K-types and Singular Spectrum

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Introduction.

In "Functions on the Shilov boundary of the generalized half plane" in the same volume, we constructed unitary representations T_{k_1,k_2} of the Lie group $Sp(n,\mathbb{R})$. T_{k_1,k_2} is realized on the space of functions $u(x)$ on the space $S(n)$ of $n \times n$ symmetric matrices such that the Fourier transform

$$\hat{u}(\xi) = \int_{S(n)} u(x)\, e^{-2i\pi \mathrm{Tr}\,\xi x}\, dx$$

is supported on the set of symmetric matrices of signature (k_1,k_2). On the same time, we observed that the K-types of T_{k_1,k_2} are distributed on a cone closely connected with (k_1,k_2).

Why there is a relation between K-types and the Fourier transform?

More generally, let us consider G a real semi-simple Lie group, K its maximal compact subgroup, P a parabolic of G such that G/P is a symmetric space. Then $P = MN$ where N is abelian. We can then write, up to a set of measure zero, $G = N^- P$ where N^- is the nilpotent subgroup opposite to N. We hence can consider the compact manifold $G/P = X$ as an homogeneous space under K, or as an "almost" homogeneous space under $\mathfrak{n}^-$, the Lie algebra of N^-. Hence we can analyze $L^2(K/M\cap P) = L^2(X)$ either by the Fourier integral $\hat{u}(\xi)$ on $\mathfrak{n}$

$$\hat{u}(\xi) = \int_{\mathfrak{n}^-} u(x) e^{-2i\pi\langle \xi, x\rangle}\, dx$$

(we identify $(\mathfrak{n}^-)^*$ with $\mathfrak{n}$ by the killing form), or via the Fourier series expansion $u(k) = \Sigma\, u_\lambda(k)$ of u with respect to the finite dimensional representations of K.

Let us consider the singular spectrum SSu of the generalized function u. This subset of T^*X indicates the directions where u can be continued as an holomorphic functions on the complexification X_C of X.

If the Fourier transform $\hat{u}(\xi)$ is supported on a closed cone Γ, then the singular spectrum of $u(x)$ is contained in the subset $N_- \times \sqrt{-1}\,\Gamma$ of $\sqrt{-1}$ times the cotangent bundle $N_- \times \eta = T^*N_-$.

In this note, we prove a similar relation between the K-types appearing in the expansion of u and the singular spectrum of u.

The determination by Birgit Speh of the K-types of solutions of mass zero equations on the Minkowski-space considered as an homogeneous space under $U(2,2)$ was our first indication that there was a strong connection between the asymptotic behavior of the K-types of a given representation T of G and the geometric realization of T as acting on functions on the Minkowski space solutions of differential equations.

These questions are also in strong relation with the orbit method: Let G be a semi-simple Lie group and K a compact subgroup of G. Let $\mathcal{O}$ an orbit of G in $\mathfrak{g}^*$ and $\rho_{\mathcal{O}}$ the representation of G which can be in numerous cases associated to $\mathcal{O}$. Then the asymptotic directions to the K-types occurring in $\rho_{\mathcal{O}}$ should be the projection of the asymptotic cone of the orbit $\mathcal{O}$. In particular we prove that the asymptotic support of the K-types of an arbitrary Harish-Chandra module is given by the projection on k^* of nilpotent orbits of G in $\mathfrak{g}^*$, hence are only among a finite number of possibilities.

We thank Dan Barbasch for several discussions on asymptotic directions of the orbits of G in $\mathfrak{g}^*$ and Sigurdur Helgason for discussions on asymptotic estimates of the spherical functions.

1. Let K be a connected compact Lie group and H a Cartan subgroup of K. Let $\mathfrak{k}$ and $\mathfrak{h}$ be the Lie algebras of K and H. We denote by $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ their complexifications. We fix a K-invariant metric on $\mathfrak{k}$, by which we identify $\mathfrak{k}$ and $\mathfrak{h}$ with their dual vector spaces. Let $(\ ,\ )$ be the hermitian metric on $\mathfrak{k}_{\mathbb{C}}$ induced by this metric on $\mathfrak{k}$.

For any $\alpha \in \sqrt{-1}\,\mathfrak{h}^*$, we define

$$\mathfrak{k}_\alpha = \{X \in \mathfrak{k}_{\mathbb{C}} \, ; \, [H,X] = \alpha(H)X \, , \text{ for } H \in \mathfrak{h}\} .$$

Then $\dim \mathfrak{k}_\alpha = 0$ or 1, for $\alpha \neq 0$, and $\mathfrak{k}_\alpha = \mathfrak{h}_{\mathbb{C}}$ for $\alpha = 0$.

Let us denote $\Delta = \{\alpha \in \sqrt{-1}\,\mathfrak{h}^* - \{0\}$ such that $\mathfrak{k}_\alpha \neq 0\}$ the set of roots, and let Δ^+ be the set of positive roots with respect to some ordering. We define $\mathfrak{n} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{k}_\alpha$. Then $\mathfrak{h}_{\mathbb{C}} + \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{k}_{\mathbb{C}}$.

We have

(1.1) $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus \sqrt{-1}\,\mathfrak{h} \oplus \mathfrak{n}$, considered as a sum of real vector spaces.

Let L be the lattice of $\sqrt{-1}\,\mathfrak{h}^*$ which comes from the character group of H, i.e. the set of $\alpha \in \sqrt{-1}\,\mathfrak{h}^*$ such that $\chi(e^H) = e^{\alpha(H)}$ $(H \in \mathfrak{h})$ defines a character on H. Let C be the Weyl chamber:

$$C = \{\lambda \in \sqrt{-1}\,\mathfrak{h}^* \, ; \, \langle\lambda,\alpha\rangle > 0 \, , \ \forall\, \alpha \in \Delta^+\} .$$

Then the set $\hat{K}$ of irreducible representations of K is isomorphic to $L \cap \overline{C}$ by the highest weight. For $\lambda \in L \cap \overline{C}$, let V_λ denote the irreducible representation with highest weight λ.

2. Let dk be the Haar measure on K normalized by $\int_K dk = 1$.

Then $L^2(K)$, considered as a $K \times K$ module by left and right translations, has the following decomposition into irreducible components:

(2.1) $$L^2(K) = \bigoplus_{\lambda \in L \cap \overline{C}} (V_\lambda \otimes V_\lambda^*)$$, the element $v \otimes f$ of $V_\lambda \otimes V_\lambda^*$ being identified with the real analytic function $k \longrightarrow f(k^{-1} \cdot v)$ on K

For φ_λ an element of $V_\lambda \otimes V_\lambda^*$, we denote by $\|\varphi_\lambda\|$ the norm induced by $L^2(K)$.

<u>2.2 Theorem [1],[10]:</u>

Let φ be a function on K and develop $\varphi = \sum_{\lambda \in L \cap \overline{C}} \varphi_\lambda$, then

1) $\varphi = \Sigma \varphi_\lambda$ is a real analytic function on K if and only if there are positive constants C and δ such that

$$\|\varphi_\lambda\| \leq Ce^{-\delta|\lambda|} .$$

2) $\varphi = \Sigma \varphi_\lambda$ is a C^∞-function, if and only if, for any positive integer m, there exists a positive number C_m such that

$$\|\varphi_\lambda\| \leq C_m(1 + |\lambda|)^{-m} .$$

3) $\varphi = \Sigma \varphi_\lambda$ is contained in $L^2(K)$ if and only if

$$\Sigma \|\varphi_\lambda\|^2 < \infty .$$

4) $\varphi = \Sigma \varphi_\lambda$ is a distribution if and only if there are positive numbers m and C such that

$$\|\varphi_\lambda\| \leq C(1 + |\lambda|)^m$$

5) $\varphi = \Sigma \varphi_\lambda$ is a hyperfunction if and only if for any $\varepsilon > 0$, there exist $C_\varepsilon > 0$ such that

$$\|\varphi_\lambda\| \leq C_\varepsilon \ e^{\varepsilon|\lambda|}$$

(For the theory of microfunctions, we refer to [2], [3], [4].)

3. Let TK and T^*K be the tangent and cotangent vector bundles of K. We shall identify TK with $K \times \mathfrak{k}$ and T^*K with $K \times \mathfrak{k}^*$ by left translations. Therefore, the right (resp. left) translation of $k_0 \in K$ on K give rise to the transformation on $TK = K \times \mathfrak{k}$: $(k,X) \longrightarrow (kk_0,X)$ (resp: $(k,X) \longrightarrow (k_0k\ ,\ \mathrm{Ad}\ k_0 \cdot X)$) and on $T^*K = K \times \mathfrak{k}^*$: $(k,\xi) \longrightarrow (kk_0,\xi)$ (resp: $(k,\xi) \longrightarrow (k_0k\ ,\ \mathrm{Ad}^*\ k_0 \cdot \xi)$). Here Ad^* is the coadjoint representation.

Let T be a closed cone of $\sqrt{-1}\,\mathfrak{h}^*$ contained in $\overline{C}$ and let $\varphi = \sum_{\lambda \in T \cap L} \varphi_\lambda$ be a hyperfunction on K such that each φ_λ is a highest weight vector of highest weight λ with respect to the left action on $L^2(K)$. Then we have

$$X \cdot \varphi_\lambda = 0 \qquad \text{for } x \in \mathfrak{n}$$

$$X \cdot \varphi_\lambda = \lambda(X)\varphi_\lambda \qquad \text{for } x \in \mathfrak{h}$$

for the left action.

Let $K_{\mathbb{C}}$ be a complexification of K, then for some $\rho > 0$ the map f_ρ defined on

$$U_\rho = \{(k,H,X) \in K \times \mathfrak{h} \times \mathfrak{n}\ ;\ |H| < \rho\ ,\ |X| < \rho\}$$

by $f_\rho(k,H,X) = \exp(\sqrt{-1}\,H + X)\,k$ is an isomorphism from U_ρ onto an open neighborhood Ω_ρ of K in $K_{\mathbb{C}}$ as follows from (1.1).

Let us extend the function φ_λ holomorphically on $K_{\mathbb{C}}$. We fix H and X and we consider the function φ_λ on the translate $\exp(\sqrt{-1}\,H + X) \cdot K$ of K in $K_{\mathbb{C}}$. Then

$$\varphi_\lambda(\exp\ (\sqrt{-1}\ H + X)\cdot k) = e^{-<\lambda,\sqrt{-1}\ H>}\ \varphi_\lambda(k)\ .$$

Hence considered as a function of k, we have:

$$\int |\varphi_\lambda(\exp\ (\sqrt{-1}\ H + X)\ k|^2\ dk = e^{-2<\lambda,\sqrt{-1}\ H>}\ \|\varphi_\lambda\|^2$$

We define $T^0 = \{H \in \mathfrak{h}\ ;\ <\lambda,\sqrt{-1}\ H> > 0, \forall\lambda \in T \subset \sqrt{-1}\ \mathfrak{h}^*\}$ and $T^0_\rho = \{(k,H,X)\ ;\ H \in T^0\} \cap U_\rho$. Then $\Sigma\ \varphi_\lambda((\exp \sqrt{-1}\ H + X)\ k)$ converges on T^0_ρ : φ_λ being an hyperfunction satisfies $\|\varphi_\lambda\| \leq C_\varepsilon e^{\varepsilon|\lambda|}$ and hence $e^{-<\lambda,\sqrt{-1}\ H>}\ \|\varphi_\lambda\|$ has exponential decay for $H \in T^0$. Hence

$$\varphi(\exp\ (\sqrt{-1}\ H + X)k) = \Sigma\ \varphi_\lambda(\exp\ (\sqrt{-1}\ H + X)k)$$

is a holomorphic function defined on $f_\rho(T^0_\rho)$. This domain is an infinitesimal neighborhood of $\{(k,\sqrt{-1}\ X \in \sqrt{-1}\ TK$, with $\sqrt{-1}\ X \in (\sqrt{-1}\ T^0 + \mathfrak{n} + \mathfrak{k}) \cap \sqrt{-1}\ \mathfrak{k}\}$. Let us consider $\mathfrak{h}^\perp$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{k}$. We have $(\mathfrak{h}^\perp)_{\mathbb{C}} = \mathfrak{h}^\perp \oplus \mathfrak{n}$ as a sum of real vector subspaces. Hence

$$(\sqrt{-1}\ T^0 + \mathfrak{n} + \mathfrak{k}) \cap \sqrt{-1}\ \mathfrak{k} = \sqrt{-1}\ T^0 + \sqrt{-1}\ \mathfrak{h}^\perp\ .$$

Hence φ converges on an infinitesimal neighborhood of (k,X), for $X \in \sqrt{-1}\ T^0 + \sqrt{-1}\ \mathfrak{h}^\perp$. By definition, the singular spectrum $SS\ \varphi$ of φ is then contained in the dual of this neighborhood in $\sqrt{-1}\ T^*K$.

We imbed $\mathfrak{h}^*$ in $\mathfrak{k}^*$ according to the decomposition $\mathfrak{k}^* = \mathfrak{h}^* \oplus (\mathfrak{h}^\perp)^*$. Then it is immediate from the definition of T^0 that the dual cone of $T^0 + \mathfrak{h}^\perp$ in $\mathfrak{k}^*$ is the convex hull of $\sqrt{-1}\ T$. Thus we obtain

$$SS\ \varphi \subset \{(k,\xi)\ ;\ -\xi \in \text{convex hull of}\ \ T\}\ .$$

3.1 Proposition: Let T be a closed cone in $\sqrt{-1}\,\mathfrak{h}^* \cap \overline{C}$. Suppose that $\varphi = \sum_{\lambda\in T} \varphi_\lambda$ is an hyperfunction on K and all the φ_λ are highest weight vectors with respect to the left action. Then $SS\,\varphi \subset -K\cdot T$, where $T \subset \sqrt{-1}\,\mathfrak{k}^* = \sqrt{-1}\,T_eK^*$ and K acts by the right action on T^*K.

Proof: If T is convex, this follows from the preceding discussion. In the general case, for any disjoint family $\{T_j\}_{1\le j\le N}$ of closed convex sets such that $T \subset \cup T_j$, we have $\varphi = \sum \varphi_j$ with $\varphi_j = \sum_{\lambda\in T_j} \varphi_\lambda$ and $SS\,\varphi_j \subset -K\cdot T_j$. Therefore

$$SS\,\varphi \subset \cup(-K\cdot T_j)$$

$$= -K\cdot(\cup T_j)\ .$$

Since $\cup T_j$ can be as close to T as we like, we obtain the result.

4. Singular support and K-types.

Let χ_λ be the character of V_λ, i.e. $\chi_\lambda(k) = tr(\tau_\lambda(k);V_\lambda)$. We know that $\{\chi_\lambda\}$ forms an orthogonal basis of the space of (Ad K)-invariant L^2-functions on K. χ_λ is the unique (Ad K)-invariant functions in $V_\lambda^* \otimes V_\lambda$.

Let us consider the δ-function $\delta(k)$ supported at e characterized by $\int \delta(k)u(k)dk = u(e)$. We have $\delta(k) = \sum_\lambda \varphi_\lambda$, with $\varphi_\lambda \in V_\lambda \otimes V_\lambda^*$. Since $\delta(k)$ is invariant by Ad K, φ_λ is proportional to $\overline{\chi}_\lambda$. Since $(\varphi_\lambda,\overline{\chi}_\lambda) = \int \delta(k)\chi_\lambda(k)dk = \chi_\lambda(e) = \dim V_\lambda$, we obtain $\varphi_\lambda = \dim V_\lambda\,\overline{\chi}_\lambda$, i.e. $\delta = \sum_{\lambda\in L\cap\overline{C}} (\dim V_\lambda)\,\chi_\lambda$.

Let u_λ be the highest weight vector of the representation V_λ, provided with a K-invariant hermitian inner product $(\ ,\ )$. We set $\psi_\lambda(k) = (\tau_\lambda(k^{-1})u_\lambda,u_\lambda)$. We shall calculate $\int \psi_\lambda(k'kk'^{-1})dk'$. For any u and v $\int (\tau_\lambda(k'k^{-1}k'^{-1})u,v)dk'$ is an Ad K-invariant function contained in $V_\lambda \otimes V_\lambda^*$ and hence is proportinal to $\overline{\chi}_\lambda$.

Hence there exist a constant c such that

$$\int (\tau_\lambda(k'k^{-1}k'^{-1})u,v)\,dk' = c\,(u,v)\,\overline{\chi}_\lambda(k) \ .$$

Setting $k = e$, we have $c = \dfrac{1}{\overline{\chi}_\lambda(e)} = \dfrac{1}{\dim V_\lambda}$ i.e.

$$\int (\tau_\lambda(k'k^{-1}k'^{-1})u,v)\,dk' = \frac{1}{\dim V_\lambda}\,(u,v)\,\overline{\chi}_\lambda(k) \ .$$

If we normalize u_λ by $\|u_\lambda\| = 1$, we have

$$(4.1) \qquad \int_K \psi_\lambda(k'kk'^{-1})\,dk' = \frac{1}{\dim V_\lambda}\,\overline{\chi}_\lambda(k) \ .$$

We define for a cone T in $\overline{C}$

$$\psi_T = \sum_{\lambda\in T\cap L} (\dim V_\lambda)^2\,\psi_\lambda$$

and

$$\delta_T = \sum_{\lambda\in T\cap L} (\dim V_\lambda)\,\overline{\chi}_\lambda \ .$$

By (2.2), ψ_T and δ_T are hyperfunctions. By (4.1), δ_T is obtained from ψ_T by

$$\delta_T(k) = \int \psi_T(k'kk'^{-1})\,dk' \ .$$

Since $SS\,\psi_T$ is contained in $-K\cdot T$ by proposition 3.1, we obtain

$$(4.2) \qquad SS\,\delta_T \subset -\,(K\times K)\cdot T \ .$$

<u>4.3. Lemma</u>: Any $K\times K$ invariant subset of $\sqrt{-1}\,T^*K$ is of the form $-(K\times K)\cdot T$ with T a subset of $\overline{C}$.

<u>Proof</u>: This is equivalent to the classification of Ad K-invariant sets of $\mathfrak{k}$, and it is well known that they are written in the form $\mathrm{Ad}\,K\cdot T$ with $T\subset\overline{C}\subset\sqrt{-1}\,\mathfrak{h}^*$ for a unique T.

Let $u(k)$ and $v(k)$ be two hyperfunctions on K. We define their convolution $u * v$ by $(u * v)(k) = \int_{h \in K} u(kh^{-1})\, v(h) dh$.

We have
$$\dim V_\lambda \, \overline{\chi}_\lambda * u = u \quad \text{for} \quad u \in V_\lambda \otimes V_\lambda^*$$
$$= 0 \quad \text{for} \quad u \in V_{\lambda'} \otimes V_{\lambda'}^*, \quad \lambda \neq \lambda' .$$

Hence for a hyperfunction $\varphi = \sum_{\lambda \in L \cap \overline{C}} \varphi_\lambda$, we have $\delta_T * \varphi = \sum_{\lambda \in L \cap T} \varphi_\lambda$

<u>4.4 Lemma</u>: Let T_1 and T_2 be two closed cones in $\overline{C}$. If $SSu \subset -(K \times K) \cdot T_1$, and if $SSv \subset -(K \times K) \cdot T_2$, then we have
$$SS(u * v) \subset - (K \times K)(T_1 \cap T_2) .$$

<u>Proof</u>: This lemma is easily derived from the behavior of the singular spectrum under integration: The singular spectrum of $u(kh^{-1})$ considered as a hyperfunction on $K \times K$ is contained in
$$\{(k,h;\xi , - Ad^*(kh^{-1})^{-1} \cdot \xi) \; ; \; (kh^{-1};\xi) \in SSu\} .$$

Therefore the singular spectrum of $u(kh^{-1})v(h)$ is contained in
$$\{(k,h;\xi , \xi' - Ad^*(kh^{-1})^{-1} \cdot \xi) \; ; \text{ with } (kh^{-1};\xi) \in SSu , \; (h,\xi') \in SSv\}.$$

Hence the singular spectrum of $\int_K u(kh^{-1}) \, v(h) dh$ is contained in
$$\{(k;\alpha) , \text{ such that there exists a } h \text{ with } (k,h;\alpha,0) \in SS(u(kh^{-1})v(h))\} .$$

This implies $\alpha = \xi \in (Ad^*K)T_1$, $\xi' = Ad^*(kh^{-1})^{-1} \cdot \xi$. Hence $\alpha \in (Ad^*K)T_1 \cap (Ad^*K)T_2 = (Ad^*K)(T_1 \cap T_2)$.

Now, we are ready to prove the following theorem.

<u>4.5 Theorem</u>: Let $\varphi = \Sigma \, \varphi_\lambda$ be a hyperfunction on K . Let T be a closed cone in $\overline{C}$. Then the following conditions are equivalent:

(1) $SS\,\varphi \subset -(K \times K)\cdot T$.

(2) For any closed cone T' in $\overline{C}$ such that $T \cap T' \subset \{0\}$, there are constants $R_{T'} > 0$, and $\varepsilon_{T'} > 0$ such that

$$\|\varphi_\lambda\| \leq R_{T'} e^{-\varepsilon_{T'}|\lambda|} \quad \text{for } \lambda \in T' .$$

Proof: Let us prove first that (2) implies (1). Take T' as in (2), then $\varphi_{T'} = \delta_{T'} * \varphi = \sum_{\lambda \in T'} \varphi_\lambda$ and $\varphi_{\overline{C}-T'} = \delta_{\overline{C}-T'} * \varphi = \sum_{\lambda \notin T'} \varphi_\lambda$. By (2.2) and the hypothesis, $\varphi_{T'}$ is real analytic; by (4.2), (4.4), $SS\,\varphi_{\overline{C}-T'}$ is contained in $-K \times K\,(\overline{C} - T')$. Since we can take $\overline{C} - T'$ as close to T as we like, we obtain (1).

Reciprocally if (1) is satisfied, $SS\,\varphi_{T'} \subset -(K \times K)(T \cap T') = \{0\}$. Hence $\varphi_{T'}$ is a real analytic function. So (2) follows from Theorem (2.2).

Remark: If we employ the wave front set in the C^∞-sense instead of the singular spectrum in condition (1), then condition (2) must be changed to: For any $m > 0$, there is $C_m > 0$ such that

$$\|\varphi_\lambda\| \leq C_m\,(1 + |\lambda|)^{-m} \quad \text{for } \lambda \in T' .$$

5. K-types of induced representations.

Let M be a subgroup of K and $\mathfrak{m}$ the Lie algebra of M . Let X be the homogeneous space K/M . We denote by 0 the coset eM . Then the left action of K induces a surjective map: $\mathfrak{k} \longrightarrow T_0(X)$ whose kernel is $\mathfrak{m}$. Hence T_0^*X is identified with the orthogonal complement $\mathfrak{m}^\perp$ in $\mathfrak{k}^*$.

Let σ be a finite dimensional unitary representation of M in the complex vector space U . We denote by $\mathcal{U}$ the corresponding homogeneous vector bundle $K \times_M U$ over X . Hence the space of section of $\mathcal{U}$ is the space of U-valued functions $u(k)$ on K satisfying

(5.1) $$u(km) = \sigma(m)^{-1}u(k) \quad \text{for} \quad k \in K\ ,\ m \in M\ .$$

The group K acts by left translations on this space. The space of L^2-sections of $\mathcal{U}$ is denoted by $L^2(K/M;\mathcal{U}) = L^2(X,U)$.

The decomposition of $L^2(X,U)$ under K is given by the Frobenius reciprocity law, i.e.:

(5.2) $$L^2(X,U) = \bigoplus_{\lambda\in\hat{K}} V_\lambda \otimes \mathrm{Hom}_M(V_\lambda, U)$$

where $v \otimes f$, for $v \in V_\lambda$, $f \in \mathrm{Hom}_M(V_\lambda,U)$ is identified with the function $(v \otimes f)(g) = f(g^{-1}v)$.

We denote by $W_\lambda = V_\lambda \otimes \mathrm{Hom}_M(V_\lambda,U)$.

We wish to determine what are the asymptotic behavior of the representations of K appearing in $L^2(X,U)$; i.e. what are the representations λ of K such that $\mathrm{Hom}_M(V_\lambda,U) \neq \{0\}$ when $|\lambda| \to \infty$. Consider the singular spectrum of a section u of $\mathcal{U}$ regarded as a U-valued function on K satisfying 5.1. Since $u(k)$ satisfies (5.1), we have:

$$SSu \subset \{(k,\xi);\ (Ad^*k^{-1})\xi \in \sqrt{-1}\ \mathfrak{m}^\perp\}\ .$$

We consider the inclusion $\mathfrak{m} \subset \mathfrak{k}$ and the corresponding map $p: \mathfrak{k}^* \to \mathfrak{m}^*$. The kernel of this map is $\mathfrak{m}^\perp$. We consider the set $(Ad^*K)\mathfrak{m}^\perp$ of orbits intersecting $\mathfrak{m}^\perp$. Let

(5.3) $$\mathfrak{h}^*_{\mathfrak{m}} = \mathfrak{h}^* \cap (Ad^*K)\mathfrak{m}^\perp\ .$$

Then every orbit intersecting $\mathfrak{m}^\perp$ intersects $\mathfrak{h}^*_{\mathfrak{m}}$

<u>5.4 Proposition</u>: For any closed cone T in $\overline{C}$ such that $T \cap \sqrt{-1}\ \mathfrak{h}^*_{\mathfrak{m}} \subset \{0\}$, there exists a constant R_T such that $\mathrm{Hom}_M(V_\lambda,V) = 0$ for $\lambda \in T$, and $|\lambda| \geq R_T$.

<u>Proof</u>: If it is not true, there is a sequence λ_j in T such that

$|\lambda_j|$ tends to infinity, when j tends to infinity, and such that $W_{\lambda_j} \neq \{0\}$. Let us take a vector φ_j in W_{λ_j} normalized by $\|\varphi_j\| = 1$. Take any sequence a_j in $\mathbb{C}$, such that $\Sigma\ |a_j|^2 < \infty$. We consider $u(k) = \Sigma\ a_j\varphi_j$ which belongs to $L^2(X,U)$. We have as u satisfies (5.1) $(SSu) \subset K\cdot(\sqrt{-1}\ \mathfrak{m}^{\perp})$ for the left action of K,

$$\subset (K \times K)(\sqrt{-1}\ \mathfrak{h}^*_{\mathfrak{m}}).$$

But by Theorem (4.5) as $T \cap \sqrt{-1}\ \mathfrak{h}^*_{\mathfrak{m}} = 0$, this would imply that there exist $R > 0$, and $\varepsilon > 0$ with $|a_j| \leq Re^{-\varepsilon|\lambda_j|}$. This cannot be true for any sequence a_j , with $\Sigma\ |a_j|^2 < \infty$, hence we obtain our result.

Remark: Let us consider $\hat{K}$ as a subset of orbits in $\sqrt{-1}\ \mathfrak{k}^*$ by $V_\lambda \longmapsto (Ad^*K)\cdot\lambda$. This is a bijection with the set of integral orbits of K in $\sqrt{-1}\ \mathfrak{k}^*$. Let us consider the projection of the orbit $\mathcal{O}_\lambda = (Ad^*K)\cdot\lambda$ on $\sqrt{-1}\ \mathfrak{m}^*$ with respect to the restriction $p: \sqrt{-1}\ \mathfrak{k}^* \to \sqrt{-1}\ \mathfrak{m}^*$. This set decomposes under M into a union of M-orbits. The "philosophy" of the orbit method would imply that the restriction of V_λ to M decomposes as a sum of representations μ_j of M corresponding to "some" integral orbits of M in $\sqrt{-1}\ \mathfrak{m}^*$ contained in the projection of $\mathcal{O}_\lambda$ on $\sqrt{-1}\ \mathfrak{m}^*$. In particular the λ's of $\hat{K}$ containing a given representation of M corresponds to orbits $\mathcal{O}_\lambda$ intersecting $p^{-1}(B)$ for B a compact subset of $\mathfrak{m}^*$. The asymptotic directions of the corresponding highest weights is $\sqrt{-1}\ \mathfrak{h}^*_{\mathfrak{m}} \cap \overline{C}$. Hence for a cone T such that $T \cap \mathfrak{h}^*_{\mathfrak{m}} = 0$, the set $(Ad^*K)\cdot T \cap p^{-1}(B)$ is a bounded set. Our result gives an "asymptotic" verification of this desired result. (We thank Donald King for discussions of the case $K \to K \times K$ via the diagonal map, i.e. of the case of decomposition of tensor products [5].)

We can reformulate our Theorem 4.5 in the following:

5.5 Theorem: Let $\varphi = \Sigma\ \varphi_\lambda$, $\varphi_\lambda \in W_\lambda$ be a hyperfunction section of $\mathcal{U}$. Let T be a closed cone in $\sqrt{-1}\ \mathfrak{h}^*_{\mathfrak{m}} \cap \overline{C}$. The following conditions are equivalent:

(1) $SS\varphi \subset -K\cdot(Ad^*K\cdot T \cap \sqrt{-1}\,\mathfrak{m}^{\perp})$ when K acts by the left.

(2) For any closed cone T' in $\sqrt{-1}\,\mathfrak{h}^*$ such that $T' \cap T = \{0\}$, there exists $R_{T'}$ and $\varepsilon_{T'}$ such that $\|\varphi_\lambda\| \leq R_{T'} e^{-\varepsilon_{T'}|\lambda|}$ for $\lambda \in T'$.

Remark: It is only necessary to investigate the condition (2) for the cones T' intersecting $\sqrt{-1}\,\mathfrak{h}^*_{\mathfrak{m}}$, as follows from 5.3.

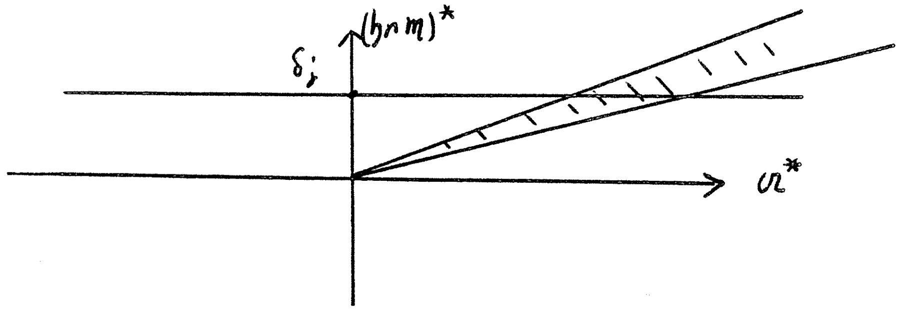
The conditions of the Theorem (5.5) will be more easily described when K/M is a symmetric space. Let $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{p}$ the decomposition of $\mathfrak{k}$ with respect to the involution σ, i.e. $\sigma|\mathfrak{m} = \mathrm{Id}$, $\sigma|\mathfrak{p} = -\mathrm{Id}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{k}$ contained in $\mathfrak{p}$. We can choose a σ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$, such that $\mathfrak{h} \cap \mathfrak{p} = \mathfrak{a}$, i.e. $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{m} \oplus \mathfrak{a}$. Let $C_{\mathfrak{a}}$ be a Weyl chamber of $\sqrt{-1}\,\mathfrak{a}^*$ and C a Weyl chamber of $\sqrt{-1}\,\mathfrak{h}^*$ compatible with $C_{\mathfrak{a}}$.

We define for $\mu \in \overline{C}_{\mathfrak{a}}$

$$F_\mu = \bigoplus_{\lambda|\mathfrak{a}=\mu} V_\lambda \otimes \mathrm{Hom}_M(V_\lambda, U) .$$

We recall that if $\mathrm{Hom}_M(V_\lambda, U) \neq 0$ then $\lambda|\mathfrak{h} \cap \mathfrak{m}$ is a weight of the representation U of M restricted to $\mathfrak{h} \cap \mathfrak{m}$ as follows from the following remark: Let f be a nonzero element of $\mathrm{Hom}_M(V_\lambda, U)$ and v be the highest weight vector of V_λ. Clearly $f(v) \in U$ transform under $\mathfrak{h} \cap \mathfrak{m}$ according to $\lambda|\mathfrak{h} \cap \mathfrak{m}$. Hence we have to see that $f(v) \neq 0$. Let $\mathfrak{n}' = \{\bigoplus_{\alpha\in\Delta^+} \mathfrak{k}_\alpha,\ \alpha|\mathfrak{a} \neq 0\}$, we have $\mathfrak{k}^C = \mathfrak{m}^C + \mathfrak{a}^C + \mathfrak{n}'$. As v is an eigenvector for $\mathfrak{a}^C + \mathfrak{n}$, for any $u \in \mathcal{U}(\mathfrak{k}^C)$, we have $u\cdot v = u_0\cdot v$ with $u_0 \in \mathcal{U}(\mathfrak{m}^C)$. Hence $f(u\cdot v) = u_0\cdot f(v)$. As $\mathcal{U}(\mathfrak{k}^C)\cdot v = V_\lambda$, $f(v) \neq 0$.

In particular, for any μ, the possible λ's occuring in F_μ are of the form $\mu + \delta_j$ for a finite choice of δ_j in $\sqrt{-1}\,(\mathfrak{h} \cap \mathfrak{m})^*$. In this case we see that the possible K-types occuring in $L^2(K,U)$ are contained in a strip along $\mathfrak{a}^*$.



Hence the Proposition (5.4) is then automatically satisfied. We remark also that F_μ is finite dimensional.

Our Theorem (5.4) is reformulated as follows:

5.8 Theorem: Let $\varphi = \sum_{\mu \in \overline{C}_{\mathfrak{a}}} \varphi_\mu$ a hyperfunction section of $\mathcal{U}$ and S be a closed cone in $\overline{C}_{\mathfrak{a}}$. Then the following conditions are equivalent:

(1) $(SS\varphi) \subset -K\cdot S$

(2) For any closed cone S' contained in $\sqrt{-1}\,\mathfrak{a}^*$ satisfying $S \cap S' = \{0\}$, there are positive numbers $R_{S'}$ and $\varepsilon_{S'}$ such that $\|\varphi_\mu\| \leq R_{S'}\, e^{-\varepsilon_{S'}|\mu|}$ when $\mu \in S'$.

Proof: This follows from (5.5) as for a symmetric space, $(Ad^*K)\mathfrak{m}^\perp \cap \overline{C} = \overline{C}_{\mathfrak{a}}$ and for any $S \subset \overline{C}_{\mathfrak{a}}$, $(Ad^*K)S \cap \sqrt{-1}\,\mathfrak{m}^\perp = (Ad^*M)S$.

Let $X = K/M$ being a symmetric space. Let us now consider H a K-invariant subspace of $L^2(X,U)$. We consider $H = \bigoplus_{\mu \in \overline{C}_{\mathfrak{a}}} H_\mu$ where

$$H_\mu = \bigoplus_{\lambda \in \hat{K}} H_\lambda \qquad \lambda|\mathfrak{a} = \mu .$$

5.9. We define the asymptotic support of H by

$T(H) = \{\mu \in \overline{C}_{\mathfrak{a}}$, such that there exist a sequence (μ_n, ε_n), $\mu_n \in \overline{C}_{\mathfrak{a}}$, $\varepsilon_n > 0$ with $|\mu_n| \to \infty$, $H_{\mu_n} \neq 0$ and $\varepsilon_n \mu_n \to \mu$.$\}$

We define $SSH = \bigcup_{u \in H} SSu \subset T^*(K/M)$. We have then:

5.10 Corollary: $SSH = K \cdot T(H)$.

Proof: Following the construction of the Proposition 5.4, it is easy to construct for every $\xi \in T(H)$ a function u in H such that $k \cdot (1, \xi) \in SSu$. As H is K-invariant the corollary follows.

6. Singular spectrum of G-modules.

Let G be a real semi-simple Lie group, K a maximal compact subgroup of G, $G = KAN$ an Iwasawa decomposition of G. We denote by M the centralizer of A in K. Let $(\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}, \mathfrak{m})$ be the Lie algebras of the groups involved.

For σ a finite dimensional representation of M in U and λ a homomorphism of A in C^*, we consider the representation $\sigma \otimes \lambda$ of MAN in the vector space U trivial on N and extending $\sigma \otimes \lambda$ on $M \times A$. We consider the G-bundle $G \times_{MAN} U$. This as a K-bundle is isomorphic to $K \times_M U$ over $X = K/M$. The decomposition under K of the associated principal series $\operatorname{Ind}_{MAN}^{G} \sigma \otimes \lambda$ is then given by the formula 5.2. The Proposition (5.4) gives us the asymptotic behavior of the K-types of the principal series associated to the parabolic MAN.

Let $\mathcal{V}$ be an irreducible Harish-Chandra $(\mathfrak{g}, K)$ module. We write $\mathcal{V} = \bigoplus_{\lambda \in \hat{K}} \mathcal{V}_\lambda$ where $\mathcal{V}_\lambda$ is the isotypic component of type λ.

Let us define the following subsets of $\overline{C}$:

6.1 Definition.

a) The K-support of $\mathcal{V}$, $S(\mathcal{V}) = \{\lambda \in \overline{C} \,;\, \mathcal{V}_\lambda \neq 0\}$.

b) The asymptotic K-support of $\mathcal{H}$

$$T(\mathcal{H}) = \{\lambda \in \overline{C} \text{ , such that there exists } \lambda_n \in S(\mathcal{H}) \text{ with } |\lambda_n| \to \infty \text{ , } t_n > 0 \text{ , and } t_n\lambda_n \to \lambda\} \text{ .}$$

It is known that $\mathcal{H}$ can be imbedded as a $\mathfrak{g}$-submodule in a principal series $\operatorname{Ind}\uparrow_{MAN}^{G} \sigma \otimes \lambda$. Let us choose such an imbedding, and let us denote by H the completion of $\mathcal{H}$ in $L^2(X,U)$. Then G acts by bounded transformations on H . We denote by $SSH = \overline{\bigcup_{u\in H} SSu}$. Hence SSH is a closed subset of $\sqrt{-1}\, T^*K$. We identify $(SSH)_e$ as a subset of $\sqrt{-1}\, T_e^*K = \sqrt{-1}\, \mathfrak{k}^*$. We have $(Ad^*K)(SSH)_e = (Ad^*K)\, T(\mathcal{H})$, as follows from 5.5 and the proof of 5.4.

As H is stable by G , SSH is a G-invariant subset of $\sqrt{-1}\, T^*(K/M) \simeq \sqrt{-1}\, T^*(G/MAN)$. We have $T_0^*(G/MAN) \simeq (\mathfrak{m} + \mathfrak{a} + \mathfrak{n})^{\perp} \simeq \mathfrak{n}$ by the Killing form, $T_0^*(K/M) \simeq \mathfrak{m}^{\perp} \subset \mathfrak{k}^*$, the isomorphism $i: \mathfrak{n} \to \mathfrak{m}^{\perp}$ being given by the Killing form $X \longrightarrow B(X,\cdot)$. Hence if we identify $\mathfrak{k}^*$ with $\mathfrak{k}$, and we write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the map i is the restriction to $\mathfrak{n}$ of the orthogonal projection π from $\mathfrak{g}$ to $\mathfrak{k}$ perpendicular to $\mathfrak{p}$.

Let Q be an MAN invariant closed cone in $\mathfrak{n}$. Then SSH is of the form $\sqrt{-1}\, G\cdot Q = \sqrt{-1}\, K\cdot i(Q)$. We have:
$Ad^*K\cdot T(\mathcal{H}) = (Ad^*K)\cdot i(Q)$.

<u>6.2 **Theorem.**</u> Let S be a closed subset of nilpotent orbits of G in $\mathfrak{g}$. Let $\pi(S)$ be the projection of S to $\mathfrak{k}^*$ by the Killing form; let us denote by $T_S = \overline{C} \cap \sqrt{-1}\, \pi(S)$. For any Harish-Chandra module $\mathcal{H}$ there exists a closed subset S of nilpotent orbits of G in $\mathfrak{g}$ such that: $T(\mathcal{H}) = T_S$.

In particular for $\mathcal{H}$ a module of the principal series associated to MAN, we have $S = Ad^*G\cdot\mathfrak{n}$ = the nilpotent cone.

Let us give a example:

6.3 Example: $G = SU(2,1)$.

We consider the group $SU(2,1)$ associated to the canonical hermitian form

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

We choose

$$\mathfrak{a} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $K \simeq S(U(2) \times U(1))$. Then the group M is given by

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix} .$$

Hence $\sqrt{-1}\,\mathfrak{m}$ has basis

$$H_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{k}$ given by

$$\sqrt{-1}\,\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} ;\ a_1 + a_2 + a_3 = 0 \ ,\ a_i \in \mathbb{R} \right\} .$$

We identify $\mathfrak{h}$ with its dual via the G-invariant form $(A,B) = \operatorname{Tr} AB$ for $A,B \in \mathfrak{su}(2,1)$. We identify the Lie algebra of $\sqrt{-1}\,\mathfrak{k}$ with the space of hermitian matrices by

$$A \to \begin{pmatrix} A & 0 \\ 0 & -\operatorname{Tr} A \end{pmatrix} :$$

then the orbits of K in $\sqrt{-1}\,\mathfrak{k}$ are classified by the eigenvalues of A . In this identification $\sqrt{-1}\,\mathfrak{m}^{\perp}$ is the subspace of matrices

$$\left\{ \begin{pmatrix} x_0 & u \\ \bar{u} & 0 \end{pmatrix} \right\}.$$

Hence an hermitian matrice

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

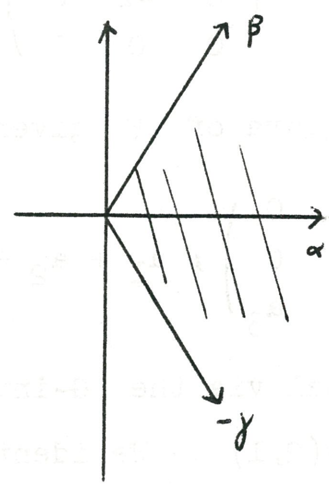
is conjugated to $\mathfrak{m}^{\perp}$ if and only if $\lambda_1\lambda_2 \leq 0$.

We have for $\Delta^+ = \{\alpha, \beta, \gamma\}$ the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, α being the compact root, $\gamma = \beta + \alpha$.

$$H_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H_\gamma = H_\alpha + H_\beta .$$

The Weyl chamber C corresponding to the system of compact positive roots $\Delta^+_{\mathfrak{k}} = \{\alpha\}$, is given by $\lambda(H_\alpha) > 0$. Hence $\overline{C} \cap \sqrt{-1}\,\mathfrak{h}^*_{\mathfrak{m}}$ is given by

$$\lambda = \{x_1\beta + x_2\gamma \quad x_1 \geq 0 \quad x_2 \leq 0\}$$



Let us consider the three possible classes of nilpotent elements for the action of G in $\mathfrak{g}$ ([6])

$$X_{\pm} = \pm \begin{pmatrix} 0 & 0 & 0 \\ 0 & i/2 & i/2 \\ 0 & -i/2 & -i/2 \end{pmatrix} \quad , \quad X_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

It is easily computed that

$$T_0 = \sqrt{-1}\, Ad^*G\cdot X_0 \cap \overline{C} = \sqrt{-1}\, \mathfrak{h}^*_{m} \cap \overline{C}$$

$$T_+ = \sqrt{-1}\, Ad^*G\cdot X_+ \cap \overline{C} = \mathbb{R}^+\cdot\beta \text{ , the half line of direction } \beta$$

$$T_- = \sqrt{-1}\, Ad^*G\cdot X_- \cap \overline{C} = -\mathbb{R}^+\cdot\gamma \text{ , the half line of direction } \gamma .$$

Let us precise our theorem (6.2) as follows: If the $(\mathfrak{g},K)$ module can be associated to an orbit Λ of Ad^*G in $\mathfrak{g}^*$ (for example, for the discrete series D_Λ, we will choose the G orbit of the elliptic element Λ) the choice of S should be given as follows: we define for an element $f \in \mathfrak{g}^*$ the asymptotic cone $S(f)$ to the orbit $G\cdot f$

i.e. $u \in S(f)$ if there exist $f_n \in G\cdot f$, $|f_n| \to \infty$ and $\varepsilon_n > 0$, such that $\varepsilon_n f_n \to u$

we then should have $T(\mathcal{X}_\Lambda) = T_{S(\Lambda)}$.[1]

It is easy to verify this conjecture in the case of $\mathfrak{g} = \mathfrak{su}(2,1)$: If Λ corresponds to an element of the holomorphic discrete series, we have $S(\Lambda) = G\cdot X_+$. If Λ corresponds to an element of the antiholomorphic discrete series, we have $S(\Lambda) = G\cdot X_-$. If Λ corresponds to the non-holomorphic discrete series, we have $S(\Lambda) = \overline{G\cdot X_0}$.

<u>Example 6.4</u>: Let $G = Sp(n,\mathbb{R})$ operating on the vector space $S(n)$ of symmetric $n \times n$ real matrices by $x \longrightarrow (ax+b)(cx+d)^{-1}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n;\mathbb{R})$. The maximal compact subgroup K of G

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[1] This conjecture has been proven recently by D. Barbasch and

is isomorphic to $U(n)$, via $a + ib \in U(n) \longrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

For P the parabolic

$$P = \left\{ \begin{pmatrix} a & 0 \\ * & {}^t a^{-1} \end{pmatrix} ; a \in GL(n;\mathbb{R}) \right\}$$

and μ a given finite dimensional representation μ of $GL(n;\mathbb{R})$, we consider the associated principal series $\mathrm{Ind}\uparrow_P^G \mu = T_\mu$ (not necessarily unitary).

We denote by $M = P \cap K = O(n)$. We realize T_μ as a space of sections of a bundle over $G/P = K/M = X$. The vector space $S(n)$ can be considered as an open subset of G/P by $x \longrightarrow \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ mod P , the corresponding action of G being given by the above formula. The corresponding identification

$$T^*(U(n)/O(n)) \simeq T^*S(n)$$

is given at the origin by $B \in S(n) \longrightarrow \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix} \in \mathfrak{m}$. The pair $(K,M) = (U(n), O(n))$ is a symmetric pair. The preceding map allows us to identify the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}$ with $S(n)$, the action of M on $\mathfrak{m}^\perp$ being given by $g \cdot X = gX{}^tg$, for $g \in O(n)$.

Let $\mathfrak{a}$ be the subspace of $\mathfrak{m}$ defined by diagonal matrices

$$\mathfrak{a} = \left\{ \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}, a_i \in \mathbb{R} \right\},$$

then every M-invariant subset of $\mathfrak{m}$ is of the form $M \cdot T$ where T is a subset of $\mathfrak{a}$.

$\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{k}$, hence every irreducible representation of K is indexed by its highest weight $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, considered as an element of $\mathfrak{a}^* - \mathfrak{a}$.

Let $\mathcal{V}$ be a $(\mathfrak{g},K)$ submodule of the space of K-finite vectors of the representation $\mathrm{Ind}\uparrow_P^G \mu$. We can analyze $SS\,H = \bigcup_{u\in H} SSu$ by analyzing the expansion of a function φ of H in terms of the K-Fourier series of $\varphi = \sum_{\lambda\in K} \varphi_\lambda$: i.e. let $\mathcal{V} = \bigoplus_{\lambda\in\hat{K}} \mathcal{V}_\lambda$, let $T(\mathcal{V})$ be the asymptotic K-support of $\mathcal{V}$ (definition 6.1). Let $M\cdot T(\mathcal{V}) \subset S(n)$ be the orbit of $T(\mathcal{V})$ under the group $O(n)$. We know that $SSH \subset T^*(K/M)$ is given by a K-invariant set of $T^*(K/M)$, and $(SSH)_e = M\cdot T(\mathcal{V})$, by 5.10.

Let us consider H as a G-module, then $SSH \subset T^*(G/P)$ is a G-invariant subspace of $T^*(G/P)$. Hence $(SSH)_e$ is given by a $GL(n,\mathbb{R})$ invariant closed subset of $S(n)$. The action of $GL(n)$ on $S(n)$ via gX^tg decomposes $S(n)$ into a union of finite number of orbits $\mathcal{O}_{k_1,k_2}$, where $\mathcal{O}_{k_1,k_2}$ is the set of symmetric matrices of signature (k_1,k_2) . Hence we have necessarily $(SSH)_e = \bigcup \overline{\mathcal{O}_{k_1,k_2}} \subset S(n)$ over a subset of orbits. Realizing H as a space of tempered distributions on the vector space $S(n)$, we may compute the singular spectrum of H using the Fourier integral $\hat{u}\ (\xi) = \int u(x)\, e^{-2i\pi \mathrm{Tr}\xi x}\, d\xi$ over the vector space $S(n)$ with respect to the bilinear form $\mathrm{Tr}\xi x$. If H is such that for every $u \in H$, $\hat{u}(\xi)$ is supported in $\bigcup\overline{\mathcal{O}_{k_1,k_2}}$, then $(SSH)_e \subset \bigcup \overline{\mathcal{O}_{k_1,k_2}}$.

Let us consider

$$\overline{C}_{k_1,k_2} = \{\lambda \in \overline{C}\ ;\ \lambda = (x_1,x_2,\cdots,x_i,0,\cdots,0,-y_j,-y_{j-1},\cdots -y_1)\},$$
$$\text{with } x_i \geq 0,\ y_j \geq 0,\ i \leq k_1,\ j \leq k_2\} .$$

We have $\overline{\mathcal{O}_{k_1,k_2}} = M\cdot\overline{C}_{k_1,k_2}$. Hence if H is a $(\mathfrak{g},K)$ module such that the asymptotic support of $\mathcal{V}$ is contained in a finite union of the sets C_{k_1,k_2} , it follows that $SSH \subset \overline{\mathcal{O}_{k_1,k_2}}$ and reciprocally.

This explains "asymptotically" the relation between the description of the spaces $H_{p,q}$ introduced in the article [7] via the support of the Fourier transform of the functions involved, and the K-support of $H_{p,q}$ given in [7].

In the similar example of the group $U(2,2)$ acting by conformal transformations on the Minkowski space, we consider sub-representations H on the space of sections of the classical spin bundles on the Minkowski-space: We have in this case to consider $K = U(2) \times U(2)$, $M = U(2)$. Our bundles can be considered either as bundles over $K/M \simeq U(2)$, either on the flat Minkowski space identified with $H(2)$ by

$$\vec{x} = (x_0, x_1, x_2, x_3) \longrightarrow x = \begin{pmatrix} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{pmatrix} .$$

The asymptotic directions of the K-types occurring in H are given by

$$T(H) = \{(m_1, m_2) \times (-m_2, -m_1) \in U(2)^\wedge \times U(2)^\wedge\} \text{ with }$$
$$(m_1, m_2) \in T \subset \overline{C} \subset H(2)\}.$$

We can similarly read on the asymptotic directions of the K-types of H the support of the Fourier transform of a function u on H considered as a classical field. For example the space H of solutions of Maxwell, Dirac or Wave equation (considered as a subspace of the appropriate bundle) will have as asymptotic support the line $T = (m,0)$ as $U(2) \cdot T \subset H(2)$ is the light cone $(x_0^2 = x_1^2 + x_2^2 + x_3^2)$. The precise description of the support of H is given in [8].

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