International Journal of Modern Physics A, Vol. 7, Suppl. 1A (1992) 449–484
Proceedings of the RIMS Research Project 1991, "Infinite Analysis".

AFFINE CRYSTALS AND VERTEX MODELS

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Received October 17, 1991

1. Introduction

The subject of this paper is on one hand the theory of crystals for the quantum affine Lie algebras, and on the other hand, is the computation of the 1 point functions for the 6 vertex model and its generalizations.

The *crystals* are so designed that by handling them one can extract useful information about the representation theory of $U_q(\mathfrak{g})$ [K1-4][KN]. The main idea therein is to consider the $U_q(\mathfrak{g})$-modules at $q = 0$, where one finds surprisingly simple behaviours of certain weight vectors. The set B of these weight vectors (considered at $q = 0$ and with a certain graph structure) is called the crystal of the $U_q(\mathfrak{g})$-module.

The 1 point functions are the basic macroscopic quantities that describe the multi-phase structure of a given lattice model of statistical mechanics. For the 2 dimensional solvable lattice models a method of computing the 1 point functions is known [B] as the corner transfer matrix method (CTM), which reduces the 2 dimensional statistical sums of the 1 point functions to the 1 dimensional statistical sums over certain *paths* [ABF]. In this paper we consider the vertex models given by the R-matrices of the quantum affine Lie algebras [J1-2]. The parameter q behaves like the temperature, and the sum over the paths leads to the series expansions at the low temperature $q = 0$ of the 1 point functions.

The aim of this paper is to identify the set $\mathcal{P}$ of the paths with the crystal B of a $U_q(\mathfrak{g})$-module under certain condition. We will find that for a given model the sets of paths corresponding to the multi-phases are in one-to-one correspondence with the crystals of the irreducible highest weight $U_q(\mathfrak{g})$-modules of a certain fixed level [DJKMO1-4]. As a result we get the closed expressions of the 1 dimensional statistical sums (and hence, of the 1 point functions) in terms of the string functions of the corresponding affine Lie algebras. Viewing from the other side, we get the explicit description of the crystals for the highest weight modules of arbitrary levels for $U_q(\mathfrak{g})$ of the following types: $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$. The details of this part is given in a separate paper.

For $U_q(\mathfrak{g})$ of types A_n, B_n, C_n, D_n the description of the crystals for the irreducible finite dimensional modules are obtained in [KN]. The basic tool therein is

◇supported in part by NSA/MSP Grant ♯ MDA 90-H-4039

a correspondence between the tensor category O_{int} of the integrable highest weight modules and the tensor category of the crystals. The present study requires to handle the crystals and their tensor products of $U_q(\mathfrak{g})$-modules which do not belong to O_{int}. Namely, we will deal with finite dimensional $U_q(\mathfrak{g})$-modules where $\mathfrak{g}$ is an affine Lie algebra.

Let us put it in the vertex model language. Each line of the 2 dimensional lattice bears a finite dimensional module. Multiple lines mean the tensor product of the associated modules. Therefore, the thermodynamic limit means the infinite tensor product. The exact treatment of this infinite product is very difficult. But if we consider only at $q = 0$, it is tractable, and the result is summarized as the isomorphisms between the set of paths $\mathcal{P}$ and the crystal B.

The idea of identifying the paths and the crystal bases have been already developed in the case of $U_q\left(\widehat{\mathfrak{sl}(n)}\right)$ in [MM] and [JMMO]. Their method relies on the Fock space realization of the highest weight representations of $U_q\left(\widehat{\mathfrak{sl}(n)}\right)$. In this paper we stand on the theory of crystals [K1-4] and further develop it, especially for quantum affine Lie algebras, so that we can get the isomorphisms for general cases without knowing explicit forms of the highest weight representations.

In this introduction we are going to describe the material in the simplest case, i.e., the 6 vertex model, then to extract several points from this case that are to be clarified in the study of the general cases, and to explain the basic ideas to resolve these issues. We assume that the reader knows the basic definitions and theorems on the crystals. If it is not the case, the reader is advised to read a few pages in Section 2. Some notations given there will be used in this introduction.

1.1. *The 6 vertex model* The Boltzmann weights of the 6 vertex model is given by

(1.1.1)
$$1\overset{1}{\underset{1}{+}}1 = 2\overset{2}{\underset{2}{+}}2 = x - q^2,$$
$$1\overset{1}{\underset{2}{+}}2 = 1 - q^2, \quad 2\overset{2}{\underset{1}{+}}1 = x(1 - q^2),$$
$$2\overset{1}{\underset{1}{+}}2 = 1\overset{2}{\underset{2}{+}}1 = q(x - 1).$$

Consider the two dimensional vector space V spanned by u_1 and u_2, and define the element R in $\mathrm{End}(V \otimes V)$ by

(1.1.2)
$$R(u_j \otimes u_k) = \sum_{l,m} l\overset{j}{\underset{m}{+}}k \; u_l \otimes u_m.$$

In the limit $q = 0$, the operator R becomes diagonal with respect to $u_j \otimes u_k$:

(1.1.3)
$$\lim_{q \to 0} R = diag\left(x^{H(u_j \otimes u_k)}\right)$$

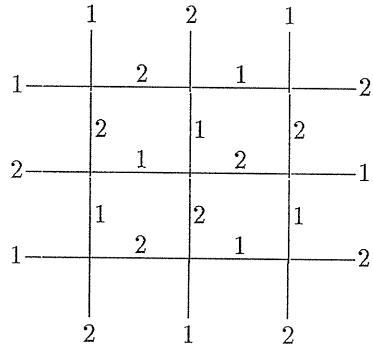
where

$$H(u_j \otimes u_k) = 1 \quad \text{if} \quad j \geq k$$
$$= 0 \quad \text{if} \quad j < k.$$

The partition function is the sum (over all the configurations of 1 or 2 on each edge of the 2 dimensional lattice) of the products (over all vertices) of the Boltzmann weights (specified by the configuration):

(1.1.4) $$Z = \sum_{\text{configurations}} P(C), \quad P(C) = \prod_{\text{vertices}} \; l \overset{j}{\underset{m}{+}} k.$$

We restrict our consideration to the region of the parameters given by $|q|, |x| < 1$. Then the following configuration C_{gr} is called a ground state configuration.



The other ground state configuration is obtained by exchanging 1 and 2. A configuration $C = \{C(k)\}_{k:\text{bond}}$ is said to be in the sector of C_{gr} if $C(k) = C_{gr}(k)$ for all but finitely many k. To each configuration C in the sector of C_{gr} we associate a configuration $W = \{W(j, C)\}_{j:\text{face}}$ as follows.

(1.1.5) $$W(j, C) \in \mathbf{Z}$$

and

(1.1.6) $$W(j', C) = W(j, C) + 1 \quad \text{if} \quad C(k) = 1$$
$$= W(j, C) - 1 \quad \text{if} \quad C(k) = 2,$$

where j and j' are the adjacent faces next to a bond k, and j' is either on the right of j or below j. We choose the boundary condition as

(1.1.7) $$W(j, C) = W(j, C_{gr}) \quad \text{for all but finitely many } j.$$

Here $W(j, C_{gr})$ is the following.

$\{W(j, C_{gr})\} =$

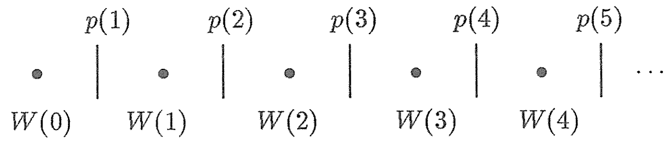
0	1	0	1
1	0	1	0
0	1	0	1
1	0	1	0

We call W the weight configuration associated with C.

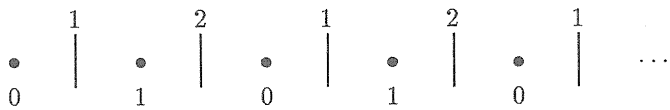
Choose and fix a face, say 0. Then, the 1 point function $F(a)$ $(a \in \mathbb{Z})$ is defined by

$$F(a) = \frac{\sum \delta_{a,W(0,C)} P(C)}{Z}. \tag{1.1.8}$$

The corner transfer matrix method (CTM) reduces this sum to a one-dimensional configuration sum. Consider the half infinite chain of vertical bonds and the faces between them.



A path $p = \left(p(k)\right)_{k=1,2,\ldots}$ is a configuration of 1 and 2 on the bonds. For example, the ground state path p_{gr} and the associated weight configuration are as follows.



The sum (1.1.8) essentially reduces to the the following sum.

$$G(a) = \sum_{\text{paths}} q^{2\sum_{k=1}^{\infty} k\left\{H\left(p(k+1),p(k)\right) - H\left(p_{gr}(k+1),p_{gr}(k)\right)\right\}}. \tag{1.1.9}$$

Here the sum is taken over the path p such that $W(0,p) = a$. It is easy to evaluate this sum. The result is

$$\begin{aligned} G(a) &= \frac{q^{a^2/2}}{\prod_{k=1}^{\infty}(1-q^{2k})} && \text{if } a \text{ is even} \\ &= 0 && \text{if } a \text{ is odd.} \end{aligned} \tag{1.1.10}$$

Consider the quantum affine Lie algebra $\widehat{\mathfrak{sl}}(2)$. Let $V(\Lambda_0)$ be the level one vacuum representation. Then we have the equality

$$G(a) = \sum_{n \geq 0} \dim V(\Lambda_0)_{\Lambda_0 - a\alpha_1/2 - n\delta} q^{2n} \tag{1.1.11}$$

where $\delta = \alpha_0 + \alpha_1$ is the null root.

1.2 *Paths and crystals for* $U_q\left(\widehat{\mathfrak{sl}}(2)\right)$ Let us discuss the role of the quantum affine Lie algebra $U_q\left(\widehat{\mathfrak{sl}}(2)\right)$ in the discussion of 1.1. It is two fold.

(i) The R-matrix appears as the set of Boltzmann weights of the 6-vartex model; (1.1.1-2).

(ii) The string functions appear as the 1 dimensional sums; (1.1.9-11).

These two are connected by the energy function H appearing in the $q = 0$ limit of R; (1.1.3).

In (i) we use the 2 dimensional irreducible representation V of $U_q(\widehat{\mathfrak{sl}}(2))$. This V is not a highest weight module, and the level of this representation is zero. In (ii) we use the infinite-dimensional representation $V(\Lambda_0)$. This is a highest weight module of level 1.

Strictly speaking, the definitions of $U_q(\widehat{\mathfrak{sl}}(2))$ employed in (i) and (ii) are different. Recall that $U_q(\widehat{\mathfrak{sl}}(2))$ is generated by e_i, f_i $(i = 0, 1)$ and q^h $(h \in P^*)$ where P^* is the dual lattice of the weight lattice P. The choice of P is not unique. One choice is such that $P^* = P^*_{cl} = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1$ and $P = P_{cl} = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1$ where $\langle \Lambda_i, h_j \rangle = \delta_{ij}$. For this, the simple roots α_0 and α_1 are not linearly independent: $\delta = \alpha_0 + \alpha_1 = 0$.

With this choice, the corresponding quantum affine Lie algebra, which we denote by $U'_q(\widehat{\mathfrak{sl}}(2))$, have non-trivial irreducible finite dimensional modules such as our 2 dimensional V.

One can define the highest weight module $V(\Lambda_0)$ as a $U'_q(\widehat{\mathfrak{sl}}(2))$-module. But it is then inevitable that the weight spaces of this module are infinite-dimensional. To avoid this inconvenience we choose the second choice of P; $P = \mathbb{Z}\delta \oplus \mathbb{Z}\Lambda_0 \oplus \Lambda_1$. Then one can apply the theory of crystals developed in [K4].

The crystal $B(\Lambda_0)$ is a set with the weight decomposition

$$B(\Lambda_0) = \sqcup_{\lambda \in P} B(\Lambda_0)_\lambda \quad \text{where} \quad \sharp B(\Lambda_0)_\lambda = \dim V(\Lambda_0)_\lambda$$

and the maps $\tilde{f}_i$, $\tilde{e}_i$ $(i = 0, 1)$

$$\begin{aligned} \tilde{f}_i &: B(\Lambda_0)_\lambda \longrightarrow B(\Lambda_0)_{\lambda - \alpha_i} \sqcup \{0\}, \\ \tilde{e}_i &: B(\Lambda_0)_\lambda \longrightarrow B(\Lambda_0)_{\lambda + \alpha_i} \sqcup \{0\}. \end{aligned}$$

The precise definition will be given later (Definition 2.2.3). The several higher weight spaces are illustrated in Figure 1.2.1, in which the dots show the elements of $B(\Lambda_0)$ and the arrows show the action of $\tilde{f}_i$.

Figure1.2.1

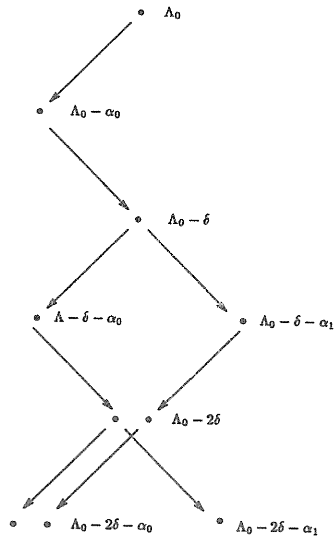
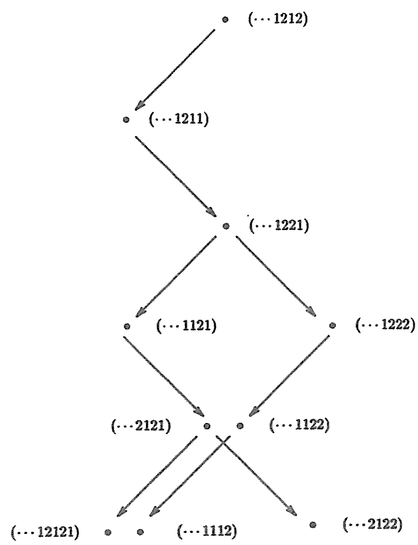


Figure1.2.2



Consider the set

$$\mathcal{P}(\Lambda_0) = \{p = \left(p(k)\right)_{k\geq 1} \mid p(k) \in \{1,2\}, p(k) = p_{gr}(k) \quad \text{for} \quad k >> 0\}.$$

Define the weight of a path $p \in \mathcal{P}(\Lambda_0)$ by

$$\text{(1.2.1)} \qquad \begin{aligned} \operatorname{wt} p = \Lambda_0 &- \sum_{k=1}^{\infty}\left(p(k) - p_{gr}(k)\right)\alpha_1 \\ &- \sum_{k=1}^{\infty} k\{H\left(p(k+1), p(k)\right) - H\left(p_{gr}(k+1), p_{gr}(k)\right)\}\delta. \end{aligned}$$

Set $n_i(a,b;p) = \sharp\{j \mid a \leq j \leq b, p(j) = i\}$ where $p \in \mathcal{P}(\Lambda_0)$, $i = 1,2$ and $1 \leq a \leq b$. Set $S_1 = \{j \mid n_1(j',j;p) > n_2(j',j;p) \text{ for any } j' \leq j\}$. Define $\tilde{f}_1 : \mathcal{P}(\Lambda_0) \longrightarrow \mathcal{P}(\Lambda_0) \sqcup \{0\}$ as follows:

(i) If $S_1 = \phi$, then set $\tilde{f}_1 p = 0$.

(ii) If $S_1 \neq \phi$, then set

$$\begin{aligned} (\tilde{f}_1 p)(k) &= p(k) \quad k \neq \max S_1 \\ &= 2 \quad k = \max S_1. \end{aligned}$$

The map $\tilde{f}_0$ is similarly defined by interchanging 1 and 2. By these definitions of the arrows we can identify $\mathcal{P}(\Lambda_0)$ with $B(\Lambda_0)$, and (1.2.1) gives the weight of the element of corresponding to a path p. Hence we obtain

$$\dim V(\Lambda_0)_\lambda = \sharp\{p \in \mathcal{P}(\Lambda_0) \mid \operatorname{wt} p = \lambda\},$$

and (1.1.11) is also its consequence. Figure 1.2.2 shows part of this identification.

1.3. *R matrix* Now we consider the generalization of the scheme explained in 1.1-2. Let $\mathfrak{g}$ be an affine Lie algebra. We consider two versions of quantum affine Lie algebras $U_q'(\mathfrak{g})$ and $U_q(\mathfrak{g})$ as in the $\widehat{\mathfrak{sl}}(2)$ case. Denote by Mod^f the category of the finite dimensional $U_q'(\mathfrak{g})$-modules. For $V \in \mathrm{Mod}^f$, one can construct canonically the $U_q(\mathfrak{g})$-module $V_x = \mathbb{Q}[x,x^{-1}] \otimes V$. The R matrix for V is an invertible $\mathbb{Q}[x,x^{-1},y,y^{-1}] \otimes U_q(\mathfrak{g})$-linear map

$$\text{(1.3.1)} \qquad R_{x,y} : V_x \otimes V_y \longrightarrow V_y \otimes V_x$$

satisfying the following properties.

(1.3.2) $R_{x,y} \in \mathbb{Q}(q)[x/y, y/x] \otimes \mathrm{End}_{\mathbb{Q}(q)}(V \otimes V)$.

(1.3.3) $(R_{y,z} \otimes 1)(1 \otimes R_{x,z})(R_{x,y} \otimes 1) = (1 \otimes R_{x,y})(R_{x,z} \otimes 1)(1 \otimes R_{y,z})$.

(1.3.4) $R_{x,y} R_{y,x} \in \mathbb{Q}(q)[x/y, y/x]$.

The universal R matrix is a candidate which gives $R_{x,y}$ on $V_x \otimes V_y$. Since we are working in the affine situation, the expression of the universal R matrix contains infinitely many non zero terms. To be able to sum up all these, we assume that $V \otimes V$ is irreducible $U_q'(\mathfrak{g})$-module. Now the problem is under which conditions on V, one can prove this irreducibility.

The idea is to use the crystal B of V. We assume the following.

(1.3.5) There exists a weight λ such that $\sharp(B \otimes B)_{2\lambda} = 1$, and
(1.3.6) $B \otimes B$ is connected.
Under these assumptions one can prove the irreducibility of $V \otimes V$ and the existence of the R matrix.

The next step is to consider the $q \longrightarrow 0$ limit of R. Set $B_x = \sqcup_n x^n \otimes B$ and define the maps $\tilde{e}_i$, $\tilde{f}_i$ $(i \in I)$: $B_x \sqcup \{0\} \longrightarrow B_x \sqcup \{0\}$ by

$$\tilde{e}_i = x^{\delta_{i0}} \otimes \tilde{e}_i, \quad \tilde{f}_i = x^{-\delta_{i0}} \otimes \tilde{f}_i.$$

With these maps B_x is the crystal for the $U_q(\mathfrak{g})$-module V_x.

A map $R : B_x \otimes B_y \longrightarrow B_y \otimes B_x$ is called a combinatorial R matrix if it commutes with the multiplications by x, y and the maps $\tilde{e}_i, \tilde{f}_i (i \in I)$. A map $H : B \otimes B \longrightarrow \mathbf{Z}$ is called an energy function if for any b, $b' \in B$ and $i \in I$ such that $\tilde{e}_i(b \otimes b') \neq 0$ it satisfies

$$(1.3.7) \qquad \begin{aligned} H(\tilde{e}_i(b \otimes b')) &= H(b \otimes b') && \text{if } i \neq 0 \\ &= H(b \otimes b') + 1 && \text{if } i = 0 \text{ and } \varphi_0(b) \geq \varepsilon_0(b') \\ &= H(b \otimes b') - 1 && \text{if } i = 0 \text{ and } \varphi_0(b) < \varepsilon_0(b'). \end{aligned}$$

The combinatorial R matrix is of the form

$$(1.3.8) \qquad R(b \otimes b') = (y/x)^{H(b \otimes b')} b \otimes b'$$

for some energy function H.

We will show in Section 5 that after a suitable normalization $R : V_x \otimes V_y \longrightarrow V_y \otimes V_x$ induces a combinatorial R matrix of the form (1.3.8) at $q = 0$, and it satisfies (1.1.3) with this H and with such a base $\{u_j\}$ that $\{u_j \bmod qL\} = B$, where L is the crystal lattice of V.

1.4 *Perfect crystals* Fix a positive integer l. Set $(P_{cl}^+)_l = \{\lambda \in \oplus_{i=0}^n \mathbf{Z}_{\geq 0} \Lambda_i \mid \langle c, \lambda \rangle = l\}$ where c is the canonical central element of $\mathfrak{g}$. For $\lambda \in (P_{cl}^+)_l$ we denote by $V(\lambda)$ the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight λ and by $B(\lambda)$ the associated crystal. Suppose that V is a finite dimensional irreducible $U_q'(\mathfrak{g})$-module and it has a crystal base (L, B).

It may happen that

$$(1.4.1) \qquad B(\lambda) \otimes B \cong B(\mu)$$

for some $\mu \in (P_{cl}^+)_l$. This is a clue to the isomorphisms between the sets of paths and the crystals —the isomorphisms we are looking for.

An element $b_1 \otimes b_2 \in B(\lambda) \otimes B$ is a highest weight element, i.e., there are no arrows in $B(\lambda) \otimes B$ which point to $b_1 \otimes b_2$, if and only if

$$(1.4.2) \qquad b_1 = u_\lambda : \text{the highest weight element in } B(\lambda),$$
$$(1.4.3) \qquad \varepsilon_i(b_2) \leq \langle h_i, \lambda \rangle \quad \text{for any } i.$$

If $B(\lambda) \otimes B$ has only one highest weight element with weight μ, then one can deduce (1.4.1). (In fact, the proof requires an auxiliary condition which applies only to the case rank $\mathfrak{g} \geq 3$. See Theorem 4.4.1.) For $b \in B$ we set $\varepsilon(b) = \sum_{i=0}^n \varepsilon_i(b) \Lambda_i$. We assume the following.

$$(1.4.4) \qquad \langle c, \varepsilon(b) \rangle \geq l \quad \text{for all } b \in B.$$

Set

$$B_l = \{b \in B \mid \langle c, \varepsilon(b)\rangle = l\}. \tag{1.4.5}$$

We call B *perfect of level* l if it satisfies (1.4.4) and the following.

$$\varphi,\ \varepsilon : B_l \longrightarrow (P_{cl}^+)_l \quad \text{are bijective.} \tag{1.4.6}$$

If B satisfies (1.4.6), then for any $\lambda_0 \in (P_{cl}^+)_l$, one can find periodic sequences $b_j \in B$ $(b_{j+N} = b_j)$ and $\lambda_j \in (P_{cl}^+)_l$ $(\lambda_{j+N} = \lambda_j)$ such that

$$\varepsilon(b_j) = \lambda_j = \varphi(b_{j+1}). \tag{1.4.7}$$

Now from (1.4.1) we have the isomorphism of crystals

$$\begin{array}{ccc} B(\lambda_k) \otimes B & \cong & B(\lambda_{k-1}) \\ \Cup & & \Cup \\ u_{\lambda_k} \otimes b_k & \mapsto & u_{\lambda_{k-1}} \end{array} \tag{1.4.8}$$

Iterating this isomorphism we have

$$\begin{array}{cccc} \psi_k : & B(\lambda_0) & \cong & B(\lambda_k) \otimes B^{\otimes k} \\ & \Cup & & \Cup \\ & u_{\lambda_0} & \mapsto & u_{\lambda_k} \otimes b_k \otimes \cdots \otimes b_1. \end{array} \tag{1.4.9}$$

We make the following definitions.

The ground state path of weight λ_0 is the sequence $\{b_k\}_{k\geq 1}$. A λ_0-path in B is a sequence $\{p(k)\}_{k\geq 1}$ such that

$$p(k) \in B, \quad p(k) = b_k \quad \text{if} \quad k \gg 0. \tag{1.4.10}$$

Denote by $\mathcal{P}(\lambda_0, B)$ the set of λ_0-paths in B. Then $B(\lambda_0)$ and $\mathcal{P}(\lambda_0, B)$ are in one to one correspondence by

$$B(\lambda_0) \ni b \longleftrightarrow p \in \mathcal{P}(\lambda_0, B)$$

where

$$\psi_k(b) = u_{\lambda_k} \otimes p(k) \otimes \cdots \otimes p(1) \quad \text{for} \quad k \gg 0.$$

The weight of b is given in terms of p by

$$\mathrm{wt}\, b = \lambda_0 + \sum_{k=1}^{\infty} (\mathrm{wt}\, p(k) - \mathrm{wt}\, b_k) - \sum_{k=1}^{\infty} k\{H\big(p(k+1) \otimes p(k)\big) - H(b_{k+1} \otimes b_k)\}\delta.$$

1.5. *Vertex models* In the setting given in 1.3-4, we can generalize the 6-vertex model. Let $\{u_j\}_{j\in S}$ be a base of the crystal lattice L of V such that $\{u_j \bmod qL\} = B$. For $j, k, l, m \in S$ define the Boltzmann weights $j \!-\!\!\!\!\!\!\!\begin{smallmatrix} k \\ | \\ l \end{smallmatrix}\!\!\!\!\!\!\!-\! m$ by (1.1.2) where R is the R-matrix for V. The sequence $\{b_k\}$ determines the ground state configuration

C_{gr} and the corresponding sector of the 2-dimensional configerations. The dual configuration $W = \{W(j,C)\}_{j:\text{face}}$ is defined by

$$W(j,C) \in P_{cl}, \tag{1.5.1}$$

$$W(j',C) = W(j,C) + \text{wt}\, C(k), \tag{1.5.2}$$

where j, j' and k are the same as (1.1.6). (We choose and fix $W_{gr} = \big(W(j,C_{gr})\big)$.)

The 1 point functions are defined by (1.1.8) where $a \in P_{cl}$. Then, the routine procedure of the CTM method gives the expressions of the 1 point functions in terms of the sums over the paths. As explained in the appendix of [ABF] (see also [DJKMO1-4]), the CTM method requires the second inversion relation . We show that it is valid in our general setting. Thus in our models, the 1 point functions are expressed in terms of the string functions (Theorem 5.2.2).

The plan of the paper is as follows. In Section 2 we give the basic definitions on crystals. Proposition 2.4.4 is a key when we prove results like (1.4.1). In Section 3 we discuss about the finite dimensional representations of the quantum affine Lie algebras. The existence of the R matrix is shown under certain conditions. In Section 4 we introduce the energy functions and the paths. The identity of the form (1.4.1) is established in Theorem 4.4.2. We define the perfect crystals. In Section 5 the second inversion relation for the R matrix is derived. Finally we give a formula for the one point function. In Section 6 examples of the perfect crystals of level one are given. An extensive study of the perfect crystals are left to a subsequent publication.

2. Crystals

2.1. *Definition of* $U_q(\mathfrak{g})$ Let us recall the definition of the quantized universal enveloping algebra $U_q(\mathfrak{g})$.

Consider the following data:

(2.1.1) a free $\mathbb{Z}$-module P of finite rank (the weight lattice),

(2.1.2) a finite set I, $\alpha_i \in P$ and $h_i \in P^* = \text{Hom}(P,\mathbb{Z})$ for $i \in I$,

(2.1.3) a $\mathbb{Q}$-valued symmetric form $(\ ,\ )$ on P.

We set $\mathfrak{t} = \mathbb{Q} \otimes P^*$ and $\mathfrak{t}^* = \mathbb{Q} \otimes P$. We assume that they satisfy the following conditions:

$$\langle h_i, \alpha_j \rangle \text{ is a generalized Cartan matrix,} \tag{2.1.4}$$

$$(\alpha_i, \alpha_i) \in \mathbb{Z}_{>0} \text{ for any } i, \tag{2.1.5}$$

$$\langle h_i, \lambda \rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i) \text{ for any } i \in I \text{ and } \lambda \in \mathfrak{t}^*. \tag{2.1.6}$$

We do not assume that $\{\alpha_i \mid i \in I\}$ or $\{h_i \mid i \in I\}$ are linearly independent. The quantized universal enveloping algebra $U_q(\mathfrak{g})$ associated to these data is the $\mathbb{Q}(q)$-algebra generated by the symbols e_i, f_i $(i \in I)$ and q^h $(h \in P^*)$ satisfying the following defining relations:

$$q^h = 1 \text{ for } h = 0 \text{ and } q^h q^{h'} = q^{h+h'}, \tag{2.1.7}$$

$$q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i \text{ and } q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i, \tag{2.1.8}$$

$$[e_i, f_j] = \delta_{ij}(t_i - t_i^{-1})/(q_i - q_i^{-1}) \tag{2.1.9}$$

where $q_i = q^{(\alpha_i,\alpha_i)}$ and $t_i = q^{(\alpha_i,\alpha_i)h_i}$,

$$(2.1.10)\qquad \sum_{n=0}^{b}(-1)^n e_i^{(n)} e_j e_i^{(b-n)} = \sum_{n=0}^{b}(-1)^n f_i^{(n)} f_j f_i^{(b-n)} = 0$$

for $i, j \in I, i \neq j$ where $b = 1 - \langle h_i, \alpha_j \rangle$.

Here we set $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{k=1}^n [k]_i$, $e_i^{(n)} = e_i^n/[n]_i!, f_i^{(n)} = f_i^n/[n]_i!$. We also use $\{x\}_i = (x - x^{-1})/(q_i - q_i^{-1})$ and $\left\{ {x \atop n} \right\}_i = \prod_{k=1}^n (\{q_i^{1-k}x\}_i/[k]_i)$.

We define the comultiplication by

$$(2.1.11)\qquad \begin{aligned} \Delta(e_i) &= e_i \otimes t_i^{-1} + 1 \otimes e_i, \\ \Delta(f_i) &= f_i \otimes 1 + t_i \otimes f_i, \\ \Delta(t_i) &= t_i \otimes t_i. \end{aligned}$$

Then $U_q(\mathfrak{g})$ is a Hopf algebra and the antipode is given by

$$(2.1.12)\qquad \begin{aligned} a(e_i) &= -e_i t_i, \\ a(f_i) &= -t_i^{-1} f_i, \\ a(q^h) &= q^{-h}. \end{aligned}$$

If we want to emphasize P and I, we write $U_q(\mathfrak{g}; P, I)$ for $U_q(\mathfrak{g})$.

2.2. *Crystal bases and Crystal pseudo-bases* For a subset J of I we denote by $U_q(\mathfrak{g}_J)$ the $\mathbf{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i $(i \in J)$ and q^h $(h \in P^*)$. A $U_q(\mathfrak{g})$-module M is called *integrable*, if it satisfies

$$(2.2.1)\qquad M = \bigoplus_{\lambda \in P} M_\lambda \text{ where } M_\lambda = \{u \in M \mid q^h u = q^{\langle h, \lambda \rangle} u \quad \forall h \in P^*\},$$

(2.2.2) M is a union of finite-dimensional $U_q(\mathfrak{g}_{\{i\}})$-modules for any $i \in I$.

For each $i \in I$, any weight vector $u \in M_\lambda$ of such a module M can be written uniquely as

$$(2.2.3)\qquad u = \sum f_i^{(n)} u_n,$$

where $u_n \in M_{\lambda + n\alpha_i} \cap \operatorname{Ker} e_i$ and n ranges over integers such that $n \geq 0$ and $\langle h_i, \lambda \rangle + n \geq 0$. Then we define endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ by

$$\tilde{e}_i u = \sum f_i^{(n-1)} u_n \quad \tilde{f}_i u = \sum f_i^{(n+1)} u_n.$$

Let A be the subring of $\mathbf{Q}(q)$ consisting of $f \in \mathbf{Q}(q)$ that is regular at $q = 0$.

Definition 2.2.1. *A crystal lattice L of an integrable $U_q(\mathfrak{g})$-module M is a free A-submodule of M such that*

$$(2.2.4)\qquad M \cong \mathbf{Q}(q) \otimes_A L,$$

$$(2.2.5)\qquad L = \bigoplus_{\lambda \in P} L_\lambda \text{ where } L_\lambda = L \cap M_\lambda,$$

$$(2.2.6)\qquad \tilde{e}_i L \subset L \text{ and } \tilde{f}_i L \subset L.$$

For a $\mathbf{Q}$-vector space V, a subset B of V is called a *pseudo-base* if there exists a base B' of V such that $B = B' \sqcup (-B')$.

Definition 2.2.2. *A crystal base (resp. crystal pseudo-base) of an integrable $U_q(\mathfrak{g})$-module M is a pair (L, B) such that*

(2.2.7) L *is a crystal lattice of* M,

(2.2.8) B *is a base (resp. pseudo-base) of* L/qL,

(2.2.9) $B = \bigsqcup_{\lambda \in P} B_\lambda$ *where* $B_\lambda = B \cap (L_\lambda / qL_\lambda)$,

(2.2.10) $\tilde{e}_i B \subset B \sqcup \{0\}$ *and* $\tilde{f}_i B \subset B \sqcup \{0\}$,

(2.2.11) *for* $b, b' \in B$ *and* $i \in I$, $b' = \tilde{f}_i b$ *if and only if* $b = \tilde{e}_i b'$.

If (L, B) is a crystal base, then $(L, B \sqcup (-B))$ is a crystal pseudo-base. Abstracting the properties of crystal bases, we introduce the following notions.

Definition 2.2.3. *A crystal B is a set B with the maps*

$$\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \to B \sqcup \{0\} \quad (i \in I)$$

satisfying the following properties:

(2.2.12) $\tilde{e}_i 0 = \tilde{f}_i 0 = 0$,

(2.2.13) *For any b and i, there is $n > 0$ such that* $\tilde{e}_i^n b = \tilde{f}_i^n b = 0$,

(2.2.14) *For* $b, b' \in B$ *and* $i \in I$, $b' = \tilde{f}_i b$ *if and only if* $b = \tilde{e}_i b'$.

When we want to emphasize I, we call B an I-crystal. A crystal may be regarded as an oriented and colored (by I) graph by defining arrows as follows :

$$\text{for} \quad b, b' \in B, \quad b \xrightarrow{i} b' \quad \text{if and only if} \quad b' = \tilde{f}_i b.$$

For an element b of a crystal B we set

(2.2.15)
$$\varepsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n b \in B\},$$
$$\varphi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i^n b \in B\}.$$

Let B_1 and B_2 be two crystals. A morphism $\varphi : B_1 \to B_2$ of crystals is defined to be a map φ from B_1 to B_2 that commutes with the actions of $\tilde{e}_i$ and $\tilde{f}_i$. Here we understand $\varphi(0) = 0$. Then the crystals and their morphisms form a category. The category of crystals has a structure of tensor category. For two crystals B_1 and B_2, we define its tensor product $B_1 \otimes B_2$ as follows. The underlying set is $B_1 \times B_2$. We write $b_1 \otimes b_2$ for (b_1, b_2). We understand $b_1 \otimes 0 = 0 \otimes b_2 = 0$. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are given by

(2.2.16)
$$\begin{aligned} \tilde{f}_i(b_1 \otimes b_2) &= \tilde{f}_i b_1 \otimes b_2 \quad \text{if} \quad \varphi_i(b_1) > \varepsilon_i(b_2) \\ &= b_1 \otimes \tilde{f}_i b_2 \quad \text{if} \quad \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{aligned}$$

(2.2.17)
$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \tilde{e}_i b_1 \otimes b_2 \quad \text{if} \quad \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ &= b_1 \otimes \tilde{e}_i b_2 \quad \text{if} \quad \varphi_i(b_1) < \varepsilon_i(b_2). \end{aligned}$$

The following lemma is immediate.

Lemma 2.2.4.
(i) *By the definition above,* $B_1 \otimes B_2$ *is a crystal.*
(ii) *We have*

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_1) + \varepsilon_i(b_2) - \varphi_i(b_1))$$
$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_2) + \varphi_i(b_1) - \varepsilon_i(b_2))$$

(iii) *Let* B_1, B_2 *and* B_3 *be crystals. Then the map* $(B_1 \otimes B_2) \otimes B_3 \to B_1 \otimes (B_2 \otimes B_3)$ *given by* $(b_1 \otimes b_2) \otimes b_3 \to b_1 \otimes (b_2 \otimes b_3)$ *is an isomorphism of crystals.*

By this lemma, the category of crystals is endowed with the structure of tensor category.

Definition 2.2.5. *A weighted crystal* B *is a crystal with the weight decomposition*

$$B = \bigsqcup_{\lambda \in P} B_\lambda$$

such that

$$\tilde{e}_i B_\lambda \subset B_{\lambda+\alpha_i} \sqcup \{0\}, \quad \tilde{f}_i B_\lambda \subset B_{\lambda-\alpha_i} \sqcup \{0\}$$

and

(2.2.18) $$\langle h_i, \mathrm{wt}\, b\rangle = \varphi_i(b) - \varepsilon_i(b).$$

Here, we write $\lambda = \mathrm{wt}(b)$ *if* $b \in B_\lambda$.

If we want to emphasize P, we say that B is a P-weighted crystal. For a crystal base (L, B) of an integrable $U_q(\mathfrak{g})$-module, B can be considered as a crystal. If (L, B) is a crystal pseudo-base, $B/\{1, -1\}$ is considered as a crystal. If B_1 and B_2 are weighted crystals , then $B_1 \otimes B_2$ is a weighted crystal by

(2.2.19) $$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2).$$

Note that by Lemma 2.2.4 (ii) we have $\varphi_i(b_1 \otimes b_2) - \varepsilon_i(b_1 \otimes b_2) = \sum_{j=1}^{2}(\varphi_i(b_j) - \varepsilon_i(b_j))$, and hence (2.2.18) and (2.2.19) are compatible. Thus the category of weighted crystals forms a tensor category.

2.3. *Linear independency of* $\{\alpha_i\}$

In [K4], we assumed that $\{\alpha_i\}$ is linearly independent. However, in this paper, we have to treat also the case where $\{\alpha_i\}$ is not linearly independent.

If we deal only with the $U_q(\mathfrak{g})$-modules in $O_{int}(\mathfrak{g})$, the linear independency is not important. In fact, let us take another data P', $\alpha'_i \in P'$, $h'_i \in P'^*$ $(i \in I)$, $(\ ,\ )$ satisfying (2.1.1-6), and a surjective homomorphism $\varphi : P' \longrightarrow P$ such that $\varphi(\alpha'_i) = \alpha_i$, $\varphi^*(h_i) = h'_i$ and $(\alpha'_i, \alpha'_i) = (\alpha_i, \alpha_i)$. Moreover assume that $\{\alpha'_i\}$ are linearly independent.
(e.g., take $P' = P \oplus (\oplus_i \mathbb{Z}\alpha'_i)$, $\varphi(\alpha'_i) = \alpha_i$, $\varphi(\lambda) = \lambda \in P$, $(\lambda \oplus \sum a_i\alpha'_i, \lambda' \oplus \sum a'_i\alpha'_i) = (\lambda, \lambda') + \sum a_i(\alpha_i, \lambda') + a'_i(\alpha_i, \lambda) + \sum a_i a'_j(\alpha_i, \alpha_j)$.)
Then we have a ring homomorphism $U_q(\mathfrak{g}; P, I) \longrightarrow U_q(\mathfrak{g}; P', I)$.

Let $V_{P'}(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g}, P', I)$-module with highest weight λ and let $V_P(\varphi(\lambda))$ be the irreducible highest weight $U_q(\mathfrak{g}; P, I)$-module with highest weight $\varphi(\lambda)$. Then, $V_{P'}(\lambda)$ and $V_P(\varphi(\lambda))$ are isomorphic as $U_q(\mathfrak{g}; P, I)$-module. Consequently, their crystal bases $B_{P'}(\lambda)$ and $B_P(\varphi(\lambda))$ are isomorphic as P-crystals. In particular, (2.3.1) even if $\{\alpha_i\}$ is not linearly independent, for $\lambda \in P_+$,

$$\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda = \tilde{f}_{j_1} \cdots \tilde{f}_{j_k} u_\lambda \in B(\lambda)$$

implies

$$\sharp\{\nu \mid i_\nu = i\} = \sharp\{\mu \mid j_\mu = i\}$$

for any $i \in I$.

However, when we treat integrable modules not belonging to $O_{int}(\mathfrak{g})$ the representation theory of $U_q(\mathfrak{g}; P, I)$ and that of $U_q(\mathfrak{g}; P', I)$ are very different. In the affine case, if $\{\alpha_i\}$ is linearly independent, any irreducible finite-dimensional representation must be one-dimemsional. But there are a lot of irreducible finite-dimensional representations when $\{\alpha_i\}$ is linearly dependent.

2.4. *Crystal base of highest weight modules* Let $O_{int}(\mathfrak{g})$ be the category of integrable $U_q(\mathfrak{g})$-modules M such that for any $u \in M$, there exists $l \geq 0$ such that $e_{i_1} \cdots e_{i_l} u = 0$ for any $i_1, \cdots, i_l \in I$. For $\lambda \in P_+ = \{\lambda \in P \mid \langle h_i, \lambda \rangle \geq 0\}$, let us denote by $V(\lambda)$ the irreducible $U_q(\mathfrak{g})$-module with highest weight λ. Then any object of $O_{int}(\mathfrak{g})$ is a direct sum of $V(\lambda)$'s. Let u_λ be the highest weight vector of $V(\lambda)$.

Let $L(\lambda)$ be the smallest A-module that contains u_λ and that is stable under $\tilde{f}_i$'s. Let us set $B(\lambda) = \{b \in L(\lambda)/qL(\lambda) \mid b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda \bmod qL(\lambda)\} \backslash \{0\}$. We denote by the same symbol u_λ the element of $B(\lambda)$ corresponding to $u_\lambda \in L(\lambda)$. The following three theorems are proved in [K4].

Theorem 2.4.1. *$(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$.*

A crystal is called a *crystal with highest weight* if it is isomorphic to $B(\lambda)$ for some $\lambda \in P_+$.

Theorem 2.4.2. *Suppose that M is in $O_{int}(\mathfrak{g})$. If (L, B) is a crystal base of M, then there is an isomorphism $M \cong \bigoplus_j V(\lambda_j)$ which induces $(L, B) \cong \bigoplus_j (L(\lambda_j), B(\lambda_j))$.*

Theorem 2.4.3. *Let M_1 and M_2 be two integrable $U_q(\mathfrak{g})$-modules, and let (L_j, B_j) be a crystal base (resp. crystal pseudo-base) of M_j. Set $L = L_1 \otimes L_2 \subset M_1 \otimes M_2$, $B = \{b_1 \otimes b_2 \mid b_j \in B_j\} \subset L/qL$. Then (L, B) is a crystal base (resp. crystal pseudo-base). Moreover $B \cong B_1 \otimes B_2$ (resp. $B/\{\pm 1\} \cong (B_1/\{\pm 1\}) \otimes (B_2/\{\pm 1\})$) as crystals.*

Proposition 2.4.4. *Let B be a weighted crystal. Assume the following.*

(2.4.1) *For any $b \in B$, there exist $l \geq 0$, $i_1, \cdots, i_l \in I$ such that $b' = \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} b \in B$ is a highest weight element, i.e., $\tilde{e}_i b' = 0$ for any i.*

(2.4.2) *For any $i, j \in I$, B regarded as the $\{i, j\}$-crystal is a disjoint union of the crystals of integrable highest weight modules over $U_q(\mathfrak{g}_{\{i,j\}})$.*

Then B is a direct sum of crystals with highest weight.

Proof. Let $H = \{b \in B \mid \tilde{e}_i b = 0 \text{ for any } i\}$. For $b_0 \in H$, let us denote by $B(b_0)$ the set of non-zero vectors of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} b_0$. We shall show that $B(b_0)$ is stable under $\tilde{e}_i$ and is isomorphic to $B(\lambda)$, where λ is the weight of b_0. For a sequence $\sigma = (\sigma_1, \cdots, \sigma_l)$ in I, we set $l = |\sigma|$, $\tilde{f}^\sigma = \tilde{f}_{\sigma_1} \cdots \tilde{f}_{\sigma_l}$. We shall show by induction on $|\sigma|$ and $|\tau|$

$$\tilde{f}^\sigma b_0 = 0 \Leftrightarrow \tilde{f}^\sigma u_\lambda = 0, \tag{2.4.3}$$

$$\tilde{e}_i \tilde{f}^\sigma b_0 = 0 \Leftrightarrow \tilde{e}_i \tilde{f}^\sigma u_\lambda = 0, \tag{2.4.4}$$

$$\tilde{f}^\sigma b_0 = \tilde{f}^\tau b_0 \Leftrightarrow \tilde{f}^\sigma u_\lambda = \tilde{f}^\tau u_\lambda. \tag{2.4.5}$$

We shall show first (2.4.3). Set $j = \sigma_1$, and $\sigma' = (\sigma_2, \cdots, \sigma_l)$. If $\tilde{f}^{\sigma'} u_\lambda = 0$ then $\tilde{f}^{\sigma'} b_0 = 0$ and there is nothing to prove. Hence we may assume $\tilde{f}^{\sigma'} u_\lambda \in B(\lambda)$. There exists k and τ such that $|\tau| = l - k - 1$

$$\tilde{f}^{\sigma'} u_\lambda = \tilde{f}_j^k \tilde{f}^\tau u_\lambda, \tilde{e}_j \tilde{f}^\tau u_\lambda = 0.$$

Hence by the induction hypothesis

$$\tilde{f}^{\sigma'} b_0 = \tilde{f}_j^k \tilde{f}^\tau b_0 \quad \text{and} \quad \tilde{e}_j \tilde{f}^\tau b_0 = 0.$$

Since $\tilde{f}^\sigma b_0 = \tilde{f}_j^{k+1} \tilde{f}^\tau b_0 = 0$ is equivalent to $k + 1 > \langle h_j, \mathrm{wt}(\tilde{f}^\tau b_0)\rangle$ and $\tilde{f}^\sigma u_\lambda = 0$ is also equivalent to $k + 1 > \langle h_j, \mathrm{wt}(\tilde{f}^\tau u_\lambda)\rangle = \langle h_j, \mathrm{wt}(\tilde{f}^\tau b_0)\rangle$. This shows (2.4.3).

Now, we shall show (2.4.4). By the same notation, we may assume $\tilde{f}^{\sigma'} u_\lambda \neq 0$. There exist σ'' and σ''' such that σ'' is a sequence in $\{i, j\}$, $|\sigma''| + |\sigma'''| = |\sigma'|$, $\tilde{f}^{\sigma'} u_\lambda = \tilde{f}^{\sigma''} \tilde{f}^{\sigma'''} u_\lambda$ and $\tilde{e}_i \tilde{f}^{\sigma'''} u_\lambda = \tilde{e}_j \tilde{f}^{\sigma'''} u_\lambda = 0$. Hence $\tilde{f}^{\sigma'} b_0 = \tilde{f}^{\sigma''} \tilde{f}^{\sigma'''} b_0$ and $\tilde{e}_i \tilde{f}^{\sigma'''} b_0 = \tilde{e}_j \tilde{f}^{\sigma'''} b_0 = 0$. Since $\tilde{f}^{\sigma'''} b_0$ and $\tilde{f}^{\sigma'''} u_\lambda$ are annihilated by $\tilde{e}_i$ and $\tilde{e}_j$ and have the same weight, (2.4.2) implies that $\tilde{e}_i \tilde{f}_j \tilde{f}^{\sigma''} \tilde{f}^{\sigma'''} b_0 = 0$ if and only if $\tilde{e}_i \tilde{f}_j \tilde{f}^{\sigma''} \tilde{f}^{\sigma'''} u_\lambda = 0$. Finally, we shall show (2.4.5). We set $\sigma_1 = j$, $\tau_1 = i$, and $\sigma' = (\sigma_2, \cdots, \sigma_l)$ and $\tau' = (\tau_2, \cdots, \tau_l)$. Then there exist sequences σ'', τ'' in $\{i, j\}$ and σ''', τ''' such that $|\sigma'| = |\sigma''| + |\sigma'''|$, $|\tau'| = |\tau''| + |\tau'''|$, $\tilde{f}^{\sigma'} u_\lambda = \tilde{f}^{\sigma''} \tilde{f}^{\sigma'''} u_\lambda$, $\tilde{f}^{\tau'} u_\lambda = \tilde{f}^{\tau''} \tilde{f}^{\tau'''} u_\lambda$ and $\tilde{e}_i \tilde{f}^{\tau'''} u_\lambda = \tilde{e}_j \tilde{f}^{\tau'''} u_\lambda = \tilde{e}_i \tilde{f}^{\sigma'''} u_\lambda = \tilde{e}_j \tilde{f}^{\sigma'''} u_\lambda = 0$. Hence $\tilde{e}_i \tilde{f}^{\tau'''} b_0 = \tilde{e}_j \tilde{f}^{\tau'''} b_0 = \tilde{e}_i \tilde{f}^{\sigma'''} b_0 = \tilde{e}_j \tilde{f}^{\sigma'''} b_0 = 0$. If $\tilde{f}^\sigma u_\lambda = \tilde{f}^\tau u_\lambda$ (resp. $\tilde{f}^\sigma b_0 = \tilde{f}^\tau b_0$), then $\tilde{f}^{\sigma'''} u_\lambda = \tilde{f}^{\tau'''} u_\lambda$ (resp. $\tilde{f}^{\sigma'''} b_0 = \tilde{f}^{\tau'''} b_0$) and hence $\tilde{f}^{\sigma'''} b_0 = \tilde{f}^{\tau'''} b_0$ (resp. $\tilde{f}^{\sigma'''} u_\lambda = \tilde{f}^{\tau'''} u_\lambda$). Hence $\tilde{f}^\sigma b_0 = \tilde{f}^\tau b_0$ and $\tilde{f}^\sigma u_\lambda = \tilde{f}^\tau u_\lambda$ are both equivalent to $\tilde{f}_j \tilde{f}^{\sigma''} w = \tilde{f}_i \tilde{f}^{\tau''} w$ where w is the highest weight element of the crystal of the integrable irreducible $U_q(\mathfrak{g}_{\{i,j\}})$-module with highest weight $\mathrm{wt}(\tilde{f}^{\sigma'''} u_\lambda)$. Hence we have (2.4.5). Now $\tilde{e}_i \tilde{f}^\sigma u_\lambda = \tilde{f}^\tau u_\lambda$ implies $\tilde{e}_i \tilde{f}^\sigma b_0 = \tilde{f}^\tau b_0$ because $\tilde{e}_i \tilde{f}^\sigma u_\lambda = \tilde{f}^\tau u_\lambda$ is equivalent to $\tilde{f}^\sigma u_\lambda = \tilde{f}_i \tilde{f}^\tau u_\lambda$. Hence $B(b_0)$ is stable under $\tilde{e}_i$ and is isomorphic to $B(\lambda)$. □

3. Finite-dimensional representations and R matrices.

3.1. *Affine and classical weights* Let $\mathfrak{g}$ be an indecomposable affine Lie algebra defined over $\mathbf{Q}$, and let $\mathfrak{t}$ be the Cartan subalgebra. Let $\{\alpha_i \mid i \in I\} \subset \mathfrak{t}^*$ be the set of simple roots and let $\{h_i \mid i \in I\} \subset \mathfrak{t}$ be the set of simple coroots. We assume $\{\alpha_i \mid i \in I\}$ and $\{h_i \mid i \in I\}$ are linearly independent and $\dim \mathfrak{t} = \#I + 1$. Set $Q = \sum_i \mathbf{Z}\alpha_i, Q_+ = \sum_i \mathbf{Z}_{\geq 0}\alpha_i$ and $Q_- = -Q_+$. Let $\delta \in Q_+$ be the generator of null roots, i.e., $\{\lambda \in Q \mid \langle h_i, \lambda\rangle = 0\} = \mathbf{Z}\delta$. Let $c \in \sum_i \mathbf{Z}_{\geq 0} h_i$ be the generator of the center, i.e., $\{h \in \sum_i \mathbf{Z}h_i \mid \langle h, \alpha_i\rangle = 0\} = \mathbf{Z}c$. Set $\mathfrak{t}_{cl} = \bigoplus_i \mathbf{Q}h_i \subset \mathfrak{t}$ and $\mathfrak{t}^*_{cl} = (\bigoplus_i \mathbf{Q}h_i)^*$ and let $cl : \mathfrak{t}^* \to \mathfrak{t}^*_{cl}$ be the canonical morphism. We have an exact sequence

$$0 \to \mathbf{Q}\delta \to \mathfrak{t}^* \to \mathfrak{t}^*_{cl} \to 0. \tag{3.1.1}$$

Then $\dim \mathfrak{t}^*_{cl} = \#I$ and

$$\{\lambda \in \mathfrak{t}^*_{cl} \mid \langle c, \lambda\rangle = 0\} = \sum_{i\in I} \mathbf{Q}cl(\alpha_i). \tag{3.1.2}$$

Let us fix $i_0 \in I$ such that $\delta - \alpha_{i_0} \in \sum_{i\neq i_0} \mathbf{Z}\alpha_i$. Let us take a map $af : \mathfrak{t}^*_{cl} \to \mathfrak{t}^*$ satisfying:

$$cl \circ af = id \tag{3.1.3}$$

$$af \circ cl(\alpha_i) = \alpha_i \quad \text{for} \quad i \neq i_0. \tag{3.1.4}$$

We have

$$af \circ cl(\alpha_{i_0}) = \alpha_{i_0} - \delta \tag{3.1.5}$$

since $af \circ cl(\alpha_{i_0} - \delta) = \alpha_{i_0} - \delta$ by (3.1.4) and $cl(\delta) = 0$.

Let Λ_i be the element of $af(\mathfrak{t}^*_{cl}) \subset \mathfrak{t}^*$ such that $\langle h_j, \Lambda_i\rangle = \delta_{ij}$. We take $\sum_i \mathbf{Z}\Lambda_i + \mathbf{Z}\delta$ as $P \subset \mathfrak{t}^*$ and we set $P_{cl} = cl(P) = \sum_i \mathbf{Z}cl(\Lambda_i) \subset \mathfrak{t}^*_{cl}$. We call an element of P_{cl} a *classical weight* and an element of P an *affine weight*. Note that $af \circ cl(\Lambda_i) = \Lambda_i$. We have

$$\alpha_i = \sum \langle h_j, \alpha_i\rangle \Lambda_j + \delta_{ii_0}\delta. \tag{3.1.6}$$

In particular, all α_i belong to P.

Remark. The map af depends on the choice of i_0. Even for a fixed i_0, it is not uniquely determined by the properties (3.1.3) and (3.1.4). If af' is another map satisfying the same conditions, then there exists $r \in \mathbf{Q}$ such that $af'(\lambda) = af(\lambda) + r\langle c, \lambda\rangle\delta$.

3.2. *Affinization of $U'_q(\mathfrak{g})$-modules* Let $U_q(\mathfrak{g})$ be the q-analogue associated with P, and let $U'_q(\mathfrak{g})$ be its subalgebra generated by e_i, f_i, q^h $(h \in (P_{cl})^*)$. Then $U'_q(\mathfrak{g})$ is also a quantized universal enveloping algebra with P_{cl} as the weight lattice. Note that the simple roots $cl(\alpha_i) \in P_{cl}$ are not linearly independent. For a $U_q(\mathfrak{g})$-module, affine weights make sense but for a $U'_q(\mathfrak{g})$-module only classical weights make sense.

Let us denote by $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ the category of $U'_q(\mathfrak{g})$-module M satisfing the following conditions:

(3.2.1) $\quad M$ has the weight decomposition $M = \bigoplus_{\lambda \in P_{cl}} M_\lambda$

and

(3.2.2) $\quad M$ is finite-dimensional over $\mathbf{Q}(q)$.

For a $U'_q(\mathfrak{g})$-module M in $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$, we define the $U_q(\mathfrak{g})$-module $\mathrm{Aff}(M)$ by

$$\mathrm{Aff}(M) = \bigoplus_{\lambda \in P} \mathrm{Aff}(M)_\lambda, \quad \mathrm{Aff}(M)_\lambda = M_{cl(\lambda)} \quad \text{for} \quad \lambda \in P.$$

The actions of e_i and f_i are defined by the commutative diagrams

$$\begin{array}{ccc} \mathrm{Aff}(M)_\lambda & \xrightarrow{e_i} & \mathrm{Aff}(M)_{\lambda+\alpha_i} \\ \wr\| & & \wr\| \\ M_{cl(\lambda)} & \xrightarrow{e_i} & M_{cl(\lambda+\alpha_i)} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Aff}(M)_\lambda & \xrightarrow{f_i} & \mathrm{Aff}(M)_{\lambda-\alpha_i} \\ \wr\| & & \wr\| \\ M_{cl(\lambda)} & \xrightarrow{f_i} & M_{cl(\lambda-\alpha_i)}. \end{array}$$

Let us denote by $cl : \mathrm{Aff}(M) \to M$ the morphism defined by

$$\mathrm{Aff}(M)_\lambda \xrightarrow{\sim} M_{cl(\lambda)} \quad (\lambda \in P),$$

and by $af : M \to \mathrm{Aff}(M)$ the morphism defined by

$$M_\lambda \xrightarrow{\sim} \mathrm{Aff}(M)_{af(\lambda)} \quad (\lambda \in P_{cl}).$$

We define the $U'_q(\mathfrak{g})$-linear automorphism T of $\mathrm{Aff}(M)$ by

$$\begin{array}{ccc} \mathrm{Aff}(M)_\lambda & \xrightarrow{T} & \mathrm{Aff}(M)_{\lambda+\delta} \\ \wr\| & & \wr\| \\ M_{cl(\lambda)} & \xrightarrow{id} & M_{cl(\lambda+\delta)}. \end{array}$$

Then we have

(3.2.3) $$\mathrm{Aff}(M) \xleftarrow{\sim} \mathbf{Q}[T, T^{-1}] \underset{\mathbf{Q}}{\otimes} M$$

by the map

$$\mathbf{Q}[T, T^{-1}] \otimes M \ni T^k \otimes u \mapsto T^k af(u) \in \mathrm{Aff}(M).$$

Note that cl is $U'_q(\mathfrak{g})$-linear but af is not $U'_q(\mathfrak{g})$-linear. By (3.1.4) and (3.1.5) we have

(3.2.4) $$\begin{aligned} e_i \, af(u) &= af(e_i u) \quad \text{if} \quad i \neq i_0 \\ &= T \, af(e_{i_0} u) \quad \text{if} \quad i = i_0, \end{aligned}$$

(3.2.5) $$\begin{aligned} f_i \, af(u) &= af(f_i u) \quad \text{if} \quad i \neq i_0 \\ &= T^{-1} \, af(f_{i_0} u) \quad \text{if} \quad i = i_0 \end{aligned}$$

for $u \in M$.

This notion can be reformulated as follows. Let K be a commutative ring containing $\mathbf{Q}(q)$ and let x be an invertible element of K. For a $K \otimes_{\mathbf{Q}(q)} U'_q(\mathfrak{g})$-module

M let us denote by $\Phi_x(M)$ the $K \otimes_{\mathbf{Q}(q)} U'_q(\mathfrak{g})$-module constructed as follows: There exists a K-linear bijective homomorphism $\phi_x : M \longrightarrow \Phi_x(M)$ such that

$$q^h \phi_x(u) = \phi_x(q^h u) \quad \text{for } h \in P^*_{cl},$$
$$e_i \phi_x(u) = x^{\delta_{ii_0}} \phi_x(e_i u),$$
$$f_i \phi_x(u) = x^{-\delta_{ii_0}} \phi_x(f_i u).$$

For a $U'_q(\mathfrak{g})$-module M in $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$, we have the following $U'_q(\mathfrak{g})$-linear isomorphism

$$\mathrm{Aff}(M) \simeq \Phi_T\big(\mathbf{Q}(q)[T, T^{-1}] \otimes_{\mathbf{Q}(q)} M\big) \tag{3.2.6}$$

by $T^n\, af(u) \leftrightarrow T^n \phi_T(u)$.

For invertible elements x, y of K and $K \otimes U'_q(\mathfrak{g})$-modules M, N, we have

$$\Phi_x \Phi_y(M) \simeq \Phi_{xy}(M) \tag{3.2.7}$$

by $\phi_x\big(\phi_y(u)\big) \leftrightarrow \phi_{xy}(u)$, and

$$\Phi_x(M \otimes N) \simeq \Phi_x(M) \otimes \Phi_x(N) \tag{3.2.8}$$

by $\phi_x(u \otimes v) \leftrightarrow \phi_x(u) \otimes \phi_x(v)$.

For a $K \otimes U'_q(\mathfrak{g})$-linear homomorphism $f : M \longrightarrow N$, we denote by $\Phi_x(f)$ the $K \otimes U'_q(\mathfrak{g})$-linear homomorphism $\Phi_x(M) \longrightarrow \Phi_x(N)$ given by $\Phi_x(f)\big(\phi_x(v)\big) = \phi_x\big(f(v)\big)$ for any $v \in M$.

Remark.

(i) This construction makes sense because $\{cl(\alpha_i)\}$ are not linearly independent. In general, M and $\Phi_x(M)$ are not isomorphic. However, if we define $\Phi_x(M)$ for $K \otimes U_q(\mathfrak{g})$-module when $\{\alpha_i\}$ are linearly independent, then $\Phi_x(M)$ is isomorphic to M whenever M has a weight decomposition.

(ii) For the sake of simplicity, we use the same simbol Φ_x even when we consider different rings of scalars. But in any case it is always obvious which scalars we use.

3.3. *Affinization of classical crystals* The investigation similarly goes through for crystals. A P_{cl}-weighted crystal is called *classical* and P-weighted crystal is called *affine*. For a classical crystal B, let us define the affine crystal $\mathrm{Aff}(B)$ by

$$\mathrm{Aff}(B) = \bigsqcup_{\lambda \in P} \mathrm{Aff}(B)_\lambda \quad \text{where} \quad \mathrm{Aff}(B)_\lambda = B_{cl(\lambda)}.$$

The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by the commutative diagrams:

$$\begin{array}{ccc} \mathrm{Aff}(B)_\lambda & \xrightarrow{\tilde{e}_i} & \mathrm{Aff}(B)_{\lambda+\alpha_i} \sqcup \{0\} \\ \wr\| & & \wr\| \\ B_{cl(\lambda)} & \xrightarrow{\tilde{e}_i} & B_{cl(\lambda+\alpha_i)} \sqcup \{0\} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Aff}(B)_\lambda & \xrightarrow{\tilde{f}_i} & \mathrm{Aff}(B)_{\lambda-\alpha_i} \sqcup \{0\} \\ \wr\| & & \wr\| \\ B_{cl(\lambda)} & \xrightarrow{\tilde{f}_i} & B_{cl(\lambda-\alpha_i)} \sqcup \{0\}. \end{array}$$

We define the automorphism T of $\mathrm{Aff}(B)$ by

$$\begin{array}{ccc} \mathrm{Aff}(B)_\lambda & \xrightarrow{T} & \mathrm{Aff}(B)_{\lambda+\delta} \\ \wr\| & & \wr\| \\ B_{cl(\lambda)} & \xrightarrow{id} & B_{cl(\lambda+\delta)}. \end{array}$$

We define $cl : \mathrm{Aff}(B) \to B$ and $af : B \to \mathrm{Aff}(B)$ similarly to the definitions of $cl : \mathrm{Aff}(M) \longrightarrow M$ and $af : M \longrightarrow \mathrm{Aff}(M)$. Then cl commutes with all $\tilde{e}_i$ and $\tilde{f}_i$ and af satisfies

$$\begin{aligned}(3.3.1)\qquad \tilde{e}_i\, af(b) &= af(\tilde{e}_i b) \quad \text{if} \quad i \neq i_0 \\ &= T\, af(\tilde{e}_{i_0} b) \quad \text{if} \quad i = i_0, \\ (3.3.2)\qquad \tilde{f}_i\, af(b) &= af(\tilde{f}_i b) \quad \text{if} \quad i \neq i_0 \\ &= T^{-1}\, af(\tilde{f}_{i_0} b) \quad \text{if} \quad i = i_0\end{aligned}$$

for $b \in B$.

Let M be a $U_q'(\mathfrak{g})$-module in $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ and let L be a crystal lattice of M. Then $\mathrm{Aff}(L) = cl^{-1}(L)$ is a crystal lattice of $\mathrm{Aff}(M)$. If (L, B) is a crystal (pseudo-)base of M, then $\mathrm{Aff}(B) = cl^{-1}(B)$ can be considered as a subset of $\mathrm{Aff}(L)/q\,\mathrm{Aff}(L)$ and $(\mathrm{Aff}(L), \mathrm{Aff}(B))$ is a crystal (pseudo-)base of $\mathrm{Aff}(M)$.

3.4. *Existence of R-matrix* Let V be a $U_q'(\mathfrak{g})$-module in $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$. We assume that

(3.4.1) $V \otimes V$ is an irreducible $U_q'(\mathfrak{g})$-module.

Under this condition V has an R-matrix.

Theorem 3.4.1. *Let V be a $U_q'(\mathfrak{g})$-module in $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ satisfying (3.4.1).*

(i) *There exists a non-zero $U_q(\mathfrak{g})$-linear endomorphism R of $\mathrm{Aff}(V) \otimes \mathrm{Aff}(V)$ such that $(T \otimes 1) \circ R = R \circ (1 \otimes T)$ and $(1 \otimes T) \circ R = R \circ (T \otimes 1)$.*

(ii) *$R^2 = f(T \otimes T^{-1})$ for some non-zero $f \in \mathbb{Q}(q)[T \otimes T^{-1}, T^{-1} \otimes T]$.*

(iii) *R satisfies the Yang-Baxter equation: $(R \otimes 1) \circ (1 \otimes R) \circ (R \otimes 1) = (1 \otimes R) \circ (R \otimes 1) \circ (1 \otimes R)$ as an endomorphism of $\mathrm{Aff}(V) \otimes \mathrm{Aff}(V) \otimes \mathrm{Aff}(V)$.*

In order to prove this theorem, let us start by the proof of the following classical result. It says that a representation with a parameter λ is irreducible for a generic λ whenever it is irreducible at a special λ.

Lemma 3.4.2. *Let C be a Noetherian commutative ring and $\mathfrak{m}$ a maximal ideal. We assume that C is an integral domain and denote by K its fraction field. Let R be a C-algebra, i.e., a non-commutative ring R with a ring homomorphism $C \to R$ whose image is contained in the center of R. Let M be an R-module free of finite rank as a C-module. If $M/\mathfrak{m}M$ is a simple $(R/\mathfrak{m}R)$-module then $K \otimes_C M$ is a simple $K \otimes_C R$-module.*

Proof. By localization, we may assume that C is a local ring. Let N be a $K \otimes_C R$-submodule of $K \otimes_C M$. We shall prove N is zero or $K \otimes_C M$. Let us set $\bar{M} = M/\mathfrak{m}M$, $\mathrm{gr}_k C = \mathfrak{m}^k/\mathfrak{m}^{k+1}$, $\mathrm{gr}_k M = \mathfrak{m}^k M/\mathfrak{m}^{k+1} M = (\mathrm{gr}_k C) \otimes_{C/\mathfrak{m}} \bar{M}$ and $\mathrm{gr}_k N = (N \cap \mathfrak{m}^k M)/(N \cap \mathfrak{m}^{k+1} M) \subset \mathrm{gr}_k M$ for $k \geq 0$. Since $\mathrm{gr}_k N$ is an $(R/\mathfrak{m}R)$-submodule of $\mathrm{gr}_k C \otimes_{C/\mathfrak{m}} \bar{M}$, there exists a vector subspace $\mathfrak{a}_k$ of $\mathrm{gr}_k C$ such that $\mathrm{gr}_k N \simeq \mathfrak{a}_k \otimes_{C/\mathfrak{m}} \bar{M}$. Set $\mathfrak{a} = \oplus \mathfrak{a}_k$. Then $\mathrm{gr}\, N = \bigoplus_{k \geq 0} \mathrm{gr}_k N = \mathfrak{a} \otimes \bar{M}$, and $\mathfrak{a}$ is an ideal of $\mathrm{gr}\, C = \bigoplus_{k \geq 0} \mathrm{gr}_k C$. If $\mathfrak{a}$ is nilpotent, then $\mathfrak{a}_k = 0$ for $k \gg 0$ and hence $N \cap \mathfrak{m}^k M = N \cap \mathfrak{m}^{k+1} M$ for $k \gg 0$, Therefore $\cap\, \mathfrak{m}^k M = 0$ implies $N \cap \mathfrak{m}^k M = 0$ for $k \gg 0$, which implies $N = 0$. If $\mathfrak{a}$ is not nilpotent, then there exists $\bar{c} \in \mathfrak{a}_k$ which is not nilpotent. Then taking a representative $c \in \mathfrak{m}^k$ of $\bar{c}$, we have

$$cM \subset N \cap \mathfrak{m}^k M + \mathfrak{m}^{k+1} M.$$

Let $\{u_i\}$ be a system of generators of M. Then there exists $a_{ij} \in \mathfrak{m}^{k+1}$ such that

$$cu_i - \sum_j a_{ij}u_j \in N \quad \text{for any} \quad i.$$

Since $\det(c\delta_{ij} - a_{ij})_{i,j}$ does not vanish in C by the choice of $\bar{c}$, all u_i belong to N and hence $N = K \otimes_C M$. □

Proof of Theorem 3.4.1. Let z be an indeterminate and let K be $\mathbf{Q}(q, z)$. We shall first prove the existence of the R-matrix $R : \Phi_z(K \otimes V) \otimes_K (K \otimes V) \longrightarrow (K \otimes V) \otimes_K \Phi_z(K \otimes V)$. By the existence theorem of the universal R-matrix due to Drinfeld ([D2]), there exists a formal R-matrix of the form $\mathfrak{R} = \sigma \sum_i (P_i \otimes Q_i)a$ where a acts on $M_\lambda \otimes N_\mu$ by $q^{2(\lambda,\mu)}$ and $P_i \in U_q^-(\mathfrak{g}), Q_i \in U_q^+(\mathfrak{g})$ are certain bases dual to each other with respect to certain coupling $U_q^-(\mathfrak{g}) \times U_q^+(\mathfrak{g}) \to \mathbf{Q}(q)$. The map σ is the permutation map $u \otimes v \mapsto v \otimes u$. We may assume that P_i and Q_i are weight vectors and if Q_i has weight $\xi_i \in Q_+$ then P_i has weight $-\xi_i$. Now, let $\hat{K}$ be the fraction field of $\mathbf{Q}(q)[[z^{-1}]]$. We shall show that $\mathfrak{R}$ gives a well-defined homomorphism from $\Phi_z(\hat{K} \otimes V) \otimes (\hat{K} \otimes V)$ into $(\hat{K} \otimes V) \otimes \Phi_z(\hat{K} \otimes V)$. Observe that since the level of any weight $cl(\lambda)$ of V is zero, we have $(\delta, \lambda) = 0$. Hence (λ, μ) is determined by $cl(\lambda)$ and $cl(\mu)$. Therefore, for $v \in V_{cl(\lambda)}$, $w \in V_{cl(\mu)}$ $(\lambda, \mu \in P)$, we have

$$(P_i \otimes Q_i)a(\phi_z(v) \otimes w) = q^{2(\lambda,\mu)} P_i\phi_z(v) \otimes Q_i w.$$

If we write $\xi_i = af \circ cl(\xi_i) + c_i\delta$, then c_i is a non-negative integer and $P_i\phi_z(v) = z^{-c_i}\phi_z(P_i v)$. Hence, we have $(P_i \otimes Q_i)a(\phi_z(v) \otimes w) \in z^{-c_i}\phi_z(V) \otimes V$. On the other hand $P_i\phi_z(v)$ and $Q_i w$ have weight $cl(\lambda - \xi_i)$ and $cl(\mu + \xi_i)$. There exist only finitely many $\xi' \in P_{cl}$ such that $cl(\lambda) - \xi'$ and $cl(\mu) + \xi'$ are both weights of V. Therefore, for any c, there exist only finitely many i such that $P_i\phi_z(v) \otimes Q_i w \neq 0$ and $c_i = c$. Hence, $\sum(P_i \otimes Q_i)a(\phi_z(v) \otimes w)$ converges in $\hat{K} \otimes V \otimes V$. Hence the universal R-matrix gives a $\hat{K} \otimes U_q'(\mathfrak{g})$-linear homomorphism $\hat{R} : \Phi_z(\hat{K} \otimes V) \otimes (\hat{K} \otimes V) \longrightarrow (\hat{K} \otimes V) \otimes \Phi_z(\hat{K} \otimes V)$.

Moreover, if we take a non-zero w such that $e_i w = 0$ $(i \neq i_0)$, then

$$\hat{R}(\phi_z(v) \otimes w) \equiv q^{2(\lambda,\mu)} w \otimes \phi_z(v) \bmod z^{-1}\mathbf{Q}(q)[[z^{-1}]](V \otimes \phi_z(V))$$

Hence $\hat{R}$ does not vanish.

Set

$$\hat{S} = \mathrm{Hom}_{\hat{K} \otimes U_q'(\mathfrak{g})}(\Phi_z(\hat{K} \otimes V) \otimes (\hat{K} \otimes V), (\hat{K} \otimes V) \otimes \Phi_z(\hat{K} \otimes V))$$

and

$$S = \mathrm{Hom}_{K \otimes U_q'(\mathfrak{g})}(\Phi_z(K \otimes V) \otimes (K \otimes V), (K \otimes V) \otimes \Phi_z(K \otimes V)).$$

Then $\hat{S} = \hat{K} \otimes_K S$. On the other hand by Lemma 3.4.2 with $C = \mathbf{Q}(q)[z, z^{-1}]$, $\mathfrak{m} = (z-1)C$, $R = C \otimes U_q'(\mathfrak{g})$, $M = \Phi_z(C \otimes_{\mathbf{Q}(q)} V) \otimes (C \otimes_{\mathbf{Q}(q)} V)$ and (3.4.1), $\Phi_z(K \otimes V) \otimes (K \otimes V)$ and $(K \otimes V) \otimes \Phi_z(K \otimes V)$ are irreducible. This implies $\dim S \leq 1$. Therefore, we have $\hat{S} = \hat{K}\hat{R}$. Thus S contains a non-zero R. By multiplying a function of z, we may assume

$$R(\Phi_z(V) \otimes_{\mathbf{Q}(q)} V) \subset \mathbf{Q}(q)[z, z^{-1}] \otimes_{\mathbf{Q}(q)} (V \otimes_{\mathbf{Q}(q)} \Phi_z(V)).$$

Let x and y be indeterminate, and set $z = x/y$. Then $\Phi_y(R)$ gives a homomorphism of $\Phi_y\big(\Phi_z(V)\otimes V\big) \simeq \Phi_x(V)\otimes\Phi_y(V)$ into $\Phi_y\big(V\otimes\Phi_z(V)\big) = \Phi_y(V)\otimes\Phi_x(V)$. Thus R gives a $\mathbf{Q}(q)[x, x^{-1}, y, y^{-1}]\otimes U_q'(\mathfrak{g})$-linear homomorphism

$$\begin{aligned}\Phi_x\big(\mathbf{Q}(q)[x,x^{-1}]\otimes V\big)&\otimes_{\mathbf{Q}(q)}\Phi_y\big(\mathbf{Q}(q)[y,y^{-1}]\otimes V\big)\\ &\longrightarrow \Phi_y\big(\mathbf{Q}(q)[y,y^{-1}]\otimes V\big)\otimes_{\mathbf{Q}(q)}\Phi_x\big(\mathbf{Q}(q)[x,x^{-1}]\otimes V\big).\end{aligned}$$

By identifying $\Phi_x\big(\mathbf{Q}(q)[x,x^{-1}]\otimes V\big)$ with $\mathrm{Aff}(V)$, we obtain R satisfying the conditions in (i).

(ii) follows from the irreducibility of $\Phi_x\big(\mathbf{Q}(q)(x,y)\otimes V\big)\otimes\Phi_y\big(\mathbf{Q}(q)(x,y)\otimes V\big)$.

(iii) follows from the fact that the universal R-matrix satisfies the Yang-Baxter equation and that R is proportional to $\hat{R}$. □

Proposition 3.4.3. *Let V be a $U_q'(\mathfrak{g})$-module in $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ satisfying the following conditions:*

(3.4.2) *V has a crystal pseudo-base (L, B).*

(3.4.3) *If we denote by $B_0 = B/\{\pm 1\}$ the associated crystal, then $B_0\otimes B_0$ is connected and there exists $\lambda_0 \in P_{cl}$ such that $\#(B_0\otimes B_0)_{2\lambda_0} = 1$.*

Then, there exists an R-matrix $R : \mathrm{Aff}(V)\otimes\mathrm{Aff}(V)\longrightarrow\mathrm{Aff}(V)\otimes\mathrm{Aff}(V)$.

Proof. Because of Theorem 3.4.1 it is enough to prove the following lemme which guarantees (3.4.1).

Lemma 3.4.4. *Let M be a $U_q'(\mathfrak{g})$-module in $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$. Assume that M has a crystal pseudo-base (L, B) such that*

(3.4.4) *there exists $\lambda \in P_{cl}$ such that $\#(B/\{\pm 1\})_\lambda = 1$,*

(3.4.5) *$B/\{\pm 1\}$ is connected.*

Then M is an irreducible $U_q'(\mathfrak{g})$-module.

Proof. First let us show that M_λ generates M. Let N be the $U_q'(\mathfrak{g})$-module generated by M_λ. Then $L\cap N$ is a crystal lattice of N. Since $(L\cap N)_\lambda = L_\lambda$, $L\cap N/qL\cap N$ contains B. Hence $L\cap N = L$ and $N = M$. Now let us prove that M is irreducible. Let N be a non-zero submodule of M. If $N_\lambda \neq 0$, then by the preceding argument, $N = M$. Suppose that $N_\lambda = 0$. Since the dual of M satisfies the similar properties to M, we have $N = 0$. □

4. Path and Energy Function

4.1. *Energy function* Let B be a classical crystal. A $\mathbf{Z}$-valued function H on $B\otimes B$ is called an *energy function* of B if for any $i\in I$ and $b\otimes b'\in B\otimes B$ such that $\tilde{e}_i(b\otimes b')\neq 0$ we have

$$\begin{aligned} H\big(\tilde{e}_i(b\otimes b')\big) &= H(b\otimes b') \quad \text{if} \quad i\neq i_0,\\ &= H(b\otimes b')+1 \quad \text{if } i=i_0 \text{ and } \varphi_{i_0}(b)\geq \epsilon_{i_0}(b'),\\ &= H(b\otimes b')-1 \quad \text{if } i=i_0 \text{ and } \varphi_{i_0}(b)< \epsilon_{i_0}(b').\end{aligned}$$

Lemma 4.1.1. *Let H and H' be $\mathbf{Z}$-valued functions on $B\otimes B$ and let R be the map from* $\mathrm{Aff}(B)\otimes\mathrm{Aff}(B)$ *into itself defined by*

$$(4.1.1)\qquad R\big(T^c\,af(b)\otimes T^{c'}\,af(b')\big)=T^{c'+H(b\otimes b')}\,af(b)\otimes T^{c+H'(b\otimes b')}\,af(b')$$

where $c,\ c'\in\mathbf{Z}$ and $b,\ b'\in B$.

Then R is a morphism of P-crystals if and only if $H+H'=0$ and H is an energy function.

Proof. Comparing the weights of the both sides of (4.1.1), we see that if R is a morphism of crystals then $H+H'=0$. Let us consider the case where $i=i_0$ and $\varphi_{i_0}(b)\geq\epsilon_{i_0}(b')$. We have

$$\begin{aligned}(4.1.2)\quad R\tilde{e}_{i_0}\big(T^c\,af(b)\otimes T^{c'}\,af(b')\big) &= R\big(T^{c+1}\,af(\tilde{e}_{i_0}b)\otimes T^{c'}\,af(b')\big)\\ &= T^{c'+H(\tilde{e}_{i_0}b\otimes b')}\,af(\tilde{e}_{i_0}b)\otimes T^{c+1-H(\tilde{e}_{i_0}b\otimes b')}\,af(b').\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(4.1.3)\quad \tilde{e}_{i_0}R\big(T^c\,af(b)\otimes T^{c'}\,af(b')\big) &= \tilde{e}_{i_0}\big(T^{c'+H(b\otimes b')}\,af(b)\otimes T^{c-H(b\otimes b')}\,af(b')\big)\\ &= T^{c'+H(b\otimes b')+1}\,af(\tilde{e}_{i_0}b)\otimes T^{c-H(b\otimes b')}\,af(b').\end{aligned}$$

Comparing (4.1.2) and (4.1.3), we obtain the assertion of the lemma in this case. Other cases can be proved similarly. □

4.2. *Combinatorial R-matrix*

Definition 4.2.1. *A combinatorial R-matrix of a classical crystal B is an endomorphism R of the P-weighted crystal* $\mathrm{Aff}(B)\otimes\mathrm{Aff}(B)$ *such that*

$$(4.2.1)\qquad (T\otimes 1)\circ R=R\circ(1\otimes T)\quad\textit{and}\quad (1\otimes T)\circ R=R\circ(T\otimes 1).$$

One can see easily that, for an energy function H,

$$(4.2.2)\qquad R\big(T^c\,af(b)\otimes T^{c'}\,af(b')\big)=T^{c'+H(b\otimes b')}\,af(b)\otimes T^{c-H(b\otimes b')}\,af(b')$$

is a combinatorial R -matrix.

Consider the following conditions for B:

(4.2.3) $B\otimes B$ is connected (and hence B is connected).

(4.2.4) There exists $\lambda_0\in P_{cl}$ such that $\#(B\otimes B)_{2\lambda_0}=1$ (and hence $\#B_{\lambda_0}=1$).

Proposition 4.2.2. *Suppose that B is a classical crystal satisfying (4.2.3), (4.2.4) and R is a combinatorial R-matrix of B. Then there is an energy function H such that (4.2.2) holds.*

Proof. Note that (4.2.3) and (4.2.4) imply that there is no endomorphism of the P_{cl}-weighted crystal $B \otimes B$ other than the identity. Hence we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Aff}(B)\otimes\mathrm{Aff}(B) & \xrightarrow{R} & \mathrm{Aff}(B)\otimes\mathrm{Aff}(B) \\ cl\otimes cl\downarrow & & cl\otimes cl\downarrow \\ B\otimes B & \xrightarrow{id} & B\otimes B. \end{array}$$

From this we have

$$R\big(af(b)\otimes af(b')\big) = T^{H(b\otimes b')}\otimes T^{H'(b\otimes b')}\big(af(b)\otimes af(b')\big). \tag{4.2.5}$$

with some functions $H, H' : B\otimes B \to \mathbf{Z}$. From (4.2.1) we have (4.1.1), and then the assertion follows from Lemma 4.1.1. □

Proposition 4.2.3. *Let H be an energy function of a classical crystal B. Let $\psi : \mathrm{Aff}(B)\otimes\mathrm{Aff}(B) \to \mathbf{Z}$ be the function defined by*

$$\psi(T^c\, af(b)\otimes T^{c'}\, af(b')) = c - c' - H(b\otimes b')$$

for b, $b' \in B$ and c, $c' \in \mathbf{Z}$. Then ψ is constant on each connected component of $\mathrm{Aff}(B)\otimes\mathrm{Aff}(B)$.

Proof. Let R be the combinatorial R-matrix associated with H. Then, for any $\tilde{b} \in \mathrm{Aff}(B)\otimes\mathrm{Aff}(B)$, we have $R(\tilde{b}) = (T^{-\psi(\tilde{b})}\otimes T^{\psi(\tilde{b})})\tilde{b}$. Since R, $T\otimes 1$ and $1\otimes T$ commute with $\tilde{e}_i$ and $\tilde{f}_i$, ψ is constant on the component containing $\tilde{b}$. □

4.3. *Existence of combinatorial R-matrix* Let us give a sufficient condition for the existence of combinatorial R-matrix.

Proposition 4.3.1. *Let V be a $U'_q(\mathfrak{g})$-module in $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ satisfying (3.4.2) and (3.4.3). Then $B_0 = B/\{\pm 1\}$ has a combinatorial R-matrix.*

Proof. By Proposition 3.4.3, V has an R-matrix $R : \mathrm{Aff}(V)\otimes\mathrm{Aff}(V) \to \mathrm{Aff}(V)\otimes\mathrm{Aff}(V)$. Let u_0 be such that $(B_0)_{\lambda_0} = \{u_0 \bmod qL\}$. Let us show after a suitable normalization of R, we have $R(u_0\otimes u_0) = u_0\otimes u_0$ and $R\big(\mathrm{Aff}(B_0)\otimes\mathrm{Aff}(B_0)\big) \subset \mathrm{Aff}(B_0)\otimes\mathrm{Aff}(B_0)$. Denoting $x = T\otimes 1$ and $y = 1\otimes T$, we have $R(u_0\otimes u_0) = g(x/y)(u_0\otimes u_0)$ for some $g \in \mathbf{Q}(q)[x/y, y/x]$. By normalizing R by a multiple of an element of $\mathbf{Q}[q]$, we may assume $g \in \mathbf{Q}[q][x/y, y/x]$ and $g_0 = g|_{q=0} \neq 0$. Set $\widetilde{L} = \mathrm{Aff}(L)\otimes\mathrm{Aff}(L)$. Then $\widetilde{L}$ is a crystal lattice of $\mathrm{Aff}(V)\otimes\mathrm{Aff}(V)$ which is stable by x and y. Since $B_0\otimes B_0$ is connected, the map $R^{-1}(\widetilde{L})\cap\widetilde{L}/R^{-1}(\widetilde{L})\cap q\widetilde{L} \longrightarrow \widetilde{L}/q\widetilde{L}$ is surjective. Hence $\widetilde{L} \subset R^{-1}(\widetilde{L}) + q\widetilde{L}$. Since $\widetilde{L}$ is a finitely generated $A[x, x^{-1}, y, y^{-1}]$-module, there exists $\varphi \in A[x, x^{-1}, y, y^{-1}]$ such that $\varphi|_{q=0} = 1$ and $\varphi\widetilde{L} \subset R^{-1}(\widetilde{L})$. Since R preserves the affine weights, we may assume $\varphi \in A[x/y, y/x]$. Thus, replacing R with φR, we have $R\widetilde{L} \subset \widetilde{L}$. If $R(b) \in g_0(x/y)\big(\mathrm{Aff}(B_0)\otimes\mathrm{Aff}(B_0)\big)$ for $b \in \mathrm{Aff}(B_0)\otimes\mathrm{Aff}(B_0)$, then $R(\tilde{f}_i b)$, $R(\tilde{e}_i b)$ and $R((T^n\otimes T^m)b)$ belong to $g_0(x/y)\big(\mathrm{Aff}(B_0)\otimes\mathrm{Aff}(B_0)\big)$. Therefore we have $R(\mathrm{Aff}(B_0)\otimes\mathrm{Aff}(B_0)) \subset g_0(x/y)\big(\mathrm{Aff}(B_0)\otimes\mathrm{Aff}(B_0)\big)$. Hence $g_0(x/y)^{-1}R$ gives a combinatorial R matrix. □

The following proposition follows from the arguments above and Proposition 4.2.2.

Proposition 4.3.2. *Take a basis* $\{u_j\}$ *of* L *such that* $u_j \bmod qL \in B$, *and set* $R(u_j \otimes u_k) = \sum r_{j,k}^{j',k'} u_{j'} \otimes u_{k'}$ *where* $r_{j,k}^{j',k'} \in A[y/x, x/y]$. *Then, we have*

$$r_{j,k}^{j',k'}\Big|_{q=0} = \delta_{j,j'}\delta_{k,k'}(y/x)^{H(u_j\otimes u_k)}g_0(x/y)$$

for some function $g_0(x/y) \in \mathbb{Q}[x/y, y/x]$.

4.4. *Tensor product of* $B(\lambda)$ *and* B Let B be a classical crystal associated with a $U_q'(\mathfrak{g})$ -module V in $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$. The product $B(\lambda) \otimes B$ may happen to be isomorphic to $B(\mu)$, although $V(\lambda) \otimes V$ is not isomorphic to $V(\mu)$ and it is even not a highest weight module.

Theorem 4.4.1. *Assume rank* $\mathfrak{g} \geq 3$. *Let* B *be a classical crystal,* $\lambda \in P_+$ *and* $b_0 \in B$ *satisfy the following properties.*
(4.4.1) *B is a finite set.*
(4.4.2) *B has an energy function H.*
(4.4.3) $\{b \in B \mid \varepsilon_i(b) \leq \langle h_i, \lambda\rangle$ *for all* $i\} = \{b_0\}$.
(4.4.4) *There exists* $b' \in B$ *such that* $\varphi_i(b') \leq \langle h_i, \lambda\rangle$ *for all* i.
(4.4.5) *For any pair* $i, j \in I$, B *regarded as* $\{i,j\}$*-crystal is a direct sum of the crystals of irreducible integrable highest weight modules over the subalgebra* $U_q(\mathfrak{g}_{\{i,j\}})$ *generated by* e_i, e_j, f_i, f_j *and* $q^h (h \in P_{cl}^*)$.

Then we have an isomorphism of classical crystals $B(\lambda) \otimes B \simeq B(\mu)$ *where* $\mu = \lambda + af(\mathrm{wt} b_0)$ *in such a way that* $u_\lambda \otimes b_0 \leftrightarrow u_\mu$.

Proof. By Proposition 2.4.4 it is enough to check the following.
(4.4.6) $\{\tilde{b} \in B(\lambda) \otimes B \mid \tilde{e}_i\tilde{b} = 0$ for all $i\} = \{u_\lambda \otimes b_0\}$
(4.4.7) For any $\tilde{b} \in B(\lambda) \otimes B$, there exists a sequence $i_1, \cdots, i_l$ $(l \geq 0)$ such that $\tilde{e}_{i_1} \cdots \tilde{e}_{i_l}\tilde{b} = u_\lambda \otimes b_0$.
(4.4.8) For any pair $i, j \in I$, $B(\lambda) \otimes B$ is a disjoint union of crystal graphs of integrable irreducible highest weight $U_q(\mathfrak{g}_{\{i,j\}})$ modules.

Note that (4.4.6) follows from (4.4.3) and (4.4.8) follows from (4.4.5). Let us prove (4.4.7). Suppose that $\tilde{b} = b' \otimes b''$ where $b' \in B(\lambda)$ and $b'' \in B$. For any $i \in I$ there exists $m \geq 1$ such that

$$\tilde{e}_i^m(b' \otimes b'') = \tilde{e}_i b' \otimes \tilde{e}_i^{m-1} b''.$$

Hence there exists $j_1, \cdots, j_l$ such that

$$\tilde{e}_{j_1} \cdots \tilde{e}_{j_l}\tilde{b} = u_\lambda \otimes b \quad \text{for some} \quad b \in B.$$

Therefore we may assume $\tilde{b} = u_\lambda \otimes b$. Now, assume that (4.4.7) does not hold for this $\tilde{b}$. Then, there exists an infinite sequence $\{i_\nu\}$ in I such that

$$\tilde{e}_{i_k} \cdots \tilde{e}_{i_1}(u_\lambda \otimes b) \neq 0.$$

Since $\tilde{e}_{i_k} \cdots \tilde{e}_{i_1}(u_\lambda \otimes b) = u_\lambda \otimes \tilde{e}_{i_k} \cdots \tilde{e}_{i_1} b$ and since B is a finite set there exists $b_1 \in B$ and $i_1, \cdots, i_k$ $(k \geq 1)$ such that

$$u_\lambda \otimes b_1 = \tilde{e}_{i_k} \cdots \tilde{e}_{i_1}(u_\lambda \otimes b_1).$$

Hence setting $b_{\nu+1} = \tilde{e}_{i_\nu} b_\nu$, we have

$$(4.4.9) \qquad \tilde{e}_{i_\nu}(u_\lambda \otimes b_\nu) = u_\lambda \otimes b_{\nu+1} \quad \text{and} \quad b_{k+1} = b_1.$$

By (2.2.17), we have $\varepsilon_{i_\nu}(b_\nu) > \varphi_{i_\nu}(u_\lambda) = \langle h_{i_\nu}, \lambda \rangle \geq \varphi_{i_\nu}(b')$, where $b' \in B$ is an element as in (4.4.4). Hence we have

$$\tilde{e}_{i_\nu}(b' \otimes b_\nu) = b' \otimes b_{\nu+1}.$$

Therefore, by the definition of energy function, we have

$$H(b' \otimes b_{\nu+1}) = H(b' \otimes b_\nu) - \delta_{i_0, i_\nu}.$$

Hence $H(b' \otimes b_{k+1}) = H(b' \otimes b_1) - \#\{\nu \mid i_\nu = i_0\}$, which implies there is no ν such that $i_\nu = i_0$. On the other hand $cl(\sum_\nu \alpha_{i_\nu}) = 0$ and hence $\sum_\nu \alpha_{i_\nu}$ is a positive multiple of δ, which contradicts $\{i_1, \cdots, i_k\} \not\ni i_0$. □

4.5. *Path* Let B be a finite classical crystal that has an energy function H and satisfies (4.4.5). We assume that there exists a sequence $\lambda_\nu \in af(P_{cl}) \cap P_+$ $(\nu \geq 0)$ and $b_\nu \in B_{cl(\lambda_{\nu-1}) - cl(\lambda_\nu)}$ $(\nu \geq 1)$ such that

(4.5.1) $\qquad \varepsilon_i(b_\nu) = \langle h_i, \lambda_\nu \rangle$ for all i and $\nu \geq 1$

(4.5.2) $\qquad \{b \in B \mid \varepsilon_i(b) \leq \langle h_i, \lambda_\nu \rangle \text{ for all } i\} = \{b_\nu\}$ for each $\nu \geq 1$.

Since $\mathrm{wt}(B) = \{\lambda \in P_{cl} \mid B_\lambda \neq \phi\}$ is a finite set invariant under the Weyl group, any weight λ has level 0 (i.e., $\langle c, \lambda \rangle = 0$).

Therefore $\{\lambda_\nu\}_\nu$ are dominant integral weights with the same level. Since there are only finitely many such weights, there exist $N \geq 1$ and $k \geq 0$ such that $\lambda_{k+N} = \lambda_k$. Hence by the uniqueness of b_ν we conclude that $b_{\nu+N} = b_\nu$ for any $\nu \geq 1$ and $\lambda_{\nu+N} = \lambda_\nu$ for any $\nu \geq 0$. Since $\langle h_i, \lambda_\nu - \lambda_{\nu+1} \rangle = \varphi_i(b_{\nu+1}) - \varepsilon_i(b_{\nu+1})$, we have $\langle h_i, \lambda_\nu \rangle = \varphi_i(b_{\nu+1})$. Thus we can apply Theorem 4.4.1 to see that there is an isomorphism between $B(\lambda_{\nu-1})$ and $B(\lambda_\nu) \otimes B$ such that $u_{\lambda_{\nu-1}} \leftrightarrow u_{\lambda_\nu} \otimes b_\nu$. Hence, repreating this, we obtain an isomorphism

$$\psi_k : B(\lambda_0) \xrightarrow{\sim} B(\lambda_k) \otimes B^{\otimes k} \quad \text{such that} \quad u_{\lambda_0} \mapsto u_{\lambda_k} \otimes b_k \otimes \cdots \otimes b_1.$$

Lemma 4.5.1. *For any $b \in B(\lambda_0)$, there exists $k > 0$ such that*

$$\psi_k(b) \in u_{\lambda_k} \otimes B^{\otimes k}.$$

Proof. It is sufficient to show that if $\psi_k(b) \in u_{\lambda_k} \otimes B^{\otimes k}$, then $\psi_{k+1}(\tilde{f}_i b) \in u_{\lambda_{k+1}} \otimes B^{\otimes(k+1)}$. If $\psi_k(b) = u_{\lambda_k} \otimes b'$ for $b' \in B^{\otimes k}$ then $\psi_{k+1}(b) = u_{\lambda_{k+1}} \otimes b_{k+1} \otimes b'$. Since $\tilde{f}_i(u_{\lambda_{k+1}} \otimes b_{k+1}) = u_{\lambda_{k+1}} \otimes \tilde{f}_i b_{k+1}$, $\psi_{k+1}(\tilde{f}_i b)$ is equal to either $u_{\lambda_{k+1}} \otimes \tilde{f}_i b_{k+1} \otimes b'$ or $u_{\lambda_{k+1}} \otimes b_{k+1} \otimes \tilde{f}_i b'$. □

The sequence $(b_1, b_2, \cdots)$ is called the *ground-state* path of weight λ_0. A λ_0-*path* in B is, by definition a sequence $\{p(n)\}_{n \geq 1}$ in B such that $p(n) = b_n$ for $n \gg 0$. Let us denote by $\mathcal{P}(\lambda_0, B)$ the set of λ_0-paths. Now, the following theorem is obvious by the preceding discussions.

Theorem 4.5.2. *$B(\lambda_0)$ is isomorphic to $\mathcal{P}(\lambda_0, B)$ by $B(\lambda_0) \ni b \mapsto p \in \mathcal{P}(\lambda_0, B)$ where $\psi_k(b) = u_{\lambda_k} \otimes p(k) \otimes \cdots \otimes p(1)$ for $k \gg 0$.*

Now, we shall calculate the weight of a path. Define a morphism of P-crystals $B(\lambda_{k-1}) \to B(\lambda_k) \otimes \mathrm{Aff}(B)$ by $u_{\lambda_{k-1}} \mapsto u_{\lambda_k} \otimes af(b_k)$. This is well-defined. By repeating this, we obtain

$$\tilde{\psi}_k : B(\lambda_0) \to B(\lambda_k) \otimes \mathrm{Aff}(B)^{\otimes k}$$

where $\tilde{\psi}_k(u_{\lambda_0}) = u_{\lambda_k} \otimes af(b_k) \otimes \cdots \otimes af(b_1)$. Evidently the following diagram commutes.

$$\begin{array}{ccc} B(\lambda_0) & \xrightarrow{\tilde{\psi}_k} & B(\lambda_k) \otimes \mathrm{Aff}(B)^{\otimes k} \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id \otimes cl\ ^{\otimes k}} \\ B(\lambda_0) & \xrightarrow{\psi_k} & B(\lambda_k) \otimes B^{\otimes k}. \end{array}$$

Lemma 4.5.3. *For $b \in B(\lambda_0)$, set*

$$\tilde{\psi}_k(b) = b' \otimes T^{c_k}\, af(b'_k) \otimes T^{c_{k-1}}\, af(b'_{k-1}) \otimes \cdots \otimes T^{c_1}\, af(b'_1)$$

with $b' \in B(\lambda_k)$, $b'_\nu \in B$. Then

$$c_\nu - c_{\nu-1} = H(b'_\nu \otimes b'_{\nu-1}) - H(b_\nu \otimes b_{\nu-1}).$$

Proof. Since $\tilde{\psi}_k(B(\lambda_0))$ is connected, both $T^{c_\nu}\, af(b'_\nu) \otimes T^{c_{\nu-1}}\, af(b'_{\nu-1})$ and $af(b_\nu) \otimes af(b_{\nu-1})$ belong to the same connected component of $\mathrm{Aff}(B) \otimes \mathrm{Aff}(B)$. Hence by Proposition 4.2.3

$$c_\nu - c_{\nu-1} - H(b'_\nu \otimes b'_{\nu-1}) = -H(b_\nu \otimes b_{\nu-1}). \qquad \square$$

Proposition 4.5.4. *If $b \in B(\lambda_0)$ corresponds to a λ_0-path $p = (p(n))_{n \geq 1}$ then*

$$\begin{aligned} \mathrm{wt}\, b =& \lambda_0 + \sum_{k=1}^{\infty} \big(af(\mathrm{wt}\, p(k)) - af(\mathrm{wt}\, b_k)\big) \\ & - \left(\sum_{k=1}^{\infty} k\big(H(p(k+1) \otimes p(k)) - H(b_{k+1} \otimes b_k)\big)\right)\delta. \end{aligned}$$

Proof. Since $p(k) = b_k$ for $k \gg 0$, the sum makes sense. Take $k \gg 0$ such that

$$\psi_k(b) = u_{\lambda_k} \otimes p(k) \otimes \cdots \otimes p(1).$$

Then $p(\nu) = b_\nu$ for $\nu > k$. Set

$$\tilde{\psi}_{k+1}(b) = u_{\lambda_{k+1}} \otimes T^{c_{k+1}}\, af(p(k+1)) \otimes T^{c_k}\, af(p(k)) \otimes \cdots \otimes T^{c_1}\, af(p(1)).$$

Then $c_{k+1} = 0$ and $p(k+1) = b_{k+1}$. Thus we have

$$\mathrm{wt}\, b = af \circ cl(\mathrm{wt}\, b) + \left(\sum_{\nu=1}^{k} c_\nu\right)\delta.$$

On the other hand we have $c_{\nu+1} - c_\nu = H(p(\nu+1) \otimes p(\nu)) - H(b_{\nu+1} \otimes b_\nu)$ for $1 \le \nu \le k$ by Lemma 4.5.3. Hence we obtain

$$\begin{aligned}
&\sum_{\nu=1}^{\infty} \nu(H(p(\nu+1) \otimes p(\nu)) - H(b_{\nu+1} \otimes b_\nu)) \\
&\quad = \sum_{\nu=1}^{k} \nu(c_{\nu+1} - c_\nu) \\
&\quad = -\sum_{\nu=1}^{k} c_\nu + kc_{k+1} = -\sum_{\nu=1}^{k} c_\nu.
\end{aligned}$$

Thus we obtain the desired result. □

4.6. *Perfect crystal* In this subsection, we assume that the rank of $\mathfrak{g}$ is greater than 2. We set $P_{cl}^+ = \{\lambda \in P_{cl} \mid \langle h_i, \lambda \rangle \ge 0 \text{ for any } i\} \cong \sum \mathbf{Z}_{\ge 0}\Lambda_i$ and $(P_{cl}^+)_l = \{\lambda \in P_{cl}^+ \mid \langle c, \lambda \rangle = l\} \cong \{\lambda \in \sum \mathbf{Z}\Lambda_i \mid \langle c, \lambda \rangle = l\}$ for $l \in \mathbf{Z}$.

Let B be a classical crystal. For $b \in B$, we set $\varepsilon(b) = \sum \varepsilon_i(b)\Lambda_i$ and $\varphi(b) = \sum \varphi_i(b)\Lambda_i$. Note that $\mathrm{wt}(b) = cl(\varphi(b) - \varepsilon(b))$. In particular $\langle c, \varepsilon(b) \rangle = \langle c, \varphi(b) \rangle$.

Definition 4.6.1. *For $l \in \mathbf{Z}_{>0}$, we say that B is a perfect crystal of level l if B satisfies the following conditions.*

(4.6.1) *$B \otimes B$ is connected.*

(4.6.2) *There exists $\lambda_0 \in P_{cl}$ such that $\mathrm{wt}(B) \subset \lambda_0 + \sum_{i \ne i_0} \mathbf{Z}_{\le 0}\alpha_i$ and that $\sharp(B_{\lambda_0}) = 1$.*

(4.6.3) *There is a $U_q'(\mathfrak{g})$-module in $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$ with a crystal pseudo-base (L, B') such that B is isomorphic to $B'/\{\pm 1\}$.*

(4.6.4) *For any $b \in B$, we have $\langle c, \varepsilon(b) \rangle \ge l$.*

(4.6.5) *The maps ε and φ from $B_l = \{b \mid \langle c, \varepsilon(b) \rangle = l\}$ to $(P_{cl}^+)_l$ are bijective.*

We call an element of B_l *minimal.*

Lemma 4.6.2. *Let B_1 and B_2 be perfect crystals of level l. Then $B_1 \otimes B_2$ is also a perfect crystal of level l. Moreover $(B_1 \otimes B_2)_l = \{b_1 \otimes b_2 \mid \varphi(b_1) = \varepsilon(b_2), \langle c, \varphi(b_1) \rangle = l\}$.*

Proof. For $b_j \in B_j$, we have $\varepsilon_i(b_1 \otimes b_2) = \varepsilon_i(b_1) + (\varepsilon_i(b_2) - \varphi_i(b_1))_+$ where $x_+ = \max(x, 0)$. Hence $\langle c, \varepsilon(b_1 \otimes b_2) \rangle \ge \langle c, \varepsilon(b_1) \rangle$. Moreover $\langle c, \varepsilon(b_1 \otimes b_2) \rangle = \langle c, \varepsilon(b_1) \rangle$ implies $\varphi_i(b_1) \ge \varepsilon_i(b_2)$ for any i. Hence $\langle c, \varepsilon(b_1 \otimes b_2) \rangle \ge l$ and if $\langle c, \varepsilon(b_1 \otimes b_2) \rangle = l$ then $\langle c, \varepsilon(b_1) \rangle = l$ and $\varphi_i(b_1) \ge \varepsilon_i(b_2)$. Hence $l = \langle c, \varphi(b_1) \rangle \ge \langle c, \varepsilon(b_2) \rangle \ge l$. Thus $\varphi(b_1) = \varepsilon(b_2)$. The conditions (4.6.2) and (4.6.3) for $B_1 \otimes B_2$ are obvious. Let us prove (4.6.1). There are $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in af((P_{cl}^+)_l)$ such that

$$\begin{aligned}
B(\lambda_1) \otimes B_1 &\simeq B(\lambda_2), \\
B(\lambda_2) \otimes B_2 &\simeq B(\lambda_3), \\
B(\lambda_3) \otimes B_1 &\simeq B(\lambda_4), \\
B(\lambda_4) \otimes B_2 &\simeq B(\lambda_5).
\end{aligned}$$

Since $B(\lambda_1) \otimes (B_1 \otimes B_2)^{\otimes 2} \simeq B(\lambda_5)$ is connected, $(B_1 \otimes B_2)^{\otimes 2}$ is connected. □

Corollary 4.6.3. *If B is a perfect crystal, then $B^{\otimes n}$ is connected for any n.*

Let B be a perfect crystal of level l. For $\lambda \in (P_{cl}^+)_l$, let $b(\lambda) \in B$ be the element such that $\varphi(b(\lambda)) = \lambda$. Let σ be the automorphism of $(P_{cl}^+)_l$ given by $\sigma\lambda = \varepsilon(b(\lambda))$. Then the conditions (4.5.1) and (4.5.2) are satisfied by taking $b_\nu = b(\sigma^{\nu-1}\lambda)$ and $\lambda_\nu = \sigma^\nu\lambda$. Hence by Theorem 4.5.2 we have the following result.

Proposition 4.6.4. *For $\lambda \in (P_{cl}^+)_l$, let $\mathcal{P}(\lambda, B)$ be the set of sequences $\{p(n)\}_{n\geq 1}$ in B such that $p(n) = b(\sigma^{n-1}\lambda)$ for $n \gg 0$. Then $B(\lambda)$ is isomorphic to $\mathcal{P}(\lambda, B)$ by $B(\lambda) \ni b \longmapsto u_{\sigma^k\lambda} \otimes p(k) \otimes \cdots \otimes p(1)$ for $k \gg 0$.*

By this correspondence, we have

$$\begin{aligned}\mathrm{wt}(b) =& \lambda + \sum_{k=1}^{\infty} af\big(\mathrm{wt}(p(k))\big) - af\big(\mathrm{wt}(b(\sigma^{k-1}\lambda))\big) \\ & - \sum_{k=1}^{\infty} k\big\{H\big(p(k+1)\otimes p(k)\big) - H\big(b(\sigma^k\lambda)\otimes b(\sigma^{k-1}\lambda)\big)\big\}\delta\end{aligned}$$

5. One point function

5.1 *Dual Module* Let us begin by the definition of dual modules.

Definition 5.1.1. *Let φ be an anti-automorphism of $U_q'(\mathfrak{g})$ and V a left $U_q'(\mathfrak{g})$-module. Then a left $U_q'(\mathfrak{g})$-module $V^{*\varphi}$ is defined by*

$$\begin{aligned}&V^{*\varphi} = \mathrm{Hom}_{\mathbb{Q}(q)}\big(V, \mathbb{Q}(q)\big),\\ &(Pf)(v) = f\big(\varphi(P)v\big) \quad \text{for } P \in U_q'(\mathfrak{g}),\ f \in V^{*\varphi} \text{ and } v \in V.\end{aligned}$$

If V is finite-dimensional, then $\big(V^{*\varphi}\big)^{*\varphi^{-1}}$ is canonically isomorphic to V. Let a be the antipode of $U_q'(\mathfrak{g})$. The following lemma is well-known.

Lemma 5.1.2. *Let $V_i (i = 1, 2, 3)$ be finite-dimensional left $U_q'(\mathfrak{g})$-modules. Then*

$$\begin{aligned}\mathrm{Hom}_{U_q'(\mathfrak{g})}(V_1 \otimes V_2, V_3) &\simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}(V_2, V_1^{*a^{-1}} \otimes V_3)\\ &\simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}(V_1, V_3 \otimes V_2^{*a}).\end{aligned}$$

Let ι be the anti-automorphism of $U_q'(\mathfrak{g})$ defined by

$$\iota(e_i) = -e_i, \quad \iota(f_i) = -f_i, \quad \iota(q^h) = q^{-h}.$$

For V in $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$, let $\langle\ ,\ \rangle_\iota$ be the pairing of V and $V^{*\iota}$ given by

$$\langle v, w\rangle_\iota = q^{-(af(\lambda), af(\lambda))}\langle v, w\rangle \quad \text{for} \quad (v, w) \in V_\lambda \times V^{*\iota},$$

where $\langle\ ,\ \rangle$ is the canonical pairing of V and V^*. For a crystal lattice L of V, let us define $L^{*\iota}$ by

$$L^{*\iota} = \{w \in V^{*\iota} | \langle L, w\rangle_\iota \subset A\}.$$

Then the pairing $\langle\ ,\ \rangle_\iota$ induces a pairing between L/qL and $L^{*\iota}/qL^{*\iota}$.

Proposition 5.1.3. *Let V be an object of $Mod^f(\mathfrak{g}, P_{cl})$ and L a crystal lattice of V. Then*
(i) $L^{*\iota}$ is a crystal lattice of $V^{*\iota}$.
(ii) Suppose that (L, B) is a crystal pseudo-base of V. Let $B^{*\iota} \subset L^{*\iota}/qL^{*\iota}$ be the dual base of B with respect to the pairing induced by $\langle\ ,\ \rangle_\iota$. Then $(L^{*\iota}, B^{*\iota})$ is a crystal pseudo-base of $V^{*\iota}$.

Proof. (i) We shall prove that $\tilde{e}_i L^{*\iota} \subset L^{*\iota}$ and $\tilde{f}_i L^{*\iota} \subset L^{*\iota}$. As a $U'_q(\mathfrak{g}_i)$-module L is a direct sum of $L(\lambda)'s$. Therefore we may assume that V is the highest weight $U'_q(\mathfrak{g}_i)$-module with highest weight λ and L is its crystal lattice. Write $L = \oplus_{j=0}^N A u_j$, where $u_j = f_i^{(j)} u_0$ for $0 \leq j \leq N$. Let $\{w_j\}$ be the dual base of $\{u_j\}$ with respect to $\langle\ ,\ \rangle_\iota$. Then

$$\begin{aligned}\langle u_k, e_i^{(k)} w_0\rangle &= \langle (-1)^k e_i^{(k)} f_i^{(k)} u_0, w_0\rangle \\ &= (-1)^k \begin{bmatrix} \langle h_i, \lambda\rangle \\ k \end{bmatrix}_i \langle u_0, w_0\rangle \\ &\in (-1)^k q^{(af(\lambda), af(\lambda))} q_i^{-k(\langle h_i, \lambda\rangle - k)}(1 + qA) \\ &= (-1)^k q^{(af(\lambda - kcl(\alpha_i)), af(\lambda - kcl(\alpha_i)))}(1 + qA).\end{aligned}$$

Hence $\langle u_k, (-1)^k e_i^{(k)} w_0\rangle_\iota \in 1 + qA$. It follows that $L^{*\iota} = \oplus A e_i^{(k)} w_0$. Then it is obvious that $\tilde{e}_i L^{*\iota} \subset L^{*\iota}$ and $\tilde{f}_i L^{*\iota} \subset L^{*\iota}$.
(ii) The proof is straightforward from the arguments above. □

Definition 5.1.4. *Let B be a weighted classical crystal. Then the weighted classical crystal B^* is defined by*

$$B^* = \sqcup_{\lambda \in P_{cl}} (B^*)_\lambda \quad \text{and} \quad (B^*)_\lambda = B_{-\lambda} \quad \text{for} \quad \lambda \in P_{cl}.$$

The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by the following commutative diagrams.

$$\begin{array}{ccc} (B^*)_\lambda & \xrightarrow{\tilde{e}_i} & (B^*)_{\lambda + cl(\alpha_i)} \sqcup \{0\} \\ \wr\| & & \wr\| \\ B_{-\lambda} & \xrightarrow{\tilde{f}_i} & B_{-\lambda - cl(\alpha_i)} \sqcup \{0\} \end{array} \quad \text{and} \quad \begin{array}{ccc} (B^*)_\lambda & \xrightarrow{\tilde{f}_i} & (B^*)_{\lambda - cl(\alpha_i)} \sqcup \{0\} \\ \wr\| & & \wr\| \\ B_{-\lambda} & \xrightarrow{\tilde{e}_i} & B_{-\lambda + cl(\alpha_i)} \sqcup \{0\}. \end{array}$$

The following lemma is immediately proved.

Lemma 5.1.5. *Let B_i $(i = 1, 2)$ be weighted classical crystals. Then the map*

$$\sigma : (B_1 \otimes B_2)^* \longrightarrow B_2^* \otimes B_1^*$$

defined by $\sigma(b_1 \otimes b_2) = b_2 \otimes b_1$ is an isomorphism of classical crystals.

We obtain the following proposition by Proposition 5.1.3 and the arguments in its proof.

Proposition 5.1.6. *Let V be an object of $Mod^f(\mathfrak{g}, P_{cl})$. Suppose that V has a crystal pseudo-base (L, B). Then $V^{*\iota}$ has a crystal pseudo-base $(L^{*\iota}, B^{*\iota})$. Moreover the associated crystal $B^{*\iota}/\{\pm 1\}$ is isomorphic to $(B/\{\pm 1\})^*$.*

Proposition 5.1.7. *Let B be a perfect crystal of level l. Then B^* is also a perfect crystal of level l.*

The proof is straightforward by using Proposition 5.1.6 and Lemma 5.1.5.

Set $\rho = \sum_{i\in I}\Lambda_i$ and $d = q^{2(\rho,\delta)}$. Note that $2(\rho,\delta)$ is an integer.

Proposition 5.1.8. *Let V be an object of $Mod^f(\mathfrak{g}, P_{cl})$. Define a linear map $F : V^{*\iota} \to \Phi_d(V^{*a})$ by $(V^{*\iota})_\lambda \ni w \mapsto q^{-(af(\lambda),af(\lambda))-2(\rho,af(\lambda))}w \in \Phi_d(V^{*a})_\lambda$. Then F is an isomorphism of $U_q'(\mathfrak{g})$-modules .*

Proof. For a weight $\lambda \in P_{cl}$ of V, let $w \in (V^{*\iota})_\lambda$ and $v \in V_{-\lambda-cl(\alpha_i)}$. Since $e_i w \in (V^{*\iota})_{\lambda+cl(\alpha_i)}$, we have

$$\begin{aligned}\langle\phi_d(v), F(e_iw)\rangle &= q^{-(af(\lambda+cl(\alpha_i)),af(\lambda+cl(\alpha_i)))-2(\rho,af(\lambda+cl(\alpha_i)))}\langle -e_iv, w\rangle \\ &= q^{-(af(\lambda),af(\lambda))-2(af(\lambda),\alpha_i)-2(\alpha_i,\alpha_i)-2(\rho,af(\lambda))+2\delta_{ii_0}(\rho,\delta)}\langle -e_iv, w\rangle.\end{aligned}$$

This follows from $\quad cl(\alpha_i) = \alpha_i - \delta_{ii_0}\delta, \quad 2(\rho,\alpha_i) = (\alpha_i,\alpha_i) \quad$ and $\quad (af(\lambda),\delta) = 0.$
On the other hand

$$\begin{aligned}\langle\phi_d(v), e_iF(w)\rangle &= q^{-(af(\lambda),af(\lambda))-2(\rho,af(\lambda))}\langle -q^{2\delta_{ii_0}(\rho,\delta)}e_it_iv, w\rangle \\ &= q^{-(af(\lambda),af(\lambda))-2(\rho,af(\lambda))+2\delta_{ii_0}(\rho,\delta)+(h_i,-\lambda-cl(\alpha_i))}\langle -e_iv, w\rangle \\ &= q^{-(af(\lambda),af(\lambda))-2(af(\lambda),\alpha_i)-2(\alpha_i,\alpha_i)-2(\rho,af(\lambda))+2\delta_{ii_0}(\rho,\delta)}\langle -e_iv, w\rangle.\end{aligned}$$

Hence $e_iF(v) = F(e_iv)$. By a similar calculation we can show that f_i commutes with F. □

Corollary 5.1.9. *Let V be an object of $Mod^f(\mathfrak{g}, P_{cl})$. Suppose that V has a crystal pseudo-base (L, B) such that $B_0 = B/\{\pm 1\}$ is a perfect crystal of level l. Then $\Phi_d(V^{*a})$ has a crystal pseudo-base such that its associated crystal is perfect of level l and isomorphic to B_0^*.*

Let us denote $V_x = \Phi_x(V)$ for a $U_q'(\mathfrak{g})$-module V in $\mathrm{Mod}^f(\mathfrak{g}, P_{cl})$. Set $c = q^{-4(\rho,\delta)}$.

Lemma 5.1.10. *Let V be a left $U_q'(\mathfrak{g})$-module in $Mod^f(\mathfrak{g}, P_{cl})$. Then we have*
(i) $(V_x)^{*a} = (V^{*a})_x$ and $(V_x)^{*a^{-1}} = (V^{*a^{-1}})_x$.
(ii) Define a linear map $\Psi : (V_x)^{*a} \to (V_{cx})^{*a^{-1}}$ by $\Psi(w) = q^{4(\rho,af(\lambda))}w$ for $w \in (V_x)^{*a}_\lambda$. Then Ψ is an isomorphism of $U_q'(\mathfrak{g})$-modules.

Proof. (i) is immediate. We shall show that Ψ commutes with e_i and f_i for all $i \in I$. Let $w \in (V_x)^{*a}_\lambda$ and $v \in (V_x)_{-\lambda-cl(\alpha_i)}$ for $\lambda \in P_{cl}$. Then $e_iw \in (V_x)^{*a}_{\lambda+cl(\alpha_i)}$. Therefore

$$\begin{aligned}\langle v, \Psi(e_iw)\rangle &= q^{4(\rho,af(\lambda+cl(\alpha_i)))}\langle -x^{\delta_{ii_0}}e_it_iv, w\rangle \\ &= x^{\delta_{ii_0}}q^{4(\rho,af(\lambda)-\delta_{ii_0}\delta)}q_i^{<h_i,-\lambda>}\langle -e_iv, w\rangle.\end{aligned}$$

Here we used $4(\rho,\alpha_i) = (\alpha_i,\alpha_i)\langle h_i,\alpha_i\rangle$.
On the other hand we have

$$\begin{aligned}\langle v, e_i\Psi(w)\rangle &= q^{4(\rho,af(\lambda))}\langle -x^{\delta_{ii_0}}q^{-4(\rho,\delta)\delta_{ii_0}}t_ie_iv, w\rangle. \\ &= x^{\delta_{ii_0}}q^{4(\rho,af(\lambda)-\delta\delta_{ii_0})}q_i^{<h_i,-\lambda>}\langle -e_iv, w\rangle.\end{aligned}$$

Hence $\Psi(e_i)(w) = e_i\Psi(w)$. Similar calculations show that f_i commutes with Ψ. □

5.2 *Second Inversion Relation* Let $R(x/y):V_x \otimes V_y \to V_y \otimes V_x$ be a $U_q'(\mathfrak{g})$-linear isomorphism. Take a base $\{v_\mu\}_{\mu\in J}$ of V in such a way that each v_μ is a weight vector. Write $R(x/y)$ by using the base $\{v_\mu\}_{\mu\in J}$ as

$$R(x/y)(v_\mu \otimes v_\nu) = \sum_{\alpha,\beta} R_{\alpha\beta\mu\nu}(x/y) v_\alpha \otimes v_\beta .$$

For $\gamma \in J$ set $g_\gamma = q^{-2(\rho, af(wt(v_\gamma)))}$ and $z = x/y$.

Proposition 5.2.1. *Suppose that $(V_y)^{*a} \otimes V_x$ is an irreducible $U_q'(\mathfrak{g})$-module. Then the following relations hold.*

$$\sum_{\alpha,\beta} g_\alpha g_\beta g_{\mu'}^{-1} g_{\nu'}^{-1} R_{\mu\alpha\nu\beta}(z) R_{\nu'\beta\mu'\alpha}(cz^{-1}) = f(z)\delta_{\mu\mu'}\delta_{\nu\nu'}, \tag{5.2.1}$$

where $f(z)$ is a function in z which is independent of μ, μ', ν and ν'. Here $c = q^{-4(\rho,\delta)}$.

Proof. By Lemma 5.1.2, we have $\mathrm{Hom}_{U_q'(\mathfrak{g})}(V_x \otimes V_y, V_y \otimes V_x) \simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}(V_x, V_y \otimes V_x \otimes (V_y)^{*a}) \simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}((V_y)^{*a} \otimes V_x, V_x \otimes (V_y)^{*a})$. By this, $R(x/y)$ gives the $U_q'(\mathfrak{g})$-linear isomorphism $R^1(z) : (V_y)^{*a} \otimes V_x \to V_x \otimes (V_y)^{*a}$. Similarly we have $\mathrm{Hom}_{U_q'(\mathfrak{g})}(V_{cy} \otimes V_x, V_x \otimes V_{cy}) \simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}(V_x \otimes (V_{cy})^{*a^{-1}}, (V_{cy})^{*a^{-1}} \otimes V_x)$, and $R(cy/x)$ gives $R^2(z) : V_x \otimes (V_{cy})^{*a^{-1}} \to (V_{cy})^{*a^{-1}} \otimes V_x$. Let $v^\mu (\mu \in J)$ be the dual base of $v_\mu (\mu \in J)$. Write

$$R^1(z)(v^\mu \otimes v_\nu) = \sum_{\alpha,\beta} R^1_{\alpha\beta\mu\nu}(z) v_\alpha \otimes v^\beta ,$$

$$R^2(z)(v_\mu \otimes v^\nu) = \sum_{\alpha,\beta} R^2_{\alpha\beta\mu\nu}(z) v^\alpha \otimes v_\beta .$$

Then $R^1_{\alpha\beta\mu\nu}(z) = R_{\mu\alpha\nu\beta}(z)$ and $R^2_{\alpha\beta\mu\nu}(z) = R_{\beta\nu\alpha\mu}(cz^{-1})$. Since $(V_y)^{*a} \otimes V_x$ is irreducible and $R^2(z)(1 \otimes \Psi)R^1(z)$ is a $U_q'(\mathfrak{g})$-linear homomorphism, $R^2(z)(1 \otimes \Psi)R^1(z) = f(z)(\Psi \otimes 1)$ for some scalar function f. Writing this relation in terms of the matrix elements of R we have the desired result. □

Suppose that V has a crystal pseudo-base (L, B) and that the associated crystal is perfect of level l. Take $v_\mu (\mu \in J)$ in such a way that $v_\mu \in L$ and the image of v_μ in L/qL is in B. By Corollary 5.1.9, Lemma 4.6.2 and Lemma 3.4.4, $(V_y)^{*a} \otimes V_x$ is an irreducible $\mathbb{Q}(q)[x, y, x^{-1}, y^{-1}] \otimes U_q'(\mathfrak{g})$-module. Therfore the R-matrix associated with V satisfies the second inversion relation (5.2.1). Thus we can apply Baxter's corner transfer matrix method (*cf.*[ABF]). Then for $a \in P_{cl}$ and $\Lambda \in (P_{cl})_{+l}$, the one point function $P(a|\Lambda)$ can be written in the form

$$P(a|\Lambda) = \frac{G(a)}{Z},$$

where

$$G(a) = q^{-4(\rho,\Lambda - af(a))} \sum_{p \in \mathcal{P}(\Lambda,B)(a)} q^{4(\rho,\delta)\omega(p)},$$

$$\mathcal{P}(\Lambda,B)(a) = \{(p(k))_{k=1}^{\infty} \in \mathcal{P}(\Lambda,B) | a + \sum_{i=1}^{k} \mathrm{wt}\, p(i) = \sum_{i=1}^{k} \mathrm{wt}\, b_i \quad \text{for} \quad k >> 0\},$$

$$\omega(p) = \sum_{k=1}^{\infty} k(H(p(k+1) \otimes p(k)) - H(b_{k+1} \otimes b_k)),$$

$$Z = \sum_{a \in P_{cl}} G(a).$$

Here $(b_k)_{k=1}^{\infty}$ is the ground state path of weight Λ. Using this expression of $P(a|\Lambda)$ and Proposition 4.6.4 we obtain

Theorem 5.2.2. *Let Λ be a dominant integral weight of level l, $V(\Lambda)$ an irreducible highest weight $\mathfrak{g}$-module with highest weight Λ and $a \in P_{cl}$. Then the one point function $P(a|\Lambda)$ is given by*

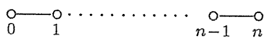
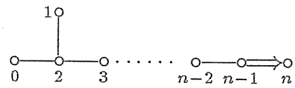
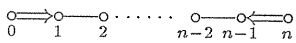
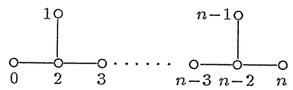
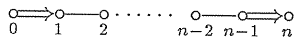
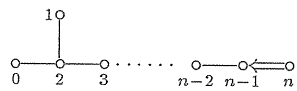
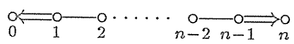
$$P(a|\Lambda) = \frac{\sum_i \dim V(\Lambda)_{\lambda_a - i\delta} q^{-4(\rho,\lambda_a - i\delta)}}{\sum_{\mu \in \mathrm{wt}(V(\Lambda))} \dim V(\Lambda)_\mu q^{-4(\rho,\mu)}},$$

where $\lambda_a = \Lambda - af(a)$.

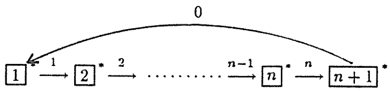
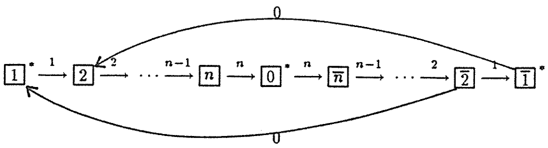
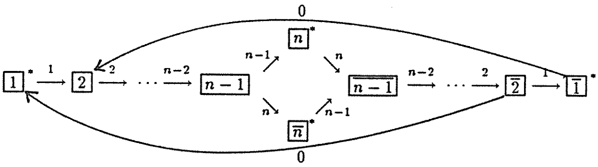
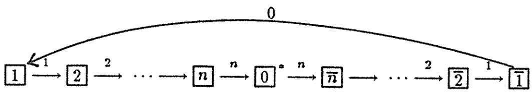
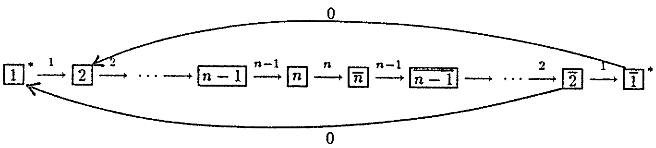
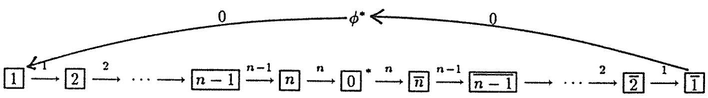
6. Level one crystals

Here we give explicit descriptions of level one perfect crystals for quantum affine Lie algebras of the following types: $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$. This is not a complete list of all the level one perfect crystals.

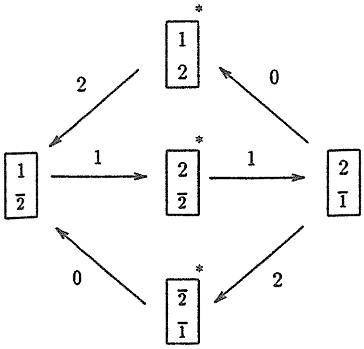
In the forthcoming paper, we will give higher level perfect crystals. Table 1 is the list of the affine Lie algebras and the perfect crystals we will treat therein. Table 2 is the level one crystal graphs except for $B(\Lambda_k)(k \neq 1)$ for $A_n^{(1)}$, $B(\Lambda_n)$ for $C_n^{(1)}$, $B(\Lambda_{n-1}), B(\Lambda_n)$ for $D_n^{(1)}$. Lower rank cases of these exceptional types are given at the bottom.

Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_{\{0\}}$		Dynkin diagram for $\mathfrak{g}$	Perfect crystal of level l
$A_n^{(1)}\ (n \geq 2)$	A_n		$B(l\Lambda_k)$, $k = 1, \ldots, n$
$B_n^{(1)}\ (n \geq 3)$	B_n		$B(l\Lambda_1)$
$C_n^{(1)}\ (n \geq 2)$	C_n		$B(l\Lambda_n)$
$D_n^{(1)}\ (n \geq 4)$	D_n		$B(l\Lambda_1), B(l\Lambda_{n-1})$ and $B(l\Lambda_n)$
$A_{2n}^{(2)}\ (n \geq 2)$	B_n		$B(l\Lambda_1) \oplus B\big((l-2)\Lambda_1\big) \oplus \cdots$
$A_{2n-1}^{(2)}\ (n \geq 3)$	C_n		$B(l\Lambda_1)$
$D_{n+1}^{(2)}\ (n \geq 2)$	B_n		$B(l\Lambda_1) \oplus B\big((l-1)\Lambda_1\big) \oplus \cdots \oplus B(0)$ and $B(l\Lambda_n)$

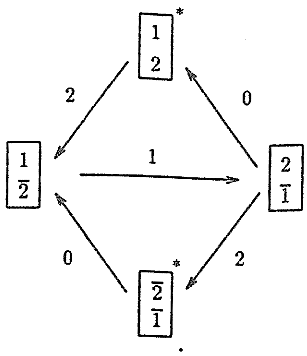
Level one perfect crystals

$A_n^{(1)}\,(n \geq 2)$	$B(\Lambda_1)$: 
$B_n^{(1)}\,(n \geq 3)$	$B(\Lambda_1)$: 
$D_n^{(1)}\,(n \geq 4)$	$B(\Lambda_1)$: 
$A_{2n}^{(2)}\,(n \geq 2)$	$B(\Lambda_1)$: 
$A_{2n-1}^{(2)}\,(n \geq 3)$	$B(\Lambda_1)$: 
$D_{n+1}^{(2)}\,(n \geq 2)$	$B(0) \oplus B(\Lambda_1)$: 

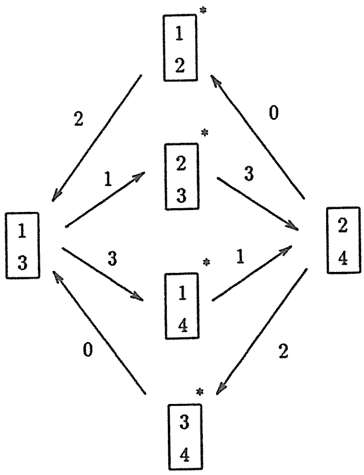
$C_2^{(1)}$ $B(\Lambda_2)$



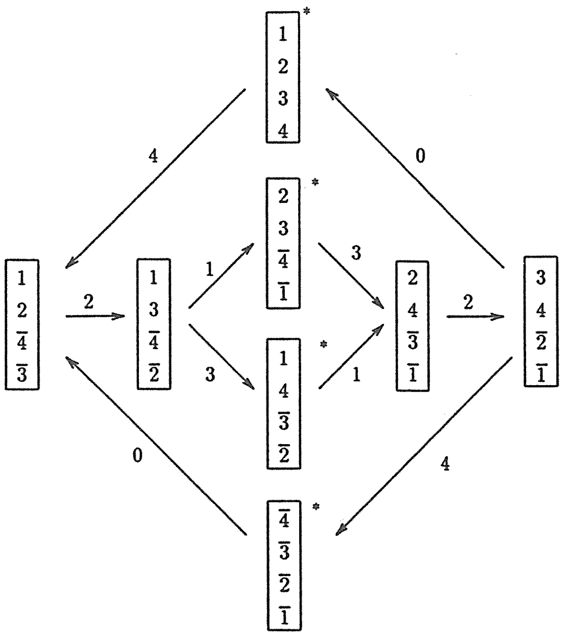
$D_3^{(2)}$ $B(\Lambda_2)$



$A_3^{(1)}$ $B(\Lambda_2)$



$D_4^{(1)}$ $B(\Lambda_4)$



Acknowledgements

The authors thank Michio Jimbo for fruitful discussions. T. M. thanks faculty members of Mathematics Deprtment of North Carolina State University for their hospitality during his stay. S. J. K and K. C. M thank their colleagues and the secretaries at RIMS, Kyoto University for their hospitality during their participation to RIMS91, Infinite Analysis.

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