Characters of irreducible modules with non-critical highest weights over affine Lie algebras

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Abstract. We shall derive Kazhdan-Lusztig type character formula for the irreducible modules with arbitrary non-critical highest weights over affine Lie algebras from the rational case by using the translation functor, the Enright functor and Bernstein's unpublished argument.

1. Introduction

The aim of this paper is to give a character formula for the irreducible modules with arbitrary non-critical highest weights over affine Lie algebras.

Let us first recall the history of the corresponding problem for finite-dimensional semisimple Lie algebras. In [**16**] Kazhdan-Lusztig proposed a conjecture describing the characters of the irreducible modules with integral highest weights over finite-dimensional semisimple Lie algebras in terms of Kazhdan-Lusztig polynomials. This conjecture was proved by Beilinson-Bernstein [**1**] and Brylinski-Kashiwara [**2**] independently using D-modules on the flag manifolds. Later its generalization to rational highest weights was obtained by combining an unpublished result of Beilinson-Bernstein and a result in Lusztig [**19**]. Finally, Bernstein proved the character formula of the irreducible modules with arbitrary highest weights by reducing it to the rational highest weight case with the help of the translation functor and a certain deformation argument (unpublished).

As for affine Lie algebras, we know already descriptions of the characters of the irreducible modules with rational non-critical highest weights by Kashiwara-Tanisaki [**14**], [**15**] (see Kashiwara (-Tanisaki) [**11**], [**12**], Kashiwara-Tanisaki [**13**], and Casian [**3**], [**4**] for the integral case). In this paper we shall derive the character formula for arbitrary non-critical highest weights over affine Lie algebras from the rational non-critical case by using the translation functor, the Enright functor and Bernstein's argument.

Let us describe our results more precisely. Let $\mathfrak{g}$ be a finite-dimensional semisimple or affine Lie algebra over the complex number field $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\{\alpha_i\}_{i\in I}$ be the set of simple roots, and let W be the Weyl group. For a real root α we denote by $s_\alpha \in W$ the corresponding reflection. Fix a W-invariant non-degenerate symmetric bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$. Set $\alpha^\vee = 2\alpha/(\alpha,\alpha)$ for a real

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root α. Fix $\rho \in \mathfrak{h}^*$ satisfying $(\alpha_i^\vee, \rho) = 1$ for any $i \in I$, and define a shifted action of W on $\mathfrak{h}^*$ by

$$w \circ \lambda = w(\lambda + \rho) - \rho \quad \text{for any } \lambda \in \mathfrak{h}^*.$$

When $\mathfrak{g}$ is affine, we denote by δ the positive imaginary root such that any imaginary root is an integral multiple of δ.

For $\lambda \in \mathfrak{h}^*$ we denote by $\Delta^+(\lambda)$ the set of positive real roots α satisfying $(\alpha^\vee, \lambda + \rho) \in \mathbb{Z}$, and by $\Pi(\lambda)$ the set of $\alpha \in \Delta^+(\lambda)$ satisfying $s_\alpha(\Delta^+(\lambda) \setminus \{\alpha\}) = \Delta^+(\lambda) \setminus \{\alpha\}$. Then the subgroup $W(\lambda)$ of W generated by $\{s_\alpha\,;\, \alpha \in \Delta^+(\lambda)\}$ is a Coxeter group with the canonical generator system $\{s_\alpha\,;\, \alpha \in \Pi(\lambda)\}$. We denote the Bruhat ordering and the length function of $W(\lambda)$ by $\geq_\lambda$ and $\ell_\lambda : W(\lambda) \to \mathbb{Z}_{\geq 0}$ respectively. For $y, w \in W(\lambda)$ we denote by $P^\lambda_{y,w}(q) \in \mathbb{Z}[q]$ the corresponding Kazhdan-Lusztig polynomial (see Kazhdan-Lusztig [**16**]), and by $Q^\lambda_{y,w}(q) \in \mathbb{Z}[q]$ the inverse Kazhdan-Lusztig polynomial defined by

$$\sum_{x \leq_\lambda y \leq_\lambda z} (-1)^{\ell_\lambda(y) - \ell_\lambda(x)} Q^\lambda_{x,y}(q) P^\lambda_{y,z}(q) = \delta_{x,z} \qquad \text{for any } x, z \in W(\lambda).$$

We denote by $W_0(\lambda)$ the subgroup of $W(\lambda)$ generated by $\{s_\alpha\,;\, \alpha \in \Delta^+,\ (\alpha^\vee, \lambda+\rho) = 0\}$.

For $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$ (resp. $L(\lambda)$) be the Verma module (resp. irreducible module) with highest weight λ. We denote the characters of $M(\lambda)$ and $L(\lambda)$ by $\mathrm{ch}(M(\lambda))$ and $\mathrm{ch}(L(\lambda))$ respectively. The aim of this paper is to give a description of $\mathrm{ch}(L(\lambda))$ for any $\lambda \in \mathfrak{h}^*$ (satisfying $(\delta, \lambda + \rho) \neq 0$ when $\mathfrak{g}$ is affine).

Set

$$\mathcal{C} = \begin{cases} \mathfrak{h}^* & \text{when } \mathfrak{g} \text{ is finite-dimensional semisimple,} \\ \{\lambda \in \mathfrak{h}^*\,;\, (\delta, \lambda + \rho) \neq 0\} & \text{when } \mathfrak{g} \text{ is affine,} \end{cases}$$

$$\mathcal{C}^+ = \{\lambda \in \mathcal{C}\,;\, (\alpha^\vee, \lambda + \rho) \geq 0 \text{ for any } \alpha \in \Delta^+(\lambda)\},$$

$$\mathcal{C}^- = \{\lambda \in \mathcal{C}\,;\, (\alpha^\vee, \lambda + \rho) \leq 0 \text{ for any } \alpha \in \Delta^+(\lambda)\}.$$

Let $\lambda \in \mathcal{C}$. Then $W_0(\lambda)$ is a finite group, and we have $(W(\lambda)\circ\lambda) \cap (\mathcal{C}^+ \cup \mathcal{C}^-) \neq \emptyset$. (see §2 below). Moreover, for any $w \in W(\lambda)$ there exists a unique $x \in wW_0(\lambda)$ such that its length $\ell_\lambda(x)$ is the largest (resp. smallest) among the elements of $wW_0(\lambda)$. We call it the longest (resp. shortest) element of $wW_0(\lambda)$.

Our main result is the following.

THEOREM 1.1. *Let $\mathfrak{g}$ be a finite-dimensional semisimple or affine Lie algebra.*

(i) *Let $\lambda \in \mathcal{C}^+$. For any $w \in W(\lambda)$ which is the longest element of $wW_0(\lambda)$ we have*

$$\mathrm{ch}(L(w \circ \lambda)) = \sum_{W(\lambda) \ni y \geq_\lambda w} (-1)^{\ell_\lambda(y) - \ell_\lambda(w)} Q^\lambda_{w,y}(1)\, \mathrm{ch}(M(y \circ \lambda)).$$

(ii) *Let $\lambda \in \mathcal{C}^-$. For any $w \in W(\lambda)$ which is the shortest element of $wW_0(\lambda)$ we have*

$$\mathrm{ch}(L(w \circ \lambda)) = \sum_{W(\lambda) \ni y \leq_\lambda w} (-1)^{\ell_\lambda(w) - \ell_\lambda(y)} P^\lambda_{y,w}(1)\, \mathrm{ch}(M(y \circ \lambda)).$$

We would like to thank J. Bernstein for informing us of his unpublished result together with its proof.

2. Integral root systems

Since the finite-dimensional case is similar and simpler, we assume in the sequel that $\mathfrak{g}$ is affine. Let $\mathfrak{g}$ be an affine Lie algebra over the complex number field $\mathbb{C}$. Let $\mathfrak{h}$ be the Cartan subalgebra, and let $\{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$ and $\{h_i\}_{i\in I} \subset \mathfrak{h}$ be the set of simple roots and the set of simple coroots respectively. We assume that $\{\alpha_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ are linearly independent and $\dim \mathfrak{h} = |I| + 1$. We denote by Δ (resp. Δ_{re}, Δ_{im}, Δ^+, Δ^-) the set of roots (resp. real roots, imaginary roots, positive roots, negative roots). Set $\Delta^{\pm}_{\mathrm{re}} = \Delta_{\mathrm{re}} \cap \Delta^{\pm}$, $\Delta^{\pm}_{\mathrm{im}} = \Delta_{\mathrm{im}} \cap \Delta^{\pm}$. There exists a unique $\delta \in \Delta^+_{\mathrm{im}}$ satisfying $\Delta^+_{\mathrm{im}} = \mathbb{Z}_{>0}\delta$. Let $c \in \sum_{i\in I} \mathbb{Z}_{>0} h_i$ be the central element of $\mathfrak{g}$ such that $\mathbb{Z}\,c = \{h \in \sum_{i\in I} \mathbb{Z} h_i ;\, \langle h, \alpha_i\rangle = 0$ for any $i \in I\}$. Here, $\langle\ ,\ \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ denotes the canonical paring. We set

$$Q = \sum_{i\in I} \mathbb{Z}\alpha_i \quad \text{and} \quad Q^+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i. \tag{2.1}$$

We fix a $\mathbb{Z}$-lattice P of $\mathfrak{h}^*$ satisfying

$$\alpha_i \in P, \quad \langle h_i, P\rangle \subset \mathbb{Z}, \tag{2.2}$$

$$\text{there exists some } \lambda \in P \text{ such that } \langle h_j, \lambda\rangle = \delta_{ij} \text{ for } j \in I \tag{2.3}$$

for any $i \in I$. Set

$$P^+ = \{\lambda \in P ;\, \langle h_i, \lambda\rangle \geq 0 \text{ for any } i \in I\}, \tag{2.4}$$

$$\mathfrak{h}^*_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} P \subset \mathfrak{h}^*, \tag{2.5}$$

$$\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} P \subset \mathfrak{h}^*. \tag{2.6}$$

We further fix a non-degenerate symmetric bilinear form $(\ ,\ ) : \mathfrak{h}^*_{\mathbb{Q}} \times \mathfrak{h}^*_{\mathbb{Q}} \to \mathbb{Q}$ satisfying

$$\langle h_i, \lambda\rangle = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \text{ for any } i \in I \text{ and } \lambda \in \mathfrak{h}^*_{\mathbb{Q}} \tag{2.7}$$

normalized by

$$\langle c, \lambda\rangle = (\delta, \lambda) \text{ for any } \lambda \in \mathfrak{h}^*_{\mathbb{Q}}. \tag{2.8}$$

Then we have

$$(\alpha, \alpha)/2 = 1/3,\ 1/2,\ 1,\ 2 \text{ or } 3 \text{ for any } \alpha \in \Delta_{\mathrm{re}}. \tag{2.9}$$

Its scalar extension to $\mathfrak{h}^*$ is also denoted by $(\ ,\ ) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$.

For $\alpha \in \Delta_{\mathrm{re}}$ we set

$$\alpha^\vee = 2\alpha/(\alpha, \alpha) \in \mathfrak{h}^*_{\mathbb{Q}}, \tag{2.10}$$

and define $s_\alpha \in GL(\mathfrak{h}^*)$ by

$$s_\alpha(\lambda) = \lambda - (\alpha^\vee, \lambda)\alpha \quad \text{for any } \lambda \in \mathfrak{h}^*. \tag{2.11}$$

The subgroup W of $GL(\mathfrak{h}^*)$ generated by $\{s_\alpha ;\, \alpha \in \Delta_{\mathrm{re}}\}$ is called the Weyl group. It is a Coxeter group with a canonical generator system $\{s_{\alpha_i} ;\, i \in I\}$. We denote its length function by $\ell : W \to \mathbb{Z}_{\geq 0}$.

Fix $\rho \in P$ satisfying $(\alpha_i^\vee, \rho) = 1$ for any $i \in I$, and define a shifted action of W on $\mathfrak{h}^*$ by

$$w \circ \lambda = w(\lambda + \rho) - \rho \quad \text{for any } \lambda \in \mathfrak{h}^*. \tag{2.12}$$

For a subset Γ of $\mathfrak{h}^*$ we denote by $\mathbb{C}\Gamma$ (resp. $\mathbb{R}\Gamma$, $\mathbb{Q}\Gamma$) the vector subspace of $\mathfrak{h}^*$ over $\mathbb{C}$ (resp. $\mathbb{R}$, $\mathbb{Q}$) spanned by Γ.

Set

$$E = \mathbb{R}\Delta_{\mathrm{re}} = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \,;\, (\delta, \lambda) = 0\}, \qquad E_{\mathrm{cl}} = E/\mathbb{R}\delta, \tag{2.13}$$

and let $\mathrm{cl} : E \to E_{\mathrm{cl}}$ denote the projection. The restriction $(\ ,\ ) : E \times E \to \mathbb{R}$ of $(\ ,\ ) : \mathfrak{h}_{\mathbb{R}}^* \times \mathfrak{h}_{\mathbb{R}}^* \to \mathbb{R}$ is positive semi-definite with radical $\mathbb{R}\delta$. Thus it induces a positive definite symmetric bilinear form $(\ ,\ ) : E_{\mathrm{cl}} \times E_{\mathrm{cl}} \to \mathbb{R}$. Set $\Delta_{\mathrm{cl}} = \mathrm{cl}(\Delta_{\mathrm{re}})$. Then Δ_{cl} is a (not necessarily reduced) finite root system in E_{cl}.

For each $\gamma \in \Delta_{\mathrm{cl}}$ there exists some $\tilde{\gamma} \in \Delta_{\mathrm{re}}$ and $r_\gamma \in \mathbb{Z}_{>0}$ satisfying

$$\mathrm{cl}^{-1}(\gamma) \cap \Delta_{\mathrm{re}} = \{\tilde{\gamma} + n r_\gamma \delta \,;\, n \in \mathbb{Z}\}, \tag{2.14}$$

$$\mathrm{cl}^{-1}(\gamma) \cap \Delta_{\mathrm{re}}^+ = \{\tilde{\gamma} + n r_\gamma \delta \,;\, n \in \mathbb{Z}_{\geq 0}\}, \tag{2.15}$$

$$\mathrm{cl}^{-1}(\gamma) \cap \Delta_{\mathrm{re}}^- = \{\tilde{\gamma} + n r_\gamma \delta \,;\, n \in \mathbb{Z}_{<0}\}. \tag{2.16}$$

Thus we have

$$\Delta_{\mathrm{re}} = \{\tilde{\gamma} + n r_\gamma \delta \,;\, \gamma \in \Delta_{\mathrm{cl}}, n \in \mathbb{Z}\}, \tag{2.17}$$

$$\Delta_{\mathrm{re}}^+ = \{\tilde{\gamma} + n r_\gamma \delta \,;\, \gamma \in \Delta_{\mathrm{cl}}, n \in \mathbb{Z}_{\geq 0}\}, \tag{2.18}$$

$$\Delta_{\mathrm{re}}^- = \{\tilde{\gamma} + n r_\gamma \delta \,;\, \gamma \in \Delta_{\mathrm{cl}}, n \in \mathbb{Z}_{<0}\}. \tag{2.19}$$

We have $\mathbb{Z} r_\gamma = \mathbb{Z} \cap \mathbb{Z}(\gamma, \gamma)/2$.

We call a subset Δ_1 of Δ_{re} a *subsystem* of Δ_{re} if $s_\alpha \beta \in \Delta_1$ for any $\alpha, \beta \in \Delta_1$ (see Kashiwara-Tanisaki [**15**] and Moody-Pianzola [**20**]). For a subsystem Δ_1 of Δ_{re} we set

$$\Delta_1^{\pm} = \Delta^{\pm} \cap \Delta_1, \tag{2.20}$$

$$\Pi_1 = \{\alpha \in \Delta_1^+ \,;\, s_\alpha(\Delta_1^+ \setminus \{\alpha\}) \subset \Delta_1^+\}, \tag{2.21}$$

$$W_1 = \langle s_\alpha \,;\, \alpha \in \Delta_1 \rangle, \tag{2.22}$$

$$S_1 = \{s_\alpha \,;\, \alpha \in \Pi_1\}. \tag{2.23}$$

We call the elements of Δ_1^+ (resp. Δ_1^-, Π_1) positive roots (resp. negative roots, simple roots) for Δ_1, and W_1 the Weyl group for Δ_1. The group W_1 is a Coxeter group with a canonical generator system S_1, and its length function $\ell_1 : W_1 \to \mathbb{Z}_{\geq 0}$ is given by $\ell_1(w) = |w\Delta_1^+ \cap \Delta_1^-|$. We have

$$(\alpha, \beta) \leq 0 \text{ for any } \alpha, \beta \in \Pi_1 \text{ such that } \alpha \neq \beta \tag{2.24}$$

(see [**15**]).

LEMMA 2.1. *The following conditions for a subsystem Δ_1 of Δ_{re} are all equivalent to each other.*

(i) $|\Delta_1| < \infty$,
(ii) $|W_1| < \infty$,
(iii) $\mathbb{C}\Delta_1 \not\ni \delta$,
(iv) $\mathbb{Q}\Delta_1 \not\ni \delta$.

PROOF. It is well-known that (i) and (ii) are equivalent, and they are also equivalent to the condition that the restriction $(\ ,\ )\,|\,\mathbb{R}\Delta_1 \times \mathbb{R}\Delta_1$ of $(\ ,\ ) : E \times E \to \mathbb{R}$ is positive definite. Thus the conditions (i) and (ii) are equivalent to $\mathbb{R}\Delta_1 \not\ni \delta$. This condition is equivalent to (iii) and (iv) because $\Delta_1 \cup \{\delta\} \subset \mathfrak{h}_{\mathbb{Q}}^* \subset \mathfrak{h}_{\mathbb{R}}^*$. □

LEMMA 2.2. *Let Δ_1 be a subsystem of Δ_{re} and let Π_1 be the set of simple roots for Δ_1. If $\mathbb{Q}\Delta_1 \ni \delta$, then we have $\delta \in \sum_{\alpha \in \Pi_1} \mathbb{Q}_{\geq 0} \alpha$.*

PROOF. Let Π_2 be a minimal subset of Π_1 such that $\mathbb{Q}\Pi_2 \ni \delta$. Write $\delta = \sum_{\alpha\in\Pi_2} c_\alpha\alpha$ with $c_\alpha \in \mathbb{Q}$. Let $\Pi_3 = \{\alpha \in \Pi_2\,;\, c_\alpha > 0\}$, and set $\gamma = \sum_{\alpha\in\Pi_3} c_\alpha\alpha = \delta + \sum_{\beta\in\Pi_2\setminus\Pi_3}(-c_\beta)\beta$. By (2.24) we have

$$0 \le (\gamma,\gamma) = \sum_{\alpha\in\Pi_3}\sum_{\beta\in\Pi_2\setminus\Pi_3} c_\alpha(-c_\beta)(\alpha,\beta) \le 0,$$

and hence $\gamma \in \mathbb{Q}\delta$. If $\gamma = 0$, then we have $\delta = \sum_{\beta\in\Pi_2\setminus\Pi_3} c_\beta\beta \in \mathbb{Q}_{\le 0}\Pi_1 \subset \sum_{i\in I}\mathbb{Q}_{\le 0}\alpha_i$. This is a contradiction. Thus $\delta \in \mathbb{Q}\gamma \subset \mathbb{Q}\Pi_3$. By the minimality of Π_2 we have $\Pi_2 = \Pi_3$, and hence we have $\delta \in \sum_{\alpha\in\Pi_2}\mathbb{Q}_{>0}\alpha \subset \sum_{\alpha\in\Pi_1}\mathbb{Q}_{\ge 0}\alpha$. □

LEMMA 2.3. *Let Π_1 be the set of simple roots for a subsystem Δ_1 of Δ_{re}. Then we have $|\Pi_1| < \infty$.*

PROOF. Let $\approx$ be the equivalence relation on Π_1 generated by

$$\alpha,\beta \in \Pi_1, (\alpha,\beta) \ne 0 \Longrightarrow \alpha \approx \beta,$$

and let $\{\Pi_{1,a}\,;\, a \in \mathcal{A}\}$ denote the set of equivalence classes with respect to $\approx$.

For $a \in \mathcal{A}$ set $V_a = \mathbb{R}\Pi_{1,a}$. Then $\mathrm{cl}(V_a)$ for $a \in \mathcal{A}$ are all non-zero and mutually orthogonal with respect to the natural positive definite symmetric bilinear form on E_{cl}. Hence $\mathcal{A}$ is a finite set. Thus it is sufficient to show that $\Pi_{1,a}$ is a finite set for each $a \in \mathcal{A}$.

If $V_a \not\ni \delta$, then $(\ ,\ )|V_a \times V_a$ is positive definite, and hence $\Delta_{\mathrm{re}} \cap V_a$ is a finite subsystem of Δ_{re}. Thus $\Pi_{1,a}$ is a finite set.

Assume that $V_a \ni \delta$. By Lemma 2.2 there exists a finite subset $\Pi_{2,a}$ of $\Pi_{1,a}$ such that $\delta = \sum_{\alpha\in\Pi_{2,a}} c_\alpha\alpha$ with $c_\alpha \in \mathbb{Q}_{>0}$. Since

$$0 = (\delta,\beta) = \sum_{\alpha\in\Pi_{2,a}} c_\alpha(\alpha,\beta) \qquad \text{for any } \beta \in \Pi_{1,a}\setminus\Pi_{2,a},$$

(2.24) implies $(\alpha,\beta) = 0$ for any $\alpha \in \Pi_{2,a}$ and $\beta \in \Pi_{1,a}\setminus\Pi_{2,a}$. Since $\Pi_{1,a}$ is an equivalence class with respect to $\approx$, we obtain $\Pi_{1,a} = \Pi_{2,a}$. Therefore, $\Pi_{1,a}$ is a finite set. □

For a subset J of I set

$$\Delta_J = \Delta \cap \sum_{i\in J}\mathbb{Z}\alpha_i. \tag{2.25}$$

If J is a proper subset of I, then Δ_J is a finite subsystem with $\{\alpha_i\,;\, i \in J\}$ as the set of simple roots.

LEMMA 2.4. *For any finite subsystem Δ_1 of Δ there exist $w \in W$ and a proper subset J of I such that $w\Delta_1 \subset \Delta_J$.*

PROOF. Set $V = \mathbb{R}\Delta_1$. By Lemma 2.1 we have $V \not\ni \delta$. Since $(\ ,\ )|V \times V$ is positive definite, $V \cap \Delta_{\mathrm{re}}$ is a finite subsystem of Δ_{re} containing Δ_1. Hence we can assume $\Delta_1 = V \cap \Delta_{\mathrm{re}}$ from the beginning.

Set $V^\perp = \{\mu \in \mathfrak{h}^*_{\mathbb{R}}; (V,\mu) = 0\}$. Since $\delta \notin V$, (δ,μ) is not identically zero on $\mu \in V^\perp$. Similarly (α,μ) $(\alpha \in \Delta_{\mathrm{re}}\setminus\Delta_1)$ is not identically zero on $\mu \in V^\perp$. Since $\Delta_{\mathrm{re}}\setminus\Delta_1$ is a countable set, there exists some $\lambda \in V^\perp$ such that $(\delta,\lambda) > 0$ and $(\alpha,\lambda) \ne 0$ for any $\alpha \in \Delta_{\mathrm{re}}\setminus\Delta_1$. Then we have $\Delta_1 = \{\alpha \in \Delta_{\mathrm{re}}\,;\, (\alpha,\lambda) = 0\}$. Since $(\delta,\lambda) > 0$, there exist only finitely many $\alpha \in \Delta^+_{\mathrm{re}}$ such that $(\alpha,\lambda) < 0$ by (2.18). Hence there exists some $w \in W$ such that $(\alpha, w\lambda) \ge 0$ for any $\alpha \in \Delta^+_{\mathrm{re}}$ by

[**9**, Proposition 3.2]. Then we obtain $w\Delta_1 = \{\alpha \in \Delta_{\mathrm{re}}\,;\,(\alpha, w\lambda) = 0\} = \Delta_J$ with $J = \{i \in I\,;\,(\alpha_i, w\lambda) = 0\}$. Since $|\Delta_J| = |\Delta_1| < \infty$, we have $J \neq I$. □

For $\lambda \in \mathfrak{h}^*$ set

$$\Delta(\lambda) = \{\alpha \in \Delta_{\mathrm{re}}\,;\,(\alpha^\vee, \lambda + \rho) \in \mathbb{Z}\}, \tag{2.26}$$

$$\Delta_0(\lambda) = \{\alpha \in \Delta_{\mathrm{re}}\,;\,(\alpha^\vee, \lambda + \rho) = 0\}. \tag{2.27}$$

They are subsystems of Δ_{re}. We denote the set of positive roots, the set of negative roots, the set of simple roots and the Weyl group for $\Delta(\lambda)$ by $\Delta^+(\lambda)$, $\Delta^-(\lambda)$, $\Pi(\lambda)$ and $W(\lambda)$ respectively. We denote those for $\Delta_0(\lambda)$ by $\Delta_0^+(\lambda)$, $\Delta_0^-(\lambda)$, $\Pi_0(\lambda)$ and $W_0(\lambda)$. The length function for $W(\lambda)$ is denoted by $\ell_\lambda : W(\lambda) \to \mathbb{Z}_{\geq 0}$.

LEMMA 2.5. *For $\lambda \in \mathfrak{h}^*$ such that $\Delta(\lambda) \neq \emptyset$, the following conditions are equivalent.*

(i) $|\Delta(\lambda)| < \infty$.
(ii) $(\delta, \lambda + \rho) \notin \mathbb{Q}$.

PROOF. (i)⇒(ii). Assume $(\delta, \lambda + \rho) \in \mathbb{Q}$ and $\Delta(\lambda) \neq \emptyset$. Take $\alpha \in \Delta(\lambda)$. By (2.14) there exists some $r \in \mathbb{Z}_{>0}$ such that $\alpha + \mathbb{Z}r\delta \subset \Delta_{\mathrm{re}}$. For $n \in \mathbb{Z}$ we have

$$((\alpha + nr\delta)^\vee, \lambda + \rho) = (\alpha^\vee, \lambda + \rho) + 2nr(\delta, \lambda + \rho)/(\alpha, \alpha),$$

and hence we have $\alpha + nr\delta \in \Delta(\lambda)$ for any $n \in \mathbb{Z}$ satisfying $2nr(\delta, \lambda + \rho)/(\alpha, \alpha) \in \mathbb{Z}$. Thus $|\Delta(\lambda)| = \infty$.

(ii)⇒(i). Assume $|\Delta(\lambda)| = \infty$. By Lemma 2.1 we have $\mathbb{Q}\Delta(\lambda) \ni \delta$. Then we have

$$(\delta, \lambda + \rho) \in \sum_{\alpha \in \Delta(\lambda)} \mathbb{Q}(\alpha^\vee, \lambda + \rho) \subset \mathbb{Q}.$$

□

Set

$$\mathcal{C} = \{\lambda \in \mathfrak{h}^*\,;\,(\delta, \lambda + \rho) \neq 0\}. \tag{2.28}$$

LEMMA 2.6. *For any $\lambda \in \mathcal{C}$ we have $|\Delta_0(\lambda)| < \infty$.*

PROOF. Since $(\delta, \lambda + \rho) \neq 0$, (2.14) implies $|\mathrm{cl}^{-1}(\gamma) \cap \Delta_0(\lambda)| \leq 1$ for any $\gamma \in \Delta_{\mathrm{cl}}$. Thus we have $|\Delta_0(\lambda)| \leq |\Delta_{\mathrm{cl}}| < \infty$. □

In the sequel, we use the following proposition on the existence of rational points of a subset defined by linear inequalities. Since the proof is elementary, we do not give the proof.

PROPOSITION 2.7. *Let $V_\mathbb{Q}$ be a finite-dimensional $\mathbb{Q}$-vector space and set $V_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Q} V_\mathbb{Q}$ and $V = \mathbb{C} \otimes_\mathbb{Q} V_\mathbb{Q}$. Let X be a subset of $V_\mathbb{Q}^*$ and $\{Y_a\}_{a \in A}$ be a family of non-empty finite subsets of $V_\mathbb{Q}^*$. Let B_x $(x \in X)$ and $C_{y,a}$ $(a \in A,\ y \in Y_a)$ be rational numbers. Set*

$$\Omega = \{\lambda \in V; \langle x, \lambda \rangle = B_x \quad \text{for any } x \in X\},$$

$$\Omega' = \{\lambda \in \Omega\,;\,\text{for any } a \in A, \text{ there exists } y \in Y_a \text{ such that } \langle y, \lambda \rangle \notin C_{y,a}\mathbb{Z}\}.$$

(i) *If A is a finite set and $\Omega' \neq \emptyset$, then $\Omega' \cap V_\mathbb{Q} \neq \emptyset$.*
(ii) *If A is a countable set and $\Omega' \neq \emptyset$, then $\Omega' \cap V_\mathbb{R} \neq \emptyset$. Moreover if $z \in V_\mathbb{Q}^*$ is not contained in the vector subspace $\mathbb{Q}X$, then there exists $\lambda \in \Omega' \cap V_\mathbb{R}$ such that $\langle z, \lambda \rangle > 1$.*

LEMMA 2.8. *For any* $\lambda \in \mathcal{C}$ *we have* $W_0(\lambda) = \{w \in W\,;\, w \circ \lambda = \lambda\}$.

PROOF. Set $W_1 = \{w \in W\,;\, w \circ \lambda = \lambda\}$. It is sufficient to show that the group W_1 is generated by the reflections contained in it. Set

$$\Omega' = \{\mu \in \mathcal{C}\,;\, w \circ \mu = \mu \text{ for any } w \in W_1,\ w \circ \mu \neq \mu \text{ for any } w \in W \setminus W_1\}.$$

Since Ω' contains λ Proposition 2.7 (ii) implies that $\Omega \cap \mathfrak{h}^*_{\mathbb{R}}$ contain a point μ such that $(\delta, \mu+\rho) > 0$. Thus replacing λ with such a μ, we may assume that $\lambda \in \mathcal{C} \cap \mathfrak{h}^*_{\mathbb{R}}$ and $(\delta, \lambda+\rho) > 0$. Then the assertion follows from [**9**, Proposition 3.2] and [**9**, Proposition 5.8]. □

By a standard argument we have the following.

LEMMA 2.9. *Set*

$$\mathfrak{h}^{*+} = \{\lambda \in \mathfrak{h}^*\,;\, (\alpha^\vee, \lambda+\rho) \geq 0 \textit{ for any } \alpha \in \Delta^+(\lambda)\},$$
$$\mathfrak{h}^{*-} = \{\lambda \in \mathfrak{h}^*\,;\, (\alpha^\vee, \lambda+\rho) \leq 0 \textit{ for any } \alpha \in \Delta^+(\lambda)\}.$$

Then for any $\lambda \in \mathfrak{h}^*$, $|(W(\lambda)\circ\lambda) \cap \mathfrak{h}^{*\pm}| \leq 1$. *Moreover,* $|(W(\lambda)\circ\lambda) \cap \mathfrak{h}^{*+}| = 1$ *(resp.* $|(W(\lambda)\circ\lambda) \cap \mathfrak{h}^{*-}| = 1$*) if and only if there exist only finitely many* $\alpha \in \Delta^+(\lambda)$ *satisfying* $(\alpha^\vee, \lambda+\rho) < 0$ *(resp.* $(\alpha^\vee, \lambda+\rho) > 0$*).*

Set

$$\mathcal{C}^+ = \{\lambda \in \mathcal{C}\,;\, (\alpha^\vee, \lambda+\rho) \geq 0 \text{ for any } \alpha \in \Delta^+(\lambda)\}, \tag{2.29}$$
$$\mathcal{C}^- = \{\lambda \in \mathcal{C}\,;\, (\alpha^\vee, \lambda+\rho) \leq 0 \text{ for any } \alpha \in \Delta^+(\lambda)\}. \tag{2.30}$$

LEMMA 2.10. *Assume* $\lambda \in \mathcal{C}$ *satisfies* $\Delta(\lambda) \neq \emptyset$.

(i) *If* $(\delta, \lambda+\rho) \notin \mathbb{Q}$*, then we have* $|(W(\lambda)\circ\lambda) \cap \mathcal{C}^+| = |(W(\lambda)\circ\lambda) \cap \mathcal{C}^-| = 1$.
(ii) *If* $(\delta, \lambda+\rho) \in \mathbb{Q}_{>0}$*, then we have* $|(W(\lambda)\circ\lambda) \cap \mathcal{C}^+| = 1$ *and* $|(W(\lambda)\circ\lambda) \cap \mathcal{C}^-| = 0$.
(iii) *If* $(\delta, \lambda+\rho) \in \mathbb{Q}_{<0}$*, then we have* $|(W(\lambda)\circ\lambda) \cap \mathcal{C}^+| = 0$ *and* $|(W(\lambda)\circ\lambda) \cap \mathcal{C}^-| = 1$.

PROOF. (i) If $(\delta, \lambda+\rho) \notin \mathbb{Q}$, then we have $|\Delta^+(\lambda)| < \infty$ by Lemma 2.5. Hence we have $|(W(\lambda)\circ\lambda) \cap \mathcal{C}^+| = |(W(\lambda)\circ\lambda) \cap \mathcal{C}^-| = 1$ by Lemma 2.9.

(ii) Assume $(\delta, \lambda+\rho) \in \mathbb{Q}_{>0}$. Set

$$\Delta_1 = \{\alpha \in \Delta^+(\lambda)\,;\, (\alpha^\vee, \lambda+\rho) > 0\},$$
$$\Delta_2 = \{\alpha \in \Delta^+(\lambda)\,;\, (\alpha^\vee, \lambda+\rho) < 0\},$$
$$\Delta_3 = \{\alpha \in \Delta^+(\lambda)\,;\, (\alpha^\vee, \lambda+\rho) \leq 0\}.$$

For each $\gamma \in \Delta_{\mathrm{cl}}$ there exist only finitely many $\alpha \in \mathrm{cl}^{-1}(\gamma) \cap \Delta^+_{\mathrm{re}}$ satisfying $(\alpha^\vee, \lambda+\rho) \in \mathbb{Z}_{\leq 0}$ by (2.15). Since $|\Delta_{\mathrm{cl}}| < \infty$, we obtain $|\Delta_3| < \infty$. Thus we have $|\Delta_2| \leq |\Delta_3| < \infty$. On the other hand we have $|\Delta^+(\lambda)| = \infty$ by Lemma 2.5, and hence $|\Delta_1| = |\Delta^+(\lambda) \setminus \Delta_3| = \infty$. Thus we obtain the desired result by Lemma 2.9.

The assertion (iii) follows from (ii) by replacing λ with $-\lambda - 2\rho$. □

COROLLARY 2.11. *For any* $\lambda \in \mathcal{C}$ *we have* $(W(\lambda)\circ\lambda) \cap (\mathcal{C}^+ \cup \mathcal{C}^-) \neq \emptyset$.

LEMMA 2.12. *Let* $\lambda \in \mathcal{C}$.

(i) *If* $\mathbb{Q}\Delta(\lambda) \ni \delta$*, then there exists some* $\mu \in \mathcal{C} \cap \mathfrak{h}^*_{\mathbb{Q}}$ *such that* $(\delta, \mu+\rho) = (\delta, \lambda+\rho)$, $\Delta(\mu) = \Delta(\lambda)$ *and* $(\alpha^\vee, \mu+\rho) = (\alpha^\vee, \lambda+\rho)$ *for any* $\alpha \in \Delta(\lambda)$.

(ii) *If* $\mathbb{Q}\Delta(\lambda) \not\ni \delta$, *then there exists some* $\mu \in \mathcal{C} \cap \mathfrak{h}^*_{\mathbb{R}}$ *such that* $(\delta, \mu + \rho) > 0$, $\Delta(\mu) = \Delta(\lambda)$ *and* $(\alpha^\vee, \mu + \rho) = (\alpha^\vee, \lambda + \rho)$ *for any* $\alpha \in \Delta(\lambda)$.

PROOF. Set

$$\Omega = \{\mu \in \mathfrak{h}^* \,;\, (\alpha^\vee, \mu + \rho) = (\alpha^\vee, \lambda + \rho) \ \text{ for any } \alpha \in \Delta(\lambda)\},$$
$$\Omega' = \{\mu \in \Omega \,;\, (\alpha^\vee, \mu + \rho) \notin \mathbb{Z} \ \text{ for any } \alpha \in \Delta_{\text{re}} \setminus \Delta(\lambda)\}.$$

Then Ω' contains λ.

(i) By the definition of Ω we have

$$(\gamma, \mu + \rho) = (\gamma, \lambda + \rho) \quad \text{for any } \gamma \in \mathbb{Q}\Delta(\lambda) \text{ and } \mu \in \Omega. \tag{2.31}$$

In particular, we have

$$(\delta, \mu + \rho) = (\delta, \lambda + \rho) \in \mathbb{Q} \quad \text{for any } \mu \in \Omega \tag{2.32}$$

by $\mathbb{Q}\Delta(\lambda) \ni \delta$. Thus $\Omega \subset \mathcal{C}$. Hence it is sufficient to show $\Omega' \cap \mathfrak{h}^*_{\mathbb{Q}} \neq \emptyset$.
Set

$$\Delta_{\text{cl},1} = \{\gamma \in \Delta_{\text{cl}} \,;\, \text{cl}^{-1}(\gamma) \cap \Delta(\lambda) = \emptyset\}. \qquad \Delta_{\text{cl},2} = \Delta_{\text{cl}} \setminus \Delta_{\text{cl},1}.$$

Let $\mu \in \Omega$. (2.14) and the assumption $\mathbb{Q}\Delta(\lambda) \ni \delta$ imply $\text{cl}^{-1}(\Delta_{\text{cl},2}) \cap \Delta_{\text{re}} \subset \mathbb{Q}\Delta(\lambda)$. Hence $(\alpha^\vee, \mu + \rho) = (\alpha^\vee, \lambda + \rho) \notin \mathbb{Z}$ for any $\alpha \in \text{cl}^{-1}(\Delta_{\text{cl},2}) \cap (\Delta_{\text{re}} \setminus \Delta(\lambda))$. Thus we have $\mu \in \Omega'$ if and only if $(\alpha^\vee, \mu + \rho) \notin \mathbb{Z}$ for any $\alpha \in \Delta_{\text{re}} \cap \text{cl}^{-1}(\Delta_{\text{cl},1})$. By (2.14) and (2.32), this condition is equivalent to

$$(\tilde{\gamma}^\vee, \mu + \rho) \notin \mathbb{Z} + \frac{2r_\gamma(\delta, \lambda + \rho)}{(\gamma, \gamma)}\mathbb{Z} \quad \text{for any } \gamma \in \Delta_{\text{cl},1}.$$

Thus we obtain

$$\Omega' = \{\mu \in \Omega \,;\, (\tilde{\gamma}^\vee, \mu + \rho) \notin q_\gamma \mathbb{Z} \ \text{ for any } \gamma \in \Delta_{\text{cl},1}\},$$

where $\{q_\gamma \,;\, \gamma \in \Delta_{\text{cl},1}\}$ is a set of positive rational numbers. Then Ω' contains λ, and Proposition 2.7 (i) implies that $\Omega' \cap \mathfrak{h}^*_{\mathbb{Q}} \neq \emptyset$.

(ii) This follows immediately from Proposition 2.7 (ii). □

LEMMA 2.13. *For any* $\lambda \in \mathcal{C}^+ \cup \mathcal{C}^-$, *there exist* $w \in W$ *and a proper subset* J *of* I *such that* $w\Delta^+(\lambda) \subset \Delta^+$ *and* $w\Delta_0(\lambda) = \Delta_J$.

PROOF. By replacing λ with $-2\rho - \lambda$ if necessary, we may assume $\lambda \in \mathcal{C}^+$ from the beginning. Let us first show that there exists some $\mu \in \mathcal{C} \cap \mathfrak{h}^*_{\mathbb{R}}$ such that $(\delta, \mu + \rho) > 0$, $\Delta(\mu) = \Delta(\lambda)$ and $(\alpha^\vee, \mu + \rho) = (\alpha^\vee, \lambda + \rho)$ for any $\alpha \in \Delta(\lambda)$. If $\mathbb{Q}\Delta(\lambda) \ni \delta$, then we have $(\delta, \lambda + \rho) > 0$ by Lemma 2.2, and Lemma 2.12 (i) implies the existence of such a μ. If $\mathbb{Q}\Delta(\lambda) \not\ni \delta$, then Lemma 2.12 (ii) implies the existence of such a μ.

By (2.18) there exist only finitely many $\alpha \in \Delta^+_{\text{re}}$ such that $(\alpha^\vee, \mu + \rho) < 0$. Thus there exists some $w \in W$ such that $(\alpha^\vee, w \circ \mu + \rho) \geq 0$ for any $\alpha \in \Delta^+_{\text{re}}$ by [**9**, Proposition 3.2]. We may assume that $\ell(w) = \min\{\ell(x) \,;\, x \in wW_0(\mu)\}$. Then we have $w(\Delta^+_0(\mu)) \subset \Delta^+$ by [**15**, Proposition 2.2.11]. For $\alpha \in \Delta^+(\mu) \setminus \Delta_0(\mu) = \Delta^+(\lambda) \setminus \Delta_0(\lambda)$ we have

$$(w\alpha^\vee, w \circ \mu + \rho) = (\alpha^\vee, \mu + \rho) = (\alpha^\vee, \lambda + \rho) > 0,$$

and hence $w\alpha \in \Delta^+$. Thus we obtain $w\Delta^+(\lambda) \subset \Delta^+$. Moreover, we have

$$w\Delta_0(\lambda) = w\Delta_0(\mu) = \Delta_0(w \circ \mu) = \Delta_J$$

with $J = \{i \in I\,;\, (\alpha_i^\vee, w \circ \mu + \rho) = 0\}$. Then J is a proper subset of I by $|\Delta_0(\lambda)| < \infty$. $\square$

3. Translation functor

In this section we shall give some properties of the translation functor (see also Deodhar-Gabber-Kac [**6**], and Kumar [**18**]).

For a Lie algebra $\mathfrak{a}$ over $\mathbb{C}$ we denote its enveloping algebra by $U(\mathfrak{a})$ and the category of $\mathfrak{a}$-modules by $\mathbb{M}(\mathfrak{a})$.

For an $\mathfrak{h}$-module M and $\mu \in \mathfrak{h}^*$ we set

$$M_\mu = \{m \in M\,;\, hm = \langle h, \mu\rangle m \text{ for any } h \in \mathfrak{h}\}. \tag{3.1}$$

An element μ of $\mathfrak{h}^*$ is called a weight of M if $M_\mu \neq 0$. For an $\mathfrak{h}$-module M satisfying

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu \qquad \text{with } \dim M_\mu < \infty \text{ for any } \mu \in \mathfrak{h}^*, \tag{3.2}$$

we define its character $\mathrm{ch}(M)$ by the formal sum

$$\mathrm{ch}(M) = \sum_{\mu \in \mathfrak{h}^*} \dim M_\mu\, \mathrm{e}^\mu. \tag{3.3}$$

We denote by $\mathbb{O}$ the full subcategory of $\mathbb{M}(\mathfrak{g})$ consisting of $M \in \mathrm{Ob}(\mathbb{M}(\mathfrak{g}))$ satisfying (3.2) and

(3.4) for any $\xi \in \mathfrak{h}^*$ there exist only finitely many $\mu \in \xi + Q^+$ such that $M_\mu \neq 0$.

For $\alpha \in \Delta$ let $\mathfrak{g}_\alpha$ denote the root space corresponding to α, and set

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+. \tag{3.5}$$

For $\lambda \in \mathfrak{h}^*$ define a $\mathfrak{g}$-module $M(\lambda)$, called the Verma module with highest weight λ, by

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \tag{3.6}$$

where $\mathbb{C}_\lambda = \mathbb{C}1_\lambda$ is the one-dimensional $\mathfrak{b}$-module given by $h1_\lambda = \lambda(h)1_\lambda$ for $h \in \mathfrak{h}$ and $\mathfrak{n}^+1_\lambda = 0$. We denote its unique irreducible quotient by $L(\lambda)$.

We have

$$\mathrm{ch}(M(\lambda)) = \frac{\mathrm{e}^\lambda}{\prod_{\alpha \in \Delta^+} (1 - \mathrm{e}^{-\alpha})^{\dim \mathfrak{g}_\alpha}}. \tag{3.7}$$

Moreover, $M(\lambda)$ and $L(\lambda)$ are objects of $\mathbb{O}$ for any $\lambda \in \mathfrak{h}^*$. For $M \in \mathrm{Ob}(\mathbb{O})$ and $\lambda \in \mathfrak{h}^*$ we denote by $[M : L(\lambda)]$ the multiplicity of $L(\lambda)$ in M (see [**9**, §9.6]).

The following result due to Kac-Kazhdan [**10**] is fundamental in the study of highest weight modules.

PROPOSITION 3.1. *Let $\lambda, \mu \in \mathfrak{h}^*$. Then the following conditions are equivalent.*

(i) *The multiplicity $[M(\lambda) : L(\mu)]$ is non-zero.*

(ii) *There exists an injective homomorphism $M(\mu) \to M(\lambda)$.*

(iii) *There exist a sequence of positive roots $\{\beta_k\}_{k=1}^l$, a sequence of positive integers $\{n_k\}_{k=1}^l$ and a sequence of weights $\{\lambda_k\}_{k=0}^l$ such that $\lambda_0 = \lambda$, $\lambda_l = \mu$ and $\lambda_k = \lambda_{k-1} - n_k\beta_k$, $2(\beta_k, \lambda_{k-1} + \rho) = n_k(\beta_k, \beta_k)$ for $k = 1, \ldots, l$.*

For a subset $\mathcal{D}$ of $\mathfrak{h}^*$ we denote by $\mathbb{O}\{\mathcal{D}\}$ the full subcategory of $\mathbb{O}$ consisting of $M \in \mathrm{Ob}(\mathbb{O})$ satisfying $[M : L(\mu)] = 0$ for any $\mu \in \mathfrak{h}^* \setminus \mathcal{D}$. For $\lambda \in \mathcal{C}$ (see (2.28) for the notation) we set $\mathbb{O}[\lambda] = \mathbb{O}\{W(\lambda) \circ \lambda\}$. We have obviously $L(\lambda) \in \mathrm{Ob}(\mathbb{O}[\lambda])$ for any $\lambda \in \mathcal{C}$.

By Proposition 3.1 we have the following.

PROPOSITION 3.2. *For any $\lambda \in \mathcal{C}$ we have $M(\lambda) \in \mathrm{Ob}(\mathbb{O}[\lambda])$.*

Define an equivalence relation $\sim$ on $\mathcal{C}$ by

$$\lambda \sim \mu \iff \mu \in W(\lambda) \circ \lambda. \tag{3.8}$$

By Kumar [**17**] we have the following.

PROPOSITION 3.3. *Any $M \in \mathrm{Ob}(\mathbb{O}\{\mathcal{C}\})$ is uniquely decomposed as*

$$M = \bigoplus_{\lambda \in \mathcal{C}/\sim} M[\lambda], \qquad M[\lambda] \in \mathrm{Ob}(\mathbb{O}[\lambda]).$$

For $\lambda \in \mathcal{C}$ let

$$P_\lambda : \mathbb{O}\{\mathcal{C}\} \to \mathbb{O}[\lambda] \tag{3.9}$$

be the projection functor given by $P_\lambda(M) = M[\lambda]$.

LEMMA 3.4. *Let $\lambda, \mu \in \mathcal{C}$, $\nu \in \mathfrak{h}^*$, $x \in W$ satisfy $\mu - \lambda = x\nu$. Then we have $M \otimes L(\nu) \in \mathrm{Ob}(\mathbb{O}\{\mathcal{C}\})$ for any $M \in \mathrm{Ob}(\mathbb{O}[\lambda])$.*

PROOF. It is easily seen that $M \otimes L(\nu) \in \mathrm{Ob}(\mathbb{O})$. Hence it is sufficient to show that if $L(\xi)$ appears as a subquotient of $M \otimes L(\nu)$, then we have $(\delta, \xi + \rho) \neq 0$.

We may assume that $M = L(w \circ \lambda)$ for $w \in W(\lambda)$. The central element c of $\mathfrak{g}$ acts on $L(\eta)$ via the multiplication of the scalar $\langle c, \eta \rangle = (\delta, \eta)$ for any $\eta \in \mathfrak{h}^*$. For $w \in W(\lambda)$ we have $(\delta, w \circ \lambda) = (\delta, \lambda)$ by the W-invariance of δ, and hence c acts on $L(w \circ \lambda)$ via the multiplication of (δ, λ). Therefore we have $cu = (\delta, \lambda + \nu)u$ for any $u \in M \otimes L(\nu)$. If $L(\xi)$ appears as a subquotient of $M \otimes L(\nu)$, then we have $(\delta, \xi) = (\delta, \lambda + \nu)$, and hence

$$(\delta, \xi + \rho) = (\delta, \lambda + \nu + \rho) = (\delta, \lambda + x\nu + \rho) = (\delta, \mu + \rho) \neq 0.$$

□

For $\lambda, \mu \in \mathcal{C}$ satisfying

$$\mu - \lambda \in WP^+, \tag{3.10}$$

we define a functor

$$T_\mu^\lambda : \mathbb{O}[\lambda] \to \mathbb{O}[\mu] \tag{3.11}$$

by $T_\mu^\lambda(M) = P_\mu(M \otimes L(\nu))$, where ν is a unique element of P^+ such that $\mu - \lambda \in W\nu$. It is obviously an exact functor.

The proofs of Lemma 3.5, Proposition 3.6 and Proposition 3.8 below are similar to those for finite-dimensional semisimple Lie algebras given in Jantzen [**8**]. We reproduce it here for the sake of completeness.

LEMMA 3.5. *Assume that we have either $\lambda, \mu \in \mathcal{C}^+$ or $\lambda, \mu \in \mathcal{C}^-$ and that $\mu - \lambda \in W\nu$ for $\nu \in P^+$. Denote by Γ the set of weights of $L(\nu)$. Then for any $w \in W(\lambda)$ satisfying $w \circ \mu - \lambda \in \Gamma$ we have $w \in W_0(\lambda) W_0(\mu)$.*

PROOF. By the assumption we have $\Delta(\lambda)=\Delta(\mu)$ and $W(\lambda)=W(\mu)$. Assume that there exists some $w\in W(\lambda)\setminus W_0(\lambda)W_0(\mu)$ satisfying $w\circ\mu-\lambda\in\Gamma$. We may assume that its length $\ell_\lambda(w)$ is the smallest among such elements. Set $\xi=w\circ\mu-\lambda\in\Gamma$.

Since w is the shortest element of $wW_0(\mu)$, [**15**, Proposition 2.2.11] implies

$$w\Delta_0^+(\mu)\subset\Delta^+(\lambda). \tag{3.12}$$

Since w is the shortest element of $W_0(\lambda)w$,

$$w^{-1}\Delta_0^+(\lambda)\subset\Delta^+(\lambda). \tag{3.13}$$

By $w\neq 1$ there exists some $\alpha\in\Delta^+(\lambda)$ satisfying $\ell_\lambda(s_\alpha w)<\ell_\lambda(w)$. Then we have $w^{-1}\alpha\in\Delta^-(\lambda)$. Hence we have $\alpha\in\Delta^+(\lambda)\setminus\Delta_0^+(\lambda)$ by (3.13). If $w^{-1}\alpha\in\Delta_0(\mu)$, then we have $-w^{-1}\alpha\in\Delta_0^+(\mu)\cap w^{-1}\Delta^-(\lambda)$. This contradicts (3.12). Thus we obtain $w^{-1}\alpha\in\Delta^-(\mu)\setminus\Delta_0^-(\mu)$. Set

$$m=(\alpha^\vee,\lambda+\rho),\quad n=-(w^{-1}\alpha^\vee,\mu+\rho)=-(\alpha^\vee,w(\mu+\rho)).$$

By $\alpha\in\Delta^+(\lambda)\setminus\Delta_0^+(\lambda)$ and $w^{-1}\alpha\in\Delta^-(\mu)\setminus\Delta_0^-(\mu)$ we have $m,n\in\mathbb{Z}_{>0}$ if $\lambda,\mu\in\mathcal{C}^+$ and $m,n\in\mathbb{Z}_{<0}$ if $\lambda,\mu\in\mathcal{C}^-$. Now we have

$$\begin{aligned}&s_\alpha w\circ\mu-\lambda=s_\alpha w(\mu+\rho)-w(\mu+\rho)+\xi=\xi+n\alpha,\\&s_\alpha\xi=\xi-(\alpha^\vee,\xi)\alpha=\xi-((\alpha^\vee,w(\mu+\rho)-(\alpha^\vee,\lambda+\rho))\alpha=\xi+(m+n)\alpha.\end{aligned}$$

Since ξ and $s_\alpha\xi=\xi+(m+n)\alpha$ are elements of Γ, we have $s_\alpha w\circ\mu-\lambda=\xi+n\alpha\in\Gamma$. By $\ell_\lambda(s_\alpha w)<\ell_\lambda(w)$ we obtain $s_\alpha w\in W_0(\lambda)W_0(\mu)$ by the minimality of $\ell_\lambda(w)$. Hence we have $s_\alpha w\circ\mu-\lambda\in W_0(\lambda)(\mu-\lambda)\subset W\nu$. It follows that $\xi+n\alpha$ is an extremal weight of $L(\nu)$. This contradicts ξ, $\xi+(m+n)\alpha\in\Gamma$, and $m,n\in\mathbb{Z}_{>0}$ or $m,n\in\mathbb{Z}_{<0}$. □

PROPOSITION 3.6. *Let $\lambda,\mu\in\mathcal{C}$ such that $\mu-\lambda\in WP^+$ and $\Delta_0(\lambda)\subset\Delta_0(\mu)$. Assume that we have either $\lambda,\mu\in\mathcal{C}^+$ or $\lambda,\mu\in\mathcal{C}^-$. Then we have $T_\mu^\lambda(M(w\circ\lambda))=M(w\circ\mu)$ for any $w\in W(\lambda)$.*

PROOF. Take $x\in W$ and $\nu\in P^+$ such that $\mu-\lambda=x\nu$. Let Γ be the set of weights of $L(\nu)$. Since

(3.14)
$$M(w\circ\lambda)\otimes L(\nu)=U(\mathfrak{g})\otimes_{U(\mathfrak{b})}(\mathbb{C}_{w\circ\lambda}\otimes L(\nu))=U(\mathfrak{n}^-)\otimes_{\mathbb{C}}(\mathbb{C}_{w\circ\lambda}\otimes L(\nu)),$$

we have

$$\operatorname{ch}(M(w\circ\lambda)\otimes L(\nu))=\sum_{\xi\in\Gamma}\dim L(\nu)_\xi\operatorname{ch}(M(w\circ\lambda+\xi)).$$

This implies

$$\begin{aligned}\operatorname{ch}(T_\mu^\lambda(M(w\circ\lambda)))&=\sum_{\xi\in\Gamma}\dim L(\nu)_\xi\operatorname{ch}(P_\mu(M(w\circ\lambda+\xi)))\\&=\sum_{\xi\in\Gamma,\,w\circ\lambda+\xi\in W(\mu)\circ\mu}\dim L(\nu)_\xi\operatorname{ch}((M(w\circ\lambda+\xi))).\end{aligned}$$

Assume that $w\circ\lambda+\xi=y\circ\mu$ for $\xi\in\Gamma$ and $y\in W(\lambda)$. Then we have $w^{-1}y\circ\mu-\lambda=w^{-1}\xi\in\Gamma$, and hence $w^{-1}y\in W_0(\lambda)W_0(\mu)=W_0(\mu)$ by Lemma 3.5. Thus we have

$$\xi=w(\mu-\lambda)=wx\nu\qquad\text{and}\quad w\circ\lambda+\xi=w\circ(\lambda+x\nu)=w\circ\mu.$$

Hence we obtain $\operatorname{ch}(T_\mu^\lambda(M(w\circ\lambda))) = \operatorname{ch}(M(w\circ\mu))$. In particular, there exists some $v \in (M(w\circ\lambda)\otimes L(\nu))_{w\circ\mu}\setminus\{0\}$ such that $\mathfrak{n}^+v = 0$. By (3.14), $M(w\circ\lambda)\otimes L(\nu)$ is a free $U(\mathfrak{n}^-)$-module. Thus the morphism $U(\mathfrak{n}^-)\to M(w\circ\lambda)\otimes L(\nu)$ given by $u\mapsto uv$ is injective. It follows that $T_\mu^\lambda(M(w\circ\lambda))$ contains $M(w\circ\mu)$ as a submodule. Hence we have $T_\mu^\lambda(M(w\circ\lambda)) = M(w\circ\mu)$. □

COROLLARY 3.7. *Let $\lambda,\mu\in\mathcal{C}$ such that $\mu-\lambda\in WP^+$ and $\Delta_0(\lambda)\subset\Delta_0(\mu)$. Assume that we have either $\lambda,\mu\in\mathcal{C}^+$ or $\lambda,\mu\in\mathcal{C}^-$. For $M\in\operatorname{Ob}(\mathbb{O}[\lambda])$ let us write*

$$\operatorname{ch} M = \sum_{w\in W(\lambda)} a_w \operatorname{ch}(M(w\circ\lambda)) \tag{3.15}$$

with integers a_w. Then we have

$$\operatorname{ch} T_\mu^\lambda(M) = \sum_{w\in W(\lambda)} a_w \operatorname{ch}(M(w\circ\mu))\,.$$

PROOF. If $\lambda\in\mathcal{C}^-$, then M has finite length. Therefore we can reduce the assertion to the case where $M = M(y\circ\lambda)$ with $y\in W(\lambda)$. Then the assertion follows from the preceding proposition.

Assume now $\lambda\in\mathcal{C}^+$. It is enough to show

$$\dim(T_\mu^\lambda(M))_\xi = \sum_{w\in W(\lambda)} a_w \dim(M(w\circ\mu)_\xi) \tag{3.16}$$

for any $\xi\in\mathfrak{h}^*$. Let $\operatorname{Wt}(M)$ be the set of weights of M. We set $\mathfrak{h}^*_N = \{\lambda - \sum_{i\in I} n_i\alpha_i; \sum n_i \ge N\}$. Since $w\circ\lambda=\lambda$ implies $w\circ\mu=\mu$ by Lemma 2.8, we may assume w ranges over $W(\lambda)/W_0(\lambda)$ in (3.15). If $\operatorname{Wt}(M)\subset\mathfrak{h}^*_N$ for a sufficiently large N, then $a_w\neq 0$ implies that $l_\lambda(w)$ is sufficiently large. Hence the both sides of (3.16) vanish. Fixing such an N we shall argue by the descending induction on m such that $\operatorname{Wt}(M)\setminus\mathfrak{h}^*_N\subset\mathfrak{h}^*_m$. Let $w\circ\lambda$ ($w\in W(\lambda)$) be a highest weight of M. Then there is an exact sequence

$$0\to M_1\to M(w\circ\lambda)^{\oplus m}\to M\to M_2\to 0\,,$$

where $\operatorname{Wt}(M_k)$ does not contain $w\circ\lambda$ ($k=1,2$). Hence by the induction hypothesis, (3.16) holds for M_1. Arguing by the induction on the cardinality of $\operatorname{Wt}(M)\setminus\mathfrak{h}^*_N$, (3.16) holds for M_2. Since $T_\mu^\lambda(M(w\circ\lambda)) = M(w\circ\mu)$ by the preceding proposition, (3.16) holds for $M(w\circ\mu)$. Then (3.16) holds for M because T_μ^λ is an exact functor. □

PROPOSITION 3.8. *Let $\lambda,\mu\in\mathcal{C}$ such that $\mu-\lambda\in WP^+$ and $\Delta_0(\lambda)\subset\Delta_0(\mu)$. Let $w\in W(\lambda)$.*

(i) *If $\lambda,\mu\in\mathcal{C}^+$, then we have*

$$T_\mu^\lambda(L(w\circ\lambda)) = \begin{cases} L(w\circ\mu) & \text{if } w(\Delta_0^+(\mu)\setminus\Delta_0^+(\lambda))\subset\Delta^-(\lambda),\\ 0 & \text{otherwise.}\end{cases}$$

(ii) *If $\lambda,\mu\in\mathcal{C}^-$, then we have*

$$T_\mu^\lambda(L(w\circ\lambda)) = \begin{cases} L(w\circ\mu) & \text{if } w(\Delta_0^+(\mu)\setminus\Delta_0^+(\lambda))\subset\Delta^+(\lambda),\\ 0 & \text{otherwise.}\end{cases}$$

PROOF. Since T_μ^λ is an exact functor, $T_\mu^\lambda(L(w\circ\lambda))$ is a quotient of $T_\mu^\lambda(M(w\circ\lambda)) = M(w\circ\mu)$. By restricting the non-degenerate contravariant form on $L(w\circ\lambda)\otimes L(\nu)$ we obtain a non-degenerate contravariant form on $T_\mu^\lambda(L(w\circ\lambda))$. Thus we have either $T_\mu^\lambda(L(w\circ\lambda)) = L(w\circ\mu)$ or $T_\mu^\lambda(L(w\circ\lambda)) = 0$.

Assume $w(\Delta_0^+(\mu)\setminus\Delta_0^+(\lambda)) \not\subset \Delta^-(\lambda)$ in the case $\lambda,\mu\in\mathcal{C}^+$ and $w(\Delta_0^+(\mu)\setminus\Delta_0^+(\lambda)) \not\subset \Delta^+(\lambda)$ in the case $\lambda,\mu\in\mathcal{C}^-$. Then there exists $\alpha\in\Delta(\lambda)$ such that $w\alpha\in\Delta^+(\lambda)$, $(\alpha^\vee,\lambda+\rho)>0$, and $(\alpha^\vee,\mu+\rho)=0$. Set $\beta=w\alpha\in\Delta^+(\lambda)$. Then we have $(\beta^\vee, w\circ\lambda+\rho)>0$ and $(\beta^\vee, w\circ\mu+\rho)=0$. By Proposition 3.1 we have exact sequences

$$0\to M(s_\beta w\circ\lambda)\to M(w\circ\lambda)\to L\to 0,$$
$$L\to L(w\circ\lambda)\to 0.$$

By applying the exact functor T_μ^λ, we obtain exact sequences

$$0\to M(s_\beta w\circ\mu)\to M(w\circ\mu)\to T_\mu^\lambda(L)\to 0,$$
$$T_\mu^\lambda(L)\to T_\mu^\lambda(L(w\circ\lambda))\to 0.$$

Since $M(s_\beta w\circ\mu)\to M(w\circ\mu)$ is an isomorphism, we have $T_\mu^\lambda(L(w\circ\lambda))=0$.

Next assume $w(\Delta_0^+(\mu)\setminus\Delta_0^+(\lambda))\subset\Delta^-(\lambda)$ in the case $\lambda,\mu\in\mathcal{C}^+$ and $w(\Delta_0^+(\mu)\setminus\Delta_0^+(\lambda))\subset\Delta^+(\lambda)$ in the case $\lambda,\mu\in\mathcal{C}^-$. Then we have

$$w\alpha\in\Delta^-(\lambda) \text{ for any } \alpha\in\Delta(\lambda) \text{ satisfying } (\alpha^\vee,\lambda+\rho)>0 \text{ and } (\alpha^\vee,\mu+\rho)=0. \tag{3.17}$$

Let M be the maximal proper submodule of $M(w\circ\lambda)$. By applying T_μ^λ to the exact sequence

$$0\to M\to M(w\circ\lambda)\to L(w\circ\lambda)\to 0\,,$$

we obtain an exact sequence

$$0\to T_\mu^\lambda(M)\to M(w\circ\mu)\to T_\mu^\lambda(L(w\circ\lambda))\to 0.$$

Thus it is sufficient to show $[T_\mu^\lambda(M):L(w\circ\mu)]=0$. Hence we have only to prove $[T_\mu^\lambda(L(z\circ\lambda)) : L(w\circ\mu)] = 0$ for any $z\in W(\lambda)$ satisfying $[M : L(z\circ\lambda)]\neq 0$. By Proposition 3.1 there exists some $\beta\in\Delta^+(\lambda)$ such that $(\beta^\vee, w(\lambda+\rho))>0$ and $[M(s_\beta w\circ\lambda) : L(z\circ\lambda)]\neq 0$. For such a β, $T_\mu^\lambda(L(z\circ\lambda))$ is a subquotient of $T_\mu^\lambda(M(s_\beta w\circ\lambda)) = M(s_\beta w\circ\mu)$. Therefore it is sufficient to show $[M(s_\beta w\circ\mu) : L(w\circ\mu)]=0$ for any $\beta\in\Delta^+(\lambda)$ such that $(\beta^\vee, w(\lambda+\rho))>0$. Set $\alpha=w^{-1}\beta$. Then we have $\alpha\in\Delta(\lambda)$, $w\alpha\in\Delta^+(\lambda)$ and $(\alpha^\vee,\lambda+\rho)>0$. Since $\alpha\in\Delta^\pm(\lambda)$ according to λ, $\mu\in\mathcal{C}^\pm$, we have $(\alpha^\vee,\mu+\rho)\geq 0$. Hence (3.17) implies $(\beta^\vee, w(\mu+\rho)) = (\alpha^\vee,\mu+\rho)>0$. Thus we obtain $[M(s_\beta w\circ\mu) : L(w\circ\mu)]=0$. □

PROPOSITION 3.9. *Let $\lambda_1,\lambda_2\in\mathcal{C}$ such that $\lambda_1-\lambda_2\in P$ and $\Delta_0(\lambda_1)=\Delta_0(\lambda_2)$. Assume that we have either $\lambda_1,\lambda_2\in\mathcal{C}^+$ or $\lambda_1,\lambda_2\in\mathcal{C}^-$. Let $w\in W(\lambda_1)$, and write*

$$\mathrm{ch}(L(w\circ\lambda_1)) = \sum_{y\in W(\lambda_1)/W_0(\lambda_1)} a_y\,\mathrm{ch}(M(y\circ\lambda_1))$$

with $a_y\in\mathbb{Z}$. Then we have

$$\mathrm{ch}(L(w\circ\lambda_2)) = \sum_{y\in W(\lambda_1)/W_0(\lambda_1)} a_y\,\mathrm{ch}(M(y\circ\lambda_2)).$$

Proof. Note that $\Delta(\lambda_1) = \Delta(\lambda_2)$, $W(\lambda_1) = W(\lambda_2)$ and $W_0(\lambda_1) = W_0(\lambda_2)$.

Case 1. $\lambda_1, \lambda_2 \in \mathcal{C}^+$.

By Lemma 2.13 there exist $x \in W$ and a proper subset J of I such that $x^{-1}\Delta^+(\lambda_k) \subset \Delta^+$ and $x^{-1}\Delta_0(\lambda_k) = \Delta_J$ for $k = 1$ (and hence also for $k = 2$). Take $\xi_1 \in P^+$ such that $(\alpha_i^\vee, \xi_1) = 0$ for $i \in J$ and $(\alpha_i^\vee, \xi_1) \in \mathbb{Z}_{>0}$ for $i \in I \setminus J$. Set $\xi_2 = \xi_1 + x^{-1}(\lambda_1 - \lambda_2)$, $\mu = \lambda_1 + x\xi_1 = \lambda_2 + x\xi_2$. Then we have

$$(\alpha_i^\vee, \xi_2) = (\alpha_i^\vee, x^{-1}(\lambda_1 - \lambda_2)) = (x\alpha_i^\vee, \lambda_1 + \rho) - (x\alpha_i^\vee, \lambda_2 + \rho) = 0 \qquad \text{for } i \in J,$$

$$(\alpha_i^\vee, \xi_2) = (\alpha_i^\vee, \xi_1) + (\alpha_i^\vee, x^{-1}(\lambda_1 - \lambda_2)) \qquad \text{for } i \in I \setminus J,$$

$$(\delta, \mu + \rho) = (\delta, \lambda_1 + \rho) + \sum_{i \in I} m_i(\alpha_i, \xi_1),$$

where $\delta = \sum_{i \in I} m_i\alpha_i$. By taking $(\alpha_i^\vee, \xi_1)$ for $i \in I \setminus J$ sufficiently large, we may assume that $\xi_2 \in P^+$ and $(\delta, \mu + \rho) \neq 0$. Moreover, we have

$$(\alpha^\vee, \mu + \rho) = (\alpha^\vee, \lambda_1 + \rho) + (x^{-1}\alpha^\vee, \xi_1) \geq 0$$

for any $\alpha \in \Delta^+(\mu) = \Delta^+(\lambda_1)$, and hence we have $\mu \in \mathcal{C}^+$ and $\Delta_0(\mu) = \Delta_0(\lambda_1) = \Delta_0(\lambda_2)$.

Thus Proposition 3.8 implies $T_\mu^{\lambda_k}(L(w \circ \lambda_k)) = L(w \circ \mu)$ for any $w \in W(\lambda_k)$ and $k = 1, 2$. The assertion then follows from Corollary 3.7.

Case 2. $\lambda_1, \lambda_2 \in \mathcal{C}^-$.

The proof is similar to the one for the case 1. Take $x \in W$ and a proper subset J of I such that $x^{-1}\Delta^+(\lambda_k) \subset \Delta^+$ and $x^{-1}\Delta_0(\lambda_k) = \Delta_J$ for $k = 1, 2$. Take $\xi_1 \in P^+$ such that $(\alpha_i^\vee, \xi_1) = 0$ for $i \in J$ and $(\alpha_i^\vee, \xi_1) \in \mathbb{Z}_{>0}$ for $i \in I \setminus J$. Set $\xi_2 = \xi_1 - x^{-1}(\lambda_1 - \lambda_2)$, $\mu = \lambda_1 - x\xi_1 = \lambda_2 - x\xi_2$. By taking $(\alpha_i^\vee, \xi_1)$ for $i \in I \setminus J$ sufficiently large, we have $\mu \in \mathcal{C}^-$, $\xi_2 \in P^+$ and $\Delta_0(\mu) = \Delta_0(\lambda_k)$ for $k = 1, 2$. Thus Proposition 3.8 implies $T_{\lambda_k}^\mu(L(w \circ \mu)) = L(w \circ \lambda_k)$ for any $w \in W(\lambda_k)$ and $k = 1, 2$. Hence we obtain the desired result by Corollary 3.7. □

Proposition 3.10. *Assume that* λ, $\mu \in \mathcal{C}^+$ (*resp.* $\lambda, \mu \in \mathcal{C}^-$) *satisfy*

$$\mu - \lambda \in P, \qquad \Delta_0(\lambda) = \emptyset. \tag{3.18}$$

Assume that $w \in W(\lambda)$ *is the longest* (*resp. shortest*) *element of* $wW_0(\mu)$. *Write*

$$\mathrm{ch}(L(w \circ \lambda)) = \sum_{y \in W(\lambda)} a_y \,\mathrm{ch}(M(y \circ \lambda)) \qquad \text{with } a_y \in \mathbb{Z}. \tag{3.19}$$

Then we have

$$\mathrm{ch}(L(w \circ \mu)) = \sum_{y \in W(\lambda)} a_y \,\mathrm{ch}(M(y \circ \mu)). \tag{3.20}$$

Proof. Let us prove first the case where $\lambda, \mu \in \mathcal{C}^+$. We first prove the following statement.

(3.21) Let $\nu \in \mathcal{C}^+$. For any $N \in \mathbb{Z}_{>0}$ there exists some $\tilde{\nu} \in \mathcal{C}^+$ such that $\tilde{\nu} - \nu \in P$, $\Delta_0(\tilde{\nu}) = \Delta_0(\nu)$, $(\alpha^\vee, \tilde{\nu} + \rho) \geq N$ for any $\alpha \in \Delta^+(\nu) \setminus \Delta_0(\nu)$, and $(\delta, \tilde{\nu} + \rho) - (\delta, \nu + \rho) \in \mathbb{Z}_{\geq N}$.

By Lemma 2.13 there exist $x \in W$ and a proper subset J of I such that $x\Delta^+(\nu) \subset \Delta^+$ and $x\Delta_0(\nu) = \Delta_J$. Take $\xi \in P^+$ such that $(\alpha_i^\vee, \xi) = 0$ for $i \in J$ and $(\alpha_i^\vee, \xi) > 0$ for $i \in I \setminus J$. Set $\tilde{\nu} = \nu + x^{-1}\xi$. Then we have $(\alpha^\vee, \tilde{\nu} + \rho) = (\alpha^\vee, \nu + \rho) + (x\alpha^\vee, \xi)$

for any $\alpha \in \Delta(\lambda)$ and $(\delta, \tilde{\nu}+\rho) = (\delta, \nu+\rho) + (\delta, \xi)$. Hence by taking $(\alpha_i^\vee, \xi) > 0$ for $i \in I \setminus J$ sufficiently large, we obtain (3.21).

Assume that $\mu \in \mathcal{C}^+$. Let $N \in \mathbb{Z}_{>0}$. By (3.21) there exists $\tilde{\mu} \in \mathcal{C}^+$ such that $\tilde{\mu} - \mu \in P$, $\Delta_0(\tilde{\mu}) = \Delta_0(\mu)$, $(\alpha^\vee, \tilde{\mu}+\rho) \geq N$ for any $\alpha \in \Delta^+(\mu) \setminus \Delta_0(\mu)$, and $(\delta, \tilde{\mu}+\rho) - (\delta, \mu+\rho) \in \mathbb{Z}_{\geq N}$. By Lemma 2.13 there exist $x \in W$ and a proper subset J of I such that $x\Delta^+(\tilde{\mu}) = x\Delta^+(\mu) \subset \Delta^+$ and $x\Delta_0(\tilde{\mu}) = x\Delta_0(\mu) = \Delta_J$. Let w_J be the longest element of W_J. Take $\nu \in P^+$ such that $(\alpha_j^\vee, \nu) > 0$ for any $j \in J$, and set $\tilde{\lambda} = \tilde{\mu} - x^{-1}w_J\nu$. Then we have

$$\tilde{\mu} - \tilde{\lambda} \in WP^+. \tag{3.22}$$

Since $(\delta, \tilde{\lambda}+\rho) = (\delta, \tilde{\mu}+\rho) - (\delta, \nu)$, we have

$$(\delta, \tilde{\lambda}+\rho) \neq 0 \tag{3.23}$$

when N is sufficiently large. For any $\alpha \in \Delta^+(\tilde{\mu})$ we have $(\alpha^\vee, \tilde{\lambda}+\rho) = (\alpha^\vee, \tilde{\mu}+\rho) - (w_Jx\alpha^\vee, \nu)$. If $\alpha \in \Delta_0^+(\tilde{\mu}) = \Delta_0^+(\mu)$, then we have $(\alpha^\vee, \tilde{\mu}+\rho) = 0$ and $w_Jx\alpha \in -\Delta_J^+$, and hence $(\alpha^\vee, \tilde{\lambda}+\rho) \in \mathbb{Z}_{>0}$. If $\alpha \in \Delta^+(\tilde{\mu}) \setminus \Delta_0^+(\tilde{\mu})$, then we have $(\alpha^\vee, \tilde{\lambda}+\rho) \in \mathbb{Z}_{>0}$ when N is sufficiently large. Since $\Pi(\tilde{\mu}) = \Pi(\mu)$ is a finite set, we have $(\alpha^\vee, \tilde{\lambda}+\rho) > 0$ for any $\alpha \in \Pi(\tilde{\mu})$ for a sufficiently large N. By $\Delta^+(\tilde{\mu}) \subset \sum_{\alpha \in \Pi(\tilde{\mu})} \mathbb{Z}_{\geq 0}\alpha$ we have

$$(\alpha^\vee, \tilde{\lambda}+\rho) \in \mathbb{Z}_{>0} \quad \text{for any } \alpha \in \Delta^+(\tilde{\mu}) = \Delta^+(\tilde{\lambda}) \tag{3.24}$$

when N is sufficiently large.

Take N satisfying (3.23), (3.24). Then we have $\tilde{\lambda} \in \mathcal{C}^+$ and $\tilde{\lambda}$ satisfies the condition (3.18) for λ. By Proposition 3.9 the integers a_y in (3.19) do not depend on the choice of λ. Hence (3.19) holds for $\tilde{\lambda}$. Since w is the longest element of $wW_0(\mu) = wW_0(\tilde{\mu})$ we have $w\Delta_0^+(\tilde{\mu}) \subset \Delta^-$, and Proposition 3.8 implies $T_{\tilde{\mu}}^{\tilde{\lambda}}(L(w \circ \tilde{\lambda})) = L(w \circ \tilde{\mu})$. Then Corollary 3.7 implies

$$\operatorname{ch}(L(w \circ \tilde{\mu})) = \sum_{y \in W(\lambda)} a_y \operatorname{ch}(M(y \circ \tilde{\mu})).$$

The desired result follows then from Proposition 3.9.

As the assertion in the case $\mu \in \mathcal{C}^-$ is proved similarly, we shall only give a sketch. By Proposition 3.9 and an analogue of (3.21) we may assume that $(\alpha^\vee, \mu+\rho)$ for $\alpha \in \Delta^+(\mu) \setminus \Delta_0(\mu)$ and $(\delta, \mu+\rho)$ are sufficiently small. Take $x \in W$ and a proper subset J of I satisfying $x\Delta^+(\mu) \subset \Delta^+$ and $x\Delta_0(\mu) = \Delta_J$. Take $\nu \in P^+$ such that $(\alpha_j^\vee, \nu) > 0$ for any $j \in J$, and set $\tilde{\lambda} = \mu - x^{-1}\nu$. Then we have $\tilde{\lambda} \in \mathcal{C}^-$ and $\tilde{\lambda}$ satisfies the condition (3.18) for λ. Hence we can take $\tilde{\lambda}$ as λ by Proposition 3.9. Then we have $T_\mu^\lambda(L(w \circ \lambda)) = L(w \circ \mu)$ by Proposition 3.8. Hence we obtain the desired result by Corollary 3.7. □

4. Enright functor

We recall certain properties of the Enright functor which will be used later (see Enright [**7**], Deodhar [**5**], Kashiwara-Tanisaki [**15**, §2.4]).

For $i \in I$ define a subalgebra $\mathfrak{g}_i$ of $\mathfrak{g}$ by $\mathfrak{g}_i = \mathfrak{h} \oplus \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{-\alpha_i}$. Take $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ such that $[e_i, f_i] = h_i$. For $a \in \mathbb{C}$ we denote by $\mathbb{M}(\mathfrak{g}_i, a)$ the full subcategory of $\mathbb{M}(\mathfrak{g}_i)$ consisting of $M \in \mathrm{Ob}(\mathbb{M}(\mathfrak{g}_i))$ satisfying

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu, \tag{4.1}$$

$$\dim M_\mu = 0 \text{ unless } \langle h_i, \mu \rangle \equiv a \bmod \mathbb{Z}, \tag{4.2}$$

$$\dim \mathbb{C}[e_i]m < \infty \quad \text{for any } m \in M. \tag{4.3}$$

For $\mu \in \mathfrak{h}^*$ let $M_i(\mu)$ be the Verma module for $\mathfrak{g}_i$ with highest weight μ. We fix a highest weight vector m_μ of $M_i(\mu)$.

LEMMA 4.1. *Assume $a \notin \mathbb{Z}$. For $M \in \mathrm{Ob}(\mathbb{M}(\mathfrak{g}_i, a))$ set $N = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu^{e_i} \otimes M_i(\mu)$, where*

$$M_\mu^{e_i} = \{m \in M_\mu\,;\, e_i m = 0\}.$$

Define a linear map $\varphi : N \to M$ by

$$\varphi(m \otimes f_i^k m_\mu) = f_i^k m \qquad \text{for } m \in M_\mu^{e_i} \text{ and } k \in \mathbb{Z}_{\geq 0}.$$

Then φ is an isomorphism of $\mathfrak{g}_i$-modules.

PROOF. By the definition of the Verma module φ is obviously a homomorphism of $\mathfrak{g}_i$-modules.

Let us show that φ is surjective. It is sufficient to show that $M_\xi \subset \mathrm{Im}(\varphi)$ for any $\xi \in \mathfrak{h}^*$. Let $m \in M_\xi$ satisfying $e_i^n m = 0$. We show by induction on n that $m \in \sum_{k=0}^\infty f_i^k M_{\xi+k\alpha_i}^{e_i}$. The case $n = 0$ is trivial. Assume $n > 0$. Since $e_i^{n-1}(e_i m) = 0$, we have $e_i m \in \sum_{k=0}^\infty f_i^k M_{\xi+(k+1)\alpha_i}^{e_i}$ by the hypothesis of induction. By $a \notin \mathbb{Z}$ the linear map $f_i^{k+1} M_{\xi+(k+1)\alpha_i}^{e_i} \to f_i^k M_{\xi+(k+1)\alpha_i}^{e_i}$ $(n \mapsto e_i n)$ is bijective. Hence there exists some $u \in \sum_{k=0}^\infty f_i^{k+1} M_{\xi+(k+1)\alpha_i}^{e_i}$ such that $e_i u = e_i m$. Then we have

$$m = (m - u) + u \in M_\xi^{e_i} + \sum_{k=0}^\infty f_i^{k+1} M_{\xi+(k+1)\alpha_i}^{e_i} = \sum_{k=0}^\infty f_i^k M_{\xi+k\alpha_i}^{e_i}.$$

Next let us show that φ is injective. Assume $\mathrm{Ker}(\varphi) \neq 0$. By $a \notin \mathbb{Z}$ the Verma module $M_i(\mu)$ is irreducible unless $M_\mu^{e_i} = 0$. Thus there exist subspaces $N(\mu)$ of $M_\mu^{e_i}$ for $\mu \in \mathfrak{h}^*$ such that $\mathrm{Ker}(\varphi) = \bigoplus_{\mu \in \mathfrak{h}^*} N(\mu) \otimes M_i(\mu)$. Hence there exists some $m \in M_\mu^{e_i} \setminus \{0\}$ such that $m \otimes M_i(\mu) \subset \mathrm{Ker}(\varphi)$. Then we have $m = \varphi(m \otimes m_\mu) = 0$. This is a contradiction. Thus we have $\mathrm{Ker}(\varphi) = 0$. □

We denote by $F : \mathbb{M}(\mathfrak{g}) \to \mathbb{M}(\mathfrak{g}_i)$ the forgetful functor. For $a \in \mathbb{C}$ let $\mathbb{M}_i(\mathfrak{g}, a)$ be the full subcategory of $\mathbb{M}(\mathfrak{g})$ consisting of $M \in \mathrm{Ob}(\mathbb{M}(\mathfrak{g}))$ satisfying $F(M) \in \mathrm{Ob}(\mathbb{M}(\mathfrak{g}_i, a))$.

For $a \in \mathbb{C}$ define a left $U(\mathfrak{g})$-module $U(\mathfrak{g})f_i^{a+\mathbb{Z}}$ by

$$U(\mathfrak{g})f_i^{a+\mathbb{Z}} = \varinjlim_n U(\mathfrak{g})f_i^{a-n}, \tag{4.4}$$

where $U(\mathfrak{g})f_i^{a-n}$ is a rank one free $U(\mathfrak{g})$-module generated by the element f_i^{a-n} and the homomorphism $U(\mathfrak{g})f_i^{a-n} \to U(\mathfrak{g})f_i^{a-n-1}$ is given by $f_i^{a-n} \mapsto f_i f_i^{a-n-1}$. Then we have a natural $U(\mathfrak{g})$-bimodule structure on $U(\mathfrak{g})f_i^{a+\mathbb{Z}}$ whose right $U(\mathfrak{g})$-module structure is given by

$$f_i^{a+m}P = \sum_{k=0}^{\infty} \binom{a+m}{k} (\mathrm{ad}(f_i)^k P) f_i^{a+m-k} \quad \text{for any } m \in \mathbb{Z} \text{ and any } P \in U(\mathfrak{g}). \tag{4.5}$$

Note that the $U(\mathfrak{g})$-bimodule $U(\mathfrak{g})f_i^{a+\mathbb{Z}}$ depends only on $(a \bmod \mathbb{Z}) \in \mathbb{C}/\mathbb{Z}$.

For $M \in \mathrm{Ob}(\mathbb{M}_i(\mathfrak{g}, a))$ we set

$$S_i(a)(M) = \{m \in U(\mathfrak{g})f_i^{a+\mathbb{Z}} \otimes_{U(\mathfrak{g})} M\,;\, \dim \mathbb{C}[e_i]m < \infty\}. \tag{4.6}$$

It defines a left exact functor

$$S_i(a) : \mathbb{M}_i(\mathfrak{g}, a) \to \mathbb{M}_i(\mathfrak{g}, -a), \tag{4.7}$$

called the Enright functor corresponding to i.

By the morphism of $U(\mathfrak{g})$-bimodules

$$U(\mathfrak{g}) \to U(\mathfrak{g})f_i^{-a+\mathbb{Z}} \otimes_{U(\mathfrak{g})} U(\mathfrak{g})f_i^{a+\mathbb{Z}} \qquad (1 \mapsto f_i^{-a} \otimes f_i^{a}) \tag{4.8}$$

we obtain a canonical morphism of functors (see [**15**, §2.4])

$$\mathrm{id}_{\mathbb{M}_i(\mathfrak{g},a)} \to S_i(-a) \circ S_i(a). \tag{4.9}$$

By [**15**, §2.4] we have the following result.

PROPOSITION 4.2. *Let $\lambda \in \mathfrak{h}^*$, and set $a = \langle h_i, \lambda \rangle$.*

(i) *If $a \notin \mathbb{Z}_{>0}$, then we have $S_i(a)(M(\lambda)) \simeq M(s_i \circ \lambda)$.*

(ii) *If $a \notin \mathbb{Z}$, then the canonical morphism $M(\lambda) \to S_i(-a) \circ S_i(a)(M(\lambda))$ induced by (4.9) is an isomorphism.*

We can similarly define a $U(\mathfrak{g}_i)$-bimodule $U(\mathfrak{g}_i)f_i^{a+\mathbb{Z}}$, and the Enright functor $\overline{S}(a) : \mathbb{M}(\mathfrak{g}_i, a) \to \mathbb{M}(\mathfrak{g}_i, -a)$ for $\mathfrak{g}_i$ is given by

$$\overline{S}(a)(M) = \{m \in U(\mathfrak{g}_i)f_i^{a+\mathbb{Z}} \otimes_{U(\mathfrak{g}_i)} M\,;\, \dim \mathbb{C}[e_i]m < \infty\}$$

for any $M \in \mathrm{Ob}(\mathbb{M}(\mathfrak{g}_i, a))$. Then we have $F \circ S_i(a) = \overline{S}(a) \circ F$ by $U(\mathfrak{g}_i)f_i^{a+\mathbb{Z}} \otimes_{U(\mathfrak{g}_i)} U(\mathfrak{g}) \simeq U(\mathfrak{g})f_i^{a+\mathbb{Z}}$.

PROPOSITION 4.3. *Assume that $a \notin \mathbb{Z}$.*

(i) *The functor $S_i(a) : \mathbb{M}_i(\mathfrak{g}, a) \to \mathbb{M}_i(\mathfrak{g}, -a)$ gives an equivalence of categories, and its inverse is given by $S_i(-a)$.*

(ii) *For $\lambda \in \mathfrak{h}^*$ such that $\langle h_i, \lambda \rangle \equiv a \bmod \mathbb{Z}$, we have*

$$S_i(a)(M(\lambda)) \simeq M(s_i \circ \lambda), \qquad S_i(a)(L(\lambda)) \simeq L(s_i \circ \lambda).$$

PROOF. (i) We have to show that the canonical morphisms $\mathrm{id}_{\mathbb{M}_i(\mathfrak{g},a)} \to S_i(-a) \circ S_i(a)$ and $\mathrm{id}_{\mathbb{M}_i(\mathfrak{g},-a)} \to S_i(a) \circ S_i(-a)$ are isomorphisms. By the symmetry we have only to show that $\mathrm{id}_{\mathbb{M}_i(\mathfrak{g},a)} \to S_i(-a) \circ S_i(a)$ is an isomorphism. Let us show that the canonical morphism $M \to S_i(-a) \circ S_i(a)(M)$ is bijective for any $M \in \mathrm{Ob}(\mathbb{M}_i(\mathfrak{g}, a))$. By $F \circ S_i(-a) \circ S_i(a)(M) = \overline{S}(-a) \circ \overline{S}(a) \circ F(M)$ it is sufficient to show that the canonical morphism $N \to \overline{S}(-a) \circ \overline{S}(a)(N)$ is bijective for any $N \in \mathrm{Ob}(\mathbb{M}(\mathfrak{g}_i, a))$. This follows from Proposition 4.2 for $\mathfrak{g}_i$ and Lemma 4.1.

(ii) We have $S_i(a)(M(\lambda)) \simeq M(s_i \circ \lambda)$ by Proposition 4.2. By (i) $S_i(a)(L(\lambda))$ is the unique irreducible quotient of $S_i(a)(M(\lambda)) \simeq M(s_i \circ \lambda)$. Thus we have $S_i(a)(L(\lambda)) \simeq L(s_i \circ \lambda)$. □

5. Proof of main theorem

In this section we shall give a proof of Theorem 1.1. We shall use different arguments according to whether $\mathbb{Q}\Delta(\lambda) \ni \delta$ or not. Assume $\lambda \in \mathcal{C}^+ \cup \mathcal{C}^-$.

Case 1. $\mathbb{Q}\Delta(\lambda) \ni \delta$.

In this case the following argument is completely similar to Bernstein's proof of the corresponding result for finite-dimensional semisimple Lie algebras.

Set

(5.1) $$\Omega(\lambda) = \{\mu \in \mathfrak{h}^* \,;\, (\alpha^\vee, \mu) = (\alpha^\vee, \lambda) \quad \text{for any } \alpha \in \Delta(\lambda)\},$$

(5.2) $$\Omega'(\lambda) = \{\mu \in \Omega(\lambda) \,;\, (\alpha^\vee, \mu) \notin \mathbb{Z} \quad \text{for any } \alpha \in \Delta_{\mathrm{re}} \setminus \Delta(\lambda)\}.$$

Then we have

(5.3) $$W(\mu) \supset W(\lambda) \text{ and } W_0(\mu) \supset W_0(\lambda) \text{ for any } \mu \in \Omega(\lambda),$$

(5.4) $$W(\mu) = W(\lambda) \text{ and } W_0(\mu) = W_0(\lambda) \text{ for any } \mu \in \Omega'(\lambda),$$

(5.5) $$w \circ \mu - y \circ \mu = w \circ \lambda - y \circ \lambda \text{ for any } \mu \in \Omega(\lambda),\ w, y \in W(\lambda),$$

(5.6) $$(\delta, \mu) = (\delta, \lambda) \text{ for any } \mu \in \Omega(\lambda).$$

For any $\mu \in \Omega'(\lambda)$ and $w \in W(\lambda)/W_0(\lambda)$ we can write uniquely

(5.7) $$\mathrm{ch}(L(w \circ \mu)) = \sum_{w \in W(\lambda)/W_0(\lambda)} a_{w,y}(\mu)\, \mathrm{ch}(M(y \circ \mu)) \qquad \text{with } a_{w,y}(\mu) \in \mathbb{Z}$$

by Proposition 3.1 and (5.4).

PROPOSITION 5.1. *For any $w, y \in W(\lambda)/W_0(\lambda)$ the function $a_{w,y}(\mu)$ defined in* (5.7) *is a constant function on $\Omega'(\lambda)$.*

PROOF. For $\mu \in \Omega'(\lambda)$ and $w \in W(\lambda)/W_0(\lambda)$ we have

$$\begin{aligned} \mathrm{ch}(L(w \circ \mu))\, \mathrm{e}^{-w\circ\mu} &= \sum_{w \in W(\lambda)/W_0(\lambda)} a_{w,y}(\mu)\, \mathrm{ch}(M(y \circ \mu))\, \mathrm{e}^{-w\circ\mu} \\ &= \sum_{w \in W(\lambda)/W_0(\lambda)} a_{w,y}(\mu) \mathrm{e}^{y\circ\mu - w\circ\mu}\, \mathrm{ch}(M(0)) \\ &= \Big(\sum_{w \in W(\lambda)/W_0(\lambda)} a_{w,y}(\mu) \mathrm{e}^{y\circ\lambda - w\circ\lambda}\Big)\, \mathrm{ch}(M(0)). \end{aligned}$$

Thus for $w \in W(\lambda)/W_0(\lambda)$ and $\mu, \mu' \in \Omega'(\lambda)$ we have $a_{w,y}(\mu) = a_{w,y}(\mu')$ for any $y \in W(\lambda)/W_0(\lambda)$ if and only if $\mathrm{ch}(L(w \circ \mu))\mathrm{e}^{-w\circ\mu} = \mathrm{ch}(L(w \circ \mu'))\mathrm{e}^{-w\circ\mu'}$. The last condition is equivalent to $\dim L(w \circ \mu)_{w\circ\mu-\xi} = \dim L(w \circ \mu')_{w\circ\mu'-\xi}$ for any $\xi \in Q^+$. Fix $w \in W(\lambda)/W_0(\lambda)$ and $\xi \in Q^+$, and consider the function

(5.8) $$F(\mu) = \dim L(w \circ \mu)_{w\circ\mu-\xi}$$

on $\Omega(\lambda)$. We have only to show that F is constant on $\Omega'(\lambda)$.

By a consideration on the contravariant forms on Verma modules we see that F is a constructible function on $\Omega(\lambda)$. In particular, it is constant on a non-empty Zariski open subset U of $\Omega(\lambda)$. Let m be the value of F on U. We have to show $F(\mu) = m$ for any $\mu \in \Omega'(\lambda)$. Let $\mu \in \Omega'(\lambda)$. By Proposition 3.9 $a_{w,y}$ is a constant function on

$$Z = \{\mu' \in \Omega'(\lambda) \,;\, \mu' - \mu \in P\}$$

for any $y \in W(\lambda)/W_0(\lambda)$. Thus we see by the above argument that F is constant on Z. Assume for the moment that

$$Z \text{ is a Zariski dense subset of } \Omega(\lambda). \tag{5.9}$$

Since $Z \cap U \neq \emptyset$, we have $F(\mu') = m$ for some $\mu' \in Z$. Since F is a constant function on Z, we have $F(\nu) = m$ for any $\nu \in Z$. In particular, we obtain $F(\mu) = m$.

It remains to show (5.9). Set

$$V = \{\xi \in \mathfrak{h}^* \,;\, (\alpha^\vee, \xi) = 0 \text{ for any } \alpha \in \Delta(\lambda)\},$$
$$V_{\mathbb{Q}} = \mathfrak{h}^*_{\mathbb{Q}} \cap V,$$
$$V_{\mathbb{Z}} = P \cap V.$$

We have $\Omega(\lambda) = \mu + V$ and $Z = \mu + V_{\mathbb{Z}}$. By the definition of V the natural morphism $\mathbb{C} \otimes_{\mathbb{Q}} V_{\mathbb{Q}} \to V$ is an isomorphism. Since $V_{\mathbb{Q}}$ is a $\mathbb{Q}$-subspace of $\mathfrak{h}^*_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} P$ we have $V_{\mathbb{Q}} \simeq \mathbb{Q} \otimes_{\mathbb{Z}} V_{\mathbb{Z}}$. Hence $V_{\mathbb{Z}}$ is a $\mathbb{Z}$-lattice of V. It follows that $Z = \mu + V_{\mathbb{Z}}$ is a Zariski dense subset of $\Omega(\lambda) = \mu + V$. □

Theorem 1.1 is already known to hold for $\lambda \in \mathfrak{h}^*_{\mathbb{Q}}$ such that $\Delta_0(\lambda) = \emptyset$ and $\{w \circ \lambda = \lambda\} = \{1\}$ by Kashiwara-Tanisaki [**14**], [**15**], and hence for any $\lambda \in \mathfrak{h}^*_{\mathbb{Q}}$ by Lemma 2.8 and Proposition 3.10. On the other hand, $\Omega'(\lambda) \cap \mathfrak{h}^*_{\mathbb{Q}} \neq \emptyset$ by Lemma 2.12. Thus the proof of Theorem 1.1 is completed in the case $\mathbb{Q}\Delta(\lambda) \ni \delta$ by virtue of Proposition 5.1.

Case 2. $\mathbb{Q}\Delta(\lambda) \not\ni \delta$.

By Lemma 2.3 $\Delta(\lambda)$ is a finite set. Thus by Lemma 2.4 there exist $x \in W$ and a proper subset J of I such that $x\Delta(\lambda) \subset \Delta_J$. We may assume that its length $\ell(x)$ is the smallest among the elements $z \in W$ satisfying $z\Delta(\lambda) \subset \Delta_J$. Choose a reduced expression $x = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_r}}$ of x. Then we have

$$(\alpha_{i_k}^\vee, s_{\alpha_{i_{k+1}}} \cdots s_{\alpha_{i_r}} \circ \lambda + \rho) \notin \mathbb{Z} \ \text{ for any } k = 1, \dots, r. \tag{5.10}$$

Indeed, if $(\alpha_{i_k}^\vee, s_{\alpha_{i_{k+1}}} \cdots s_{\alpha_{i_r}} \circ \lambda + \rho) \in \mathbb{Z}$, then we have $\beta = s_{\alpha_{i_r}} \cdots s_{\alpha_{i_{k+1}}} \alpha_{i_k} \in \Delta(\lambda)$, and hence

$$x\Delta(\lambda) = xs_\beta \Delta(\lambda) = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_{k-1}}} s_{\alpha_{i_{k+1}}} \cdots s_{\alpha_{i_r}} \Delta(\lambda).$$

This contradicts the minimality of $\ell(x)$.

Set $\lambda' = x \circ \lambda$. Then we have $x\Delta(\lambda) = \Delta(\lambda')$, $x\Delta_0(\lambda) = \Delta_0(\lambda')$ by the definition, and $x\Pi(\lambda) = \Pi(\lambda')$ by [**15**, Lemma 2.2.2]. In particular, $w \mapsto xwx^{-1}$ induces an isomorphism $W(\lambda) \to W(\lambda')$ of Coxeter groups. Moreover, by Proposition 4.3 the functor $S = S_{i_1}(a_1) \circ \cdots \circ S_{i_r}(a_r)$ with $a_k = \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_r} \circ \lambda \rangle$ induces a category equivalence $\mathbb{M}_i(\mathfrak{g}, \langle h_i, \lambda \rangle) \to \mathbb{M}_i(\mathfrak{g}, \langle h_i, \lambda' \rangle)$ and we have $S(M(w \circ \lambda)) = M(xwx^{-1} \circ \lambda')$, $S(L(w \circ \lambda)) = L(xwx^{-1} \circ \lambda')$. Thus the proof of Theorem 1.1 in the case $\mathbb{Q}\Delta(\lambda) \not\ni \delta$ is reduced to the case where $\Delta(\lambda) \subset \Delta_J$ for a proper subset J of I.

Set

$$\mathfrak{l}_J = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta_J} \mathfrak{g}_\alpha), \quad \mathfrak{n}_J^+ = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_J} \mathfrak{g}_\alpha, \quad \mathfrak{n}_J^- = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_J} \mathfrak{g}_{-\alpha}, \quad \mathfrak{p}_J = \mathfrak{l}_J \oplus \mathfrak{n}_J^+.$$

Note that we have $\dim \mathfrak{l}_J < \infty$ since J is a proper subset of I. For $\mu \in \mathfrak{h}^*$ let $M_J(\mu)$ be the Verma module for $\mathfrak{l}_J$ with highest weight μ and let $L_J(\mu)$ be its irreducible quotient. We can regard them as $\mathfrak{p}_J$-modules with trivial actions of $\mathfrak{n}_J^+$. By the definition we have $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_J)} M_J(\mu) \simeq M(\mu)$ for any $\mu \in \mathfrak{h}^*$. Hence Theorem 1.1

in the case $\mathbb{Q}\Delta(\lambda) \not\ni \delta$ follows from the character formula for the irreducible highest weight modules over finite-dimensional semisimple Lie algebras, which is already known (see the comments at the end), and the following result.

LEMMA 5.2. *For any $\lambda \in \mathcal{C}$ satisfying $\Delta(\lambda) \subset \Delta_J$ we have $U(\mathfrak{g})\otimes_{U(\mathfrak{p}_J)} L_J(\lambda) \simeq L(\lambda)$.*

PROOF. Set $M = U(\mathfrak{g})\otimes_{U(\mathfrak{p}_J)} L_J(\lambda)$. It is a highest weight module with highest weight λ. Set $M^{\mathfrak{n}^+} = \{m \in M\,;\, \mathfrak{n}^+ m = 0\}$. It is sufficient to show $M^{\mathfrak{n}^+} \cap M_{\lambda-\xi} = 0$ for any $\xi \in Q^+ \setminus \{0\}$. Assume that $M^{\mathfrak{n}^+} \cap M_{\lambda-\xi} \neq \{0\}$ for some $\xi \in Q^+ \setminus \{0\}$. By $\Delta(\lambda) \subset \Delta_J$ and Proposition 3.1 we have $\xi \in \sum_{\alpha\in\Delta_J} \mathbb{Z}\alpha$. Hence under the isomorphism $M \simeq U(\mathfrak{n}_J^-) \otimes_{\mathbb{C}} L_J(\lambda)$ we have $M_{\lambda-\xi} = 1 \otimes L_J(\lambda)_{\lambda-\xi}$. It follows that $L_J(\lambda)_{\lambda-\xi} \cap L_J(\lambda)^{\mathfrak{n}^+\cap\mathfrak{l}_J} \neq \{0\}$. This contradicts the irreducibility of $L_J(\lambda)$. □

The proof of Theorem 1.1 is complete in the case $\mathbb{Q}\Delta(\lambda) \not\ni \delta$.

We finally give comments on the proof of the character formula for the irreducible highest weight modules over finite-dimensional semisimple Lie algebras which we have used in our proof in Case 2. The unpublished result in the rational highest weight case due to Beilinson-Bernstein (in particular, the part relating some twisted D-modules with the twisted intersection cohomology groups of the Schubert varieties) is recovered as a special case of the result in Kashiwara-Tanisaki [**14**] (and also of the result in Kashiwara-Tanisaki [**15**]). The proof of Bernstein's result reducing the general case to the rational highest weight case is exactly the same as the one presented in this section in the case $\mathbb{Q}\Delta(\lambda) \ni \delta$.

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