RIMS-450

Microlocal Analysis of Theta Functions

By

Mikio SATO, Masaki KASHIWARA and

Takahiro KAWAI

July 1983

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

Microlocal Analysis of Theta Functions

Mikio SATO

Research Institute for Mathematical Sciences

Kyoto University, Kyoto 606 Japan

Masaki KASHIWARA

Research Institute for Mathematical Sciences

Kyoto University, Kyoto 606 Japan

and

Takahiro KAWAI

Research Institute for Mathematical Sciences

Kyoto University, Kyoto 606 Japan

§0. Introduction

The purpose of this report is to show how the theta-zerovalue and some related functions can be controlled micro-locally. To be specific, we first prove a result on the constructibility (in particular, finiteness) of the complex of $\Gamma_Z(\varphi_* \mathcal{O})$-solution sheaves for a class of microdifferential equations, which we call $\mathbb{R}$-holonomic. (Theorem 1.2 and Theorem 1.3. See (1.8) for the precise definition of $\Gamma_Z(\varphi_* \mathcal{O})$-solutions.) The constructibility of microfunction solution sheaves will then follow from that of $\Gamma_Z(\varphi_* \mathcal{O})$-solution sheaves. (Theorem 1.4.) This finiteness result generalizes the result for holonomic systems ([6]), giving us the hope that $\mathbb{R}$-holonomic complexes will be effectively used in application. Our expectation is augmented by the validity of the "Reconstruction Theorem" for $\mathbb{R}$-holonomic complexes. (Theorem 1.5.) Furthermore it is really confirmed by the fact that the finiteness theorem for theta-zerovalues, or Jacobi functions, which was announced in Sato [12], follows from the general result for $\mathbb{R}$-holonomic complexes. (Theorem 2.8.) Note that the finiteness theorem for theta-zerovalues is closely tied up with their autormorphic property. (See §2.)

We refer the reader to Sato-Kashiwara-Kawai [13], particularly its introduction, for the background of the finiteness theorem given in this report. At the same time, it is worth while noting the following two distinctions between the prosentation of [13] and that of this article:

First, in this article we do concentrate our attention on the theta-zerovalues, not the theta functions. The discussion on theta functions can be done in exactly the same manner as in [13], once we get the finiteness theorem (Theorem 2.8) for theta-zerovalues. So, we do not repeat it here. Note, however, that the way of the reasoning in [13] is quite different from the reasoning given here in that the former one first shows some finiteness theorem for theta functions and then deduces from it the finiteness theorem for the theta-zerovalue. Apparently such an approach is cumbersome in discussing microfunction solutions as we do in this article.

Second, in discussing Jacobi structure (Definition 2.1), we consider the problem without subsidiary conditions in the sense of [13]. This is simply because we microlocalize the problem in this article: micro-localization enables us to eliminate the subsidiary conditions, generically speaking. But, then, we have to introduce microdifferential operators of fractional order such as $\sqrt{4\pi\sqrt{-1}\partial/\partial t}$.

For the convenience of the reader, let us briefly recall the definition of the sheaf of microdifferential operators of fractional order. In what follows, X denotos a complex manifold and T^*X denotes its cotangent bundle. As usual, we denote $T^*X-T^*_XX$ by $\mathring{T}^*X$. We also denote by π(resp., $\mathring{\pi}$) the projection from T^*X (resp., $\mathring{T}^*X$) to X. Let P^*X denote $\mathring{T}^*X/\mathbb{C}^\times$ and let γ denote the projection from $\mathring{T}^*X$ to P^*X. Then, for any point p in P^*X, $\gamma^{-1}(p)=\mathbb{C}-\{0\}$. Let m be a positive integer and let $\pi_m: \mathfrak{X}_m \to \mathring{T}^*X$ be an m-fold covering of $\mathring{T}^*X$ with respect to the action of $\mathbb{C}^\times$

on each fiber of $\overset{\circ}{T}{}^*X$. Let γ_m denote the projection from $\mathfrak{X}_m$ to P^*X. Then, in analogy with the definition $\mathcal{E}_X^\infty = \gamma^{-1}\gamma_*\mathcal{E}^{\mathbb{R}}$, we define a sheaf $\mathcal{E}^\infty_{(1/m)}$ by

$$(0.1) \qquad \gamma^{-1}\gamma_{m*}\pi_m^{-1}\mathcal{E}_X^{\mathbb{R}},$$

where $\mathcal{E}_X^{\mathbb{R}}$ denotes the sheaf of holomorphic microlocal operators(*). Note that $\mathcal{E}^\infty_{(1/m)}$ is a subsheaf of $\mathcal{E}_X^{\mathbb{R}}$. Let x^* be a point in $\overset{\circ}{T}{}^*X$ and let r be in $\mathcal{E}_{X,x^*}$ such that $\sigma(r)(x^*)\neq 0$(**). Then it immediately follows from the definition of $\mathcal{E}^\infty_{(1/m)}$ that we have

$$(0.2) \qquad \mathcal{E}^\infty_{(1/m),x^*} = \sum_{k=0}^{m-1}\mathcal{E}^\infty_{X,x^*}r^{k/m} = \sum_{k=0}^{m-1}r^{k/m}\mathcal{E}^\infty_{X,x^*}.$$

If some p in $\mathcal{E}^\infty_{(1/m),x^*}$ actually belongs to $\sum_{k=0}^{m-1}\mathcal{E}_{X,x^*}r^{k/m}$, its principal symbol $\sigma(p)$ is, by definition, the term of the highest degree with respect to the fiber coordinate in the symbol sequence corresponding to p. As

---

(*) The notation used here is slightly different from that used in [8].

(**) Here and in what follows, $\mathcal{E}_X$ denotes the sheaf of microdifferential operators of finite (integral) order, and $\sigma(r)$ denotes the principal symbol of an operator r of finite order.

in the case of operators of integral order, the degree of the principal symbol is called the order of the operator. It is clear that these notions are independent of the choice of reference operator r.

The importance of introducing the sheaf of microdifferential operators of fractional order lies in the fact that $\exp p$ belongs to $\mathcal{E}_X^{\mathbb{R}}$ only when the order of p is strictly smaller than 1, whilst operators of this type provide us with most interesting and important examples of systems of microdifferential equations of infinite order. (Cf. [12],[13].) Of course, $\exp p$ is of order at most 0 if the order of p is of order at most 0. Hence our main interest lies in the case where the order of p is bigger than 0 and smaller than 1. Thus the introduction of operators of fractional order is indispensable for our purpose.

§1. Finiteness theorems for $\mathbb{R}$-holonomic complexes.

In this section we present basic finiteness results for $\mathbb{R}$-holonomic complexes of $\mathcal{E}^{\mathbb{R}}$-Modules to be defined below. In finding finiteness results for systems of microdifferential equations of infinite order, we have to use the symplectic structure of T^*X regarded as a real manifold. (Cf. [11]) So, in order to be specific, let us denote by $(T^*X)^{\mathbb{R}}$ the real homogeneous symplectic manifold endowed with the real canonical 1-form $\omega^{\mathbb{R}} \underset{\mathrm{def}}{=} \omega + \bar{\omega}$, where ω denotes the holomorphic canonical 1-form on T^*X and $\bar{\omega}$ denotes its complex conjugate.

Let us now specify the class of systems to be considered in this report. Let W be an open subset of $\mathring{T}^*X$ and let $\mathcal{K}.$ be a complex of $\mathcal{E}^{\mathbb{R}}$-Modules defined on W. We say $\mathcal{K}.$ is a good complex (of $\mathcal{E}^{\mathbb{R}}$-Modules), if $\mathcal{K}.$ is locally quasi-isomorphic to a bounded complex of free $\mathcal{E}^{\mathbb{R}}$-Modules of finite rank. For a good complex $\mathcal{K}.$, its characteristic set $\mathrm{Ch}(\mathcal{K}.)$ is, by definition, the union of the closure of $\mathrm{Supp}\ \mathcal{H}_{\ell}(\mathcal{K}.)$ for all ℓ's.

<u>Definition 1.1.</u> A good complex $\mathcal{K}.$ of $\mathcal{E}^{\mathbb{R}}$-Modules is said to be $\mathbb{R}$-holonomic, if $\mathrm{Ch}(\mathcal{K}.)$ is contained in a subanalytic subset of $(\mathring{T}^*X)^{\mathbb{R}}$ which is Lagrangian (with respect to the canonical 1-form $\omega^{\mathbb{R}}$).

See Hironaka [5] for the definition and basic properties of a subanalytic set.

Interesting examples of $\mathbb{R}$-holonomic complexes will be given in §2 by using $\exp p$'s $(\operatorname{ord} p < 1)$.

As we said in §0, our main concern is the structure of $\mathbb{R}\Gamma_Z(\mathbb{R}\mathcal{S}_*\mathcal{O}_{\mathbb{C}^n})$-solutions of $\mathbb{R}$-holonomic complexes. So we shall first prepare several notations needed to discuss $\mathbb{R}\Gamma_Z(\mathbb{R}\mathcal{S}_*\mathcal{O}_{\mathbb{C}^n})$-solutions.

Let us now suppose that X is an open subset of $\mathbb{C}^n$ and use the $\mathbb{R}$-vector space structure of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Here and in what follows we use the same notions and notations as in [9]; for example, for a closed and properly convex cone G in $\mathbb{R}^{2n}$ and an open subset D of X, we say that D is G-round (resp., G-open) if $(D+G)\cap(D+G^a) = D$ holds (resp., if $D+G\subset D$ holds), where $G^a = \{x \in X;\ -x \in G\}$, and for such a cone G and a G-round open subset D of X, we denote by $E(G,D)$ the relative cohomology group $H^n_{Z(G)}(D\times D, \mathcal{O}^{(0,n)}_{X\times X})$, where $Z(G) = \{(x,y) \in X\times X;\ y-x \in G\}$. Note that there exists a canonical ring homomorphism

$$E(G,D) \to \Gamma(D\times G^{\circ a}, \mathcal{E}^{\mathbb{R}}).$$

Here G° denotes the polar set of G, i.e., $\{\xi \in \mathbb{R}^{2n};\ \langle x,\xi\rangle > 0$ for every x in $G\}$.

Let $\mathcal{K}_\bullet$ denote an $\mathbb{R}$-holonomic complex of $\mathcal{E}^{\mathbb{R}}$-Modules defined on a neighborhood of x^* in $\mathring{T}^*X$. Since $\mathcal{E}^{\mathbb{R}}_{X,x^*}$ is the inductive limit of $E(G,D)$'s with G being a properly convex closed cone in $\mathbb{R}^{2n}$ and D being a G-round open subset such that $D\times G^{\circ a}$ is a neighborhood of x^*, we can find a cone G and an open subset D of X which satisfy the following conditions (1.1), (1.2) and (1.3).

(1.1) G is a closed and properly convex cone in $\mathbb{R}^{2n}$.

(1.2) $D\times G^{\circ a}$ is a neighborhood of x^*.

(1.3) If we define a complex $K_{\bullet}$ of $E(G,D)$-modules by

$$0 \leftarrow E(G,D)^{N_0} \xleftarrow{P_0} E(G,D)^{N_1} \xleftarrow{P_1} \cdots \xleftarrow{P_{d-1}} E(G,D)^{N_d} \leftarrow 0,$$

then $\mathcal{E}_X^{\mathbb{R}} \underset{E(G,D)}{\otimes} K_{\bullet}$ is quasi-isomorphics to $\mathcal{K}_{\bullet}$ on $D\times G^{\circ a}$.

Now, for a subset S of $\mathbb{C}^n$, we denote by S_G the topological space S endowed with the G-topology; it means, by definition, that a subset Ω of S is open as a subset of S_G, if it is an intersection of S and a G-open subset of $\mathbb{C}^n$. In what follows we denote by φ_G the canonical continuous map from $\mathbb{C}^n$ to $\mathbb{C}^n_G$. In passing, we know ([9], Corollary 3.2.5) that there exists a G-round open neighborhood U of $\overset{\circ}{\pi}(x^*)$ such that, for any G-open sets $\Omega_1 \supset \Omega_0$ satisfying $\Omega_1 \setminus \Omega_0 \Subset U$, $\mathbb{R}\Gamma_{\Omega_1\setminus\Omega_0}(\mathbb{R}\varphi_{G*}\mathcal{O}_{\mathbb{C}^n})|_{\Omega_1}$ is well-defined in the derived category of the abelian category of sheaves of $E(G,D)$-modules defined on $(\Omega_1)_G$. Here and in what follows, $\Omega_1\setminus\Omega_0$ denotes the set $\{x\in\Omega_1;\ x\notin\Omega_0\}$.

Now, our first result is the following finiteness theorem.

<u>Theorem 1.2.</u> Suppose that $\mathcal{E}^{\mathbb{R}} \underset{E(G,D)}{\otimes} K_{\bullet}$ is $\mathbb{R}$-holonomic on $U\times G^{\circ a}$. Then, for each point (x_0,ξ_0) in $U\times G^{\circ a}$, there exist a properly convex, subanalytic and closed cone G' and G'-open subanalytic sets Ω_1' and Ω_0' which satisfy the following conditions:

(1.4) G is properly contained in G',

(1.5) $G' \subset \{0\} \cup \{x\in\mathbb{R}^{2n};\ \langle x,\xi_0\rangle < 0\}$,

(1.6) $x_0 \in \Omega_1'\setminus\overline{\Omega_0'} \Subset U$,

(1.7) $\dim_{\mathbb{C}} \mathcal{H}^j(\varphi_{G'}^{-1}\,\mathbb{R}\mathcal{H}om_{E(G,D)}(K_{\bullet},\ \mathbb{R}\Gamma_{\Omega_1'\setminus\Omega_0'}(\mathbb{R}\varphi_{G*}\mathcal{O}_{\mathbb{C}^n})))_x < \infty$

for any x in $\Omega_1'\setminus\overline{\Omega_0'}$.

Once this finiteness result is obtained, we can immediately obtain the following Theorem 1.3 and Theorem 1.4 on the constructibility of the solution sheaves.

<u>Theorem 1.3.</u> In the same situation as in Theorem 1.2, we let F' denote

(1.8) $\varphi_{G'}^{-1}\,\mathbb{R}\mathcal{H}om_{E(G,D)}(K_{\bullet},\ \mathbb{R}\Gamma_{\Omega_1'\setminus\Omega_0'}(\mathbb{R}\varphi_{G'*}\mathcal{O}_{\mathbb{C}^n}))$.

Then $F'|_{\Omega_1'}$ is an $\mathbb{R}$-constructible complex on Ω_1'; that is, there exists a locally finite and decreasing family $\{N_k\}_{k=0,1,2,\dots}$ of subanalytic subsets of Ω_1' for which the following conditions are satisfied:

(1.9) $\mathcal{H}^j(F')|_{\Omega_1'} = 0$ holds except for finitely many j's,

(1.10) $N_0 = \Omega_1'$,

(1.11) $\bigcap_k N_k = \emptyset$,

(1.12) $\mathcal{H}^j(F')|_{N_k \setminus N_{k+1}}$ is a locally constant sheaf for each j and k.

(1.13) $\dim_{\mathbb{C}} \mathcal{H}^j(F')_x < \infty$ for every x in Ω_1' and for each j.

Proof. Taking subanalytic G'-open subsets Ω_1 and Ω_0 so that

(1.14) $\Omega_1 \supset \Omega_1'$ and $\Omega_0' \supset \Omega_0$

and

(1.15) $\Omega_1' \setminus \Omega_0' \Subset \Omega_1 \setminus \Omega_0 \Subset U$

may hold, we set

$$F = \varphi_G^{-1} \mathbb{R}\mathcal{H}om_{\mathcal{E}(G,D)}(\mathcal{K}_\cdot, \mathbb{R}\Gamma_{\Omega_1 \setminus \Omega_0}(\mathbb{R}\varphi_{G*}\mathcal{O}_{\mathbb{C}^n})).$$

Then we know ([10])

$$SS(F) \cap ((\Omega_1 \setminus \bar{\Omega}_0) \times G^{\circ a}) \subset Ch(\mathcal{K}_\cdot).$$

(See [10] for the definition of $SS(F)$ for a complex F of sheaves on X.) Let q_1 (resp., q_2) denote the first (resp., second) projection from $\Omega_1' \times \Omega_1'$ to Ω_1'. Then we find by the definition

$$(1.16)\quad F'|_{\Omega_1'} = \mathbb{R}_{q_{1*}}(\mathbb{C}_{Z(G')} \otimes q_2^{-1}\mathbb{R}\Gamma_{\Omega_1'\setminus\Omega_0'}(F)).$$

Denoting by p_1 (resp., p_2) the first (resp., second) projection from $T^*(\Omega_1' \times \Omega_1')$ to $T^*\Omega_1'$ and denoting by π the projection from $T^*\Omega_1'$ onto Ω_1', we obtain from (1.16) the following:

$$\begin{aligned}(1.17)\quad & SS(F'|_{\Omega_1'}) \\ & \subset p_1(N^*(Z(G')) \cap p_2^{-1}((SS(F)\hat{+}N^*(\Omega_0')^a) \cap \pi^{-1}(\Omega_1'\setminus\Omega_0'))) \\ & \subset p_1(N^*(Z(G')) \cap p_2^{-1}((Ch(\mathcal{K}_\cdot) \cap (((\Omega_1'\setminus\Omega_0')\times G^{\circ a}) \\ & \qquad \hat{+}N^*(\Omega_0')^a)) \cap \pi^{-1}(\Omega_1\setminus\Omega_0'))).\end{aligned}$$

Here and in what follows, for a subset S of X, $N^*(S)$ denotes the conormal set of S (in the sense of Definition 3.4.3 of [9]). See Definition 1.4.2 of [10] for the definition of the symbol $\hat{+}$. Since the last set in (7.17) is a closed isotropic subanalytic subset of $\pi^{-1}(\Omega_1')$ ([10]), F' is weakly $\mathbb{R}$-constructible on Ω_1', that is, F' satisfies conditions (1.9) $\sim$ (1.12). Since the preceding theorem guarantees (1.13), this completes the proof of the theorem.

Theorem 1.4. Let X be a complexification of a real analytic manifold M. Let W be an open subset of $\overset{\circ}{T}{}^*X$ and let $\mathcal{K}_\bullet$ be an $\mathbb{R}$-holonomic complex of $\mathcal{E}^{\mathbb{R}}$-Modules defined on W. Then, for each j,

$$\mathcal{E}xt^{j}_{\mathcal{E}^{\mathbb{R}}_X}(\mathcal{K}_\bullet, \mathcal{C}_M)$$

is an $\mathbb{R}$-constructible sheaf on $W \cap T^*_M X$. Here $\mathcal{C}_M$ denotes the sheaf of microfunctions.

Proof.. Let us first recall the definition of the sheaf of microfunctions, that is,

$$\mathcal{C}_M = \mathcal{H}^n(\mu_M(\mathcal{O}_X)) \otimes \omega_M.$$

Here μ_M denotes the microlocalization functor ([10]) and ω_M denotes the orientation sheaf of M. Hence it follows from the definition (1.8) of F' that

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}^{\mathbb{R}}_X}(\mathcal{K}_\bullet, \mathcal{C}_M) = \mu_M(F') \otimes \omega_M[-n]$$

holds on $((\Omega_1' \setminus \overline{\Omega_0'}) \times G'^{\circ a}) \cap T^*_M X$. Since the microlocalization functor preserves the $\mathbb{R}$-constructibility, Theorem 1.4 is an immediate consequence of Theorem 1.3. Q.E.D.

In addition to these finiteness results, we can prove the following reconstruction theorem for $\mathbb{R}$-holonomic complexes.

Theorem 1.5. Let $\mathcal{K}_\bullet$ be an $\mathbb{R}$-holonomic complex of $\mathcal{E}^{\mathbb{R}}$-

Modules defined on a neighborhood of x^* in $\overset{\circ}{T}{}^*X$ and let F' be given by (1.8). Then

$$\mathcal{K}_{\cdot} = \mu\,\mathrm{hom}\,(F', \mathcal{O}_X)$$

holds on a neighborhood of x^*. Here $\mu\,\mathrm{hom}\,(F', \mathcal{O}_X)$ denotes, by definition, $\mu_\Delta(\mathbb{R}\mathcal{H}om_{\mathbb{C}}(p_2^{-1}F', p_1^{!}\mathcal{O}_X))$ $(=\mu_\Delta(\mathbb{R}\mathcal{H}om_{\mathbb{C}}(p_2^{-1}F', p_1^{-1}\mathcal{O}_X))[2n])$, where Δ is the diagonal set of $X \times X$ and p_1 and p_2 respectively denote the first and the second projection from $X \times X$ to X.

This theorem generalizes the reconstruction theorem for holonomic $\mathcal{D}_X$-Modules ([7], Chap. I, §4) to an $\mathbb{R}$-holonomic complex of $\mathcal{E}^{\mathbb{R}}$-Modules.

In order to prove Theorem 1.2 let us first note the following

Lemma 1.6. Let Λ and Λ' be closed Lagrangian subanalytic subsets of $(T^*X)^{\mathbb{R}}$. Suppose that $\pi(\Lambda')$ is compact. Let v be a vector such that $\langle\xi, v\rangle$ never vanishes for any (x,ξ) in $\Lambda' \cap \overset{\circ}{T}{}^*X$, and let $\Lambda'(\varepsilon v)$ denote the translation of Λ' by εv, where ε is a real number. Then we find

$$(1.18) \qquad \Lambda \cap \Lambda'(\varepsilon v) \subset T^*_X X$$

for $0 < |\varepsilon| \ll 1$.

Proof. Let us prove the lemma by a reduction to absurdity. If (1.18) were false, we could find a real analytic function $(x(t), \xi(t), \varepsilon(t))$ $(0 \leq t \leq 1)$ so that

$$(1.19) \quad (x(t),\xi(t)) \in \Lambda \cap \Lambda'(\varepsilon(t)v) \cap \mathring{T}^*X \quad (0 < t \leq 1)$$

and

$$(1.20) \quad \varepsilon(0) = 0, \ \varepsilon(t) \not\equiv 0$$

might hold. Since Λ and Λ' are Lagrangian by the assumption, (1.19) should entail

$$(1.21) \quad \langle \xi(t), \frac{dx(t)}{dt} \rangle = 0$$

and

$$(1.22) \quad \langle \xi(t), \frac{dx(t)}{dt} \rangle + \langle \xi(t), \frac{d\varepsilon(t)}{dt} v \rangle = 0.$$

Hence we should have

$$(1.23) \quad \langle \xi(t), \frac{d\varepsilon(t)}{dt} v \rangle = 0.$$

On the other hand, $\langle \xi(t), v \rangle \neq 0$ by the assumption on v. Hence

$$(1.24) \quad \frac{d\varepsilon(t)}{dt} = 0$$

should hold. This contradicts (1.20), completing the proof of Lemma 1.6.

Let us now embark on the proof of Theorem 1.2. We choose complex coordinates $z = (z_1, \ldots, z_n)$ on X $(\subset \mathbb{C}^n)$ so that $(x_0; \xi_0) = (0; d(\mathrm{Re}\, z_1))$. Let $a_2, \ldots, a_n$ and c be some (sufficiently large) positive numbers such that the cone $G(a_2, \ldots, a_n; c)$ defined below properly contains G.

(1.25) $$G(a_2, \ldots, a_n; c)$$

$$= \{z \in \mathbb{C}^n;\ -c\,\mathrm{Re}\, z_1 \leqq |\mathrm{Im}\, z_1|,\ -a_j(1+c^2)^{1/2}\mathrm{Re}\, z_1 \geqq |z_j| \quad (j=2, \ldots, n)\}.$$

We choose this cone as G'. (We have chosen such a particular cone in order that we may use the results in [7], Chap. III, §1.) For a point x in X, we denote by $\Omega(x)$ the set $\{z \in X;\ z-x \in \mathrm{Int}G'$, the interior of $G'\}$. Let ω_ρ denote $\{z \in X;\ \mathrm{Re}\, z_1 < -\rho\}$. Then it follows from Lemma 1.6 that there exists a positive number ρ_1 such that

(1.26) $$\mathrm{Ch}(\mathcal{K}_\bullet) \cap N^*(\omega_\rho) \cap \pi^{-1}(U) \subset T^*_X X$$

holds for $0 < \rho < \rho_1$. Let us choose ω_{ρ_0} $(0 < \rho_0 < \rho_1)$ as Ω_0' required in Theorem 1.2. Choosing a point x_1 so that

$$\Omega(x_1) \setminus \overline{\Omega_0'} \ni 0$$

and

$$\Omega(x_1) \setminus \overline{\omega_{\rho_1}} \Subset U$$

holds, we choose $\Omega(x_1)$ as Ω_1'. Let F' denote

$$\varphi_{G'}^{-1}\,\mathbb{R}\mathcal{H}om_{E(G,D)}(K_\cdot\,,\ \mathbb{R}\Gamma_{\Omega_1'\setminus\Omega_0'}(\mathbb{R}\varphi_{G^*}\mathcal{O}_{\mathbb{C}^n})).$$

Then for any point x in $\Omega_1'\setminus\overline{\Omega_0'}$, we have

$$(1.27)\qquad \mathcal{H}^j(F')_x$$

$$= \varinjlim_{\varepsilon\downarrow 0} H^j(\mathbb{R}\mathrm{Hom}_{E(G,D)}(K_\cdot\,,\ \mathbb{R}\Gamma_{\Omega(x(\varepsilon))\setminus\Omega_0'}(\Omega(x(\varepsilon)),\ \mathcal{O}_X))),$$

where $x(\varepsilon)$ denotes the point $x+\varepsilon(1,0,\ldots,0)$.

Now, it follows from (1.26) that $Ch(\mathcal{K}_\cdot)+N^*(\omega_{\rho_0})^a$ is Lagrangian. Hence we find by Lemma 1.6 that

$$(1.28)\qquad Ch(\mathcal{K}_\cdot)\cap N^*(\Omega(x(\varepsilon)))\cap\pi^{-1}(\Omega_1'\setminus\overline{\omega_{\rho_1}})\subset T_X^*X$$

and

$$(1.29)\qquad (Ch(\mathcal{K}_\cdot)+N^*(\omega_{\rho_0})^a)\cap N^*(\Omega(x(\varepsilon))\cap\pi^{-1}(\Omega_1'\setminus\overline{\omega_{\rho_1}})\subset T_X^*X$$

hold for $0<\varepsilon\ll 1$. Since (1.29) implies

$$(1.30)\qquad Ch(\mathcal{K}_\cdot)\cap(N^*(\Omega(x(\varepsilon))+N^*(\omega_{\rho_0}))\cap\pi^{-1}(\Omega_1'\setminus\overline{\omega_{\rho_1}})\subset T_X^*X,$$

we can find a strictly positive number ε_0 and a continuous function $\rho(\varepsilon)$ of ε for which the following two conditions are satisfied:

$$(1.31)\qquad \rho(\varepsilon) < \rho(\varepsilon') < \rho_0$$

holds for any ε and ε' such that $0<\varepsilon'<\varepsilon<\varepsilon_0$,

$$(1.32)\qquad Ch(\mathcal{K}_\cdot)\cap N^*(\Omega(x(\varepsilon))\cap\omega_\rho)\cap\pi^{-1}(\Omega_1'\setminus\overline{\omega_{\rho_1}})\subset T_X^*X$$

holds for $0<\varepsilon<\varepsilon_0$ and $\rho(\varepsilon)\leqq\rho\leqq\rho_0$.

Hence it follows from Theorem 4.5.1 of [9] that

$$(1.33)\qquad H^j(\mathbb{R}Hom_{E(G,D)}(K_\cdot, \mathbb{R}\Gamma_{\Omega(x(\varepsilon))\setminus\Omega_0'}(\Omega(x(\varepsilon)), \mathcal{O}_X)))$$

$$\xrightarrow{\sim} H^j(\mathbb{R}Hom_{E(G,D)}(K_\cdot, \mathbb{R}\Gamma_{\Omega(x(\varepsilon'))\setminus\Omega_0'}(\Omega(x(\varepsilon')), \mathcal{O}_X)))$$

holds for $0<\varepsilon'\leqq\varepsilon<\varepsilon_0$ for every j. (Cf. the note added in proof in p.54 of [9] and a remark in p.909 of [7].) Hence we have

$$(1.34)\qquad \mathcal{N}^j(F')_x \xleftarrow{\sim} H^j(\mathbb{R}Hom_{E(G,D)}(K_\cdot, \mathbb{R}\Gamma_{\Omega(x(\varepsilon))\setminus\Omega_0'}(\Omega(x(\varepsilon)), \mathcal{O}_X)))$$

for $0<\varepsilon<\varepsilon_0$. Hence it suffices for us to show the finite-dimensionality of the right hand side of (1.34). On the other hand, again by (1.32), we find

$$(1.35)\qquad H^j(\mathbb{R}Hom_{E(G,D)}(K_\cdot, \mathbb{R}\Gamma_{\Omega(x(\varepsilon))\setminus\Omega_0'}(\Omega(x(\varepsilon)), \mathcal{O}_X)))$$

$$\xrightarrow{\sim} H^j(\mathbb{R}Hom_{E(G,D)}(K_\cdot, \mathbb{R}\Gamma_{\Omega(x(\varepsilon))\setminus\omega_{\rho(\varepsilon)}}(\Omega(x(\varepsilon)), \mathcal{O}_X)))$$

for $0<\varepsilon<\varepsilon_0$. Hence we obtain

$$(1.36)\qquad H^j(\mathbb{R}\mathrm{Hom}_{E(G,D)}(K_\cdot, \mathbb{R}\Gamma_{\Omega(x(\varepsilon))\setminus\Omega_0'}(\Omega(x(\varepsilon)), \mathcal{O}_X)))$$

$$\underset{\to}{\sim} H^j(\mathbb{R}\mathrm{Hom}_{E(G,D)}(K_\cdot, \mathbb{R}\Gamma_{\Omega(x(\varepsilon'))\setminus\omega_{\rho(\varepsilon')}}(\Omega(x(\varepsilon')), \mathcal{O}_X)))$$

for $0<\varepsilon'<\varepsilon<\varepsilon_0$. Since $\Omega(x(\varepsilon))$, Ω_0', $\Omega(x(\varepsilon'))$ and $\omega_{\rho(\varepsilon')}$ are all convex, (1.33) and (1.35) entail

$$(1.37)\qquad H^j(\mathrm{Hom}_{E(G,D)}(K_\cdot, H^1_{\Omega(x(\varepsilon))\setminus\Omega_0'}(\Omega(x(\varepsilon)), \mathcal{O}_X)))$$

$$\underset{\to}{\sim} H^j(\mathrm{Hom}_{E(G,D)}(K_\cdot, H^1_{\Omega(x(\varepsilon'))\setminus\omega_{\rho(\varepsilon')}}(\Omega(x(\varepsilon')), \mathcal{O}_X)))$$

holds for every j. Hence the following two complexes $c_0^\bullet(\varepsilon)$ and $c^\bullet(\varepsilon')$ $(\varepsilon<\varepsilon')$ give rise to isomorphic cohomology groups:

$$(1.38)\qquad c_0^\bullet(\varepsilon):\ 0 \to (H^1_{\Omega(x(\varepsilon))\setminus\Omega_0'}(\Omega(x(\varepsilon)), \mathcal{O}_X))^{N_0}$$

$$\xrightarrow{P_0} (H^1_{\Omega(x(\varepsilon))\setminus\Omega_0'}(\Omega(x(\varepsilon)), \mathcal{O}_X))^{N_1}$$

$$\xrightarrow{P_1} \cdots\cdots$$

$$\xrightarrow{P_{d-1}} (H^1_{\Omega(x(\varepsilon))\setminus\Omega_0'}(\Omega(x(\varepsilon)), \mathcal{O}_X))^{N_d}$$

$$\longrightarrow 0.$$

$$(1.39)\qquad c^{\bullet}(\varepsilon'):\ 0 \longrightarrow (H^1_{\Omega(x(\varepsilon'))\setminus\omega_{\rho(\varepsilon')}}(\Omega(x(\varepsilon')),\mathcal{O}_X))^{N_0}$$

$$\xrightarrow{P_0} (H^1_{\Omega(x(\varepsilon'))\setminus\omega_{\rho(\varepsilon')}}(\Omega(x(\varepsilon')),\mathcal{O}_X))^{N_1}$$

$$\xrightarrow{P_1} \cdots\cdots$$

$$\xrightarrow{P_{d-1}} (H^1_{\Omega(x(\varepsilon'))\setminus\omega_{\rho(\varepsilon')}}(\Omega(x(\varepsilon')),\mathcal{O}_X))^{N_d}$$

$$\longrightarrow\ 0.$$

Thus our problem is reduced to the verification of the finite-dimensionality of $H^j(c_0^{\bullet}(\varepsilon))$ ($\simeq H^j(c^{\bullet}(\varepsilon'))$).

Now, by the results in [7], Chap. III, §1, we can understand the action of P_j's on $H^1_{\Omega(x(\varepsilon))\setminus\Omega_0'}(\Omega(x(\varepsilon)),\mathcal{O}_X)$ etc. as that of integral operators $K^{\beta}_{\alpha}(P_j)$'s introduced in p.875 of [7]. Let us choose complex numbers α_j and β_j ($j=0,\ldots,d$) and G'-round open subsets D_j and D_j' ($j=0,\ldots,d+1$) of D so that the following conditions (1.40), (1.41) and (1.42) are satisfied. If we choose from the first Ω_1' and Ω_0' so that $\Omega_1'\setminus\Omega_0'$ is sufficiently small, the recipe in p.875 of [7] assures the existence of such numbers and sets.

(1.40) $\quad K^{\beta_j}_{\alpha_j}(P_j)$ sends $\mathcal{O}_X(\Omega\cap D_j)$ (resp., $\mathcal{O}_X(\Omega\cap D_j')$) into $\mathcal{O}_X(\Omega\cap D_{j+1})$ (resp., $\mathcal{O}_X(\Omega\cap D_{j+1}')$) for any G'-open set Ω,

(1.41) $\quad D_{d+1}'\Supset \Omega_1'\setminus\bar{\Omega}_0'$, (1.42) $\quad D_j' \Subset D_j \quad (j=0,\ldots,d+1).$

Then Proposition 3.1.5 of [7] tells us that the action of P_j on $H^1_{\Omega(x(\varepsilon))\setminus\Omega_0'}(\Omega(x(\varepsilon)),\mathcal{O}_X)$ coincides with the action of the integral operator $K_{\alpha_j}^{\beta_j}(P_j)$ which sends $\mathcal{O}_X((\Omega(x(\varepsilon))\cap\Omega_0')\cap D_j)/\mathcal{O}_X(\Omega(x(\varepsilon))\cap D_j)$ to $\mathcal{O}_X((\Omega(x(\varepsilon))\cap\Omega_0')\cap D_{j+1})/\mathcal{O}_X(\Omega(x(\varepsilon))\cap D_{j+1})$ for each j. Hence, for each j, we obtain the following two commutative diagrams (1.43) and (1.44) with the additional constraint (1.45).

(1.43)

$$\begin{array}{ccc}
(\mathcal{O}_X(\Omega(x(\varepsilon))\cap D_j))^{N_j} & \xrightarrow{K_{\alpha_j}^{\beta_j}(P_j)} & (\mathcal{O}_X(\Omega(x(\varepsilon))\cap D_{j+1}))^{N_{j+1}} \\
\big\downarrow & & \big\downarrow \\
(\mathcal{O}_X((\Omega(x(\varepsilon))\cap\Omega_0')\cap D_j))^{N_j} & \xrightarrow{K_{\alpha_j}^{\beta_j}(P_j)} & (\mathcal{O}_X((\Omega(x(\varepsilon))\cap\Omega_0')\cap D_{j+1}))^{N_{j+1}}
\end{array}$$

(1.44)

$$\begin{array}{ccc}
(\mathcal{O}_X(\Omega(x(\varepsilon))\cap D_{j+1}))^{N_{j+1}} & \xrightarrow{K_{\alpha_{j+1}}^{\beta_{j+1}}(P_{j+1})} & (\mathcal{O}_X((\Omega(x(\varepsilon))\cap D_{j+2}))^{N_{j+2}} \\
\big\downarrow & & \big\downarrow \\
(\mathcal{O}_X((\Omega(x(\varepsilon))\cap\Omega_0')\cap D_{j+1}))^{N_{j+1}} & \xrightarrow{K_{\alpha_{j+1}}^{\beta_{j+1}}(P_{j+1})} & (\mathcal{O}_X(\Omega(x(\varepsilon))\cap\Omega_0')\cap D_{j+2}))^{N_{j+2}}
\end{array}$$

(1.45)

$$K_{\alpha_{j+1}}^{\beta_{j+1}}(P_{j+1})K_{\alpha_j}^{\beta_j}(P_j)(\mathcal{O}_X((\Omega(x(\varepsilon))\cap\Omega_0')\cap D_j))^{N_j}\subset\mathcal{O}_X(\Omega(x(\varepsilon))\cap D_{j+2})^{N_{j+2}}.$$

We note that $K_{\alpha_{d+1}}^{\beta_{d+1}}(P_{d+1})$ is understood to be zero here and in what follows. We also find similar diagrams with $\Omega(x(\varepsilon))$, $\Omega(x(\varepsilon))\cap\Omega_0'$ and D_j being replaced by $\Omega(x(\varepsilon'))$, $\Omega(x(\varepsilon'))\cap\omega_{\rho(\varepsilon')}$ and D_j', respectively.

Since it is cumbersome to treat the diagrams (1.43) and (1.44) as they stand, we prepare a lemma which enables us to reduce the situation to a simpler one.

Lemma 1.7. Let $\{X^j\}$, $\{Y^j\}$ and $\{Z^j\}$ be objects in an Abelian category $\mathcal{A}$. Suppose the following data (1.46), (1.47) and (1.48) are given:

(1.46) Exact sequences $0 \to X^j \xrightarrow{f^j} Y^j \xrightarrow{h^j} Z^j \to 0$,

(1.47) $\varphi^j : X^j \to X^{j+1}$, $\psi^j : Y^j \to Y^{j+1}$

and $\phi^j : Z^j \to Z^{j+1}$ which

give rise to the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & X^j & \longrightarrow & Y^j & \longrightarrow & Z^j & \longrightarrow & 0 \\ & \varphi^j \downarrow & & \psi^j \downarrow & & \phi^j \downarrow & & \\ 0 \to & X^{j+1} & \longrightarrow & Y^{j+1} & \longrightarrow & Z^{j+1} & \longrightarrow & 0, \end{array}$$

(1.48) $g^j : Y^j \to X^{j+2}$ such that

$\psi^{j+1} \circ \psi^j = f^{j+2} \circ g^j$.

Then we find the following:

(i) $\phi^{j+1} \circ \phi^j = 0$.

(ii) Set $W^j = X^{j+1} \oplus Y^j$ and

define $d_W^j : W^j \to W^{j+1}$ by

$$\begin{bmatrix} -\varphi^{j+1} & g^j \\ -f^{j+1} & \psi^j \end{bmatrix}$$

Then $d_W^{j+1} \circ d_W^j = 0$ holds.

(iii) Let $p^j \colon W^j \to Z^j$ be given by $(0, h^j)$. Then $\{p^j\}$ is a quasi-isomorphism of complexes $W^\bullet \to Z^\bullet$.

Since the proof of this lemma is straightforward, we leave it to the reader. We apply Lemma 1.7 to our situation by the following correspondence:

$$X^j = (\mathcal{O}_X(\Omega(x(\varepsilon)) \cap D_j))^{N_j},$$

$$Y^j = (\mathcal{O}_X((\Omega(x(\varepsilon)) \cap \Omega_0') \cap D_j))^{N_j},$$

f^j: the natural injection,

$$\begin{aligned} Z^j &= Y^j/X^j \\ &\cong (H^1_{\Omega(x(\varepsilon))\setminus\Omega_0'}(\Omega(x(\varepsilon)), \mathcal{O}_X))^{N_j}, \end{aligned}$$

$$\varphi^j = \psi^j = K_{\alpha_j}^{\beta_j}(P_j),$$

$$\phi^j = P_j.$$

It is then clear that all the assumptions in Lemma 1.7 are satisfied. We denote the resulting complex $W^\bullet$ by $W_0^\bullet(\varepsilon)$. Note that the complex $Z^\bullet$ in this case is nothing but $c_0^\bullet(\varepsilon)$. We obtain a similar complex $W(\varepsilon')$ by replacing $\Omega(x(\varepsilon))$, Ω_0' and D_j respectively with $\Omega(x(\varepsilon'))$, $\omega_{\rho(\varepsilon')}$ and D_j' in the above correspondence. Since $H^j(c_0^\bullet(\varepsilon)) \tilde{\to} H^j(c^\bullet(\varepsilon'))$ holds, Lemma 1.7 implies

(1.49) $\quad H^j(W_0^\bullet(\varepsilon)) \tilde{\to} H^j(W(\varepsilon'))$

for every j. Hence, in view of (1.42), we can conclude from a classical result on compact perturbation in functional analysis (see e.g. [4] and references cited therein) that $H^j(W_0^{\bullet}(\varepsilon))$ is finite-dimensional for every j. Hence we find $H^j(c_0^{\bullet}(\varepsilon))$ is finite-dimensional. This completes the proof of Theorem 1.2.

Next let us prove Theorem 1.5. In what follows we denote by φ(resp., φ_a) the canonical map from X to $X_{G'}$(resp., X_{G^a}). Let $\tilde{\varphi}$ denote $1\times\varphi_a$. It then follows from the definition of the sheaf $\mathcal{E}_X^{\mathbb{R}}$ and the complex $K_\bullet$ that

$$(1.50)\qquad K_\bullet = \mu_\Delta(\tilde{\varphi}^{-1}(\mathbb{R}\Gamma_{X\times(\tilde{\Omega}_1\setminus\tilde{\Omega}_0)}(\mathbb{R}\tilde{\varphi}_*\mathcal{O}_{X\times X}^{(0,n)})\overset{\mathbb{L}}{\underset{E(G,D)}{\otimes}}K_\bullet))[n]$$

holds on $(\tilde{\Omega}_1\setminus\overline{\tilde{\Omega}}_0)\times G'^{\circ a}$ for G'^a-open sets $\tilde{\Omega}_1$ and $\tilde{\Omega}_0$ such that $\tilde{\Omega}_1\setminus\overline{\tilde{\Omega}}_0\Subset U$. Here $\mathcal{O}_{X\times X}^{(0,n)}$ denotes the sheaf of holomorphic n-forms with respect to the second variable. In what follows, we abbreviate $E(G,D)$ to E for the simplicity of notations. Let us first note the following

<u>Lemma 1.8</u>. Let $\mathcal{O}_X^{(n)}$ denote the sheaf of holomorphic n-forms on X. Then we have the following isomorphism i on $X\times\tilde{\Omega}_1$:

$$(1.51)\qquad p_1^{-1}\mathcal{O}_X\underset{\mathbb{C}}{\otimes}p_2^{-1}(\varphi_a^{-1}(\mathbb{R}\Gamma_{\tilde{\Omega}_1\setminus\tilde{\Omega}_0}(\mathbb{R}\varphi_{a*}\mathcal{O}_X^{(n)})\overset{\mathbb{L}}{\underset{E}{\otimes}}K_\bullet))$$

$$\overset{i}{\to}\tilde{\varphi}^{-1}(\mathbb{R}\Gamma_{X\times(\tilde{\Omega}_1\setminus\tilde{\Omega}_0)}(\mathbb{R}\tilde{\varphi}_*\mathcal{O}_{X\times X}^{(0,n)})\overset{\mathbb{L}}{\underset{E}{\otimes}}K_\bullet)$$

Proof. Since the existence of the homomorphism i is clear, it suffices to verify

$$(1.52)\quad \mathcal{O}_{X,x} \underset{\mathbb{C}}{\otimes} \mathcal{H}^j(\varphi_a^{-1}(\mathbb{R}\Gamma_{\tilde{\Omega}_1\setminus\tilde{\Omega}_0}(\mathbb{R}\varphi_{a*}\mathcal{O}_X^{(n)}) \overset{\mathbb{L}}{\underset{E}{\otimes}} K_\cdot))_y$$

$$= \mathcal{H}^j(\tilde{\varphi}^{-1}(\mathbb{R}\Gamma_{X\times(\tilde{\Omega}_1\setminus\tilde{\Omega}_0)}(\mathbb{R}\tilde{\varphi}_*\mathcal{O}_{X\times X}^{(0,n)}) \overset{\mathbb{L}}{\underset{E}{\otimes}} K_\cdot))_{(x,y)}$$

for every j and every x in X and every y in $\tilde{\Omega}_1$. In proving (1.52) we may suppose without loss of generality that both $\tilde{\Omega}_1$ and $\tilde{\Omega}_0$ are pseudo-convex. Then, by using Lemma 1.7 as in the proof of Theorem 1.2, we find that

$$\mathcal{H}^j(\varphi_a^{-1}(\mathbb{R}\Gamma_{\tilde{\Omega}_1\setminus\tilde{\Omega}_0}(\mathbb{R}\varphi_{a*}\mathcal{O}_X^{(n)}) \overset{\mathbb{L}}{\underset{E}{\otimes}} K_\cdot))_y$$

is isomorphic to the j-th cohomology group of a complex of nuclear Fréchet spaces. Since it is finite-dimensional, we find (1.52) by making the topologically completed tensor product of $\mathcal{O}_X(\omega)$ and the complex, where ω is an open neighborhood of x, and then taking the inductive limit with respect to ω. Q.E.D.

In view of this lemma, we next study

$$\mathcal{A} \underset{\text{def}}{=} (\mathbb{R}\Gamma_{\tilde{\Omega}_1\setminus\tilde{\Omega}_0}(\mathbb{R}\varphi_{a*}\mathcal{O}_X^{(n)}) \overset{\mathbb{L}}{\underset{E}{\otimes}} K_\cdot)[n]$$

$$= (\mathbb{R}\varphi_{a*}\mathbb{R}\Gamma_{\tilde{\Omega}_1\setminus\tilde{\Omega}_0}(\mathcal{O}_X^{(n)}) \overset{\mathbb{L}}{\underset{E}{\otimes}} K_\cdot)[n].$$

In order to rewrite $\mathcal{A}$, let us note the following

Lemma 1.9. There exists a canonical right $\mathcal{D}^{\infty}$-linear homomorphism from $\mathcal{O}_X^{(n)}$ to $\mathbb{R}\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X,\mathbb{C})[n]=\mathcal{E}xt^n_{\mathbb{C}}(\mathcal{O}_X,\mathbb{C})$,

Proof. By the Poincaré duality we have

$$(1.53)\qquad \mathbb{R}\Gamma(\omega, \mathbb{R}\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X,\mathbb{C})) = \mathbb{R}\mathrm{Hom}(\mathbb{R}\Gamma_c(\omega,\mathcal{O}_X), \mathbb{C})[-2n]$$

for any open subset ω of an n-dimensional complex manifold X. On the other hand, if ω is pseudo-convex,

$$H^k_c(\omega, \mathcal{O}_X) = 0$$

holds for $k\neq n$, and hence

$$(1.54)\qquad H^j(\omega, \mathbb{R}\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X,\mathbb{C}))[n] \cong \mathrm{Hom}_{\mathbb{C}}(H^{n-j}_c(\omega, \mathcal{O}_X), \mathbb{C})$$
$$= \begin{cases} \mathrm{Hom}(H^n_c(\omega; \mathcal{O}_X), \mathbb{C}) & (j = 0) \\ 0 & (j \neq 0). \end{cases}$$

Since we have a canonical map from $H^n_c(\omega, \mathcal{O}_X^{(n)})$ to $\mathbb{C}$, we have a canonical map from $\mathcal{O}_X^{(n)}(\omega)$ to $\mathrm{Hom}_{\mathbb{C}}(H^n_c(\omega; \mathcal{O}_X), \mathbb{C})$. Thus we have a canonical map from $\mathcal{O}_X^{(n)}$ to $\mathbb{R}\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X,\mathbb{C})[n]$ $=\mathcal{E}xt^n_{\mathbb{C}}(\mathcal{O}_X, \mathbb{C})$. Q.E.D.

By this lemma we obtain a canonical map δ from $\mathcal{A}$ to

$$\mathcal{B} \underset{\mathrm{def}}{=} (\mathbb{R}\varphi_{a*}\mathbb{R}\Gamma_{\tilde{\Omega}_1\setminus\tilde{\Omega}_0}\mathbb{R}\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X,\mathbb{C}) \underset{E}{\otimes} K_\bullet)[2n].$$

On the other hand, for a $G^{\prime a}$-open convex set ω, we have

$$(1.55)\qquad \mathbb{R}\Gamma(\omega,\mathcal{A}) = ((H^0(\omega\cap\tilde{\Omega}_0,\mathcal{O}_X^{(n)})/H^0(\omega,\mathcal{O}_X^{(n)}))\underset{E}{\otimes} K_{\bullet})[n]$$

and

$$(1.56)\qquad \mathbb{R}\Gamma(\omega,\mathcal{B}) = (\mathrm{Hom}_{\mathbb{C}}((H_c^n(\omega,\mathcal{O}_X)/H_c^n(\omega\cap\tilde{\Omega}_0,\mathcal{O}_X),\mathbb{C})\underset{E}{\otimes} K_{\bullet})[n]$$

$$= \mathrm{Hom}_{\mathbb{C}}(\mathbb{R}\mathrm{Hom}_E(K_{\bullet},(H_c^n(\omega,\mathcal{O}_X)/H_c^n(\omega\cap\tilde{\Omega}_0,\mathcal{O}_X))),\mathbb{C})[n].$$

Hence, with the aid of Lemma 1.7, Serre's duality theorem tells us

$$(1.57)\qquad \mathbb{R}\Gamma(\omega,\mathcal{A}) \tilde{\to} \mathbb{R}\Gamma(\omega,\mathcal{B})$$

Therefore the canonical map $\delta:\mathcal{A}\to\mathcal{B}$ is an isomorphism.

In order to rewrite $\varphi_a^{-1}\mathcal{B}$ further, we note the following

Lemma 1.10. Let F be a complex of sheaves on X. Then there exists another complex R of sheaves on X which satisfies the following two conditions:

(1.58) R forms the following distinguished triangle:

$$\begin{tikzcd} & \varphi_a^{-1}\mathbb{R}\varphi_{a*}\mathbb{R}\mathcal{H}om_{\mathbb{C}}(F,\mathbb{C}) \arrow[dl] & \\ \mathbb{R}\mathcal{H}om_{\mathbb{C}}(\varphi^{-1}\mathbb{R}\varphi_*F,\mathbb{C}) \arrow[rr] & & R, \arrow[ul, "+1"'] \end{tikzcd}$$

$$(1.59)\qquad SS(R)\cap(X\times G'^{\circ}) = \emptyset.$$

Proof. Since the existence of the triangle (1.58) is obvious, it suffices to verify (1.59). For this purpose let us first note the existence of the following distinguished triangles (1.60) and (1.61) for which (1.62) and (1.63) hold.

(1.60)
$$\begin{tikzcd} & \varphi_a^{-1}\mathbb{R}\varphi_{a*}\mathbb{R}\mathcal{H}om_{\mathbb{C}}(F,\mathbb{C}) \arrow[dl] & \\ \mathbb{R}\mathcal{H}om_{\mathbb{C}}(F,\mathbb{C}) \arrow[rr] & & R', \arrow[ul, "+1"'] \end{tikzcd}$$

(1.61)
$$\begin{tikzcd} & \varphi^{-1}\mathbb{R}\varphi_{*}F \arrow[dl] & \\ F \arrow[rr] & & F', \arrow[ul, "+1"'] \end{tikzcd}$$

(1.62) $SS(R') \cap (X \times G^{\circ}) = \emptyset$,

(1.63) $SS(F') \cap (X \times G^{\circ a}) = \emptyset$.

The triangle (1.61) induces

$$\begin{tikzcd} & \mathbb{R}\mathcal{H}om_{\mathbb{C}}(\varphi^{-1}\mathbb{R}\varphi_{*}F,\mathbb{C}) \arrow[dr, "+1"] & \\ \mathbb{R}\mathcal{H}om_{\mathbb{C}}(F,\mathbb{C}) \arrow[ur] & & \mathbb{R}\mathcal{H}om_{\mathbb{C}}(F',\mathbb{C}), \arrow[ll] \end{tikzcd}$$

where

$$SS(\mathbb{R}\mathcal{H}om_{\mathbb{C}}(F',\mathbb{C})) \subset SS(F')^{a}.$$

Then the hexagon axiom assures the existence of the following distinguished triangle

$$
\begin{tikzcd}
 & R \arrow[dl] & \\
\mathbb{R}\mathcal{H}om(F', \mathbb{C}) \arrow[rr] & & R', \arrow[ul]
\end{tikzcd}
$$

which implies (1.59). This completes the proof of Lemma 1.10.

Since

$$(1.64)\qquad \mathbb{R}\Gamma_{\tilde{\Omega}_1\setminus\tilde{\Omega}_0}(\mathbb{R}\mathcal{H}om_{\mathbb{C}}(F,\mathbb{C})) = \mathbb{R}\mathcal{H}om_{\mathbb{C}}(F_{\tilde{\Omega}_1\setminus\tilde{\Omega}_0},\mathbb{C})$$

holds by the definition of $F_{\tilde{\Omega}_1\setminus\tilde{\Omega}_0}$, we find

$$(1.65)\qquad \varphi_a^{-1}\mathcal{B} = \varphi_a^{-1}(\mathbb{R}\varphi_{a*}\mathbb{R}\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_{\tilde{\Omega}_1\setminus\tilde{\Omega}_0},\mathbb{C}) \underset{E}{\otimes} K_\cdot)[2n].$$

Hence it follows from Lemma 1.10 that

$$
\begin{aligned}
(1.66)\qquad \varphi_a^{-1}\mathcal{B} &= (\mathbb{R}\mathcal{H}om_{\mathbb{C}}(\varphi^{-1}\mathbb{R}\varphi_*\mathcal{O}_{\tilde{\Omega}_1\setminus\tilde{\Omega}_0},\mathbb{C}) \overset{\mathbb{L}}{\underset{E}{\otimes}} K_\cdot)[2n] \\
&= \mathbb{R}\mathcal{H}om_{\mathbb{C}}(\varphi^{-1}\mathbb{R}\mathcal{H}om_E(K_\cdot,\mathbb{R}\varphi_*\mathcal{O}_{\tilde{\Omega}_1\setminus\tilde{\Omega}_0}),\mathbb{C})[2n]
\end{aligned}
$$

holds on $(\tilde{\Omega}_1\setminus\bar{\tilde{\Omega}}_0)\times G'^{\circ}$. Let us now take G'-open sets Ω_1' and Ω_0' so that the following three conditions may be satisfied:

$$(1.67)\qquad \tilde{\Omega}_1\setminus\bar{\tilde{\Omega}}_0 \supset \Omega_1'\setminus\Omega_0',$$

(1.68) $(\Omega_1'\setminus\Omega_0')\times G'^{\circ a}$ is a neighborhood of x^*,

(1.69) $$F' \underset{\text{def}}{=} \varphi^{-1}\,\mathbb{R}\mathcal{H}om_E(K_\bullet, \mathbb{R}\Gamma_{\Omega_1'\setminus\Omega_0'}(\mathbb{R}\varphi_*\mathcal{O}_X))$$

is $\mathbb{R}$-constructible.

Then we obtain

(1.70) $$\mathbb{R}\mathcal{H}om_{\mathbb{C}}(\varphi^{-1}\,\mathbb{R}\mathcal{H}om_E(K_\bullet, \mathbb{R}\varphi_*\mathcal{O}_{\tilde{\Omega}_1\setminus\tilde{\Omega}_0}),\mathbb{C})\Big|_{\tilde{\Omega}_1\cap\Omega_1'}[2n]$$

$$\cong \mathbb{R}\mathcal{H}om_{\mathbb{C}}(\varphi^{-1}\,\mathbb{R}\mathcal{H}om_E(K_\bullet, \mathbb{R}\Gamma_{\Omega_1\setminus\Omega_0'}(\mathbb{R}\varphi_*\mathcal{O}_X)),\mathbb{C})\Big|_{\tilde{\Omega}_1\cap\Omega_1'}[2n]$$

on $(\Omega_1'\setminus\overline{\Omega}_0')\times G^\circ$. Hence we find

(1.71) $$\mathcal{K}_\bullet = \mu_\Delta(p_1^{-1}\mathcal{O}_X \underset{\mathbb{C}}{\otimes} p_2^{-1}\,\mathbb{R}\mathcal{H}om_{\mathbb{C}}(F',\mathbb{C}))[2n]$$

holds on a neighborhood of x^*. Therefore, by the assumption (1.69) on F', we finally obtain

(1.72) $$\mathcal{K}_\bullet = \mu_\Delta(\mathbb{R}\mathcal{H}om_{\mathbb{C}}(p_2^{-1}F', p_1^{-1}\mathcal{O}_X))[2n]$$

on a neighborhood of x^*. This completes the proof of Theorem 1.5.

§2. Applications to the study of theta-functions.

In this section we show how Theorem 1.4 is used to prove the finite-dimensionality of the space of microfunction solutions (=Jacobi (micro)functions) of a particular system of microdifferential equations associated with a Jacobi structure to be defined below. As we explained in the introduction of [13], the classical theta-zerovalue $\sum_{\nu\in\mathbb{Z}} \exp(\pi\sqrt{-1}\nu^2 t)$ (regarded as a microfunction) is one of the simplest examples of Jacobi functions.

<u>Definition 2.1</u>. (Jacobi structure). Let (V,E) be a symplectic vector space over $\mathbb{C}$ and L a lattice in V. Let X be a complex manifold and W an open subset of $\overset{\circ}{T}{}^*X$. Let $\varphi : V \to \mathcal{E}_{(1/2)}(W)$ be a linear map. Then the pair P of φ and (V,L,E) is called a Jacobi structure (of restricted type) on W if the following conditions (2.1), (2.2) and (2.3) are satisfied:

(2.1) $$[\varphi(v),\ \varphi(v')] = E(v,v')$$

holds for any v and v' in V,

(2.2) E is $2\pi\sqrt{-1}\mathbb{Z}$-valued on $L\times L$,

(2.3) $$\operatorname{ord} \varphi(v) \leqq 1/2$$

holds for every v.

Remark 2.2. (i) In this article we restrict our consideration to the case where $\varphi(v)$ is of order at most $1/2$. This is the reason why we use the terminology "Jacobi structure (of restricted type)". (Cf. [12],[13]) In what follows, we will omit "of restricted type".

(ii) Although $\varphi(v)$'s are not commutative, (2.1) and (2.2) guarantee that $\exp \varphi(v)$ $(v \in L)$ *do* commute. In fact, the Campbell-Hausdorff formula (cf. [2]) tells us

$$
\begin{aligned}
&\exp \varphi(u) \exp \varphi(v) \\
&\qquad = \exp(\varphi(u) + \varphi(v) + \tfrac{1}{2}E(u,v)) \\
&\qquad = \exp \varphi(v) \exp \varphi(u) \exp(E(u,v)) \\
&\qquad = \exp \varphi(v) \exp \varphi(u)
\end{aligned}
$$

holds, if u and v belong to L.

(iii) The condition (2.3) guarantees that $\exp \varphi(u)$ belongs to $\mathcal{E}^{\infty}_{(1/2)}$ for every u in V. (Cf. [14], p.438 ~ p.442 and [3].)

In order to define a complex of $\mathcal{E}^{\mathbb{R}}$-Modules associated with a Jacobi structure, let us further introduce a function $c : L \to \mathbb{C}^{\times} (\underset{\text{def}}{=} \mathbb{C} - \{0\})$ which satisfies the following condition:

$$
(2.4) \qquad c(u + v) = \exp(E(u,v)/2)c(u)c(v).
$$

It is obvious that such a function c really exists. (Actually there exist infinitely many, although they are all equivalent in the sense to be explained in Remark 2.3 below.) If we define $J_c(u)$ by $c(u)\exp\varphi(u)$, then (2.4) combined with Campbell-Hausdorff formula entails

$$(2.5) \qquad J_c(u+v) = J_c(u)J_c(v)$$

for any u and v in L. We then let the group algebra $\mathbb{C}[L]$ to act upon $\mathcal{E}^\infty_{(1/2)}$ from the right by $p\circ u = pJ_c(u)$ for p in $\mathcal{E}^\infty_{(1/2)}$ and u in L. Let us denote by $(\mathcal{E}^\infty_{(1/2)},c)$ the $(\mathcal{E}^\infty_{(1/2)},\mathbb{C}[L])$-bi-Module thus obtained.

<u>Remark 2.3</u>. For each $c' : L \to \mathbb{C}^\times$ that satisfies (2.4), $(\mathcal{E}^\infty_{(1/2)},c')$ is isomorphic to $(\mathcal{E}^\infty_{(1/2)},c)$ as $(\mathcal{E}^\infty_{(1/2)},\mathbb{C}[L])$-bi-Module. In fact, there exists u_0 in V which satisfies

$$c'(u) = c(u)\exp E(u,u_0)$$

for any u in L. Hence we find

$$\begin{aligned} &(p\ \exp\varphi(u_0))c'(u)\exp\varphi(u) \\ &= p\ c(u)\exp\varphi(u)\exp\varphi(u_0), \end{aligned}$$

that is, the map : $p \mapsto p\exp\varphi(u_0)$ is an isomorphism from $(\mathcal{E}^\infty_{(1/2)},c)$ to $(\mathcal{E}^\infty_{(1/2)},c')$. Therefore we will abbreviate $(\mathcal{E}^\infty_{(1/2)},c)$ to $\mathcal{E}^\infty_{(1/2)}$ for short.

Now let us define a complex $\mathcal{K}_{\bullet}(P)$ associated with the Jacobi structure P by the following:

$$\mathcal{K}_{\bullet}(P) = \mathcal{E}^{\infty}_{(1/2)} \overset{\mathbb{L}}{\underset{\mathbb{C}[L]}{\otimes}} \mathbb{C}. \tag{2.6}$$

Note that $\mathbb{C}$ has a free resolution of finite length as a $\mathbb{C}[L]$-module, e.g., the so-called Koszul complex. Hence, for a basis $(u_1,\dots,u_d)$ of L, $\mathcal{K}_{\bullet}(P)$ is quasi-isomorphic to the following complex:

$$0 \leftarrow \mathcal{E}^{\infty}_{(1/2)} \xleftarrow{\begin{bmatrix} J_c(u_1) - 1 \\ \vdots \\ J_c(u_d) - 1 \end{bmatrix}} (\mathcal{E}^{\infty}_{(1/2)})^d \longleftarrow$$

$$\cdots \longleftarrow (\mathcal{E}^{\infty}_{(1/2)})^d \xleftarrow[(J_c(u_1) - 1,\dots,J_c(u_d) - 1)]{} \mathcal{E}^{\infty}_{(1/2)} \leftarrow 0.$$

We now introduce the definition of Jacobi functions. Here and in what follows, M denotes a real analytic manifold, X its complexification and W an open subset of $\overset{\circ}{T}{}^*X$.

<u>Definition 2.4</u>. A Jacobi (micro)function (with respect to a Jacobi structure P and $c : L \to \mathbb{C}^{\times}$ satisfying (2.4)) is, by definition, a microfunction f on $W \cap T^*_M X$ that satisfies

$$c(u)\exp(\varphi(u))f = f$$

for any u in L.

Remark 2.5. It immediately follows from the above definition that the sheaf of Jacobi functions is isomorphic to $\mathcal{H}om(\mathcal{K}_{\cdot}(P), \mathcal{C}_M)$.

Remark 2.6. Since there is a one-to-one correspondence between a Jacobi function and the zerovalue of a theta function in the sense of [12],[13], a Jacobi function is sometimes called a theta-zerovalue. (Cf. [12], Theorem, [13], Theorem 2.10.)

Remark 2.7. In order to make the notation more symmetric, we have changed the definition of c_j used in [12],[13], that is, c_j used there corresponds to $c(u_j)^{-1}$ in the notation used here, where $(u_1, \ldots, u_d)$ is a basis of L.

Now we have the following

Theorem 2.8. Let $P = (\varphi, V, L, E)$ be a Jacobi structure on $W \subset \overset{\circ}{T}{}^*X$. Suppose

$$\text{(2.7)} \qquad \dim V = 2\dim X.$$

Then the sheaf of Jacobi functions, i.e., $\mathcal{H}om(\mathcal{K}_{\cdot}(P), \mathcal{C}_M)$ is an $\mathbb{R}$-constructible sheaf on $W \cap T^*_M X$.

In order to deduce Theorem 2.8 from Theorem 1.4, let us first recall the following Lemma 2.9 due to Aoki. As a matter of fact, Aoki's result on the invertibility of microdifferential operators is a quite general one and the following lemma is a very special case of it. ([1], see also [2].)

Lemma 2.9. Let x^* be a point in $\mathring{T}^*X$ and let p be in $\mathcal{E}_{(\lambda),x^*}$ for some λ. Suppose that $\operatorname{ord} p < 1$. Then, for any complex number c, there exists R in $\mathcal{E}^{\mathbb{R}}_{X,x^*}$ such that

$$(2.8) \qquad R(c \exp p - 1) = (c \exp p - 1)R = 1,$$

if $\sigma(p)(x^*)$ is not a purely imaginary number. Here $\sigma(p)$ denotes the principal symbol of p.

Lemma 2.9 guarantees that all the cohomology groups of $\mathcal{K}_{\bullet}(P)$ vanish outside

$$\Lambda = \{(z,\zeta) \in W \ ; \ \operatorname{Re}\sigma_{1/2}(\varphi(v))(z,\zeta) = 0 \quad \text{for any } v \text{ in } L\}.$$

This implies

$$(2.9) \qquad \operatorname{Ch}(\mathcal{K}_{\bullet}(P)) \subset \Lambda.$$

We now want to show that Λ is a Lagrangian subset of $(T^*X)^{\mathbb{R}}$. In order to prove this, let us introduce a map $\chi : W \to V^*$ defined by

$$(2.10) \qquad \chi(y^*)(v) = \sigma_{1/2}(\varphi(v))(y^*)$$

for y^* in W and v in V. Since

$$(2.11) \qquad E(v,v') = \{\sigma_{1/2}(\varphi(v)),\sigma_{1/2}(\varphi(v'))\}$$

follows from (2.1) and (2.3), (2.7) guarantees that χ is a symplectic transformation. Here the right hand side of (2.11) denotes the Poisson bracket of $\sigma_{1/2}(\varphi(v))$ and $\sigma_{1/2}(\varphi(v'))$. Furthermore, if we define $V^*_{\mathbb{R}}$ by $\{\lambda \in V^* ; \lambda(L) \subset \mathbb{R}\}$, it is a real Lagrangian subspace of real symplectic space $(V^*, E^* + \bar{E}^*)$, where E^* denotes the dual form of E and $\bar{E}^*$ denotes the complex conjugate of E^*. Since $\Lambda = \chi^{-1}(V^*_{\mathbb{R}})$ by the definition, Λ is a Lagrangian subset of $(T^*X)^{\mathbb{R}}$. Hence (2.9) implies that $Ch(\mathcal{K}_{\bullet}(P))$ is contained in a Lagrangian subanalytic subset of $(T^*X)^{\mathbb{R}}$. Hence Theorem 1.4 entails Theorem 2.8.

Now, Theorem 2.8 tells us that Jacobi structures provides us with good examples of $\mathbb{R}$-holonomic complexes. For example, using Theorem 1.5 and the results in [11], we can verify the sheaf $\mathcal{C}_M$ of microfunctions regarded as an $\mathcal{E}^{\mathbb{R}}$-Module is an $\mathbb{R}$-holonomic complex obtained through a Jacobi structure.

Furthermore, the similarity of the assertions for holonomic systems and those for $\mathbb{R}$-holonomic complexes indicates that a very wild function such as the Jacobi function, or the theta-zerovalue $\sum_{\nu \in \mathbb{Z}} \exp(\pi\sqrt{-1}\nu^2 t)$ may be controlled microlocally by a system of microdifferential equations. In order to exemplify this expectation, let us indicate how we can deduce from the Jacobi structure the automorphy property of a Jacobi function under the action of $Sp(n;\mathbb{Z})$. In order to discuss such a global problem, we generalize the notion of Jacobi structure so that φ is a map from V to

$$\mathcal{E}_{(1/2)}(\mathcal{L}) \underset{\mathrm{def}}{=} \pi^{-1}\mathcal{L} \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \mathcal{E}_{(1/2)} \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \pi^{-1}\mathcal{L}^{\otimes -1}$$

for a line bundle $\mathcal{L}$ on X. In what follows, $\mathcal{E}^{\infty}_{(1/2)}(\mathcal{L})$ denotes $\pi^{-1}\mathcal{L} \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \mathcal{E}^{\infty}_{(1/2)} \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \pi^{-1}\mathcal{L}^{\otimes -1}$ and $\mathcal{C}_M(\mathcal{L})$ denotes $\pi^{-1}\mathcal{L} \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \mathcal{C}_M$. Let $S^2(V)$ denote the symmetric product of V and define a linear map $\varepsilon : S^2(V) \to gl(V)$ by

$$\text{(2.12)} \qquad \varepsilon(v_1 \cdot v_2)(v) = E(v_1, v)v_2 + E(v_2, v)v_1$$

for $v_1 \cdot v_2$ in $S^2(V)$ and v in V. Then ε is an isomorphism between $S^2(V)$ and $sp(V)$. Define a map s from $S^2(V)$ to $\mathcal{E}_{(1/2)}(\mathcal{L})(W)$ by assigning

$$\tfrac{1}{2}(\varphi(v_1)\varphi(v_2) + \varphi(v_2)\varphi(v_1))$$

to $v_1 \cdot v_2$ in $S^2(V)$ and let Q denote $s \circ \varepsilon^{-1}$. Using

(2.12) we can easily verify that Q is a Lie algebra homomorphism from $sp(V)$ to $\mathcal{E}_{(1/2)}(\mathcal{L})(W)$ which satisfies

$$\varphi(av) = [Q(a), \varphi(v)]$$

for any a in $sp(V)$ and any v in V. The fact that Q is a Lie algebra homomorphism implies

$$\{\sigma_1(Q(a)), \sigma_1(Q(a'))\} = \sigma_1(Q([a,a']))$$

for any a and a' in $sp(V)$. Hence $sp(V)$ acts upon W by the Hamiltonian map $H_{\sigma_1(Q(a))}$ $(a \in sp(V))$. In order to discuss the global structure of microfunction solutions, we suppose a reality condition as follows :

Let $V_{\mathbb{R}}$ denote $\mathbb{R} \underset{\mathbb{Z}}{\otimes} L$ and let $\widetilde{Sp}(V_{\mathbb{R}})$ denote the universal covering space of the symplectic group $Sp(V_{\mathbb{R}})$. We suppose that $\widetilde{Sp}(V_{\mathbb{R}})$ acts upon $W \cap T^*_M X$ so that its infinitesimal action is given by the Hamiltonian map $H_{\sigma_1(Q(a))}$.

For g in $\widetilde{Sp}(V_{\mathbb{R}})$, we denote by $\psi(g)$ the action described above, and by $(\Psi(g), \Phi(g))$ the corresponding quantized contact transformation. To be more precise, for g in $\widetilde{Sp}(V_{\mathbb{R}})$, $\Psi(g)$ is a $\mathbb{C}$-Algebra isomorphism from $\mathcal{E}^{\infty}_{(1/2)}(\mathcal{L})\big|_{W \cap T^*_M X}$ to $\psi(g)_*(\mathcal{E}^{\infty}_{(1/2)}(\mathcal{L})\big|_{W \cap T^*_M X})$ and $\Phi(g)$ is an isomorphism from $\mathcal{C}_M(\mathcal{L})\big|_{W \cap T^*_M X}$ to $\psi(g)_*\mathcal{C}_M(\mathcal{L})\big|_{W \cap T^*_M X}$ for which the following conditions (2.13)~(2.16) are satisfied.

(2.13) $\Phi(g)(pf) = (\Psi(g)(p))(\Phi(g)f)$ for p in $\mathcal{E}^{\infty}_{(1/2)}(\mathcal{L})$ and f in $\mathcal{C}_M(\mathcal{L})$,

(2.14) $\Psi(g_1 g_2) = \Psi(g_1)\Psi(g_2)$

and

$\Phi(g_1 g_2) = \Phi(g_1)\Phi(g_2)$

hold for any g_1 and g_2 in $\widetilde{Sp}(V_{\mathbb{R}})$,

(2.15) $\frac{d}{dt}\Psi(\exp(ta))(p)\Big|_{t=0} = [Q(a),p]$ holds for any a in $sp(V_{\mathbb{R}})$ and any p in $\mathcal{E}^{\infty}_{(1/2)}(\mathcal{L})\Big|_W$,

(2.16) $\frac{d}{dt}\Phi(\exp(ta))f\Big|_{t=0} = Q(a)f$ holds for any a in $sp(V_{\mathbb{R}})$ and any f in $\mathcal{C}_M(\mathcal{L})\Big|_{W\cap T^*_M X}$.

Note that (2.15) implies

(2.17) $\Psi(g)\exp\varphi(v) = \exp\varphi(gv)$

for g in $\widetilde{Sp}(V_{\mathbb{R}})$ and v in $V_{\mathbb{R}}$.

Now, let $\Gamma(c)$ denote $\{g \in \widetilde{Sp}(V_{\mathbb{R}}); gL = L$ and $c(gv) = c(v)$ for any v in $L\}$. It then follows from (2.13) and (2.17) that, for any Jacobi function f, we find

(2.18) $\exp\varphi(v)(\Phi(g)f)$

$= (\Psi(g)\exp\varphi(g^{-1}v))(\Phi(g)f)$

$$= \Phi(g)(\exp \varphi(g^{-1}v)f)$$

$$= c(g^{-1}v)^{-1}\Phi(g)f$$

$$= c(v)^{-1}\Phi(g)f$$

for any g in $\Gamma(c)$. Hence $\Phi(g)f$ is again a Jacobi function. This implies an automorphic property of a Jacobi functions, if the space of Jacobi functions is finite-dimensional. We also note that, in view of Remark 2.3, we find an automorphic property of a Jacobi function under the action of $\Gamma_0 \underset{\text{def}}{=} \{g \in \widetilde{Sp}(V_{\mathbb{R}}) ; gL = L\}$ if we allow as "automorphic factors" microdifferential operators of infinite order, not only constant matrices.

Thus we have seen how global properties of Jacobi functions can be derived from microlocal information, namely, the fact that Jacobi functions are characterized as solutions of microdifferential equations (of infinite order). This gives us a hope that the theory of $\mathbb{R}$-holonomic complexes will turn out to be fruitful in application as that of holonomic systems has already shown to be.

References

[1] Aoki, T.: Invertibility for microdifferential operators of infinite order. Publ. RIMS, Kyoto Univ., 18 (1982), 421-449.

[2] ———: Calcul exponentiel des opérateurs microdifférentiels d'ordre infini, I. To appear in Ann. Inst. Fourier, Grenoble, 33-4 (1983).

[3] ———: Symbols and formal symbols of pseudodifferential operators. To appear.

[4] Bony, J.M. and P. Schapira: Existence and prolongement des solutions holomorphes des équations aux dériveés partielles. Inventiones math., 17 (1972), 95-105.

[5] Hironaka, H.: Subanalytic sets. Number Theory, Algebraic Geometry and Commutative Algebra in honor of Y. Akizuki, Kinokuniya, Tokyo, 1975, pp.453-495.

[6] Kashiwara, M. and T. Kawai: Finiteness theorem for holonomic systems of micro-differential equations. Proc. Japan Acad., 52 (1976), 341-343.

[7] ———: On holonomic systems of micro-differential equations. III. Publ. RIMS, Kyoto Univ., 17 (1981), 813-979.

[8] Kashiwara, M., T. Kawai and T. Kimura: Foundations of Algebraic Analysis. Kinokuniya, Tokyo, 1980. (In Japanese.)

[9] Kashiwara, M. and P. Schapira: Micro-hyperbolic systems. Acta Math., 142 (1979), 1-55.

[10] ——————: Microlocal study of sheaves I. To appear.

[11] Kawai, T.: An example of a complex of linear differential operators of infinite order. Proc. Japan Acad., 59, Ser. A. (1983), 113-115.

[12] Sato, M.: Pseudo-differential equations and theta functions. Astérisque, 2 et 3 (1973), 286-291.

[13] Sato, M., M. Kashiwara and T. Kawai: Linear differential equations of infinite order and theta functions. Adv. in Math., 47 (1983), 300-325.

[14] Sato, M., T. Kawai and M. Kashiwara: Microfunctions and pseudo-differential equations. Lecture Notes in Math., No.287, Springer, Berlin-Heidelberg-New York, 1973, pp.265-529.