C}_{0,0} &= v_0 \otimes z^{-2} v_0 + q^2 [2] v_{-n} \otimes z^{-2} v_n \\
&\quad + q^2 [2] z^{-2} v_n \otimes v_{-n} + q^2 z^{-2} v_0 \otimes v_0, \\
\tilde{C}_{n,-n} &= v_n \otimes v_{-n} + q v_0 \otimes v_0 + q^4 v_{-n} \otimes v_n, \\
\tilde{C}_{-1,1} &= v_{-1} \otimes z^{-2} v_1 + q z^{-1} v_\phi \otimes z^{-1} v_\phi + q^4 z^{-2} v_1 \otimes v_{-1}, \\
\tilde{C}_{\phi,\phi} &= v_\phi \otimes z^{-2} v_\phi + q^2 [2] z^{-1} v_1 \otimes z^{-1} v_{-1} \\
&\quad + q^2 [2] z^{-1} v_{-1} \otimes z^{-1} v_1 + q^2 z^{-2} v_\phi \otimes v_\phi.
\end{aligned}$$

Notice that each $\tilde{C}_{i,j}$ has $v_i \otimes z^{-H(i,j)} v_j$ as its first term and a term in $z^{-H(i,j)} v_j \otimes v_i$.

Define the following elements in N.

$$C_{i,j} := \begin{cases} \tilde{C}_{i,j} & \text{for } (i,j) \in J^2 \setminus \{(k,-k)\}_{k \in J}, \\ \sum_{k=i}^{n} (-q^2)^{k-i} \tilde{C}_{k,-k} & \text{for } (i,j) \in \{(k,-k)\}_{k \in [1,n]}, \\ \sum_{k=1}^{j} (-q^2)^{j-k} \tilde{C}_{-k,k} & \text{for } (i,j) \in \{(-k,k)\}_{k \in [1,n]}, \\ \tilde{C}_{0,0} - q^2 [2] C_{-n,n} & \text{for } (i,j) = (0,0), \\ \tilde{C}_{\phi,\phi} - q^2[2](z^{-1} \otimes z^{-1}) C_{1,-1} & \text{for } (i,j) = (\phi,\phi). \end{cases}$$

Explicitly for $(i,j) \in \{(k,-k)\}_{k \in J} \cup \{\phi\}$, we have

$$\begin{aligned} C_{j,-j} &= v_j \otimes v_{-j} + q^4 v_{-j} \otimes v_j + q(-q^2)^{n-j} v_0 \otimes v_0 \\ &\quad - (1-q^4) \sum_{k=j+1}^{n} (-q^2)^{k-j} v_{-k} \otimes v_k \qquad (j \in [1,n]), \\ C_{-j,j} &= v_{-j} \otimes z^{-2} v_j + q^4 z^{-2} v_j \otimes v_{-j} + q(-q^2)^{j-1} z^{-1} v_\phi \otimes z^{-1} v_\phi \\ &\quad - (1-q^4) \sum_{k=1}^{j-1} (-q^2)^{j-k} z^{-2} v_k \otimes v_{-k} \qquad (j \in [1,n]), \\ C_{0,0} &= v_0 \otimes z^{-2} v_0 + q^2 z^{-2} v_0 \otimes v_0 + q[2](-q^2)^n z^{-1} v_\phi \otimes z^{-1} v_\phi \\ &\quad - [2](1-q^4) \sum_{k=1}^{n} (-q^2)^{n+1-k} z^{-2} v_k \otimes v_{-k}, \\ C_{\phi,\phi} &= v_\phi \otimes z^{-2} v_\phi + q^2 z^{-2} v_\phi \otimes v_\phi + q[2](-q^2)^n z^{-1} v_0 \otimes z^{-1} v_0 \\ &\quad - [2](1-q^4) \sum_{k=1}^{n} (-q^2)^j z^{-1} v_{-j} \otimes z^{-1} v_j. \end{aligned}$$

Proposition 5.7.1. *Identify $C_{i,j}$ with $C_{b_i, z^{-H(i,j)} b_j}$. Then $\{z^m \otimes z^m \cdot C_{i,j}\}_{m \in \mathbb{Z}; i,j \in J}$ with the function l satisfies condition (R) of subsection 3.3.*

5.7.5. Fock space. For $U_q(D_{n+1}^{(2)})$ we have

$$B_{\min} = \{b_0, b_\phi\},$$
$$(P_{\mathrm{cl}}^+)_1 = \{\Lambda_n^{\mathrm{cl}}, \Lambda_0^{\mathrm{cl}}\},$$

with

$$\varepsilon(b_0) = \Lambda_n^{\mathrm{cl}}, \qquad \varepsilon(b_\phi) = \Lambda_0^{\mathrm{cl}},$$
$$\varphi(b_0) = \Lambda_n^{\mathrm{cl}}, \qquad \varphi(b_\phi) = \Lambda_0^{\mathrm{cl}}.$$

Since $H(b_0 \otimes z^{-1}b_0) = 1$ and $H(b_\phi \otimes z^{-1}b_\phi) = 1$, there are two ground state sequences (see also the remark at the end of §6.6):

$$\begin{aligned} b_m^\circ &= z^{-m}b_0 && (m \in \mathbb{Z}), \\ \mathrm{cl}(\lambda_m) &= \Lambda_n && (m \in \mathbb{Z}), \end{aligned}$$

and

$$\begin{aligned} b_m^\circ &= z^{-m}b_\phi && (m \in \mathbb{Z}), \\ \mathrm{cl}(\lambda_m) &= \Lambda_0 && (m \in \mathbb{Z}). \end{aligned}$$

The vacuum vectors of $\mathcal{F}_m$ are respectively

$$|m\rangle := z^{-m}v_0 \wedge z^{-m-1}v_0 \wedge z^{-m-2}v_0 \wedge \cdots\cdots \qquad (m \in \mathbb{Z}),$$

with $\mathrm{wt}\,(|m\rangle) = \Lambda_n$, and

$$|m\rangle := z^{-m}v_\phi \wedge z^{-m-1}v_\phi \wedge z^{-m-2}v_\phi \wedge \cdots\cdots \qquad (m \in \mathbb{Z}),$$

with $\mathrm{wt}\,(|m\rangle) = \Lambda_0$.

6. Level 1 two point functions

In this section we calculate the boson commutation relations using the decomposition of the Fock space vertex operator into a product of a $U_q(\mathfrak{g})$-vertex operator and a bosonic vertex operator (Theorem 4.5.1), for level 1 $A_{2n}^{(2)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$. The two point functions of the level 1 $U_q(\mathfrak{g})$-vertex operators that we use are due to Date and Okado [DO] (except for type $D_{n+1}^{(2)}$ which is given in Appendix C).

6.1. Summary

In the following table we list the dual Coxeter number $h^\vee := \sum_{i\in I} a_i^\vee$, $p := q^{(\alpha_0,\alpha_0)/(2a_0^\vee)}$ and $\xi := (-)^{r-1}p^{h^\vee}$ for $\mathfrak{g} = X_n^{(r)}$ of types $A_n^{(1)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$ and $D_n^{(1)}$.

$\mathfrak{g}$	$A_n^{(1)}$	$A_{2n}^{(2)}$	$B_n^{(1)}$	$A_{2n-1}^{(2)}$	$D_n^{(1)}$
$h^\vee$	$n+1$	$2n+1$	$2n-1$	$2n$	$2n-2$
p	q	q^2	q^2	q	q
ξ	q^{n+1}	$-q^{2(2n+1)}$	$q^{2(2n-1)}$	$-q^{2n}$	q^{2n-2}

Proposition 6.1.1 [KMS]. *For $A_n^{(1)}$ at level 1, we have*

$$\gamma_m = m\frac{1-\xi^{2m}}{1-q^{2m}}.$$

See [KMS] §2 for the proof.

Let $\mathfrak{g} = X_n^{(r)}$ be of type $A_{2n}^{(2)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$ or $D_n^{(1)}$. Let $|\mathfrak{g}\rangle$ be one of the vacuum vectors of the $U_q(\mathfrak{g})$-Fock modules described in the previous section. For each type, direct calculations of $B_m \cdot B_{-m} \cdot |\mathfrak{g}\rangle$ for small m, suggest the following result.

Theorem 6.1.2. *For $\mathfrak{g} = X_n^{(r)} \in \{A_{2n}^{(2)}, B_n^{(1)}, A_{2n-1}^{(2)}, D_n^{(1)}\}$ at level 1, we have*

$$\gamma_m = m\frac{1+\xi^m}{1-p^{2m}}.$$

In this section we prove this theorem case-by-case using Proposition 4.5.2. We also give a corresponding result for level 1 $D_{n+1}^{(2)}$.

For this boson commutation relation, the boson two point function (4.5.7) is

$$\theta(w_2/w_1) = \frac{(w_2/w_1;\xi^2)_\infty(p^2\xi w_2/w_1;\xi^2)_\infty}{(p^2w_2/w_1;\xi^2)_\infty(\xi w_2/w_1;\xi^2)_\infty}. \tag{6.1.1}$$

Let us introduce the operator $Z(t,d) \in \mathrm{End}\,(V_{\mathrm{aff}} \otimes V_{\mathrm{aff}})$ defined by

$$Z(t,d) := z^t \otimes z^{d-t} + \delta(2t > d)z^{d-t} \otimes z^t - \delta(2t < d)z^t \otimes z^{d-t} \qquad (t,d \in \mathbb{Z}).$$

Note that $Z(t,d)$ is a symmetric Laurent polynomial in $z \otimes 1$ and $1 \otimes z$, so we have

Lemma 6.1.3. *$Z(t,d) \cdot N \subset N$ $(t,d \in \mathbb{Z})$.*

6.2. Type $A_{2n}^{(2)}$

Recall we have $\lambda_m = \Lambda_n$ and $b_m^\circ = b_0$ $(m \in \mathbb{Z})$. So the level 1 intertwiner maps

$$\Phi_m : V_{\mathrm{aff}} \otimes V(\Lambda_n) \to V(\Lambda_n) \qquad (m \in \mathbb{Z}).$$

From [DO], (up to a factor of a constant power in w_2/w_1) we have in our notation (4.5.5)

$$\phi_{v_0,v_0}(w_2/w_1) = (1-p^{h^\vee+1}w_2/w_1) \\ \times \frac{(p^{2h^\vee+2}w_2/w_1;p^{2h^\vee})_\infty(-p^{3h^\vee}w_2/w_1;p^{2h^\vee})_\infty}{(-p^{h^\vee+2}w_2/w_1;p^{2h^\vee})_\infty(p^{2h^\vee}w_2/w_1;p^{2h^\vee})_\infty}.$$

Define

$$g_j(t) := \langle m-1|z^t v_j \wedge z^{-t}v_{-j} \wedge |m+1\rangle \qquad (j\in J; t\in\mathbb{Z}).$$

Note that $g_0(0)=1$, $g_{-j}(0)=0$ ($j\in[1,n]$) and $g_j(t)=0$ ($j\in J$; $t\in\mathbb{Z}_{<0}$).

Proposition 6.2.1. *$g_0(t)$ satisfies the following recurrence relation*

$$g_0(t)-(p^2-p^{h^\vee})g_0(t-1)-p^{h^\vee+2}g_0(t-2) = \\ \delta_{t,0}-(1+p^{h^\vee+1})\delta_{t,1}+p^{h^\vee+1}\delta_{t,2}. \tag{6.2.1}$$

Proof. The proof for $n>1$ goes as follows (the exceptional case $n=1$ is similar). First note that any element in N, that is generated by $C_{j,-j}$ ($j\in J$), gives rise to a linear relation of some $g_k(t)$ ($k\in J$, $t\in\mathbb{Z}$). For example $(z\otimes 1+1\otimes z)\cdot C_{0,0}$ gives

$$g_0(1)+q^2g_0(0)+q^2[2](1-q^4)\sum_{k=1}^{n}(-q^2)^{n-k}g_k(0)+g_0(0)=0.$$

From $C_{k,-k}$ ($k\in[1,n]$) we get

$$g_k(0)+q(-q^2)^{n-k}g_0(0)=0.$$

Combining these two relations, we get

$$g_0(1)+(1-q^4+q^{4n+2}+q^{4n+4})g_0(0)=0,$$

which is (6.2.1) with $t=1$.

The recurrence relation in the general case ($t\in\mathbb{N}$) comes from

$$\begin{aligned}\mathcal{A}_t =& \big(Z(t,1)-p^{h^\vee+1}Z(t-1,1)\big)\cdot C_{0,0} \\ &+[2](1-p^2)(-p)^{n+1}\sum_{j=1}^{n}\big((-p)^{-j}Z(t-1,0)\cdot C_{j,-j} \\ &\qquad -(-p)^jZ(t-1,1)\cdot C_{-j,j}\big).\end{aligned}$$

Let $\mathcal{A}_t^\wedge$ denote the image of $\mathcal{A}_t$ in $V_{\mathrm{aff}} \wedge V_{\mathrm{aff}}$. We have:

$$
\begin{aligned}
&\langle m-1|\mathcal{A}_t^\wedge \wedge|m+1\rangle = \\
&\qquad g_0(t) - (p^{h^\vee+1} - p)g_0(t-1) - p^{h^\vee+2}g_0(t-2) \\
&\qquad +[2](1-p^2)\Big\{ -\sum_{j=1}^{n}(-p)^{n+1-j}\big(g_j(t-1) - (-p)^{h^\vee+1}g_j(t-2)\big) \\
&\qquad\qquad + \sum_{j=1}^{n}(-p)^{n+1-j}\big(g_j(t-1) + p^2 g_{-j}(t-1)\big) \\
&\qquad\qquad - q\frac{(-p)^{h^\vee} - (-p)}{1-p^2} g_0(t-1) \\
&\qquad\qquad - (1-p^2)p^{n+1}\sum_{j=1}^{n}(-)^j[j-1]_p g_{-j}(t-1) \\
&\qquad\qquad - \sum_{j=1}^{n}(-p)^{n+1+j}\big(g_{-j}(t-1) + p^2 g_j(t-2)\big) \\
&\qquad\qquad - (1-p^2)p^{h^\vee+1}\sum_{j=1}^{n}(-)^{n-j}[n-j]_p g_j(t-2)\Big\} + \\
&\qquad + \delta(2t>1)g_0(1-t) - \delta(2t<1)g_0(t) \\
&\qquad - p^{h^\vee+1}\big(\delta(2t>3)g_0(2-t) - \delta(2t<3)g_0(t-1)\big) = 0.
\end{aligned}
$$

All terms in g_k $(k \in J \setminus \{0\})$ cancel and the proposition follows. $\square$

The two point function of Fock intertwiners (4.5.4) is given by

Corollary 6.2.2.

$$\omega_{v_0,v_0}(w_2/w_1) = \frac{(1-w_2/w_1)(1-p^{h^\vee+1}w_2/w_1)}{(1-p^2w_2/w_1)(1+p^{h^\vee}w_2/w_1)}.$$

Proof. Note that $\omega_{v_0,v_0}(w) = \sum_{t\in\mathbb{N}} w^t g_0(t)$. Multiplying both sides of (6.2.1) by w^t $(w := w_2/w_1)$ and summing over non-negative t we get

$$\Big(1 - (p^2 - p^{h^\vee})w - p^{h^\vee+2}w^2\Big)\sum_{t\in\mathbb{N}} w^t g_0(t) = 1 - (1+p^{h^\vee+1})w + p^{h^\vee+1}w^2,$$

from which the result follows.

Hence

$$\frac{\omega_{v_0,v_0}(w_2/w_1)}{\phi_{v_0,v_0}(w_2/w_1)} = \frac{(-p^{h^\vee+2}w_2/w_1;p^{2h^\vee})_\infty (w_2/w_1;p^{2h^\vee})_\infty}{(p^2w_2/w_1;p^{2h^\vee})_\infty(-p^{h^\vee}w_2/w_1;p^{2h^\vee})_\infty},$$

in agreement with (6.1.1).

6.3. Type $B_n^{(1)}$

Recall that we have two ground state sequences ($\kappa = 0, 1$)

$$b_m^\circ = \begin{cases} b_\kappa & \text{for } m \in 2\mathbb{Z}, \\ z^\kappa b_{-\kappa} & \text{for } m \in 2\mathbb{Z}+1, \end{cases}$$

and

$$\lambda_m = \begin{cases} (1-\kappa)\Lambda_n + \kappa(\Lambda_1 - \frac{m}{2}\delta) & \text{for } m \in 2\mathbb{Z}, \\ (1-\kappa)\Lambda_n + \kappa(\Lambda_0 - \frac{m-1}{2}\delta) & \text{for } m \in 2\mathbb{Z}+1. \end{cases}$$

From [DO], (up to a factor of a constant power in w_2/w_1) we have in our notation (4.5.5)

$$\begin{aligned} \phi_{v^\circ_{m-1}, v^\circ_m}(w_2/w_1) = {} & (1 + p^{h^\vee+1} w_2/w_1)^{\delta_{\kappa,0}} \\ & \times \frac{(p^{2h^\vee+2} w_2/w_1; p^{2h^\vee})_\infty (p^{3h^\vee} w_2/w_1; p^{2h^\vee})_\infty}{(p^{h^\vee+2} w_2/w_1; p^{2h^\vee})_\infty (p^{2h^\vee} w_2/w_1; p^{2h^\vee})_\infty}. \end{aligned}$$

By a diagram automorphism, it is sufficient just to consider the case when m is even. Let $m \in 2\mathbb{Z}$. Define

$$g_j(t) := \langle m-1 | z^{t+\kappa} v_j \wedge z^{-t} v_{-j} \wedge | m+1 \rangle \qquad (j \in J; t \in \mathbb{Z}).$$

Note that $g_{-\kappa}(0) = 1$, $g_{-j}(-\kappa) = 0$ ($j \in [1, n]$) and $g_j(t) = 0$ ($j \in J$; $t \in \mathbb{Z}_{<0}$).

Proposition 6.3.1. *$g_{-\kappa}(t)$ satisfies the following recurrence relation*

$$\begin{aligned} g_{-\kappa}(t) - (p^2 - p^{h^\vee}) g_{-\kappa}(t-1) - p^{h^\vee+2} g_{-\kappa}(t-2) = {} & \\ \delta_{t,0} - (1 - \delta_{\kappa,0} p^{h^\vee+1})\delta_{t,1} - \delta_{\kappa,0} p^{h^\vee+1} \delta_{t,2}. & \qquad (6.3.1) \end{aligned}$$

Proof. The proof is like the proof of Proposition 6.2.1 for type $A_{2n}^{(2)}$, but using

$$\begin{aligned} \mathcal{A}_t := {} & \big(Z(t, 1+\kappa) + p^{1+h^\vee \delta_{\kappa,0}} Z(t+\kappa-1, 1+\kappa)\big) C_{0,0} \\ & + [2] Z(t-1, \kappa)\Big((-p)^n C_{1,-1} + (1-p^2)\sum_{j=2}^{n} (-p)^{n+1-j} C_{j,-j}\Big) \\ & + [2] p^{-h^\vee \delta_{\kappa,1}} \Big((1-p^2) Z(t+\kappa-1, 1+\kappa) \sum_{j=2}^{n} (-p)^{n+j-1} C_{-j,j} \\ & \qquad + (-p)^n Z(t+\kappa, 2+\kappa) C_{-1,1}\Big). \end{aligned}$$

□

The two point function of Fock intertwiners (4.5.4) is given by

Corollary 6.3.2. *Let $m \in 2\mathbb{Z}$.*

$$\omega_{v^\circ_{m-1},v^\circ_m}(w_2/w_1) = \frac{(1-w_2/w_1)(1+p^{h^\vee+1}w_2/w_1)^{\delta_{\kappa,0}}}{(1-p^2w_2/w_1)(1-p^{h^\vee}w_2/w_1)}.$$

Proof. Note that $\omega_{v^\circ_{m-1},v^\circ_m}(w) = \sum_{t\in\mathbb{N}} w^t g_{-\kappa}(t)$. Multiplying both sides of (6.3.1) by w^t ($w := w_2/w_1$) and summing over non-negative t we get

$$\left(1-(p^2-p^{h^\vee})w - p^{h^\vee+2}w^2\right)\sum_{t\in\mathbb{N}} w^k g_{-\kappa}(t) = \\ 1-(1-\delta_{\kappa,0}p^{h^\vee+1})w - \delta_{\kappa,0}p^{h^\vee+1}w^2,$$

from which the result follows. □

Hence

$$\frac{\omega_{v^\circ_{m-1},v^\circ_m}(w_2/w_1)}{\phi_{v^\circ_{m-1},v^\circ_m}(w_2/w_1)} = \frac{(p^{h^\vee+2}w_2/w_1;p^{2h^\vee})_\infty(w_2/w_1;p^{2h^\vee})_\infty}{(p^2w_2/w_1;p^{2h^\vee})_\infty(p^{h^\vee}w_2/w_1;p^{2h^\vee})_\infty},$$

in agreement with (6.1.1).

6.4. Type $A^{(2)}_{2n-1}$

Recall that we have

$$b^\circ_m = \begin{cases} b_1 & \text{for } m\in 2\mathbb{Z},\\ zb_{-1} & \text{for } m \in 2\mathbb{Z}+1,\end{cases} \quad \text{and} \quad \lambda_m = \begin{cases} \Lambda_1 - \frac{m}{2}\delta & \text{for } m\in 2\mathbb{Z},\\ \Lambda_0 - \frac{m-1}{2}\delta & \text{for } m\in 2\mathbb{Z}+1.\end{cases}$$

From [DO], (up to a factor of a constant power in w_2/w_1) we have in our notation (4.5.5).

$$\phi_{v^\circ_{m-1},v^\circ_m}(w_2/w_1) = \frac{(q^{2h^\vee+2}w_2/w_1;q^{2h^\vee})_\infty(-q^{3h^\vee}w_2/w_1;q^{2h^\vee})_\infty}{(-q^{h^\vee+2}w_2/w_1;q^{2h^\vee})_\infty(q^{2h^\vee}w_2/w_1;q^{2h^\vee})_\infty}.$$

By a diagram automorphism it is sufficient just to consider the case when m is even. Let $m \in 2\mathbb{Z}$. Define

$$g_j(t) := \langle m-1|z^{t+1}v_j \wedge z^{-t}v_{-j}\wedge |m+1\rangle \qquad (j\in J; t\in\mathbb{Z}).$$

Note that $g_{-1}(0) = 1$ and $g_j(t) = 0$ ($j \in J$; $t \in \mathbb{Z}_{<0}$).

Proposition 6.4.1. *$g_{-1}(t)$ satisfies the following recurrence relation*

$$g_{-1}(t) - (q^2 - q^{h^\vee})g_{-1}(t-1) - q^{h^\vee+2}g_{-1}(t-2) = \delta_{t,0} - \delta_{t,1}.$$

Proof. The proof is like the proof of Proposition 6.2.1, but using

$$\begin{aligned}\mathcal{A}_t := {} & Z(t,1)(z\otimes z)C_{-1,1} \\ & + Z(t,2)(1-q^2)\sum_{j=2}^{n}(-q)^{j-1}C_{-j,j} \\ & - Z(t-1,1)q^{h^\vee}\left(C_{1,-1} + (1-q^2)\sum_{j=2}^{n}(-q)^{1-j}C_{j,-j}\right).\end{aligned}$$

□

The two point function of Fock intertwiners (4.5.4) is given by

Corollary 6.4.2. *Let $m \in 2\mathbb{Z}$.*

$$\omega_{v^\circ_{m-1},v^\circ_m}(w_2/w_1) = \frac{(1-w_2/w_1)}{(1-q^2w_2/w_1)(1+q^{h^\vee}w_2/w_1)}.$$

Proof. Note that $\omega_{v^\circ_{m-1},v^\circ_m}(w) = \sum_{t\in\mathbb{N}} w^t g_{-1}(t)$. Multiplying both sides of (6.4.1) by w^t ($w := w_2/w_1$) and summing over non-negative t we get

$$\left(1 - (q^2 - q^{h^\vee})w - q^{h^\vee+2}w^2\right)\sum_{t\in\mathbb{N}} w^k g_{-1}(t) = 1 - w,$$

from which the result follows. □

Hence

$$\frac{\omega_{v^\circ_{m-1},v^\circ_m}(w_2/w_1)}{\phi_{v^\circ_{m-1},v^\circ_m}(w_2/w_1)} = \frac{(-q^{h^\vee+2}w_2/w_1;q^{2h^\vee})_\infty(w_2/w_1;q^{2h^\vee})_\infty}{(q^2w_2/w_1;q^{2h^\vee})_\infty(-q^{h^\vee}w_2/w_1;q^{2h^\vee})_\infty},$$

in agreement with (6.1.1).

6.5. Type $D_n^{(1)}$

Recall that we have two ground state sequences ($\kappa = 0,1$)

$$b^\circ_m = \begin{cases} b_{n\delta_{\kappa,0}+1\delta_{\kappa,1}} & \text{for } m \in 2\mathbb{Z}, \\ z^\kappa b_{n\delta_{\kappa,0}-1\delta_{\kappa,1}} & \text{for } m \in 2\mathbb{Z}+1, \end{cases}$$

From [DO], (up to a factor of a constant power in w_2/w_1) we have in our notation (4.5.5)

$$\phi_{v^\circ_{m-1},v^\circ_m}(w_2/w_1) = \frac{(q^{2h^\vee+2}w_2/w_1;q^{2h^\vee})_\infty(q^{3h^\vee}w_2/w_1;q^{2h^\vee})_\infty}{(q^{h^\vee+2}w_2/w_1;q^{2h^\vee})_\infty(q^{2h^\vee}w_2/w_1;q^{2h^\vee})_\infty}.$$

By diagram automorphisms, it is sufficient just to consider the case when $\kappa = 0$ and m is odd. Let $\kappa = 0$ and $m \in 2\mathbb{Z}+1$. Define

$$g_j(t) := \langle m-1|z^t v_j \wedge z^{-t}v_{-j} \wedge |m+1\rangle \qquad (j \in J; t \in \mathbb{Z}).$$

Note that $g_n(0) = 1$, $g_{-j}(0) = 0$ ($j \in [1,n]$) and $g_j(t) = 0$ ($j \in J$; $t \in \mathbb{Z}_{<0}$).

Proposition 6.5.1. *$g_n(t)$ satisfies the following recurrence relation*

$$g_n(t) - (q^2 - q^{h^\vee})g_n(t-1) - q^{h^\vee+2}g_n(t-2) = \delta_{t,0} - \delta_{t,1}.$$

Proof. The proof is like the proof of Proposition 6.2.1, but using

$$\begin{aligned}\mathcal{A}_t := {} & Z(t,1)C_{n,-n} - q^{h^\vee}Z(t-1,1)C_{-n,n} \\ & + Z(t-1,0)\big((-q)^{n-1}C_{1,-1} + (1-q^2)\sum_{j=2}^{n-1}(-q)^{n-j}C_{j,-j}\big) \\ & - (-q)^{n-1}\big(Z(t-1,0)(z\otimes z)C_{-1,1} \\ & \quad + (1-q^2)Z(t-1,1)\sum_{j=2}^{n-1}(-q)^{j-1}C_{-j,j}\big).\end{aligned}$$

□

The two point function of Fock intertwiners (4.5.4) is given by

Corollary 6.5.2. *Let $m \in 2\mathbb{Z}+1$.*

$$\omega_{v^\circ_{m-1},v^\circ_m}(w_2/w_1) = \frac{(1-w_2/w_1)}{(1-q^2w_2/w_1)(1-q^{h^\vee}w_2/w_1)}.$$

Proof. Note that $\omega_{v^\circ_{m-1},v^\circ_m}(w) = \sum_{t\in\mathbb{N}} w^t g_n(t)$. Multiplying both sides of (6.5.1) by w^t ($w := w_2/w_1$) and summing over non-negative t we get

$$\Big(1-(q^2-q^{h^\vee})w - q^{h^\vee+2}w^2\Big)\sum_{t\in\mathbb{N}} w^k g_n(t) = 1-w,$$

from which the result follows. □

Hence

$$\frac{\omega_{v^\circ_{m-1},v^\circ_m}(w_2/w_1)}{\phi_{v^\circ_{m-1},v^\circ_m}(w_2/w_1)} = \frac{(q^{h^\vee+2}w_2/w_1;q^{2h^\vee})_\infty(w_2/w_1;q^{2h^\vee})_\infty}{(q^2w_2/w_1;q^{2h^\vee})_\infty(q^{h^\vee}w_2/w_1;q^{2h^\vee})_\infty},$$

in agreement with (6.1.1).

6.6. Type $D_{n+1}^{(2)}$

This type is somewhat special because of the fact that V_{aff} is not irreducible. The dual Coxeter number is $h^\vee = 2n$. We define $p = q^2$ and $\xi^2 = p^{h^\vee}$.

Recall that we have two ground state sequences ($\kappa = 0, \phi$)

$$b_m^\circ = z^{-m} b_\kappa.$$

By a diagram automorphism, it is sufficient just to consider one of the two cases $\kappa \in \{0, \phi\}$. We choose $\kappa = 0$.

The boson commutator $\gamma_m = [B_m, B_{-m}]$ is given as follows:

Proposition 6.6.1.

$$\gamma_m = \begin{cases} m\frac{(1+\xi^m)}{(1-2p^m-\xi^m)} & \text{for } m \in 2\mathbb{Z}, \\ m & \text{for } m \in 2\mathbb{Z}+1. \end{cases} \tag{6.6.1}$$

This corresponds to the following boson two point function

$$\theta(w) = (1-w)\frac{(\xi^4 w^2;\xi^4)_\infty (p^2\xi^2 w^2;\xi^4)_\infty}{(\xi^2 w^2;\xi^4)_\infty (p^2 w^2;\xi^4)_\infty}. \tag{6.6.2}$$

From Appendix C we have

Lemma 6.6.2. *Let* $w = w_2/w_1$.

$$\phi_{z^{1-m}v_0, z^{-m}v_0}(w_2/w_1) = (1+p\xi^2 w^2)\frac{(\xi^6 w^2;\xi^4)_\infty (p^2\xi^4 w^2;\xi^4)_\infty}{(\xi^4 w^2;\xi^4)_\infty (p^2\xi^2 w^2;\xi^4)_\infty}.$$

It is sufficient just to consider the case $m = 0$. Define

$$g_j(t) := \langle -1| z^{t+1} v_j \wedge z^{-t} v_{-j} \wedge |1\rangle \qquad (j \in J; t \in \mathbb{Z}).$$

Note that $g_0(0) = 1$, $g_{-j}(0) = 0$ ($j \in [1,n]$) and $g_j(t) = 0$ ($j \in J$; $t \in \mathbb{Z}_{<0}$).

Proposition 6.6.3. $g_0(t)$ *satisfies the following recurrence relation*

$$g_0(t) - (p^2+\xi^2)g_0(t-2) + p^2\xi^2 g_0(t-4) = \\ \delta_{t,0} - \delta_{t,1} + p\xi^2\delta_{t,2} - p\xi^2\delta_{t,3}. \tag{6.6.3}$$

Proof. The proof is like the proof of Proposition 6.2.1, but using

$$\begin{aligned} \mathcal{A}_t := &\big(Z(t,1)(z\otimes z) + p^{h^\vee+1} Z(t-1,3)\big)C_{0,0} \\ &+ Z(t-1,1)[2]\big(-q(-p)^n (z\otimes z)C_{\phi,\phi} \\ &\qquad + (1-p^2)\sum_{j=1}^n (-p)^{n+1-j} C_{j,-j}\big) \\ &- Z(t-1,3)[2](1-p^2)\sum_{j=1}^n (-p)^{n+j} C_{-j,j}. \end{aligned}$$

□

The two point function of Fock intertwiners (4.5.4) is given by

Corollary 6.6.4.

$$\omega_{zv_0,v_0}(w_2/w_1)=\frac{(1-w_2/w_1)(1+p\xi^2(w_2/w_1)^2)}{(1-p^2(w_2/w_1)^2)(1-\xi^2(w_2/w_1)^2)}.$$

Proof. Note that $\omega_{zv_0,v_0}(w)=\sum_{t\in\mathbb{N}}w^tg_0(t)$. Multiplying both sides of (6.6.3) by w^t ($w:=w_2/w_1$) and summing over non-negative t we get

$$(1-(p^2+\xi^2)w^2-p^2\xi^2w^4)\sum_{t\in\mathbb{N}}w^kg_0(t)=1-w+p\xi^2w^2-p\xi^2w^3,$$

from which the result follows. □

Finally we have $\frac{\omega_{zv_0,v_0}(w)}{\phi_{zv_0,v_0}(w)}=$ (6.6.2), which proves Proposition 6.6.1.

Remark. It is possible to work in an irreducible component of V_{aff}, say $V_{\mathrm{aff}}^{\mathrm{even}}=V_0\otimes\mathbb{C}[z^2,z^{-2}]+v_\phi\otimes z\mathbb{C}[z^2,z^{-2}]$. On $V_{\mathrm{aff}}^{\mathrm{even}}\otimes V_{\mathrm{aff}}^{\mathrm{even}}$ the energy function takes only even values. The condition $H(b_m^\circ\otimes b_{m+1}^\circ)=1$ for a ground state sequence $\{b_m^\circ\}_{m\in Z}$ should then be replaced by $H(b_m^\circ\otimes b_{m+1}^\circ)=2$ for all $m\in Z$.

The ground state sequence in $B_{\mathrm{aff}}^{\mathrm{even}}$ is given by $b_m^\circ=b_0$ for all $m\in Z$. The Fock two-point function can be shown to be given by

$$\omega_{v_0,v_0}(w)=\frac{(1-w^2)(1+p\xi^2w^2)}{(1-\xi^2w^2)(1-p^2w^2)},$$

where $w=w_1/w_2$. Comparing with Lemma 6.6.2, we find that γ_m is now given by the same formula as in Theorem 6.1.2

$$\gamma_m=m\frac{1+\xi^{2m}}{1-p^{2m}}.$$

7. Higher level example: level k $A_1^{(1)}$

7.1. Cartan datum

$I=\{0,1\}$. The Dynkin diagram for $A_1^{(1)}$ is

$$0 \Longleftrightarrow 1 \quad .$$

We have

$$\delta=\alpha_0+\alpha_1,$$
$$c=h_0+h_1,$$
$$(\alpha_i,\alpha_i)=2 \qquad (i\in I).$$

7.2. Perfect crystal

Fix $k \in \mathbb{Z}_{>0}$. Let $J := [0, k]$. Let V be the $(k+1)$-dimensional $U'_q(A_1^{(1)})$-module with the level k perfect crystal $B := \{b_j\}_{j \in J}$ and crystal graph:

$$b_0 \underset{0}{\overset{1}{\rightleftarrows}} b_1 \underset{0}{\overset{1}{\rightleftarrows}} b_2 \underset{0}{\overset{1}{\rightleftarrows}} \cdots \underset{0}{\overset{1}{\rightleftarrows}} b_k .$$

The elements of B have the following weights

$$\begin{aligned} \mathrm{wt}\,(b_j) &= (k/2 - j)\alpha_1 \\ &= (k-2j)(\Lambda_1 - \Lambda_0) \qquad (j \in J). \end{aligned}$$

Let $v_j := G(b_j)$ $(j \in J)$. The action of $U'_q(A_1^{(1)})$ on $v_j \in V$ obeys (5.1.1).

7.3. Energy function

The energy function H has the following values on $B \otimes B$

$$H(b_i \otimes b_j) = \min(i, k-j) \qquad (i, j \in J).$$

Write $H(i,j)$ for $H(b_i \otimes b_j)$ $(i, j \in J)$.

The Coxeter number of $A_1^{(1)}$ is $h = 2$. We take

$$l(z^m b_j) = 2m - j \qquad (m \in \mathbb{Z}, j \in J).$$

H and l satisfy condition (L). Note that $l(z^m b_j) = l(z^{m+1} b_{j+2})$ ($m \in \mathbb{Z}$ and $j \in [0, k-2]$), so l gives a partial ordering of B_{aff} for $k > 1$.

7.4. q-binomials

Define the q-binomial coefficient $\begin{bmatrix} m \\ n \end{bmatrix}$ $(m, n \in \mathbb{Z})$ by

$$\begin{bmatrix} m \\ n \end{bmatrix} = \begin{cases} \dfrac{[m][m-1]\cdots[m-n+1]}{cl[n][n-1]\cdots[1]} & : m \geq n \geq 0 \\ 0 & : \text{otherwise.} \end{cases}$$

We will often write sums involving q-binomial coefficients as sums over all integers. The advantage is that we can then freely change variables without worrying about the range of summation. The following result is widely used in the sequel:

Lemma 7.4.1.

(i) *For any $\eta \in \mathbb{C}(q)$ and $n \in \mathbb{Z}_{>0}$, we have*

$$\sum_{j \in \mathbb{Z}} (-\eta)^j \begin{bmatrix} n \\ j \end{bmatrix} = \prod_{i=0}^{n-1} (q^{n-1-2i} - \eta).$$

(ii) *The sum in (i) vanishes if $\eta = q^m$ with m an integer lying in the range $[-n+1, n-1]_2$. Here $[a, a+2b]_2$ means $\{a + 2i; 0 \leq i \leq b\}$.*

7.5. Wedge relations

Define the vectors $C_{i,j} \in N$ $(i, j \in J)$ by

$$C_{i,j} = \begin{cases} (z \otimes z)^{-i} e_0^{(i)} f_1^{(j)} (v_0 \otimes v_0) & : i + j \leq k, \\ e_1^{(k-i)} f_0^{(k-j)} (v_k \otimes v_k) & : i + j > k. \end{cases}$$

Explicitly, we have

$$C_{i,j} = \begin{cases} \sum_{i',j',a,b} q^{(k-j')(i'-b)+(k-i')a} \begin{bmatrix} j' \\ a \end{bmatrix} \begin{bmatrix} i' \\ b \end{bmatrix} z^{-a} v_{i'} \otimes z^{-b} v_{j'} & : i' + j' \leq k \\ \sum_{i',j',a,b} q^{i'(k-j'-b)+j'a} \begin{bmatrix} k - i' \\ a \end{bmatrix} \begin{bmatrix} k - j' \\ b \end{bmatrix} z^{-a} v_{i'} \otimes z^{-b} v_{j'} & : i' + j' > k. \end{cases} \tag{7.5.1}$$

The summation in both cases is over $i', j' \in J$ and $a, b \geq 0$ with

$$i' + j' = i + j,$$
$$a + b = H(i, j).$$

Proposition 7.5.1. *Identify $C_{i,j}$ with $C_{b_i, z^{-H(i,j)} b_j}$. Then $\{z^m \otimes z^m \cdot C_{i,j}\}_{m \in \mathbb{Z}; i,j \in J}$ with the function l satisfies condition (R) of subsection 3.3.*

7.6. Fock space

We have

$$B_{\min} = B,$$
$$(P_{\mathrm{cl}}^+)_k = \{j\Lambda_0^{\mathrm{cl}} + (k - j)\Lambda_1^{\mathrm{cl}}\}_{j \in J},$$

with the bijections

$$\varepsilon(b_j) = (k - j)\Lambda_0^{\mathrm{cl}} + j\Lambda_1^{\mathrm{cl}},$$
$$\varphi(b_j) = j\Lambda_0^{\mathrm{cl}} + (k - j)\Lambda_1^{\mathrm{cl}}.$$

Fix $\kappa \in J$ and let $\kappa' = k - \kappa$. We have $H(z^{-\ell(k-2)} b_\kappa \otimes z^{-\ell(k-2)-\kappa+1} b_{\kappa'}) = H(b_\kappa \otimes b_{\kappa'}) - \kappa + 1 = 1$. $H(z^{-\ell(k-2)-\kappa+1} b_{\kappa'} \otimes z^{-(\ell+1)(k-2)} b_\kappa) = 1$ and so the following is a ground state sequence $(\ell \in \mathbb{Z})$

$$b_{2\ell-1}^\circ = z^{-\ell(k-2)} b_\kappa,$$
$$b_{2\ell}^\circ = z^{-\ell(k-2)-\kappa+1} b_{\kappa'}, \tag{7.6.1}$$

with

$$\mathrm{cl}(\lambda_m) = \begin{cases} \kappa\Lambda_0 + \kappa'\Lambda_1 & : \text{if } m \text{ is odd}, \\ \kappa'\Lambda_0 + \kappa\Lambda_1 & : \text{if } m \text{ is even}. \end{cases}$$

With $v_m^\circ = G(b_m^\circ)$, the vacuum vector of $\mathcal{F}_m$ is

$$|m\rangle = v_m^\circ \wedge v_{m+1}^\circ \wedge v_{m+2}^\circ \wedge \cdots\cdots$$

with weight λ_m.

7.7. Two point functions

A priori $\gamma_n = [B_n, B_{-n}]$ may depend on the choice of κ. However, we find that it is independent of κ.

Theorem 7.7.1.

$$\gamma_n = n\frac{1-q^{4n}}{1-q^{2n}-q^{4n}+q^{2(k+1)n}}.$$

The theorem follows by applying Proposition 4.5.2 to Proposition 7.7.2 and Corollary 7.7.4 below.

From [IIJMNT] we have

Proposition 7.7.2.

$$\phi_{v^\circ_{2\ell-1},v^\circ_{2\ell}}(w) = \frac{(q^{2(k+2)}w;q^4)_\infty}{(q^4w;q^4)_\infty}\sum_{p=0}^{\infty}\left(q^{k+2}w\right)^p \begin{bmatrix}\kappa\\ p\end{bmatrix}\begin{bmatrix}\kappa'\\ p\end{bmatrix},$$

where $w = w_2/w_1$.

Without loss of generality, we can choose $\ell = 0$. Define

$$g_\kappa(t) := \langle -1|z^t v_\kappa \wedge z^{-t+1-\kappa}v_{\kappa'} \wedge |+1\rangle \qquad (t \in \mathbb{Z}; j \in J).$$

Note that $g_\kappa(t) = \delta_{t,0}$ for $t \in \mathbb{Z}_{\le 0}$ by Theorem 4.2.5.

Proposition 7.7.3. $g_\kappa(t)$ *satisfies the following recurrence relation*

$$\sum_{\alpha\in\mathbb{Z}}\left(-q^{k+1}\right)^\alpha \begin{bmatrix}k\\ \alpha\end{bmatrix} g_\kappa(t-\alpha) - \left(q^{k+2}\right)^t \begin{bmatrix}\kappa\\ t\end{bmatrix}\begin{bmatrix}\kappa'\\ t\end{bmatrix} + \left(q^{k+2}\right)^{t-1}\begin{bmatrix}\kappa\\ t-1\end{bmatrix}\begin{bmatrix}\kappa'\\ t-1\end{bmatrix} = 0 \quad (7.7.1)$$

Corollary 7.7.4.

$$\omega_{v^\circ_{-1},v^\circ_0}(w) = \frac{(1-w)}{\prod_{j=1}^{k}(1-q^{2j}w)}\sum_{p=0}^{\infty}\left(q^{k+2}w\right)^p\begin{bmatrix}\kappa\\ p\end{bmatrix}\begin{bmatrix}\kappa'\\ p\end{bmatrix},$$

where $w = w_2/w_1$.

Proof. We have

$$\omega_{v^\circ_{-1},v^\circ_0}(w) = \sum_{j\in\mathbb{Z}}\left(\frac{w_2}{w_1}\right)^j g_\kappa(j). \qquad (7.7.2)$$

Multiply both sides of (7.7.1) by w^t and sum over all $t \geq 0$. After relabelling of t and using (7.7.2) we obtain

$$\omega_{v^\circ_{-1},v^\circ_0}(w) \sum_{\alpha\in\mathbb{Z}} \left(-q^{k+1}w\right)^\alpha \begin{bmatrix} k \\ \alpha \end{bmatrix} = (1-w) \sum_{t=0}^{\infty} \left(q^{k+2}w\right)^t \begin{bmatrix} \kappa \\ t \end{bmatrix} \begin{bmatrix} \kappa' \\ t \end{bmatrix}. \tag{7.7.3}$$

From Lemma 7.4.1 (i) we have

$$\sum_{\alpha\in\mathbb{Z}} \left(-q^{k+1}w\right)^\alpha \begin{bmatrix} k \\ \alpha \end{bmatrix} = \prod_{j=1}^{k} (1-q^{2j}w),$$

thereby proving the result. □

Only Proposition 7.7.3 remains to be proved.

7.8. Proof of recurrence relation

Let $Z(t,d)$ be the operator defined in §6.1:

$$Z(t,d) = z^t \otimes z^{d-t} + \delta(2t > d) z^{d-t} \otimes z^t - \delta(2t < d) z^t \otimes z^{d-t} \qquad (t,d \in \mathbb{Z}).$$

For $t \in \mathbb{Z}$, define

$$\mathcal{A}_t := \sum_{\substack{i\in J\\ \gamma\in\mathbb{Z}}} (-q^{\kappa+1})^{k-i-\kappa} q^{\gamma(k+2)} \begin{bmatrix} i \\ \gamma \end{bmatrix} \begin{bmatrix} k-i \\ \kappa-\gamma \end{bmatrix} Z(t-\gamma, -i+\kappa'+1) C_{k-i,i}.$$

We split the proof into three parts. Define

$$\begin{aligned} Z^{(1)}(t,d) &= z^t \otimes z^{d-t}, \\ Z^{(2)}(t,d) &= -z^t \otimes z^{d-t}\delta(2t < d), \\ Z^{(3)}(t,d) &= z^{d-t} \otimes z^t \delta(2t > d). \end{aligned} \tag{7.8.1}$$

Then $Z(t,d) = Z^{(1)}(t,d) + Z^{(2)}(t,d) + Z^{(3)}(t,d)$. Define $\mathcal{A}_t^{(i)}$ ($i \in \{1,2,3\}$) by replacing Z by $Z^{(i)}$ in the definition of $\mathcal{A}_t$. Then $\mathcal{A}_t = \mathcal{A}_t^{(1)} + \mathcal{A}_t^{(2)} + \mathcal{A}_t^{(3)}$. We will deal with each $\mathcal{A}_t^{(i)}$ separately. Note that $\mathcal{A}_t^{(i)} \notin N$ ($i \in \{1,2,3\}$), only $\mathcal{A}_t \in N$.

7.8.1. $\mathcal{A}_t^{(1)}$**.** From (7.5.1) we obtain

$$C_{k-i,i} = \sum_{\substack{j\in J\\ b\in\mathbb{Z}}} q^{j^2-j(k+i)+k(k-b)} \begin{bmatrix} j \\ k-i-b \end{bmatrix} \begin{bmatrix} k-j \\ b \end{bmatrix} z^{b+i-k} v_{k-j} \otimes z^{-b} v_j.$$

Substituting into (7.8) and performing a change of variable $b \to \gamma + k - i - \alpha$, followed by $i \to i + \gamma$ we obtain

$$\mathcal{A}_t^{(1)} = \sum_{\substack{i,\gamma,\alpha\in\mathbb{Z}\\ j\in J}} \left(-q^{\kappa+1}\right)^{k-i-\kappa-\gamma} q^{2\gamma+k\alpha+(i-j+\gamma)(k-j)} \begin{bmatrix} i+\gamma \\ \gamma \end{bmatrix} \begin{bmatrix} k-i-\gamma \\ \kappa-\gamma \end{bmatrix}$$
$$\times \begin{bmatrix} j \\ \alpha-\gamma \end{bmatrix} \begin{bmatrix} k-j \\ k-i-\alpha \end{bmatrix} z^{t-\alpha} v_{k-j} \otimes z^{-t+\alpha-\kappa+1} v_j. \quad (7.8.2)$$

Let us now argue that *only* the $j = k - \kappa$ terms contribute in the above sum. Recall that our convention for q-binomial coefficients implicitly defines for us the upper and lower limits of summation in formulae like $\mathcal{A}_t^{(1)}$. For instance, the constraints on i are

$$\max(0, j-\alpha) \le i \le \min(k-\kappa, k-\alpha). \quad (7.8.3)$$

Let us assume first that $j \le k - \kappa - 1$. The strategy is to recast the sum over i in (7.8.2), more specifically,

$$I_j = \begin{bmatrix} j \\ \alpha-\gamma \end{bmatrix} \sum_{i\in\mathbb{Z}} \left(-q^{k-j-\kappa-1}\right)^i \begin{bmatrix} i+\gamma \\ \gamma \end{bmatrix} \begin{bmatrix} k-i-\gamma \\ \kappa-\gamma \end{bmatrix} \begin{bmatrix} k-j \\ k-i-\alpha \end{bmatrix} \quad (7.8.4)$$

into a form such that Lemma 7.4.1 applies. Consider the case $j \le \alpha \le \kappa$, so that according to (7.8.3) we have $0 \le i \le k - \kappa$. By manipulating the q-binomial coefficients we obtain

$$I_j = \frac{[k-j]![j]!}{[k-\kappa]![\kappa-\gamma]![\gamma]!}$$
$$\times \sum_{i=0}^{k-\kappa} \left(-q^{k-j-\kappa-1}\right)^i \begin{bmatrix} k-\kappa \\ i \end{bmatrix} \begin{bmatrix} i+\gamma \\ j-\alpha+\gamma \end{bmatrix} \begin{bmatrix} k-i-\gamma \\ \alpha-\gamma \end{bmatrix}. \quad (7.8.5)$$

Now treat the product of the last two q-binomial coefficients in I_j together with $\left(-q^{k-j-\kappa-1}\right)^i$ as a polynomial in q^i; the powers of q^i which appear can be seen to lie in the range $[k-\kappa-1-2j, k-\kappa-1]_2$. In fact, due to the assumption on j the range is $[1-k+\kappa, k-\kappa-1]_2$. Therefore I_j is a finite sum of sums for which Lemma 7.4.1 (ii) applies and thus vanishes.

For the other three remaining cases (a) $j, \kappa \le \alpha$, (b) $j, \kappa \ge \alpha$ and (c) $j \ge \alpha \ge \kappa$

we use, respectively, the identities for I_j:

$$I_j = \frac{[k-j]![j]!}{[k-\alpha]![\alpha-\gamma]![\gamma]!} \sum_{i=0}^{k-\alpha} \left(-q^{k-j-\kappa-1}\right)^i \begin{bmatrix} k-\alpha \\ i \end{bmatrix} \begin{bmatrix} i+\gamma \\ j-\alpha+\gamma \end{bmatrix} \times \begin{bmatrix} k-i-\gamma \\ \kappa-\gamma \end{bmatrix}$$

$$I_j = \frac{[k-j]![j]!}{[j-\alpha+\gamma]![\kappa-\gamma]![k-\kappa-j+\alpha]!} \sum_{i=j-\alpha}^{k-\kappa} \left(-q^{k-j-\kappa-1}\right)^i \times \begin{bmatrix} k-\kappa-j+\alpha \\ i-j+\alpha \end{bmatrix} \begin{bmatrix} i+\gamma \\ \gamma \end{bmatrix} \begin{bmatrix} k-i-\gamma \\ \alpha-\gamma \end{bmatrix}$$

$$I_j = \sum_{i=j-\alpha}^{k-\alpha} \left(-q^{k-j-\kappa-1}\right)^i \begin{bmatrix} k-j \\ i-j+\alpha \end{bmatrix} \begin{bmatrix} i+\gamma \\ \gamma \end{bmatrix} \begin{bmatrix} k-i-\gamma \\ \kappa-\gamma \end{bmatrix}.$$

In each case I_j vanishes by application of Lemma 7.4.1 (ii).

We have proved that the sum over $j < k-\kappa$ in (7.8.2) vanishes. The sum over $j > k-\kappa$ vanishes for similar reasons. Keeping only the $j = k-\kappa$ term we arrive at

$$\mathcal{A}_t^{(1)} = \sum_{\gamma,\alpha\in\mathbb{Z}} (-q)^{k-\kappa-\gamma} q^{2\gamma+k\alpha} I_{k-\kappa}\, z^{t-\alpha} v_\kappa \otimes z^{-t+\alpha-\kappa+1} v_{\kappa'} \tag{7.8.6}$$

where $I_{k-\kappa}$ is given by (7.8.4). Once again, we have the constraint (7.8.3) and have to treat the four cases separately. We consider in detail only the case $k-\kappa \le \alpha \le \kappa$, using the form (7.8.5) for $I_{k-\kappa}$. The other three cases are similar. We proceed as before but now find that the powers of q^i lie in the range $[-1-k+\kappa, -1+k-\kappa]_2$. By Lemma 7.4.1 (ii) only the term whose power of q^i is $-1-k+\kappa$ survives. In other words,

$$I_{k-\kappa} = \begin{bmatrix} \kappa \\ \gamma \end{bmatrix} \sum_{i=0}^{k-\kappa} (-q^{-1})^i \begin{bmatrix} k-\kappa \\ i \end{bmatrix} \frac{q^{-(i+\gamma)} q^{-(i+\gamma-1)} \dots}{[k-\kappa-\alpha+\gamma]!} \times \frac{q^{k-i-\gamma} q^{k-i-\gamma-1} \dots}{[\alpha-\gamma]!} \times \frac{(-)^{k-\kappa-\alpha+\gamma}}{(q-q^{-1})^{k-\kappa}}.$$

Applying Lemma 7.4.1 (i) and simplifying we find

$$I_{k-\kappa} = \begin{bmatrix} \kappa \\ \gamma \end{bmatrix} \begin{bmatrix} k-\kappa \\ \alpha-\gamma \end{bmatrix} \left(-q^{\kappa+1}\right)^\alpha \left(-q^{k+1}\right)^{-\gamma} (-q)^{\kappa-k}.$$

Substituting into (7.8.6) we obtain

$$\mathcal{A}_t^{(1)} = \sum_{\gamma,\alpha\in\mathbb{Z}} \left(-q^{k+\kappa+1}\right)^{\alpha-\gamma} \left(-q^{\kappa+1}\right)^\gamma \begin{bmatrix} \kappa \\ \gamma \end{bmatrix} \begin{bmatrix} k-\kappa \\ \alpha-\gamma \end{bmatrix} z^{t-\alpha} v_\kappa \otimes z^{-t+\alpha-\kappa+1} v_{\kappa'}. \tag{7.8.7}$$

We now note the identity

$$\left(\sum_{\beta\in\mathbb{Z}}(-q^{k+\kappa+1}x)^{\beta}\begin{bmatrix}k-\kappa\\ \beta\end{bmatrix}\right)\left(\sum_{\gamma\in\mathbb{Z}}(-q^{\kappa+1}x)^{\gamma}\begin{bmatrix}\kappa\\ \gamma\end{bmatrix}\right)=$$
$$\prod_{i=1}^{k}(1-q^{2i}x)=\sum_{\alpha\in\mathbb{Z}}(-q^{k+1}x)^{\alpha}\begin{bmatrix}k\\ \alpha\end{bmatrix},$$

which follows from the ubiquitous Lemma 7.4.1, to perform the γ-sum in (7.8.7) with the result

$$\mathcal{A}_t^{(1)}=\sum_{\alpha\in\mathbb{Z}}\left(-q^{k+1}\right)^{\alpha}\begin{bmatrix}k\\ \alpha\end{bmatrix}z^{t-\alpha}v_{\kappa}\otimes z^{-t+\alpha+1-\kappa}v_{\kappa'}.$$

7.8.2. $\mathcal{A}_t^{(2)}$. The only difference between $\mathcal{A}_t^{(1)}$ and $\mathcal{A}_t^{(2)}$ is that the latter has a negative sign and an additional constraint

$$i<\gamma-2t+k-\kappa+1 \tag{7.8.8}$$

on the sum (denoted by prime) due to the definition of $Z^{(2)}$:

$$\mathcal{A}_t^{(2)}=-{\sum}'_{\substack{i,\gamma,\alpha\in\mathbb{Z}\\ j\in J}}(-q^{\kappa+1})^{k-i-\kappa-\gamma}q^{2\gamma+(i-j+\gamma)(k-j)}\begin{bmatrix}i+\gamma\\ \gamma\end{bmatrix}\begin{bmatrix}k-i-\gamma\\ \kappa-\gamma\end{bmatrix}$$
$$\times\begin{bmatrix}j\\ \alpha-\gamma\end{bmatrix}\begin{bmatrix}k-j\\ k-i-\alpha\end{bmatrix}z^{t-\alpha}v_{k-j}\otimes z^{-t+\alpha-\kappa+1}v_j. \tag{7.8.9}$$

Furthermore we are now interested in dropping terms that annihilate the vacuum. Using Theorem 3.5 this means that we require

$$\begin{aligned}&H\left(z^{-t+\alpha-m(k-2)-\kappa+1}b_j\otimes z^{-(m+1)(k-2)}b_{\kappa}\right)\\ &\quad=t-\alpha+\kappa-k+1+\min(j,k-\kappa)>0.\end{aligned} \tag{7.8.10}$$

Let us assume first that $j>k-\kappa$. From (7.8.10) we need $\alpha-t<0$. Now from the last q-binomial in (7.8.9) and (7.8.8) we have

$$j\le i+\alpha<(\alpha-t)+(\gamma-t)+k-\kappa+1. \tag{7.8.11}$$

Thus we have $k-\kappa<j<\gamma-t+k-\kappa+1$ and so $\gamma-t>0$. But this means $\gamma>t\ge\alpha$ which contradicts the requirement $\gamma\le\alpha$ coming from the third q-binomial in (7.8.9).

Next assume that $j < k-\kappa$. From (7.8.10) we now need $j \geq k-\kappa+\alpha-t$ and thus $\alpha - t < 0$. But again we have (7.8.11), and so

$$j \leq (\alpha-t)+(\gamma-t)+k-\kappa < (\gamma-t)+k-\kappa.$$

Thus we have $k-\kappa+\alpha-t \leq j < \gamma-t+k-\kappa$ and so $\alpha < \gamma$ which again contradicts $\gamma \leq \alpha$. Hence we must have

$$j = k-\kappa. \tag{7.8.12}$$

According to (7.8.10) we need $\alpha - t < 0$. But again (7.8.11) is required, which leads to $0 \leq (\alpha-t)+(\gamma-t) \leq \gamma - t$. Therefore $\alpha \leq t \leq \gamma$ which together with $\gamma \leq \alpha$ from the third q-binomial in (7.8.9) makes mandatory

$$\alpha = \gamma = t. \tag{7.8.13}$$

This means that (7.8.8) can be rephrased as $i \leq -t+k-\kappa$. But from the last q-binomial in (7.8.9), together with (7.8.12) and (7.8.13) we must have also

$$i = -t+k-\kappa. \tag{7.8.14}$$

Substituting (7.8.12), (7.8.13) and (7.8.14) into (7.8.9) we arrive at

$$\mathcal{A}_t^{(2)} = -\left(q^{k+2}\right)^t \begin{bmatrix} \kappa \\ t \end{bmatrix} \begin{bmatrix} \kappa' \\ t \end{bmatrix} v_\kappa \otimes z^{-\kappa+1} v_{k-\kappa} + \cdots .$$

7.8.3. $\mathcal{A}_t^{(3)}$. One argues in the same way that

$$\mathcal{A}_t^{(3)} = \left(q^{k+2}\right)^{t-1} \begin{bmatrix} \kappa \\ t-1 \end{bmatrix} \begin{bmatrix} \kappa' \\ t-1 \end{bmatrix} v_\kappa \otimes z^{-\kappa+1} v_{k-\kappa} + \cdots .$$

Let $\mathcal{A}_t^\wedge$ denote the image of $\mathcal{A}_t$ in $V_{\mathrm{aff}}^{\wedge 2}$. Adding the three parts together, the relation $\langle -1|\mathcal{A}_t^\wedge \wedge |1\rangle = 0$ gives us Proposition 7.7.3.

Appendix A. Perfect crystal

Let V be an integrable finite-dimensional $U_q'(\mathfrak{g})$-module with a perfect crystal base (L, B) of level l. We assume that it has a lower global base (i.e. satisfies (G)). In [KMN1], we proved that the "semi-infinite tensor product" $B \otimes B \otimes \cdots$ is isomorphic to the crystal base $B(\lambda)$ of the highest irreducible module, provided that the rank of $\mathfrak{g}$ is greater than 2. In this appendix, we prove the same statement for any rank. In [KMN1], the proof is combinatorial, and here it is by the use of a vertex operator. Let us take a ground state sequence $(\cdots, b_m^\circ, b_{m+1}^\circ, \cdots)$ in B_{aff}. Set $v_k^\circ = G(b_k^\circ)$. For an integral dominant weight λ, we denote by $V(\lambda)$ the irreducible $U_q(\mathfrak{g})$-module with highest weight λ and highest weight vector u_λ, and by $\big(L(\lambda), B(\lambda)\big)$ its crystal base.

Proposition A.1. $B \otimes B(\lambda_m) \cong B(\lambda_{m-1})$.

This proposition implies the following result.

Proposition A.2.

$$B(\lambda_m) \cong \{(b_m, b_{m+1}, \cdots);\ b_k \in B_{\mathrm{aff}}, H(b_k \otimes b_{k+1}) = 1 \textit{ for any } k \geq m \textit{ and } b_k = b_k^\circ \textit{ for } k \gg m\}.$$

The following lemma is proved in [DJO].

Lemma A.3. $\mathrm{Hom}_{U_q(\mathfrak{g})}(V_{\mathrm{aff}} \otimes V(\lambda_m), V(\lambda_{m-1})) = K$.

Let $\Phi : V_{\mathrm{aff}} \otimes V(\lambda_m) \to V(\lambda_{m-1})$ be a $U_q(\mathfrak{g})$-linear homomorphism. We normalize it by

$$\Phi(v_{m-1}^\circ \otimes u_{\lambda_m}) = u_{\lambda_{m-1}}.$$

Then the following lemma is also proved in [DJO] in the dual form.

Lemma A.4. $\Phi(L_{\mathrm{aff}} \otimes_A L(\lambda_m)) \subset L(\lambda_{m-1})$.

Let $\bar{\Phi} : (L_{\mathrm{aff}} \otimes_A L(\lambda_m))/q(L_{\mathrm{aff}} \otimes_A L(\lambda_m)) \to L(\lambda_{m-1})/qL(\lambda_{m-1})$ be the induced homomorphism.

The following two lemmas are easily proved.

Lemma A.5. *Let M_j be an integrable $U_q(\mathfrak{g})$-module, and (L_j, B_j) a crystal base of M_j for $j = 1, 2$. Let $\psi : M_1 \to M_2$ be a $U_q(\mathfrak{g})$-linear homomorphism sending L_1 to L_2. Let $\bar{\psi} : L_1/qL_1 \to L_2/qL_2$ be the induced homomorphism. Set $\tilde{B} = \{b \in B_1 | \bar{\psi}(b) \in B_2\}$. Then $\tilde{B}$ has a crystal structure such that $\iota : \tilde{B} \to B_1$ and $\bar{\psi} : \tilde{B} \to B_2$ are strict morphism of crystals.*

Here a strict morphism means a morphism commuting with $\tilde{e}_i$ and $\tilde{f}_i$.

Lemma A.6. *Let λ be a dominant integral weight. Let B be a semi-regular crystal (i.e. $\varepsilon_i(b) = \max\{n \in \mathbb{Z}_{\geq 0} | \tilde{e}_i^n b \neq 0\}$ and $\varphi_i(b) = \max\{n \in \mathbb{Z}_{\geq 0} | \tilde{f}_i^n b \neq 0\}$). We assume further that B is connected.*

(i) *If $\psi : B(\lambda) \to B$ is a strict morphism such that $\psi(B(\lambda)) \subset B$, then ψ is an isomorphism.*
(ii) *If $\psi : B \to B(\lambda)$ is a strict morphism such that $\psi(B) \subset B(\lambda)$, then ψ is an isomorphism.*

Let B' be the connected component of $B_{\mathrm{aff}} \otimes B(\lambda_m)$ containing $b_{m-1}^\circ \otimes u_{\lambda_m}$. Then Φ sends B' to $B(\lambda_{m-1})$. Hence B' is a subcrystal of $B_{\mathrm{aff}} \otimes B(\lambda_m)$, and Lemma A.5 implies $B' \to B_{\mathrm{aff}} \otimes B(\lambda_m)$ and $B' \to B(\lambda_{m-1})$ are strict morphisms. Moreover any $b \in B'$ is not mapped to 0 by the morphism $B' \to B(\lambda_{m-1})$. Hence by Lemma A.6, B' is isomorphic to $B(\lambda_{m-1})$. Hence we obtain a strict morphism $B(\lambda_{m-1}) \to B_{\mathrm{aff}} \otimes B(\lambda_m)$. Composing it with $B_{\mathrm{aff}} \to B$, we obtain a strict morphism $B(\lambda_{m-1}) \to B \otimes B(\lambda_m)$.

The following lemma is proved in [KMN1].

Lemma A.7. *$B \otimes B(\lambda_m)$ is connected.*

Thus $B(\lambda_{m-1}) \to B \otimes B(\lambda_m)$ is an isomorphism by Lemma A.6.

Appendix B. Serre relations

Let $\overline{U}_q(\mathfrak{g})$ be the algebra associated to a symmetrizable Kac-Moody algebra with the same generators and the defining relations as the quantized universal enveloping algebra except the Serre relations. Let $U_q(\mathfrak{g})_i$ be its subalgebra generated by e_i, f_i and $t_i^{\pm 1}$. In this appendix, we prove the following proposition.

Proposition B.1. *Let M be a $\overline{U}_q(\mathfrak{g})$-module. Assume that M is an integrable $U_q(\mathfrak{g})_i$-module for every i. Then the action of $\overline{U}_q(\mathfrak{g})$ on M satisfies the Serre relations.*

Hence M has the structure of a $U_q(\mathfrak{g})$-module.

Let M and N be integrable $U_q(\mathfrak{g})_i$-modules. We endow the structure of $U_q(\mathfrak{g})_i$-module on $\mathrm{Hom}\,(M,N)$ such that $\mathrm{Hom}\,(M,N) \otimes M \to N$ is $U_q(\mathfrak{g})_i$-linear. Namely for $x \in U_q(\mathfrak{g})_i$ with $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$, x acts on $\psi \in \mathrm{Hom}\,(M,N)$ by $x_{(1)}\psi a(x_{(2)})$.

Recall that an element ψ of $\mathrm{Hom}\,(M,N)$ is called locally $U_q(\mathfrak{g})_i$-finite, if it is contained in an integrable $U_q(\mathfrak{g})_i$-submodule.

Lemma B.2. *Let M and N be integrable $U_q(\mathfrak{g})_i$-modules. Assume that a weight vector $\psi \in \mathrm{Hom}\,(M,N)$ satisfies*

$$f_i^{n+1}\psi = 0 \quad \textit{for some } n \geq 0.$$

Then ψ is locally $U_q(\mathfrak{g})_i$-finite.

Proof. Assume $t_i\psi = q_i^m\psi$. It is enough to show

$$e_i^s\psi = 0. \tag{B.1}$$

Here $s = \max(n - m + 1, 0)$. In order to see this, we may assume that M is finite-dimensional. Replacing N with the $U_q(\mathfrak{g})_i$-module generated by $\psi(M)$, we may assume that N is also finite-dimensional. Hence $\mathrm{Hom}\,(M,N)$ is finite-dimensional and hence integrable. In this case it is a well-known fact that $f_i^{n+1}\psi = 0$ implies (B.1). □

Proof of Proposition B.1.. Let $\mathrm{ad} : \overline{U}_q(\mathfrak{g}) \to \mathrm{End}\,(\overline{U}_q(\mathfrak{g}))$ be a $\overline{U}_q(\mathfrak{g})$-module structure on $\overline{U}_q(\mathfrak{g})$ such that the multiplication $\overline{U}_q(\mathfrak{g}) \otimes M \to M$ is $\overline{U}_q(\mathfrak{g})_q$-linear. We have

$$\mathrm{ad}\,(t_i)(a) = t_i a t_i^{-1} \tag{B.2}$$

$$\mathrm{ad}\,(e_i)(a) = e_i a - t_i^{-1} a t_i e_i \tag{B.3}$$

$$\mathrm{ad}\,(f_i)(a) = [f_i, a] t_i^{-1} \tag{B.4}$$

for $a \in \overline{U}_q(\mathfrak{g})$. Let $X : \overline{U}_q(\mathfrak{g}) \to \mathrm{End}(M)$ be the homomorphism given by the $\overline{U}_q(\mathfrak{g})$-module structure on M. Let $i \neq j$. Since $[f_i, e_j] = 0$, $f_i X(e_j) = 0$. Since $X(e_j)$ has weight $\langle h_i, \alpha_j \rangle$ with respect to $U_q(\mathfrak{g})_i$, the preceding lemma implies

$$e_i^b X(e_j) = 0, \tag{B.5}$$

where $b = 1 - \langle h_i, \alpha_j \rangle$. On the other hand

$$\begin{aligned} e_i^b X(e_j) &= X(\mathrm{ad}\,(e_i^b) e_j) \\ &= X\left(\sum_{k=0}^{b} (-1)^k q_i^{-k(b-k)} \begin{bmatrix} b \\ k \end{bmatrix}_i e_i^{b-k} t_i^{-k} e_j (t_i e_i)^k \right) \\ &= X\left(\sum_{k=0}^{b} (-1)^k \begin{bmatrix} b \\ k \end{bmatrix}_i e_i^{b-k} e_j e_i^k \right). \end{aligned}$$

This along with (B.5) gives the Serre relation

$$X\left(\sum_{k=0}^{b} (-1)^k \begin{bmatrix} b \\ k \end{bmatrix}_i e_i^{b-k} e_j e_i^k \right) = 0.$$

By applying the automorphism $e_i \mapsto f_i$, $f_i \mapsto e_i$ $q^h \mapsto q^{-h} (h \in P^*)$ of $\overline{U}_q(\mathfrak{g})$, we obtain the other Serre relations

$$X\left(\sum_{k=0}^{b} (-1)^k \begin{bmatrix} b \\ k \end{bmatrix}_i f_i^{b-k} f_j f_i^k \right) = 0.$$

Appendix C. Two-point function for $D_{n+1}^{(2)}$

In this appendix we will sketch the calculation for level 1 $D_{n+1}^{(2)}$, of the two-point function $\Psi(z_1/z_2) = \langle \Lambda_n | \Phi_{\Lambda_n}^{\Lambda_n V_2}(z_2) \Phi_{\Lambda_n}^{\Lambda_n V_1}(z_1) | \Lambda_n \rangle$, for the intertwiner $\Phi_\lambda^{\mu V}(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z$, by solving the corresponding q-KZ equation it must satisfy. The corresponding calculations for the other cases in this paper have been done in [IIJMNT] and [DO]. For the theoretical background the appendix in [IIJMNT] should be consulted. To conform with their conventions, we will use here the upper global base and corresponding coproduct Δ_+, in contrast to the main text of this paper.

Recall the total order $\succ$ on the index set J defined in (5.7.1). Extend the natural definition of minus on $J \setminus \{\phi\}$ to all of J by defining $-\phi = \phi$. Let

$$\begin{aligned}
&\overline{j} = j, \qquad \overline{-j} = 2n+1-j, \qquad j = 1,\dots,n \\
&\overline{0} = n, \qquad \overline{\phi} = 2n.
\end{aligned}$$

Denote, as usual, by E_{jk} the matrix acting on $\{v_j\}_{j\in J}$ as $E_{jk}v_i = \delta_{ki}v_j$. The R-matrix $\overline{R}(z)$ with normalization $\overline{R}(z)v_1 \otimes v_1 = v_1 \otimes v_1$ is then given by

$$\begin{aligned}
\overline{R}(z) = &\sum_{i\neq 0,\phi} E_{ii}\otimes E_{ii} + \frac{q^2(1-z^2)}{(1-q^4z^2)} \sum_{i\neq j,-j} E_{ii}\otimes E_{jj} \\
&+ \frac{(1-q^4)}{(1-q^4z^2)}\Big(\sum_{i\succ j, i\neq -j} z^{\alpha_{ij}} E_{ij}\otimes E_{ji} + z^2 \sum_{i\succ j, i\neq -j} z^{-\alpha_{ij}} E_{ij}\otimes E_{ji}\Big) \\
&+ \sum a_{ij}(z) E_{ij}\otimes E_{-i,-j}
\end{aligned}$$

where

$$\alpha_{ij} = \begin{cases} 1 & : \text{if } i=\phi \text{ or } j=\phi, \\ 0 & : \text{otherwise}, \end{cases}$$

$$a_{ij}(z) = \begin{cases} (1-z^2)(q^4-\xi^2z^2) + \delta_{i,-i}(1-q^2)(q^2+z^2)(1-\xi^2z^2) & : i=j, \\ (1-q^4)\{z^{\alpha_{ij}}(z^2-1)s_{ij}(-q^2)^{\overline{j}-\overline{i}} + \delta_{i,-j}(1-\xi^2z^2)\} & : i\succ j, \\ (1-q^4)z^2\{\xi^2 z^{-\alpha_{ij}}(z^2-1)s_{ij}(-q^2)^{\overline{j}-\overline{i}} + \delta_{i,-j}(1-\xi^2z^2)\} & : i\prec j, \end{cases}$$

and

$$\begin{aligned}
s_{i0} &= -\frac{1}{[2]}\mathrm{sgn}(i) \quad (i\neq\phi), & s_{0j} &= -[2]\mathrm{sgn}(j) \quad (j\neq\phi), \\
s_{i\phi} &= \frac{1}{[2]}\mathrm{sgn}(i) \quad (i\neq 0), & s_{\phi j} &= [2]\mathrm{sgn}(j) \quad (j\neq 0), \\
s_{0\phi} &= s_{\phi 0} = -1, & s_{ij} &= \mathrm{sgn}(i)\,\mathrm{sgn}(j)(i,j\neq 0,\phi).
\end{aligned}$$

Also we have $\xi^2 = q^{4n}$. The expression for $\overline{R}(z)$ is given in [J] in a different basis.

Let $\{v_j^*\}_{j\in J}$ be the canonical dual basis of the upper global base. The following isomorphism of $U_q(\mathfrak{g})$-modules

$$\begin{aligned}
C : V_{\xi^{-1}z} &\longrightarrow (V_z)^{*\,a} \\
v_j &\mapsto \mathrm{sgn}(j)\,(-q^2)^{\overline{j}-1}v_{-j}^* \quad (j\in J/\{0,\phi\}) \\
v_0 &\mapsto -\frac{1}{[2]}(-q^2)^{\overline{0}-1}v_0^* \\
v_\phi &\mapsto \frac{1}{[2]}\xi^{-1}(-q^2)^{\overline{\phi}-1}v_\phi^*
\end{aligned}$$

gives rise to crossing-symmetry for the R-matrix

$$\left(R^{-1}(z)\right)^{t_1} = \beta(z)(C \otimes 1)R(z\xi^{-1})(C \otimes 1)^{-1}, \tag{C.1}$$

with

$$\beta(z) = q^{-4}\frac{(1-z^2)(1-q^{-4n+4}z^2)}{(1-q^{-4n}z^2)(1-q^{-4}z^2)}. \tag{C.2}$$

The image $R^+(z_1/z_2) = \pi_{V_{z_1}} \otimes \pi_{V_{z_2}}(\mathcal{R}')$ of the modified universal R-matrix $\mathcal{R}'$ also satisfies (C.1) with $z = z_1/z_2$ and $\beta(z) = 1$. Therefore we have

$$R^+(z) = q^{-2}\frac{(q^4z^2;\xi^4)_\infty(\xi^2z^2;\xi^4)_\infty^2(q^{-4}\xi^4z^2;\xi^4)_\infty}{(z^2;\xi^4)_\infty(q^{-4}\xi^2z^2;\xi^4)_\infty(q^4\xi^2z^2;\xi^4)_\infty(\xi^4z^2;\xi^4)_\infty}\overline{R}(z).$$

The two-point function satisfies the q-KZ equation

$$\Psi(q^{2(h^\vee+k)}z) = R^+(q^{2(h^\vee+k)}z)(q^{-\phi}\otimes 1)\Psi(z), \tag{C.3}$$

where $k = 1$ is the level and $\phi = 2\Lambda_n^{\mathrm{cl}} + 2\rho^{\mathrm{cl}}$ and, as a consequence, also

$$(\pi_{V_{z_1}} \otimes \pi_{V_{z_2}})\Delta'(e_i)^{\langle h_i,\Lambda_n\rangle+1}\Psi(z) = 0, \qquad \text{wt } \Psi(z) = 0. \tag{C.4}$$

It can be shown that

$$\begin{aligned} w(z) =& (1+z^2q^2\xi^2)v_0\otimes v_0 + q[2](-q^2)^n z v_\phi \otimes v_\phi \\ & - q\sum_{i=1}^n (-q^2)^{n-i}(v_i\otimes v_{-i} + z^2q^{4i-2}v_{-i}\otimes v_i) \end{aligned} \tag{C.5}$$

solves (C.4) and satisfies

$$\overline{R}(q^2\xi^2z)(q^{-\phi}\otimes 1)w(z) = q^2\frac{(1-q^4\xi^2z^2)(1-\xi^4z^2)}{(1-q^8\xi^4z^2)(1-q^4\xi^6z^2)}w(q^2\xi^2z). \tag{C.6}$$

Letting $\Psi(z) = \phi(z)w(z)$ and substituting (C.6) into (C.3) one gets a scalar q-difference equation for $\phi(z)$ which can be solved to obtain the final result

$$\Psi(z) = \frac{(q^4\xi^4z^2;\xi^4)_\infty(\xi^6z^2;\xi^4)_\infty}{(q^4\xi^2z^2;\xi^4)_\infty(\xi^4z^2;\xi^4)_\infty}w(z). \tag{C.7}$$

Appendix D. The limit $q \to 1$ for the $U_q(A_{2n}^{(2)})$ Fock space

In this appendix we will show how to recover the known classical (i.e. at $q = 1$) Fock space $\mathcal{F}_{\text{class}}$ for $\mathfrak{g} = A_{2n}^{(2)}$ at level 1. This involves reduction of the Fock space $\mathcal{F}$ defined for generic q by means of an invariant inner product on $\mathcal{F}$. To facilitate the discussion we shall make a transcription from the semi-infinite wedge description of $\mathcal{F}$ to one involving Young diagrams or, synonymously, partitions (the so-called "combinatorial description").

Define the following subspace of V_{aff}:

$$V_{\text{aff}}^{+} = \left(z^{-1}\mathbb{Q}[z^{-1}] \otimes V\right) \oplus \mathbb{Q}\langle v_{-1}, \ldots, v_{-n}, v_0\rangle. \tag{D.1}$$

In any normally ordered pure wedge in $\mathcal{F}$ it is clear that only bases in V_{aff}^{+} appear as components. Recall the single-valued function l on B_{aff} in (5.3.3). To the normally ordered pure wedge $u = G(u_1) \wedge G(u_2) \wedge \cdots \wedge G(u_k) \wedge v_0 \wedge v_0 \wedge \cdots$ let us associate the sequence $Y(u) = [-l(u_1), -l(u_2), \ldots, -l(u_k), 0, 0, \ldots]$, whose tail of zeros we shall ignore. Now, $-l$ takes non-negative values on V_{aff}^{+}. Also, the sequence $Y(u)$ is non-increasing because of the normal-ordering rules. Furthermore, the only integers allowed to repeat belong to $h\mathbb{N}$, where $h = 2n + 1$, because of the rule $v_i \wedge v_i = 0$ if $i \neq 0$. Thereby we have the identification

$$\mathcal{F} \simeq \mathbb{Q}(q)\langle Y\rangle_{Y \in \mathrm{DP}_h}, \tag{D.2}$$

where DP_k is the set of Young diagrams whose rows are allowed to repeat only if their length is $0 \mod k$. In this notation, DP_∞ is the set of Young diagrams with *no* repeating rows, i.e., the set of Distinct Partitions.

The action of $U_q(\mathfrak{g})$ on $\mathcal{F}$ can be transcribed to the Young diagram setting. The generators t_i act diagonally, of course, while f_i (respectively e_i) act by adding (removing) one box in the following manner. Let the Young diagram Y be denoted by $[y_1, \ldots, y_m]$. For $y \in \mathbb{Z}_{>0}$, let $\alpha_Y(y)$ denote the number of occurences of y in Y. Define the functions β_i for $i = 0, 1, \ldots, n$ by

$$\beta_0(y) = \begin{cases} \pm 4 & : y \in h\mathbb{Z} \pm n \\ 0 & : \text{otherwise} \end{cases}$$

$$\beta_i(y) = \begin{cases} \pm 2 & : y \in h\mathbb{Z} \mp n \pm (i-1) \\ \mp 2 & : y \in h\mathbb{Z} \mp n \pm i \\ 0 & : \text{otherwise} \end{cases} \qquad (i = 1, \ldots, n-1)$$

$$\beta_n(y) = \begin{cases} \pm 2 & : y \in h\mathbb{Z} \mp 1 \\ 0 & : \text{otherwise} \end{cases}$$

Then the action of $U_q(\mathfrak{g})$ on Y is given explicitly by

$$
\begin{aligned}
t_i \cdot Y &= q^{\sum_j \beta_i(y_j)+\delta_{i,n}}\, Y \\
f_i \cdot Y &= \sum_{\substack{y_k \in h\mathbb{N}+n\pm i \\ y_{k-1}\neq y_k+1}} q^{\sum_{j>k}\beta_i(y_j)}\, [y_1, \dots, y_k+1, \dots, y_m] \qquad\qquad (i \neq n) \\
f_n \cdot Y &= \sum_{\substack{y_k \in h\mathbb{N}-1 \\ y_{k-1}\neq y_k+1}} q^{\sum_{j>k}\beta_n(y_j)+1}\, [y_1, \dots, y_k+1, \dots, y_m] \\
&\quad + \sum_{\substack{y_k \in h\mathbb{N} \\ y_{k-1}\neq y_k+1, y_k}} q^{\sum_{j>k}\beta_n(y_j)} \left(1-(-q^2)^{\alpha_Y(y_k)}\right) [y_1, \dots, y_k+1, \dots, y_m] \\
&\quad + \delta(y_m \neq 1)[y_1, \dots, y_m, 1] \\
e_i \cdot Y &= \sum_{\substack{y_k \in h\mathbb{N}+n+1\pm i \\ y_{k+1}\neq y_k-1}} q^{-\sum_{j<k}\beta_i(y_j)}\, [y_1, \dots, y_k-1, \dots, y_m] \qquad\qquad (i \neq n) \\
e_n \cdot Y &= \sum_{\substack{y_k \in h\mathbb{N}+1 \\ y_{k+1}\neq y_k-1}} q^{-\sum_{j<k}\beta_n(y_j)}\, [y_1, \dots, y_k-1, \dots, y_m] \\
&\quad + \sum_{\substack{y_k \in h\mathbb{N} \\ y_{k+1}\neq y_k-1, y_k}} q^{-\sum_{j<k}\beta_n(y_j)-1} \left(1-(-q^2)^{\alpha_Y(y_k)}\right) [y_1, \dots, y_k-1, \dots, y_m].
\end{aligned}
$$

Note that all Young diagrams appearing on the right-hand side belong to DP_h. In other words, the corresponding pure wedges are already normally ordered. The factors $\left(1-(-q^2)^{\alpha_Y(y_k)}\right)$ come from normal ordering and summing up Young diagrams which arise when Y has repeated rows. Note also that the vacuum vector is the empty Young diagram $\emptyset$ and $f_n \cdot \emptyset = [1]$. This combinatorial description is in the same spirit as that for $U_q(A_n^{(1)})$ given in [MM].

Let us now introduce an inner product $(\ ,\ )$ on $\mathcal{F}$. We shall require that the normally ordered pure wedges, or equivalently Young diagrams in DP_h, form an orthogonal basis with respect to $(\ ,\ )$. We shall also require that with respect to $(\ ,\ )$ the adjoints of the generators satisfy

$$
\begin{aligned}
f_i^\dagger &= q_i\, e_i\, t_i, \\
e_i^\dagger &= q_i\, f_i\, t_i^{-1}, \\
t_i^\dagger &= t_i.
\end{aligned}
$$

These conditions are natural for a $U_q(\mathfrak{g})$-module V because then on the module $V \otimes V$ with induced inner product given by $(v_1 \otimes v_2, u_1 \otimes u_2) = (v_1, u_1)(v_2, u_2)$ we have $\Delta(f_i)^\dagger = q_i \Delta(e_i)\Delta(t_i)$, etc. Using the explicit description of the $U_q(\mathfrak{g})$ action on $\mathcal{F}$ one can show that the squared norm of an arbitrary Young diagram Y in DP_h is given by

$$||Y||^2 = (Y,Y) = \prod_{y \in h\mathbb{Z}_{>0}} \prod_{i=1}^{\alpha_Y(y)} \left(1 - (-q^2)^i\right). \tag{D.3}$$

From calculations for small k we conjecture that the boson operators satisfy

$$B_{-k}^\dagger = B_k.$$

As at the end of §4.3, we denote by $\mathcal{F}^{\mathbb{Q}}$ the $\mathbb{Q}[q,q^{-1}]$-vector space generated by the pure wedges. Set $\mathcal{F}_1 = \mathcal{F}^{\mathbb{Q}}/(q-1)\mathcal{F}^{\mathbb{Q}}$. Then the action of $U_q(\mathfrak{g})$ on $\mathcal{F}$ induces an action of $U_q(\mathfrak{g})$ on $\mathcal{F}_1$. The inner product $(\ ,\ )$ on $\mathcal{F}^{\mathbb{Q}}$ induces a $\mathbb{Q}$-valued inner product on $\mathcal{F}_1$, which we also denote by $(\ ,\ )$. The adjoint of operators in $\mathfrak{g}$ is then given by $e_i^\dagger = f_i$, $f_i^\dagger = e_i$ and $h_i^\dagger = h_i$. Define the subspace $\mathcal{F}_0 = \{u \in \mathcal{F}_1 : (u, \mathcal{F}_1) = 0\}$. The reduced Fock space $\mathcal{F}_{\mathrm{red}} = \mathcal{F}_1/\mathcal{F}_0$ is a $U(\mathfrak{g})$-module. From (D.3) we note that $\mathcal{F}_0$ is the $\mathbb{Q}$-span of Young diagrams with *some* repeated rows. It follows that $\mathcal{F}_{\mathrm{red}}$ is the $\mathbb{Q}$-span of Young diagrams in DP_∞. This is isomorphic to the well-known classical Fock space $\mathcal{F}_{\mathrm{class}} \simeq \mathbb{Q}[x_k]_{k \in \mathbb{N}_{\mathrm{odd}}}$ [KKLW], [DJKM]. In fact, the action of the generators on $\mathcal{F}_{\mathrm{red}}$ and at $q = 1$ reduces to a known classical action [JY]. Furthermore we recover the known decomposition of $\mathcal{F}_{\mathrm{class}} \simeq \mathbb{Q}[x_k]_{k \in \mathbb{N}_{\mathrm{odd}} \setminus h\mathbb{N}} \otimes \mathbb{Q}[x_{hk}]_{k \in \mathbb{N}_{\mathrm{odd}}}$ as a $U(\mathfrak{g}) \otimes \mathbb{Q}[H_-]$-module. Here we identify bosons $x_{hk} \sim B_{-k}$ for $k \in \mathbb{N}_{\mathrm{odd}}$. The even boson commutators γ_k for $k \in \mathbb{N}_{\mathrm{even}}$ have a pole at $q = 1$. After appropriately rescaling we find that such B_{-k} act as 0 on $\mathcal{F}_{\mathrm{red}}$ at $q = 1$.

In most of the cases considered in this paper, the boson commutator γ_k has a pole at $q = 1$ for some k. We take this to indicate that similar Fock space reductions to the one considered in this Appendix are necessary to recover any known classical Fock spaces.

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