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Crystal Graphs for Representations of the q-Analogue of Classical Lie Algebras

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The explicit form of the crystal graphs for the finite-dimensional representations of the q-analogue of the universal enveloping algebras of type A, B, C, and D is given in terms of semi-standard tableaux and its analogues.

Contents.

0. INTRODUCTION

0.1. In [K1], the notion of a crystal base is introduced and its existence and uniqueness are proved in the case of A_n, B_n, C_n and D_n. This result is generalized to any integrable representation with highest weight of the q-analogue of arbitrary symmetrizable Kac–Moody Lie algebra [K2]. This is also studied by G. Lusztig [L1, L2] in a different point of view.

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In this paper, we give their explicit description for A_n, B_n, C_n and D_n. In particular, in the A_n-case, the crystal bases are labelled by the semi-standard tableaux.

0.2. We explain here the labelling of crystal bases used in our article. Since the other cases are similar, we explain this by taking A_n as an example. Let $\alpha_1, \cdots, \alpha_n$ be the simple roots of A_n such that $(\alpha_i, \alpha_i) = 1$ and $(\alpha_i, \alpha_j) = -1/2$ for $|i-j| = 1$ and let Λ_i be the fundamental weights.

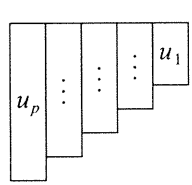
(1) The crystal graph $B(\Lambda_1)$ of the vector representation $V(\Lambda_1)$ is easily obtained by the explicit construction of $V(\Lambda_1)$. They are labelled by $\boxed{i} \in \left\{ \boxed{i} \mid 1 \leqslant i \leqslant n+1 \right\}$, and the crystal graph is

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{n+1}\,.$$

(2) Let $V(\Lambda_N)$ be the irreducible representation with the fundamental weight Λ_N $(1 \leqslant N \leqslant n)$ as highest weight. We embed $V(\Lambda_N)$ into $V(\Lambda_1)^{\otimes N}$. Accordingly, $B(\Lambda_N)$ is embedded into $B(\Lambda_1)^{\otimes N}$. Then we see that $B(\Lambda_N)$ consists of $\boxed{i_1} \otimes \cdots \otimes \boxed{i_N}$ with $1 \leqslant i_1 < \cdots < i_N \leqslant n+1$. We write

$$\begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \quad \text{for} \quad \boxed{i_1} \otimes \cdots \otimes \boxed{i_N}\,.$$

(3) In general, for $\lambda = \sum_{i=1}^{p} \Lambda_{m_i}$, with $1 \leqslant m_1 \leqslant \cdots \leqslant m_p \leqslant n$, let $Y(\lambda)$ be the corresponding Young diagram, namely the Young diagram with p columns with $m_1, ..., m_p$ as their lengths. By the embedding of $V(\lambda)$ into $V(\Lambda_{m_1}) \otimes \cdots \otimes V(\Lambda_{m_p})$, $B(\lambda)$ is also embedded into $B(\Lambda_{m_1}) \otimes \cdots \otimes B(\Lambda_{m_p})$. We write for an element $u_1 \otimes \cdots \otimes u_p$ in $B(\Lambda_{m_1}) \otimes \cdots \otimes B(\Lambda_{m_p})$ the Young tableaux



THEOREM. *$B(\lambda)$ coincides with the set of semi-standard tableaux with shape $Y(\lambda)$.*

This statement is proved by using the relation,

$$B(\lambda) \cong \bigcap_{i=1}^{p-1} B(\Lambda_{m_1}) \otimes \cdots \otimes B(\Lambda_{m_{i-1}}) \otimes B(\Lambda_{m_i} + \Lambda_{m_{i+1}}) \otimes B(\Lambda_{m_{i+2}}) \otimes \cdots \otimes B(\Lambda_{m_p}),$$

which permits the reduction to the case $\lambda = \Lambda_i + \Lambda_j$.

0.3. The labelling of crystal bases in the C_n-case is similar to the A_n-case. Let $(\varepsilon_1, ..., \varepsilon_n)$ be the orthonormal base of the dual of the Cartan subalgebra of C_n such that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leqslant i < n)$ and $\alpha_n = 2\varepsilon_n$ are simple roots. Hence, α_n is the long root and $\alpha_1, ..., \alpha_{n-1}$ are short roots. Let $\{\Lambda_i\}_{1 \leqslant i \leqslant n}$ be the dual base of $\{h_i\}_{1 \leqslant i \leqslant n}$. Hence $\Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i$ $(1 \leqslant i \leqslant n)$.

Then the crystal graph of $V(\Lambda_1)$ is given by

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}.$$

Here, $\boxed{j}$ has weight ε_j and $\boxed{\bar{j}}$ has weight $-\varepsilon_j$.

The representation $V(\Lambda_N)$ with highest weight Λ_N $(1 \leqslant N \leqslant n)$ is embedded into $V(\Lambda_1)^{\otimes N}$. We give the linear order $\prec$ on $\{i, \bar{i}; 1 \leqslant i \leqslant n\}$ by $1 \prec 2 \prec \cdots n \prec \bar{n} \prec \cdots \prec \bar{2} \prec \bar{1}$. Similarly to the A_n-case, the crystal base of $B(\Lambda_N)$ is given as

$$B(\Lambda_N) = \left\{ \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \in B(\Lambda_1)^{\otimes N};\ \begin{array}{l} (1)\ 1 \preccurlyeq i_1 \prec \cdots \prec i_N \preccurlyeq \bar{1}, \\ (2)\ \text{if } i_k = p \text{ and } i_l = \bar{p}, \text{ then } k + (N - l + 1) \leqslant p \end{array} \right\}.$$

By embedding $V(\Lambda_M + \Lambda_N)$ into $V(\Lambda_M) \otimes V(\Lambda_N)$, $B(\Lambda_M + \Lambda_N)$ is the connected component of $B(\Lambda_M) \otimes B(\Lambda_N)$ containing $u_{\Lambda_M} \otimes u_{\Lambda_N}$.

For

$$u = \begin{array}{|c|} \hline j_1 \\ \hline \vdots \\ \hline j_M \\ \hline \end{array} \in B(\Lambda_M) \text{ and } v = \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \in B(\Lambda_N),\ u \otimes v \text{ will be denoted by } \begin{array}{|c|c|} \hline i_1 & j_1 \\ \hline \vdots & \vdots \\ & j_M \\ \hline i_N & \\ \hline \end{array}.$$

Definition. For $1 \leqslant a \leqslant b \leqslant n$ and u, v as above, we say that $u \otimes v \in B(\Lambda_M) \otimes B(\Lambda_N)$ is in the (a, b)-configuration if $u \otimes v$ satisfies the following: There exist $1 \leqslant p \leqslant q < r \leqslant s \leqslant M$ such that $i_p = a$, $j_q = b$, $j_r = \bar{b}$, $j_s = \bar{a}$ or $i_p = a$, $i_q = b$, $i_r = \bar{b}$, $j_s = \bar{a}$.

This definition includes the case $a=b$, $p=q$, and $r=s$. Now, we define $p(a, b;\ u\otimes v)=(q-p)+(s-r)$. *Then we have*

$$B(\Lambda_M+\Lambda_N)=\left\{ w=\begin{array}{|c|c|}\hline i_1 & j_1 \\ \hline \vdots & \vdots \\ & j_M \\ \hline i_N & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \in B(\Lambda_M)\otimes B(\Lambda_N);\ w \text{ satisfies the conditions (M.N.1) and (M.N.2)} \right\}.$$

(M.N.1) $\quad i_k \preccurlyeq j_k$ for $1\leqslant k\leqslant M$.

(M.N.2) $\quad$ if w is in the (a, b)-configuration, then $p(a, b;\ w)<b-a$.

In general, for a dominant integral weight $\lambda=\sum_{i=1}^{p}\Lambda_{l_i}$ $(1\leqslant l_1\leqslant l_2\leqslant\cdots\leqslant l_p\leqslant n)$, an element

$$u_1\otimes\cdots\otimes u_p=\begin{array}{|c|c|c|c|c|}\hline & & & \vdots & t_l^1 \\ \cline{5-5} & \vdots & \vdots & & \multicolumn{1}{c}{} \\ \cline{4-4} & & & \multicolumn{2}{c}{} \\ \cline{3-3} t_{l'}^p & \vdots & \multicolumn{3}{c}{} \\ \cline{2-2} & \multicolumn{4}{c}{} \\ \cline{1-1}\end{array} \in B(\Lambda_{l_1})\otimes\cdots\otimes B(\Lambda_{l_p})$$

is called a *semi-standard C-tableau* of shape λ if $u_k\otimes u_{k+1}\in B(\Lambda_{l_k}+\Lambda_{l_{k+1}})$ for any k.

THEOREM. *The crystal graph $B(\lambda)$ of the irreducible $U_q(C_n)$-module $V(\lambda)$ with highest weight λ coincides with the set of semi-standard C-tableaux of shape λ.*

For the cases B_n and D_n, see Theorem 5.7.1 and Theorem 6.7.1. Note that analogues of semi-standard tableaux are already known (e.g., cf. [K-E, S]). But ours are different from theirs.

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1. CRYSTAL BASE

1.1. *Definitions*

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra with a Cartan subalgebra $\mathfrak{t}$, the set of simple roots $\{\alpha_i\in\mathfrak{t}^*\}_{i\in I}$, and the set of simple coroots

$\{h_i \in \mathfrak{t}\}_{i\in I}$, where I is a finite index set. We take an inner product $(\ ,\ )$ on $\mathfrak{t}^*$ such that $(\alpha_i, \alpha_i) \in \mathbf{Z}_{>0}$ and $\langle h_i, \lambda\rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for $\lambda \in \mathfrak{t}^*$. Let $\{\Lambda_i\}_{i\in I}$ be the dual base of $\{h_i\}$ and set $P = \sum \mathbf{Z}\Lambda_i$ and $P^* = \sum \mathbf{Z}h_i$. Then the q-analogue $U_q(\mathfrak{g})$ is the algebra over $\mathbf{Q}(q)$ generated by e_i, f_i, and q^h $(h \in P^*)$ satisfying the relations

$$q^h = 1 \quad \text{if} \quad h = 0 \quad \text{and} \quad q^h q^{h'} = q^{h+h'}, \tag{1.1.1}$$

$$\begin{aligned} q^h e_j q^{-h} &= q^{\langle h, \alpha_j\rangle} e_j, \\ q^h f_j q^{-h} &= q^{-\langle h, \alpha_j\rangle} f_j, \end{aligned} \tag{1.1.2}$$

$$[e_i, f_j] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}, \quad \text{where} \quad q_i = q^{(\alpha_i, \alpha_i)} \text{ and } t_i = q^{(\alpha_i, \alpha_i)h_i}. \tag{1.1.3}$$

For $i \neq j$, we have, setting $b = 1 - \langle h_i, \alpha_j\rangle$,

$$\sum_{\mu=0}^{b} e_i^{(\mu)} e_j e_i^{(b-\mu)} = \sum_{\mu=0}^{b} f_i^{(\mu)} f_j f_i^{(b-\mu)} = 0. \tag{1.1.4}$$

Here $e_i^{(k)} = e_i^k/[k]_i!$, $f_i^{(k)} = f_i^k/[k]_i!$, $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$ and $[k]_i! = \prod_{n=1}^k [n]_i$.

For a finite-dimensional $U_q(\mathfrak{g})$-module M and $\lambda \in P$, we set $M_\lambda = \{u \in M; q^h u = q^{\langle h, \lambda\rangle} u\}$. We call M integrable if $M = \bigoplus M_\lambda$. Then we have

$$M_\lambda = \bigoplus_{k\geq 0,\ -\langle h_i, \lambda\rangle} f_i^{(k)}(M_{\lambda + k\alpha_i} \cap \operatorname{Ker} e_i). \tag{1.1.5}$$

We define the operators $\tilde{e}_i$, $\tilde{f}_i$ acting on M by

$$\tilde{e}_i f_i^{(k)} u = f_i^{(k-1)} u \quad \text{and} \quad \tilde{f}_i f_i^{(k)} u = f_i^{(k+1)} u, \tag{1.1.6}$$

for $u \in M_\lambda \cap \operatorname{Ker} e_i$ and (λ, k) as above.

DEFINITION 1.1.1. Let A be the ring of rational functions regular at $q = 0$. A pair (L, B) is called a crystal base of a finite-dimensional integrable representation M if the following conditions are satisfied:

(1) L is a free sub-A-module of M such that $\mathbf{Q}(q) \otimes_A L \cong M$,
(2) B is a base of the $\mathbf{Q}$-vector space L/qL,
(3) $L = \bigoplus L_\lambda$, $B = \coprod B_\lambda$, where $L_\lambda = L \cap M_\lambda$ and $B_\lambda = B \cap (L_\lambda/qL_\lambda)$,
(4) $\tilde{f}_i L \subset L$, and $\tilde{e}_i L \subset L$,
(5) $\tilde{f}_i B \subset B \cup \{0\}$ and $\tilde{e}_i B \subset B \cup \{0\}$,
(6) For $u, v \in B$ and $i \in I$, $u = \tilde{e}_i v$ if and only if $v = \tilde{f}_i u$.

Then the following results are proved in [K1] for $\mathfrak{g}=A_n$, B_n, C_n and D_n and in [K2] in the general case. Let $\lambda \in P_+ = \{\lambda \in t^*; \langle h_i, \lambda\rangle \in \mathbf{Z}_{\geqslant 0}\}$ and $V(\lambda)$ be the irreducible integrable $U_q(\mathfrak{g})$-module generated by the highest weight vector u_λ of weight λ. Let $L(\lambda)$ be the sub-A-module generated by the vectors in the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\lambda$ and let $B(\lambda)$ be the subset of $L(\lambda)/qL(\lambda)$ consisting of the non-zero vectors in the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} u_\lambda$ mod $qL(\lambda)$.

THEOREM 1.1.2. *$(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$.*

THEOREM 1.1.3. *If (L, B) is a crystal base of an integrable $U_q(\mathfrak{g})$-module M, then there is an isomorphism*

$$M \cong \bigoplus_j V(\lambda_j) \qquad \text{by which} \quad (L, B) \cong \bigoplus_j (L(\lambda_j), B(\lambda_j)).$$

THEOREM 1.1.4. *Let (L_j, B_j) be a crystal base of an integrable $U_q(\mathfrak{g})$-module M_j $(j=1, 2)$. Set $L = L_1 \otimes_A L_2 \subset M_1 \otimes M_2$ and $B = \{b_1 \otimes b_2; b_j \in B_j$ $(j=1, 2)\} \subset L/qL$. Then we have*

(1) *(L, B) is a crystal base of $M_1 \otimes M_2$.*

(2) $$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if} \quad \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if} \quad \varphi_i(b_1) \leqslant \varepsilon_i(b_2), \end{cases}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{e}_i b_2 & \text{if} \quad \varphi_i(b_1) < \varepsilon_i(b_2), \\ \tilde{e}_i b_1 \otimes b_2 & \text{if} \quad \varphi_i(b_1) \geqslant \varepsilon_i(b_2). \end{cases}$$

Here, $\varepsilon_i(b) = \max\{k \geqslant 0; \tilde{e}_i^k b \neq 0\}$ and $\varphi_i(b) = \max\{k \geqslant 0; \tilde{f}_i^k b \neq 0\}$.

COROLLARY 1.1.5. *For $b_j \in B_j$ $(j=1, 2)$,*

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) + \varepsilon_i(b_1) - \varphi_i(b_1)),$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \varphi_i(b_2) - \varepsilon_i(b_2)).$$

DEFINITION 1.1.6. The crystal graph of a crystal base (L, B) is the colored and oriented graph B, with the arrows

$$u \xrightarrow{i} v \qquad \text{if and only if} \quad v = \tilde{f}_i u.$$

EXAMPLE 1.1.7. Let $\mathfrak{g} = \mathfrak{sl}_2$ and let V_l be the $(l+1)$-dimensional irreducible integrable $U_q(\mathfrak{sl}_2)$-module.

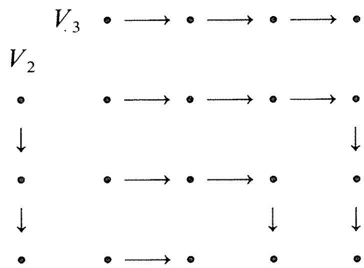
(i) The crystal graph of V_2 is given as

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

(ii) The crystal graph of V_3 is given as

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

(iii) By Theorem 1.1.4, the crystal graph of $V_3 \otimes V_2$ is visualized as



2. Remarks on Crystal Graphs

2.1. *The Actions of $\tilde{e}_i$ and $\tilde{f}_i$ on Tensor Products*

Let us investigate the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on the tensor product of several crystal bases.

Let (L_j, B_j) be a crystal base of an integrable $U_q(\mathfrak{g})$-module M_j $(j = 1, \ldots, N)$.

Proposition 2.1.1. *For $i \in I$ and $b_j \in B_j$ $(j = 1, \ldots, N)$, we set*

$$a_k = \sum_{1 \leqslant j < k} (\varphi_i(b_j) - \varepsilon_i(b_{j+1})) \qquad \textit{for} \quad 1 \leqslant k \leqslant N. \tag{2.1.1}$$

Remark that $a_1 = 0$.

(0) *We have*

$$\varepsilon_i(b_1 \otimes \cdots \otimes b_N) = \varepsilon_i(b_1) - \min\{a_k; 1 \leqslant k \leqslant N\}$$
$$= \max\left\{\sum_{1 \leqslant j \leqslant k} \varepsilon_i(b_j) - \sum_{1 \leqslant j < k} \varphi_i(b_j); 1 \leqslant k \leqslant N\right\},$$

$$\varphi_i(b_1 \otimes \cdots \otimes b_N) = \max\left\{\varphi_i(b_N) + \sum_{k \leqslant j < N} (\varphi_i(b_j) - \varepsilon_i(b_{j+1})); 1 \leqslant k \leqslant N\right\}.$$

(i) *If $a_\nu \geqslant a_k$ for $1 \leqslant \nu < k$ and $a_\nu > a_k$ for $k < \nu \leqslant N$ (i.e., k is the largest element such that $a_k = \min\{a_j; 1 \leqslant j \leqslant N\}$) then we have*

$$\tilde{f}_i(b_1 \otimes \cdots \otimes b_N) = b_1 \otimes \cdots \otimes b_{k-1} \otimes \tilde{f}_i b_k \otimes b_{k+1} \otimes \cdots \otimes b_N.$$

(ii) *If* $a_\nu > a_k$ *for* $1 \leqslant \nu < k$ *and* $a_\nu \geqslant a_k$ *for* $k < \nu \leqslant N$ (*i.e.*, k *is the smallest element such that* $a_k = \min\{a_j;\ 1 \leqslant j \leqslant N\}$) *then we have*

$$\tilde{e}_i(b_1 \otimes \cdots \otimes b_N) = b_1 \otimes \cdots \otimes b_{k-1} \otimes \tilde{e}_i b_k \otimes b_{k+1} \otimes \cdots \otimes b_N.$$

Proof. The proof being similar, we only give the proof of (0) and (i).

Let us prove (i) by the induction on N. Let us take k_1 such that $a_\nu \geqslant a_{k_1}$ for $2 \leqslant \nu < k_1$ and $a_\nu > a_{k_1}$ for $k_1 < \nu \leqslant N$. Then, by the hypothesis of the induction we obtain

$$\tilde{f}_i(b_2 \otimes \cdots \otimes b_N) = b_2 \otimes \cdots \otimes \tilde{f}_i b_{k_1} \otimes \cdots \otimes b_N,$$

$$\varepsilon_i(b_2 \otimes \cdots \otimes b_N) = \varepsilon_i(b_2) - \sum_{2 \leqslant j < k_1} (\varphi_i(b_j) - \varepsilon_i(b_{j+1})) = \varphi_i(b_1) - a_{k_1}.$$

Hence

$$\tilde{f}_i(b_1 \otimes \cdots \otimes b_N) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 \otimes \cdots \otimes b_N & \text{if} \quad a_{k_1} > 0, \\ b_1 \otimes \tilde{f}_i(b_2 \otimes \cdots \otimes b_N) & \text{if} \quad a_{k_1} \leqslant 0. \end{cases}$$

Since $k = 1$ or k_1 according to $a_{k_1} > 0$ or $a_{k_1} \leqslant 0$, we obtain (i) and $a_k = \min(0, a_{k_1})$. Then we have

$$\begin{aligned} \varepsilon_i(b_1 \otimes \cdots \otimes b_N) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_2 \otimes \cdots \otimes b_N) + \varepsilon_i(b_1) - \varphi_i(b_1)) \\ &= \varepsilon_i(b_1) + \max(0, -a_{k_1}) \\ &= \varepsilon_i(b_1) - a_k. \end{aligned}$$

Finally we have

$$\begin{aligned} &\varphi_i(b_1 \otimes \cdots \otimes b_N) \\ &\quad = \max(\varphi_i(b_N), \varphi_i(b_1 \otimes \cdots \otimes b_{N-1}) + \varphi_i(b_N) - \varepsilon_i(b_N)) \\ &\quad = \max\Bigg(\varphi_i(b_N), \max\Bigg\{\varphi_i(b_{N-1}) + \varphi_i(b_N) - \varepsilon_i(b_N) \\ &\qquad + \sum_{k \leqslant j < N-1} (\varphi_i(b_j) - \varepsilon_i(b_{j+1}));\ 1 \leqslant k \leqslant N-1\Bigg\}\Bigg). \\ &\quad = \max\Bigg\{\varphi_i(b_N) + \sum_{k \leqslant j < N} (\varphi_i(b_j) - \varepsilon_i(b_{j+1}));\ 1 \leqslant k \leqslant N\Bigg\}. \end{aligned}$$

Q.E.D.

Remark 2.1.2. We can restate the preceding proposition when b_j is either u_+, u_-, u_0, where $u_+ \xrightarrow{i} u_-$, $\tilde{e}_i u_+ = \tilde{f}_i u_- = \tilde{e}_i u_0 = \tilde{f}_i u_0 = 0$.

(0) We neglect u_0.

(1) If there is $u_+ \otimes u_-$ in $u = b_1 \otimes \cdots \otimes b_N$, then we neglect such a pair. We continue this procedure as far as we can.

(2) Then $\tilde{e}_i$ changes u_- in the rightmost to u_+ and $\tilde{f}_i$ changes u_+ in the leftmost to u_-. If there is no such u_- (resp. u_+), then $\tilde{e}_i u = 0$ (resp. $\tilde{f}_i u = 0$).

Example. For

$$u = u_- \otimes u_0 \otimes \underbrace{u_+ \otimes \overbrace{u_+ \otimes u_-} \otimes u_-} \otimes u_+,$$

$$\tilde{e}_i u = u_+ \otimes u_0 \otimes u_+ \otimes u_+ \otimes u_- \otimes u_- \otimes u_+,$$

$$\tilde{f}_i u = u_- \otimes u_0 \otimes u_+ \otimes u_+ \otimes u_- \otimes u_- \otimes u_-,$$

$$\tilde{e}_i^2 u = \tilde{f}_i^2 u = 0.$$

Remark 2.1.3. In the case when b_j is either u_0, u_1, u_2, u_3 where $u_1 \xrightarrow{i} u_2 \xrightarrow{i} u_3$, $\tilde{e}_i u_1 = \tilde{f}_i u_3 = \tilde{e}_i u_0 = \tilde{f}_i u_0 = 0$, we can describe the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on their tensor products by the identifications

$$u_1 \leftrightarrow u_+ \otimes u_+, \qquad u_2 \leftrightarrow u_- \otimes u_+, \qquad u_3 \leftrightarrow u_- \otimes u_-, \tag{2.1.2}$$

since we have the crystal graph

$$u_+ \otimes u_+ \xrightarrow{\ i\ } u_- \otimes u_+ \xrightarrow{\ i\ } u_- \otimes u_-. \tag{2.1.3}$$

Example. For $u = u_2 \otimes u_1 \otimes u_3 \otimes u_0 \otimes u_3 \otimes u_0 \otimes u_+$, by (2.1.2), u can be identified with

$$(u_- \otimes u_+) \otimes (u_+ \otimes u_+) \otimes (u_- \otimes u_-) \otimes u_0 \otimes (u_- \otimes u_-) \otimes u_0 \otimes u_+.$$

By Remark 2.1.2, we obtain

$$\begin{aligned} &\tilde{e}_i((u_- \otimes u_+) \otimes (u_+ \otimes u_+) \otimes (u_- \otimes u_-) \otimes u_0 \otimes (u_- \otimes u_-) \otimes u_0 \otimes u_+) \\ &\qquad = (u_- \otimes u_+) \otimes (u_+ \otimes u_+) \otimes (u_- \otimes u_-) \otimes u_0 \otimes (u_- \otimes u_+) \otimes u_0 \otimes u_+, \\ &\tilde{f}_i((u_- \otimes u_+) \otimes (u_+ \otimes u_+) \otimes (u_- \otimes u_-) \otimes u_0 \otimes (u_- \otimes u_-) \otimes u_0 \otimes u_+) \\ &\qquad = (u_- \otimes u_+) \otimes (u_+ \otimes u_+) \otimes (u_- \otimes u_-) \otimes u_0 \otimes (u_- \otimes u_-) \otimes u_0 \otimes u_-. \end{aligned}$$

Hence

$$\begin{aligned} \tilde{e}_i(u_2 \otimes u_1 \otimes u_3 \otimes u_0 \otimes u_3 \otimes u_0 \otimes u_+) &= u_2 \otimes u_1 \otimes u_3 \otimes u_0 \otimes u_2 \otimes u_0 \otimes u_+, \\ \tilde{f}_i(u_2 \otimes u_1 \otimes u_3 \otimes u_0 \otimes u_3 \otimes u_0 \otimes u_+) &= u_2 \otimes u_1 \otimes u_3 \otimes u_0 \otimes u_3 \otimes u_0 \otimes u_-. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\tilde{e}_i^2((u_- \otimes u_+) \otimes (u_+ \otimes u_+) \otimes (u_- \otimes u_-) \otimes u_0 \otimes (u_- \otimes u_-) \otimes u_0 \otimes u_+) \\ &\qquad = (u_+ \otimes u_+) \otimes (u_+ \otimes u_+) \otimes (u_- \otimes u_-) \otimes u_0 \otimes (u_- \otimes u_+) \otimes u_0 \otimes u_+, \\ &\tilde{f}_i^2((u_- \otimes u_+) \otimes (u_+ \otimes u_+) \otimes (u_- \otimes u_-) \otimes u_0 \otimes (u_- \otimes u_-) \otimes u_0 \otimes u_+) = 0. \end{aligned}$$

Hence,

$$\tilde{e}_i^2(u_2 \otimes u_1 \otimes u_3 \otimes u_0 \otimes u_3 \otimes u_0 \otimes u_+) = u_1 \otimes u_1 \otimes u_3 \otimes u_0 \otimes u_2 \otimes u_0 \otimes u_+,$$

$$\tilde{f}_i^2(u_2 \otimes u_1 \otimes u_3 \otimes u_0 \otimes u_3 \otimes u_0 \otimes u_+) = 0.$$

2.2. Now let $U_q(\mathfrak{g})$ be the q-analogue as in Section 1 and let $\lambda_1, ..., \lambda_N \in P_+$ and $\lambda = \sum_i \lambda_i$. Then there is a unique embedding $V(\lambda) \subsetneq V(\lambda_1) \otimes \cdots \otimes V(\lambda_N)$ sending u_λ to $u_{\lambda_1} \otimes \cdots \otimes u_{\lambda_N}$. Hence $B(\lambda)$ is embedded into $\bigotimes_{j=1}^N B(\lambda_j)$.

PROPOSITION 2.2.1. *Assume the following condition for any k $(1 \leqslant k < N)$. If $u \in B(\lambda_{k+1})$ satisfies*

$$\begin{aligned} &\text{(i)} \quad u_{\lambda_k} \otimes u \in B(\lambda_k + \lambda_{k+1}) \text{ and} \\ &\text{(ii)} \quad \tilde{e}_i u = 0 \text{ for any } i \text{ such that } \langle h_i, \lambda_\nu \rangle = 0 \text{ for } \nu \leqslant k, \\ &\text{then } u = u_{\lambda_{k+1}}. \end{aligned} \tag{2.2.1}$$

Then we have

$$V(\lambda) \cong \bigcap_{k=1}^{N-1} V(\lambda_1) \otimes \cdots \otimes V(\lambda_{k-1}) \otimes V(\lambda_k + \lambda_{k+1}) \otimes V(\lambda_{k+2}) \otimes \cdots \otimes V(\lambda_N), \tag{2.2.2}$$

$$B(\lambda) \cong \bigcap_{k=1}^{N-1} B(\lambda_1) \otimes \cdots \otimes B(\lambda_{k-1}) \otimes B(\lambda_k + \lambda_{k+1}) \otimes B(\lambda_{k+2}) \otimes \cdots \otimes B(\lambda_N). \tag{2.2.3}$$

Proof. Let W be the right hand side of (2.2.2), and $H = \{u \in W; e_i u = 0$ for any $i\}$. In order to prove (2.2.2) it is enough to show

$$(H \cap L)/q(H \cap L) = \mathbf{Q}(u_{\lambda_1} \otimes \cdots \otimes u_{\lambda_N}). \tag{2.2.4}$$

In fact, (2.2.4) implies $H = V(\lambda)_\lambda$ and hence $W = V(\lambda)$. Let $v \in (H \cap L)/q(H \cap L) \subset L/qL = \bigotimes_i (L(\lambda_i)/qL(\lambda_i))$. Then, $\tilde{e}_i v = 0$ for any i. On the other hand for any k,

$$v \in (L(\lambda_1)/qL(\lambda_1)) \otimes \cdots \otimes (L(\lambda_k + \lambda_{k+1})/qL(\lambda_k + \lambda_{k+1})) \otimes \cdots \otimes (L(\lambda_N)/qL(\lambda_N)).$$

Hence when we write v as a linear combination of vectors in $B(\lambda_1)\otimes \cdots \otimes B(\lambda_N)$, any component belongs to $B(\lambda_1)\otimes \cdots \otimes B(\lambda_k+\lambda_{k+1})\otimes \cdots \otimes B(\lambda_N)$ for any k, and it is annihilated by all $\tilde{e}_i$. Hence in order to prove (2.2.4) it is enough to show

If $u\in \bigcap_{k=1}^{N-1} B(\lambda_1)\otimes \cdots \otimes B(\lambda_k+\lambda_{k+1})\otimes \cdots \otimes B(\lambda_N)$ is annihilated by all $\tilde{e}_i$ then $u=u_{\lambda_1}\otimes \cdots \otimes u_{\lambda_N}$. (2.2.5)

We prove this by induction on N. Writing $u=v\otimes w$ with $v\in B(\lambda_1)\otimes \cdots \otimes B(\lambda_{N-1})$ and $w\in B(\lambda_N)$, we have $\tilde{e}_i v=0$. Hence, $v=u_{\lambda_1}\otimes \cdots \otimes u_{\lambda_{N-1}}$ by the hypothesis of the induction. If i satisfies $\langle h_i, \lambda_\nu\rangle=0$ for $\nu\leqslant N-1$, then $\tilde{e}_i v=\tilde{f}_i v=0$ and hence $\tilde{e}_i w=0$. Since $u_{\lambda_{N-1}}\otimes w$ belongs to $B(\lambda_{N-1}+\lambda_N)$, (2.2.1) implies $w=u_{\lambda_N}$. The property (2.2.3) follows immediately from (2.2.5). Q.E.D.

3. Crystal Graphs for $U_q(A_n)$-Modules

3.1. *Notations*

We define $U_q(A_n)$. Let $\mathfrak{t}=\bigoplus_{i=1}^n \mathbf{Q}h_i$ and let $\{\Lambda_i\in \mathfrak{t}^*;\ i=1, ..., n\}$ be the dual base. We set $\varepsilon_i=\Lambda_i-\Lambda_{i-1}$ for $2\leqslant i\leqslant n$ and $\varepsilon_1=\Lambda_1$ and $\varepsilon_{n+1}=-(\varepsilon_1+\cdots+\varepsilon_n)$. Define $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$. We give the inner product (,) of $\mathfrak{t}^*$ by

$$(\alpha_i, \alpha_j)=\begin{cases}1, & i=j,\\ -1/2, & |i-j|=1,\\ 0, & |i-j|>1.\end{cases} \tag{3.1.1}$$

Then (3.1.1) is satisfied and we define $U_q(A_n)$ as in Section 1 from these data by (1.1.1)–(1.1.4).

3.2. *The Crystal Graph of the Vector Representation*

First let us construct the vector representation $V_\square$. Let $V_\square$ be the $(n+1)$-dimensional $\mathbf{Q}(q)$-vector space with $\{\boxed{i};\ i=1, ..., n+1\}$ as a basis. We give the $U_q(A_n)$-module structure on $V_\square$ as

$$\begin{aligned} q^h\boxed{j}&=q^{\langle h,\varepsilon_j\rangle}\boxed{j},\\ e_i\boxed{j}&=\delta_{i,j-1}\boxed{j-1} \qquad (1\leqslant i\leqslant n,\ 1\leqslant j\leqslant n+1)\\ f_i\boxed{j}&=\delta_{i,j}\boxed{j+1}. \end{aligned} \tag{3.2.1}$$

Here we understand $\boxed{j}=0$ unless $1 \leqslant j \leqslant n+1$. Then $\boxed{1}$ is the highest weight vector of $V_{\square}$ with highest weight Λ_1. The crystal base $(L(V_{\square}), B(V_{\square}))$ of $V_{\square}$ is given by

$$L(V_{\square}) = \bigoplus_{i=1}^{n+1} A\boxed{i}, \qquad B(V_{\square}) = \left\{\boxed{i} \bmod qL(V_{\square});\ 1 \leqslant i \leqslant n+1\right\}.$$

Note that

$$\tilde{e}_i\boxed{j} = \delta_{i,j-1}\boxed{j-1}, \qquad \text{and} \qquad \tilde{f}_i\boxed{j} = \delta_{i,j}\boxed{j+1}. \tag{3.2.2}$$

Therefore the crystal graph $B(V_{\square})$ of $V_{\square}$ is given by

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{n+1}. \tag{3.2.3}$$

3.3. *The Crystal Graph of the Fundamental Representations*

For $1 \leqslant N \leqslant n$, the representation $V(\Lambda_N)$ appears once in $V_{\square}^{\otimes N}$. Hence, $B(\Lambda_N)$ is contained in $B(V_{\square})^{\otimes N}$ in a unique way. The base $u_{\Lambda_N} = \boxed{1} \otimes \boxed{2} \otimes \cdots \otimes \boxed{N}$ is annihilated by all $\tilde{e}_i$ and it has weight $\Lambda_N = \varepsilon_1 + \cdots + \varepsilon_N$, the crystal graph $B(\Lambda_N)$ is the connected component of $B(V_{\square})^{\otimes N}$ containing $\boxed{1} \otimes \cdots \otimes \boxed{N}$. We write

$$\begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_N \\ \hline\end{array} \quad \text{for} \quad \boxed{i_1} \otimes \cdots \otimes \boxed{i_N}.$$

PROPOSITION 3.3.1. (i) *By the identification above,*

$$B(\Lambda_N) = \left\{ u = \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_N \\ \hline\end{array}\ ;\ 1 \leqslant i_1 < \cdots < i_N \leqslant n+1 \right\}. \tag{3.3.1}$$

(ii) *The action of $\tilde{e}_i$ and $\tilde{f}_i$ on $u \in B(\Lambda_N)$ is given as follows.*

(a) *If i appears in u and $i+1$ does not, then $\tilde{f}_i u$ is the one obtained by replacing i with $i+1$. Otherwise, $\tilde{f}_i u = 0$.*

(b) *If $i+1$ appears in u and i does not, then $\tilde{e}_i u$ is the one obtained by replacing $i+1$ with i. Otherwise, $\tilde{e}_i u = 0$.*

Proof. Let I be the right hand side of (3.3.1). Then, for $u \in I$, the rules in (ii) follow from Remark 2.1.2. Then, one can check easily that $I \cup \{0\}$ is stable by $\tilde{e}_i$ and $\tilde{f}_i$. Hence it remains to prove that if $u = \boxed{i_1} \otimes \cdots \otimes \boxed{i_N} \in I$ satisfies $\tilde{e}_i u = 0$ for any i, then $u = \boxed{1} \otimes \cdots \otimes \boxed{N}$. If there is k $(1 \leqslant k \leqslant N)$ such that $i_\nu = \nu < k$ and $i_k > k$, then $\tilde{e}_{i_k - 1} u \neq 0$, which is a contradiction.

Q.E.D.

Remark 3.3.2. We have $\tilde{e}_i^2 u = \tilde{f}_i^2 u = 0$ on $B(\Lambda_N)$.

3.4. *The Crystal Graph of the Irreducible Representation* V_Y

Let Y be a Young diagram with size $(l_1, ..., l_n)$, i.e., l_j is the length of the jth row of Y. Let V_Y be the $U_q(A_n)$-module with highest weight $\sum_{j=1}^n l_j \varepsilon_j = \sum_{j=1}^n (l_j - l_{j+1}) \Lambda_j$ $(l_{n+1} = 0)$.

Before discussing the general case, we treat a Young diagram with two columns, which means, with weight $\Lambda_M + \Lambda_N$ $(1 \leqslant M \leqslant N \leqslant n)$. In this case, $V(\Lambda_M + \Lambda_N)$ is contained in $V(\Lambda_M) \otimes V(\Lambda_N)$ and hence $B(\Lambda_M + \Lambda_N)$ is the connected component of $B(\Lambda_M) \otimes B(\Lambda_N)$ containing $u_{\Lambda_M} \otimes u_{\Lambda_N}$, where u_{Λ_M} is the highest weight element of $B(\Lambda_M)$. We write

$$\begin{array}{|c|c|} \hline i_1 & j_1 \\ \hline \vdots & \vdots \\ \cline{2-2} & j_M \\ \cline{2-2} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} i_N & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \text{for} \quad \begin{array}{|c|} \hline j_1 \\ \hline \vdots \\ \hline j_M \\ \hline \end{array} \otimes \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \in B(\Lambda_M) \otimes B(\Lambda_N).$$

PROPOSITION 3.4.1. (1) *Let*

$$B(\Lambda_M + \Lambda_N) = \left\{ \begin{array}{|c|c|} \hline i_1 & j_1 \\ \hline \vdots & \vdots \\ \cline{2-2} & j_M \\ \cline{2-2} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} i_N & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} ; \quad \begin{array}{l} 1 \leqslant i_1 < \cdots < i_N \leqslant n+1, \\ 1 \leqslant j_1 < \cdots < j_M \leqslant n+1, \\ i_k \leqslant j_k \quad \textit{for} \quad 1 \leqslant k \leqslant M \end{array} \right\}.$$

(2) *If* $u \in B(\Lambda_N)$ *satisfies* $\tilde{e}_i u = 0$ *for* $M < i \leqslant n$, *and if* $u_{\Lambda_M} \otimes u \in B(\Lambda_M + \Lambda_N)$, *then* $u = u_{\Lambda_N}$.

Proof. Let J be the right hand side in (1). We show first that $J \cup \{0\}$ is closed by $\tilde{e}_i$ and $\tilde{f}_i$. By Remark 3.3.2 and Theorem 1.1.4 we have, for any $u \in B(\Lambda_M)$ and $v \in B(\Lambda_N)$,

$$\tilde{f}_i(u \otimes v) = \begin{cases} \tilde{f}_i u \otimes v & \text{if } \tilde{f}_i u \neq 0 \text{ and } \tilde{e}_i u = 0 \\ u \otimes \tilde{f}_i v & \text{otherwise,} \end{cases} \tag{3.4.1}$$

$$\tilde{e}_i(u \otimes v) = \begin{cases} u \otimes \tilde{e}_i v & \text{if } \tilde{e}_i v \neq 0 \text{ and } \tilde{f}_i u = 0 \\ \tilde{e}_i u \otimes v & \text{otherwise.} \end{cases} \tag{3.4.2}$$

Now assume $u \otimes v \in J$ and $\tilde{f}_i(u \otimes v) \notin J \cup \{0\}$ or $\tilde{e}_i(u \otimes v) \notin J \cup \{0\}$. Then one of the following cases occurs.

(I) There exists $1 \leqslant k \leqslant M$ such that $i_k = j_k = i$ and $i_k = i$ is changed to $i+1$ by the action of $\tilde{f}_i$.

(II) There exists $1 \leqslant k \leqslant M$ such that $i_k = j_k = i+1$ and $j_k = i+1$ is changed to i by the action of $\tilde{e}_i$.

Case (I). In this case, $\tilde{f}_i(u \otimes v) = u \otimes \tilde{f}_i v$. Since there is i in v, $\tilde{e}_i v = 0$, then we have $\tilde{f}_i u = 0$ by (3.4.1). Then $i+1$ occupies the box under i in u. Hence, by the definition of J, $i+1$ also occupies the box under i in v. Hence $\tilde{f}_i v = 0$, which is a contradiction.

Case (II). The proof is similar. We have $\tilde{e}_i(u \otimes v) = \tilde{e}_i u \otimes v$. Since $\tilde{f}_i u = \tilde{f}_i^2 \tilde{e}_i u = 0$, we have $\tilde{e}_i v = 0$. Hence, i appears above $i+1$ in v, and hence i appears above $i+1$ in u. This contradicts $\tilde{e}_i u \neq 0$.

Next, we show that

$$\text{If } u \otimes v \in J \text{ satisfies } \tilde{e}_i(u \otimes v) = 0 \text{ for any } i, \text{ then } u = u_{\Lambda_M} \text{ and } v = u_{\Lambda_N}. \tag{3.4.3}$$

Since $\tilde{e}_i(u \otimes v) = 0$ implies $\tilde{e}_i u = 0$, u must be u_{Λ_M}. Then $\tilde{f}_i u = 0$ for $i > M$. If $\tilde{e}_i v \neq 0$ for such i, we have $\tilde{e}_i(u \otimes v) = u \otimes \tilde{e}_i v \neq 0$ by (3.4.2), this contradicts the assumption of (3.4.3). Therefore $\tilde{e}_i v = 0$ for $i > M$. Thus, we reduce (3.4.3) to

$$\text{If } v \in B(\Lambda_N) \text{ satisfies } \tilde{e}_i v = 0 \text{ for } i > M \text{ and } u_{\Lambda_M} \otimes v \in J, \text{ then } v = u_{\Lambda_N}. \tag{3.4.4}$$

Let $v = \boxed{i_1} \otimes \cdots \otimes \boxed{i_N}$, then $u_{\Lambda_M} \otimes v \in J$ implies $i_\nu = \nu$ for $\nu \leqslant M$. If k $(M \leqslant k \leqslant N)$ satisfies that $i_\nu = \nu$ for $\nu < k$ and $i_k > k$, then $k > M$. Therefore $i_k \geqslant k > M$ and $\tilde{e}_{i_k - 1} v \neq 0$.

This is a contradiction and thus we obtain (3.4.4). This implies (1), i.e., J coincides with the connected component of $B(\Lambda_M) \otimes B(\Lambda_N)$ containing $u_{\Lambda_M} \otimes u_{\Lambda_N}$ and (3.4.4) shows (2). Q.E.D.

For a general Young diagram Y with p columns, let m_j $(1 \leqslant j \leqslant p)$ be the length of the jth column. Then the corresponding $U_q(A_n)$-module V_Y has a highest weight $\lambda = \Lambda_{m_1} + \cdots + \Lambda_{m_p}$ $(m_1 \leqslant \cdots \leqslant m_p)$. Then, we embed V_Y into $V(\Lambda_{m_1}) \otimes \cdots \otimes V(\Lambda_{m_p})$. In this case by Proposition 2.2.1 and Proposition 3.4.1 (2), we have

$$B(\lambda) = \bigcap_{k=1}^{p-1} B(\Lambda_{m_1}) \otimes \cdots \otimes B(\Lambda_{m_{k-1}}) \otimes B(\Lambda_{m_k} + \Lambda_{m_{k+1}}) \otimes B(\Lambda_{m_{k+2}}) \otimes \cdots \otimes (\Lambda_{m_p}). \tag{3.4.5}$$

For $u_1 \otimes \cdots \otimes u_p$, we write

$$\begin{array}{|c|c|c|c|c|} \hline & & & \vdots & u_1 \\ \cline{5-5} & & \vdots & & \\ \cline{4-4} u_p & \vdots & & & \\ \cline{3-3} & & & & \\ \cline{2-2} & & & & \\ \hline \end{array} .$$

Then by Proposition 3.4.1 and (3.4.5), we obtain the following.

THEOREM 3.4.2. (i) *For any Young diagram Y, the crystal graph* $B(V_Y)$ *of the corresponding* $U_q(A_n)$*-module* V_Y *is the set of semi-standard tableaux with shape Y;*

$$B(V_Y) = \left\{ u = \begin{array}{|c|c|c|c|c|} \hline & & & \vdots & t^1_{l'} \\ \cline{5-5} & \vdots & t^k_l & \vdots & \\ \cline{4-4} t^p_{l''} & \vdots & & & \\ \cline{3-3} & & & & \\ \cline{2-2} & & & & \\ \hline \end{array} \;; \quad \begin{array}{c} t^k_l \in \{1, 2, \dots, n, n+1\} \text{ satisfies} \\ (\text{where } 1 \leqslant k \leqslant p,\ 1 \leqslant l \leqslant m_k) \\ t^{k+1}_l \leqslant t^k_l \text{ and } t^k_l < t^k_{l+1} \end{array} \right\}.$$

(ii) *The actions of* $\tilde{e}_i$ *and* $\tilde{f}_i$ *are described by Remark* 2.1.2. *An element u of* $B(V_Y)$ *can be expressed as in the form* $\boxed{t^1_1} \otimes \cdots \otimes \boxed{t^k_l} \otimes \cdots \otimes \boxed{t^p_{m_p}} \in B(V_\square)^{\otimes(\Sigma m_\nu)}$ *and apply Remark* 2.1.2 *by identifying* $\boxed{i}$ *with* u_+, $\boxed{i+1}$ *with* u_- *and others with* u_0.

EXAMPLE 3.4.3. For $\mathfrak{g} = A_3$ and $\lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$, we consider the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on

$$v = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \in B(\Lambda_1 + \Lambda_2 + \Lambda_3).$$

By the construction of crystal graphs, it can be expressed

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} = \boxed{3} \otimes \begin{array}{|c|} \hline 2 \\ \hline 4 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline \end{array} = \boxed{3} \otimes \boxed{2} \otimes \boxed{4} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{4}.$$

(1) If we consider the actions of $\tilde{e}_2$ and $\tilde{f}_2$, the vector v can be identified with $u_- \otimes u_+ \otimes u_0 \otimes u_0 \otimes u_+ \otimes u_0$ and by Remark 2.1.2,

$$\tilde{e}(u_- \otimes u_+ \otimes u_0 \otimes u_0 \otimes u_+ \otimes u_0) = u_+ \otimes u_+ \otimes u_0 \otimes u_0 \otimes u_+ \otimes u_0,$$

$$\tilde{f}(u_- \otimes u_+ \otimes u_0 \otimes u_0 \otimes u_+ \otimes u_0) = u_- \otimes u_- \otimes u_0 \otimes u_0 \otimes u_+ \otimes u_0.$$

Thus,

$$\tilde{e}_2(\boxed{3}\otimes\boxed{2}\otimes\boxed{4}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{4})=\boxed{2}\otimes\boxed{2}\otimes\boxed{4}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{4},$$
$$\tilde{f}_2(\boxed{3}\otimes\boxed{2}\otimes\boxed{4}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{4})=\boxed{3}\otimes\boxed{3}\otimes\boxed{4}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{4}.$$

Hence, we obtain

$$\tilde{e}_2\ \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 2&4\\ \cline{1-2} 4\\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 1&2&2\\ \hline 2&4\\ \cline{1-2} 4\\ \cline{1-1}\end{array}, \qquad \tilde{f}_2\ \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 2&4\\ \cline{1-2} 4\\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 1&3&3\\ \hline 2&4\\ \cline{1-2} 4\\ \cline{1-1}\end{array}.$$

(2) If we consider the actions of $\tilde{e}_3$ and $\tilde{f}_3$, the vector v can be identified with $u_+\otimes u_0\otimes u_-\otimes u_0\otimes u_0\otimes u_-$ and by Remark 2.1.2,

$$\tilde{e}(u_+\otimes u_0\otimes u_-\otimes u_0\otimes u_0\otimes u_-)=u_+\otimes u_0\otimes u_-\otimes u_0\otimes u_0\otimes u_+,$$
$$\tilde{f}(u_+\otimes u_0\otimes u_-\otimes u_0\otimes u_0\otimes u_-)=0.$$

Thus,

$$\tilde{e}_3(\boxed{3}\otimes\boxed{2}\otimes\boxed{4}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{4})=\boxed{3}\otimes\boxed{2}\otimes\boxed{4}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{3},$$
$$\tilde{f}_3(\boxed{3}\otimes\boxed{2}\otimes\boxed{4}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{4})=0.$$

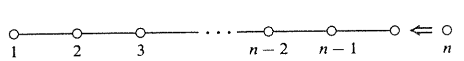
Hence, we obtain

$$\tilde{e}_3\ \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 2&4\\ \cline{1-2} 4\\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 2&4\\ \cline{1-2} 3\\ \cline{1-1}\end{array}, \qquad \tilde{f}_3\ \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 2&4\\ \cline{1-2} 4\\ \cline{1-1}\end{array} = 0.$$

4. Crystal Graphs for $U_q(C_n)$-Modules

4.1. *Notation*

We treat the C_n-case in this section. Let $(\varepsilon_1, ..., \varepsilon_n)$ be the orthonormal base of the dual of the Cartan subalgebra of C_n such that $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$ $(1\leqslant i<n)$ and $\alpha_n=2\varepsilon_n$ are simple roots. The Dynkin diagram is



Hence, α_n is the long root and $\alpha_1, ..., \alpha_{n-1}$ are short roots. Let $\{\Lambda_i\}_{1\leqslant i\leqslant n}$ be the dual base of $\{h_i\}_{1\leqslant i\leqslant n}$. Hence $\Lambda_i=\varepsilon_1+\cdots+\varepsilon_i$ $(1\leqslant i\leqslant n)$.

4.2. *The Crystal Graph of the Vector Representation*

First let us construct the vector representation $V_\square$. Let $V_\square$ be the $2n$-dimensional $\mathbf{Q}(q)$ vector space with $\{\boxed{i}, \boxed{\bar{i}}; 1 \leqslant i \leqslant n\}$ as a basis. We give $V_\square$ the $U_q(C_n)$-module structure as

$$\begin{aligned} &q^h\boxed{j} = q^{\langle h, \varepsilon_j\rangle}\boxed{j}, && q^h\boxed{\bar{j}} = q^{\langle h, -\varepsilon_j\rangle}\boxed{\bar{j}}, \\ &e_i\boxed{j} = \delta_{i+1,j}\boxed{j-1}, && e_i\boxed{\bar{j}} = \delta_{i,j}\boxed{\overline{j+1}} \qquad (1 \leqslant i < n,\ 1 \leqslant j \leqslant n) \quad (4.2.1) \\ &f_i\boxed{j} = \delta_{i,j}\boxed{j+1}, && f_i\boxed{\bar{j}} = \delta_{i+1,j}\boxed{\overline{j-1}} \end{aligned}$$

and

$$\begin{aligned} &e_n\boxed{j} = 0, && e_n\boxed{\bar{j}} = \delta_{j,n}\boxed{n}, \\ &f_n\boxed{j} = \delta_{j,n}\boxed{\bar{n}}, && f_n\boxed{\bar{j}} = 0, \end{aligned} \qquad (1 \leqslant j \leqslant n) \qquad (4.2.2)$$

Here, we understand $\boxed{j} = \boxed{\bar{j}} = 0$ unless $1 \leqslant j \leqslant n$.

Then the crystal base $(L(V_\square), B(V_\square))$ is given by

$$\begin{aligned} L(V_\square) &= \bigoplus_{i=1}^{n} \left(A\boxed{i} \oplus A\boxed{\bar{i}}\right) \\ B(V_\square) &= \left\{\boxed{i}, \boxed{\bar{i}}; 1 \leqslant i \leqslant n\right\} \end{aligned} \qquad (4.2.3)$$

and the crystal graph $B(V_\square)$ of $V_\square$ is given by

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}. \qquad (4.2.4)$$

Remark 4.2.1.

$$\tilde{e}_i^2 = \tilde{f}_i^2 = 0 \qquad \text{on } B(V_\square). \qquad (4.2.5)$$

Hence, the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $B(V_\square)^{\otimes m}$ are given by identifying $\boxed{i}$ and $\boxed{\overline{i+1}}$ with u_+, $\boxed{i+1}$ and $\boxed{\bar{i}}$ with u_-, and others with u_0 in Remark 2.1.2.

4.3. *The Crystal Graph of the Fundamental Representation*

The representation $V(\Lambda_N)$ with highest weight Λ_N $(1 \leqslant N \leqslant n)$ is embedded into $V_\square^{\otimes N}$. Similarly to the A_n-case, the connected component of the crystal graph $B(V_\square)^{\otimes N}$ containing $\boxed{1} \otimes \boxed{2} \otimes \cdots \otimes \boxed{N}$ is isomorphic to $B(\Lambda_N)$.

We write

$$\begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \quad \text{for} \quad \boxed{i_1} \otimes \cdots \otimes \boxed{i_N}.$$

We denote by u_{Λ_N} the highest weight vector $\boxed{1} \otimes \cdots \otimes \boxed{N}$. We give the linear order $\prec$ on $\{i, \bar{i}; 1 \leqslant i \leqslant n\}$ by

$$1 \prec 2 \prec \cdots \prec n \prec \bar{n} \prec \cdots \prec \bar{2} \prec \bar{1}. \tag{4.3.1}$$

This ordering is derived from the crystal graph (4.2.4). We set

$$I_N^{(C)} = \left\{ \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \in B(V_{\square})^{\otimes N}; \quad \begin{array}{l} (1)\ 1 \preccurlyeq i_1 \prec \cdots \prec i_N \preccurlyeq \bar{1}, \\ (2)\ \text{if } i_k = p \text{ and } i_l = \bar{p}, \text{ then } k + (N - l + 1) \leqslant p \end{array} \right\}. \tag{4.3.2}$$

LEMMA 4.3.1. *Assume* $\boxed{i_1} \otimes \cdots \otimes \boxed{i_N} \in I_N^{(C)}$. *If* $i_k = a$ *and* $i_l = \bar{b}$ ($1 \leqslant k < l \leqslant N$ *and* $a, b \in \{1, ..., n\}$), *then* $k + (N - l + 1) \leqslant \max(a, b)$.

Proof. If there is $p \in \{1, ..., n\}$ such that $i_{k'} = p$, $i_{l'} = \bar{p}$ for $1 \leqslant k' \leqslant k$, $l \leqslant l' \leqslant N$, then take the largest p among such p. If there is no such p, then we set $p = 0$, $k' = 0$, and $l' = N + 1$. Then by the choice of p, $\{\overline{i_{k'+1}}, ..., \overline{i_k}\}$ and $\{i_l, ..., i_{l'-1}\}$ have no intersection. Hence their union has the cardinality $k - k' + l' - l$. On the other hand, this union is contained in $\{\overline{p+1}, ..., \overline{\max(a, b)}\}$. Thus we obtain

$$k - k' + l' - l \leqslant \max(a, b) - p. \tag{4.3.3}$$

When $p > 0$, the definition of $I_N^{(C)}$ implies

$$k' + (N - l' + 1) \leqslant p. \tag{4.3.4}$$

If $p = 0$, $k' = 0$, $l' = N + 1$, this is trivially satisfied. Then (4.3.3) and (4.3.4) imply the desired result. Q.E.D.

PROPOSITION 4.3.2. *$B(\Lambda_N)$ coincides with $I_N^{(C)}$ $(1 \leqslant N \leqslant n)$.*

Proof. In order to see this, we have to show the following statements.

(a) $I_N^{(C)} \cup \{0\}$ is stable by $\tilde{f}_i$ and $\tilde{e}_i$.

(b) If $u \in I_N^{(C)}$ satisfies $\tilde{e}_i u = 0$ for any i, then $u = u_{\Lambda_N} = \boxed{1} \otimes \cdots \otimes \boxed{N}$.

In order to prove (a), assuming that $u \in I_N^{(C)}$ and $\tilde{f}_i u \notin I_N^{(C)} \cup \{0\}$, we derive a contradiction. Under this assumption, there are the following possibilities, because $\tilde{f}_i u$ always satisfies the condition (1) of (4.3.2).

(I) $1 \leqslant i < n$ and there exists $1 \leqslant k < l \leqslant N$ such that $i_k = i$, $j_l = \overline{i+1}$, $k + (N - l + 1) > i + 1$ and $i_k = i$ changes to $i+1$ by the action of $\tilde{f}_i$.

(II) $1 \leqslant i < n$ and there exists $1 \leqslant k < l \leqslant N$ such that $i_k = i$, $j_l = \overline{i+1}$, $k + (N - l + 1) > i$ and $i_l = \overline{i+1}$ changes to $\bar{i}$ by the action of $\tilde{f}_i$.

The case (I) cannot occur by Lemma 4.3.1. In the case (II), since $\overline{i+1}$ changes to $\bar{i}$ under the action of $\tilde{f}_i$, u contains $i+1$, i.e., $i_{k+1} = i+1$ by Remark 4.2.1. Therefore $k + 1 + (N - l + 1) > i + 1$, which contradicts $u \in I_N^{(C)}$. Thus, we conclude that $I_N^{(C)} \cup \{0\}$ is stable by $\tilde{f}_i$. A similar argument shows the stability of $I_N^{(C)} \cup \{0\}$ under the action of $\tilde{e}_i$. This shows (a).

Let us prove (b). Assume that $u \in I_N^{(C)}$ satisfies $\tilde{e}_i u = 0$ $(1 \leqslant i \leqslant n)$. If $u = \boxed{i_1} \otimes \cdots \otimes \boxed{i_N} \neq \boxed{1} \otimes \cdots \otimes \boxed{N}$, then there is k $(1 \leqslant k \leqslant N)$ such that $i_\nu = \nu$ for $\nu < k$ and $i_k \succ k$. If $i_k \in \{k+1, \ldots, n\}$, then $\tilde{e}_p u \neq 0$ with $p = i_k - 1$. If $i_k \in \{\bar{k}, \ldots, \bar{n}\}$, then $\tilde{e}_p u \neq 0$ with $i_k = \bar{p}$. If $i_k \in \{\bar{1}, \ldots, \overline{k-1}\}$ and $k \geqslant 1$, setting $i_k = \bar{p}$ then we have $p + (N - k + 1) > p$, which contradicts the condition (2). Therefore $i_k = k$, which contradicts the hypothesis. Q.E.D.

Remark 4.3.3. (i) $\tilde{e}_i^3 = \tilde{f}_i^3 = 0$ for $1 \leqslant i < n$ and $\tilde{e}_n^2 = \tilde{f}_n^2 = 0$ on $I_N^{(C)}$.

(ii) If $u \in I_N^{(C)}$ satisfies $\tilde{f}_i^2 u \neq 0$, then u contains i and $\overline{i+1}$ but neither $i+1$ nor $\bar{i}$. If $u \in I_N^{(C)}$ satisfies $\tilde{e}_i^2 u \neq 0$, then u contains $i+1$ and $\bar{i}$ but neither i nor $\overline{i+1}$.

(iii) If $u \in I_N^{(C)}$ satisfies $\tilde{f}_i u \neq 0$ and $\tilde{e}_i u \neq 0$, then u contains $i+1$, $\overline{i+1}$ but neither i nor $\bar{i}$.

4.4. *The Crystal Graph of $V(\Lambda_M + \Lambda_N)$*

Now, we investigate the crystal graph of $V(\Lambda_M + \Lambda_N)$ with $1 \leqslant M \leqslant N \leqslant n$. By embedding $V(\Lambda_M + \Lambda_N)$ into $V(\Lambda_M) \otimes V(\Lambda_N)$, $B(\Lambda_M + \Lambda_N)$ is the connected component of $B(\Lambda_M) \otimes B(\Lambda_N)$ containing $u_{\Lambda_M} \otimes u_{\Lambda_N}$.

For

$$u=\begin{array}{|c|}\hline j_1\\ \hline \vdots \\ \hline j_M\\ \hline\end{array}\in I_M^{(C)} \qquad\text{and}\qquad v=\begin{array}{|c|}\hline i_1\\ \hline \vdots \\ \hline i_N\\ \hline\end{array}\in I_N^{(C)},$$

$$u\otimes v\in I_M^{(C)}\otimes I_N^{(C)} \text{ will be denoted by } \begin{array}{|c|c|}\hline i_1 & j_1\\ \hline & \vdots \\ \vdots & \\ \cline{2-2} & \multicolumn{1}{|c|}{j_M} \\ \hline i_N & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}.$$

DEFINITION 4.4.1. For $1\leqslant a\leqslant b\leqslant n$ and u, v as above, we say that $u\otimes v\in I_M^{(C)}\otimes I_N^{(C)}$ is in the (a, b)-configuration if $u\otimes v$ satisfies the following: There exist $1\leqslant p\leqslant q<r\leqslant s\leqslant M$ such that $i_p=a, j_q=b, j_r=\bar{b}, j_s=\bar{a}$ or $i_p=a,\ i_q=b,\ i_r=\bar{b},\ j_s=\bar{a}$.

This definition includes the case $a=b$, $p=q$, and $r=s$. Now, we define

$$p(a, b; u\otimes v)=(q-p)+(s-r). \tag{4.4.1}$$

Let us set

$$I_{(M,N)}^{(C)}=\left\{ w=\begin{array}{|c|c|}\hline i_1 & j_1\\ \hline & \vdots \\ \vdots & \\ \cline{2-2} & \multicolumn{1}{|c|}{j_M} \\ \hline i_N & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}\in I_M^{(C)}\otimes I_N^{(C)};\quad \begin{array}{l}w \text{ satisfies the conditions}\\ \text{(M.N.1) and (M.N.2)}\end{array}\right\}. \tag{4.4.2}$$

Here the conditions (M.N.1) and (M.N.2) are

(M.N.1) $i_k\preceq j_k$ for $1\leqslant k\leqslant M$.

(M.N.2) If w is in the (a, b)-configuration, then $p(a, b; w)<b-a$.

Note that any vector of $I_{(M,N)}^{(C)}$ is not in the (a, a)-configuration, because $p(a, a; w)\geqslant 0$.

LEMMA 4.4.2. *Assume that a vector w in $I_{(M,N)}^{(C)}$ satisfies (with the notation above) $i_k=a$, $j_l=\bar{b}$ and assume either $i_{k'}=a'$, $i_{l'}=\overline{b'}$ or $j_{k'}=a'$, $j_{l'}=\overline{b'}$ with $k\leqslant k'\leqslant l'\leqslant l$, $1\leqslant a\leqslant a'\leqslant n$, $1\leqslant b\leqslant b'\leqslant n$. Then $(k'-k)+(l-l')<\max(a', b')-\min(a, b)$.*

The proof being similar to that of Lemma 4.3.1, we omit it.

PROPOSITION 4.4.3. *$B(\Lambda_M+\Lambda_N)$ coincides with $I_{(M,N)}^{(C)}$.*

Proof. First, we show that $I^{(C)}_{(M,N)} \cup \{0\}$ is closed by the actions of $\tilde{e}_i$ and $\tilde{f}_i$. In order to see this, we remark the following.

If $w = u \otimes v \in I^{(C)}_M \otimes I^{(C)}_N$ and $\tilde{f}_i(u \otimes v) = u \otimes \tilde{f}_i v \neq 0$, then by Theorem 1.1.4 and Remark 4.3.3 there are two possible cases (4.4.3a) and (4.4.3b):

$$\tilde{f}_i u = 0 \tag{4.4.3a}$$

$$\tilde{f}_i^2 u = 0, \qquad \tilde{f}_i u \neq 0, \qquad \text{and} \qquad \tilde{e}_i v \neq 0. \tag{4.4.3b}$$

The remaining cases cannot happen because $\tilde{f}_i^3 = 0$ on $I^{(C)}_M$ and $\tilde{e}_i^2 v \neq 0$ imply $\tilde{f}_i v = \tilde{f}_i^3 \tilde{e}_i^2 u = 0$.

Now, assuming that $w = u \otimes v \in I^{(C)}_{(M,N)}$ and $\tilde{f}_i w \neq 0$, we prove that (M.N.1) holds for $\tilde{f}_i w$. Otherwise there are the following possible cases:

(1-i) $i < n$ and there exists $1 \leqslant k \leqslant M$ such that $i_k = j_k = i$ and $i_k = i$ changes to $i+1$ by the action of $\tilde{f}_i$.

(1-ii) $i < n$ and there exists $1 \leqslant k \leqslant M$ such that $i_k = j_k = \overline{i+1}$ and $i_k = \overline{i+1}$ changes to $\bar{i}$ by the action of $\tilde{f}_i$.

(1-iii) $i = n$ and there exists $1 \leqslant k \leqslant M$ such that $i_k = j_k = n$ and $i_k = n$ changes to $\bar{n}$ by the action of $\tilde{f}_n$.

In those cases, v changes by the action of $\tilde{f}_i$ and therefore (4.4.3a) or (4.4.3b) holds.

Consider the case (1-i). If (4.4.3a) occurs, u contains $i+1$ or $\bar{i}$. Hence $j_{k+1} = i+1$ or there exists $1 \leqslant l \leqslant n$ such that $j_l = \bar{i}$. The last case does not occur because of (M.N.2). In the first case, if we set $i_{k+1} = \alpha$, then by the definition of $I^{(C)}_{(M,N)}$, $i \prec \alpha \preccurlyeq i+1$ and hence $\alpha = i+1$. This contradicts the fact that i in v changes to $i+1$ by $\tilde{f}_i$. If (4.4.3b) occurs, this implies that $\tilde{e}_i v \neq 0$, $\tilde{f}_i v \neq 0$ then by Remark 4.3.3 (iii), v does not contain i. This fact contradicts $i_k = i$. Then the case (4.4.3b) does not occur. Thus the case (1-i) cannot occur. Similar arguments show that the cases (1-ii) and (1-iii) do not occur. Therefore, if $w \in I^{(C)}_{(M,N)}$ and $\tilde{f}_i w \neq 0$, then $\tilde{f}_i w$ satisfies (M.N.1).

Next, we assume that $w = u \otimes v \in I^{(C)}_{(M,N)}$ and $\tilde{f}_i w = u' \otimes v' \neq 0$ for some i, and we show that $\tilde{f}_i w = u' \otimes v'$ satisfies the condition (M.N.2). In this situation, we assume that $u' \otimes v'$ is in the (a, b)-configuration.

We set

$$u \otimes v = \begin{array}{|c|c|} \hline i_1 & j_1 \\ \hline \vdots & \vdots \\ \cline{2-2} & j_M \\ \hline i_N & \\ \cline{1-1} \end{array} \quad \text{and} \quad u' \otimes v' = \begin{array}{|c|c|} \hline i'_1 & j'_1 \\ \hline \vdots & \vdots \\ \cline{2-2} & j'_M \\ \hline i'_N & \\ \cline{1-1} \end{array}\,.$$

Then $i'_p = a$ and $j'_s = \bar{a}$ for some p and s and the following cases can occur:

(2-i) $i_p = a$ and $j_s = \bar{a}$.

(2-ii) $i_p = a$ and $j_s = \overline{a+1}$.

(2-iii) $i_p = a-1$ and $j_s = \bar{a}$.

First let us consider the cases (2-i) and (2-ii). In these cases, if $i < a$, then $p(a, b; u' \otimes v') < b - a$ because $u \otimes v$ satisfies (M.N.2), therefore we may assume that $i \geqslant a$.

If we set $v^* = \boxed{i_p} \otimes \cdots \otimes \boxed{i_s}$ and $u^* = \boxed{j_p} \otimes \cdots \otimes \boxed{j_s}$, then both v^* and u^* are elements of $I^{(C)}_{s-p+1}$ for the subalgebra $U_q(C_{n-a+1})$ of $U_q(C_n)$, generated by e_i and f_i $(a \leqslant i \leqslant n)$. Note that $s - p + 1 \leqslant n - a + 1$ by Lemma 4.4.2.

In this notation, by (4.3.2), a certain vector $\boxed{l_p} \otimes \cdots \otimes \boxed{l_s}$ is an element of $I^{(C)}_{s-p+1}$ if and only if

(1) $a \preccurlyeq l_p \prec l_{p+1} \prec \cdots \prec l_s \preccurlyeq \bar{a}$,

(2) If $i_k = b$ and $l_m = \bar{b}$, then $(k - p + 1) + (q - m + 1) \leqslant b - a + 1$.

By the fact that $i \geqslant a$, $\tilde{f}_i u^*$ and $\tilde{f}_i v^*$ are in $I^{(C)}_{s-p+1}$ for $U_q(C_{n-a+1})$. Since $\boxed{i'_p} \otimes \cdots \otimes \boxed{i'_s}$ is equal to v^* or $\tilde{f}_i v^*$ and $\boxed{j'_p} \otimes \cdots \otimes \boxed{j'_s}$ is equal to u^* or $\tilde{f}_i u^*$, hence $u' \otimes v'$ satisfies (M.N.2) by (2).

Finally, let us consider the case (2-iii). In this case, not $\bar{a}$ in u but $a-1$ in v changes to a by $\tilde{f}_i$. This implies that there exists $\overline{a-1}$ just under $\bar{a}$ in u, i.e., $j_{q+1} = \overline{a-1}$. Hence, considering $\boxed{i_p} \otimes \cdots \otimes \boxed{i_{q+1}}$, $\boxed{j_p} \otimes \cdots \otimes \boxed{j_{q+1}}$ and $U_q(C_{n-a})$ instead of v^*, u^*, and $U_q(C_{n-a+1})$ in the preceding case, respectively, $u' \otimes v'$ satisfies (M.N.2). Therefore, we obtain that $I^{(C)}_{(M,N)} \cup \{0\}$ is closed by $\tilde{f}_i$. Similarly to the $\tilde{f}_i$ case, we can also obtain that $I^{(C)}_{(M,N)} \cup \{0\}$ is closed by $\tilde{e}_i$.

It remains to prove that if $w = u \otimes v \in I^{(C)}_{(M,N)}$ satisfies $\tilde{e}_i w = 0$ $(1 \leqslant i \leqslant n)$ then $w = u_{\Lambda_M} \otimes u_{\Lambda_N}$. Since $\tilde{e}_i w = 0$ implies $\tilde{e}_i u = 0$, u must be u_{Λ_M}. Hence this follows from the following lemma.

LEMMA 4.4.4. *If* $u_{\Lambda_M} \otimes v \in I^{(C)}_{(M,N)}$ *satisfies* $\tilde{e}_i v = 0$ *for* $M < i \leqslant n$, *then* $v = u_{\Lambda_N}$.

Proof. We set $v = \boxed{i_1} \otimes \cdots \otimes \boxed{i_N} \in I^{(C)}_N$. Then by (M.N.1) we have $i_\nu = \nu$ for $\nu \leqslant M$. Since $v \in I^{(C)}_N$, v does not contain $\bar{1}, \ldots, \bar{M}$. If there is $\bar{j}$ $(M + 1 \preccurlyeq j \preccurlyeq n)$ in v, take the largest j among such j, then $\tilde{e}_j v \neq 0$, which is a contradiction. If $j + 1$ is in v and j is not in v $(M + 1 \leqslant j < n)$, then $\tilde{e}_j v \neq 0$, which is a contradiction. Hence if v contains $j + 1$, then v contains j. Therefore $v = u_{\Lambda_N}$. Q.E.D.

This completes the proof of Proposition 4.4.3. Q.E.D.

4.5. *The Crystal Graph of $V(\lambda)$*

Let $\lambda=\sum_{i=1}^{p}\Lambda_{l_i}$ $(1\leqslant l_1\leqslant l_2\leqslant\cdots\leqslant n)$ be a dominant integral weight. Let us consider $B(\lambda)$.

For

$$u_k=\begin{array}{|c|}\hline t_1^k\\ \hline \vdots\\ \hline t_{l_k}^k\\ \hline\end{array}\in I_{l_k}^{(C)},$$

we denote

$$u_1\otimes\cdots\otimes u_p=\begin{array}{|c|c|c|c|c|}\hline & & & \vdots & t_l^1\\ \cline{5-5} t_{l'}^p & \vdots & \vdots & & \\ & & & & \\ \hline\end{array}\in I_{l_1}^{(C)}\otimes\cdots\otimes I_{l_p}^{(C)}.$$

We define

$$I_\lambda^{(C)}=\left\{u_1\otimes\cdots\otimes u_p=\begin{array}{|c|c|c|c|c|}\hline & & & \vdots & t_l^1\\ \cline{5-5} t_{l'}^p & \vdots & \vdots & & \\ & & & & \\ \hline\end{array}\in I_{l_1}^{(C)}\otimes\cdots\otimes I_{l_p}^{(C)};\right.$$

$$\left.\begin{array}{l}\text{for any } k=1,\ \cdots,p-1,\\ u_k\otimes u_{k+1}\in I_{(l_k,l_{k+1})}^{(C)}\end{array}\right\}. \tag{4.5.1}$$

An element of $I_\lambda^{(C)}$ is called a *semi-standard C-tableau* of shape λ.

With this definition, the following theorem is an immediate consequence of Proposition 2.2.1, Proposition 4.4.3, and Lemma 4.4.4.

THEOREM 4.5.1. *Let $\lambda\in P_+$.*

(i) *$B(\lambda)$ coincides with the set of semi-standard C-tableaux of shape λ.*

(ii) *The actions of $\tilde{e}_i$ and $\tilde{f}_i$ $(1\leqslant i<n)$ are given by identifying $\boxed{i}$ and $\boxed{\overline{i+1}}$ with u_+, and $\boxed{i+1}$ and $\boxed{\bar{i}}$ with u_-, and others with u_0 in Remark* 2.1.2. *The actions of $\tilde{e}_n$ and $\tilde{f}_n$ are given by identifying $\boxed{n}$ with u_+, $\boxed{\bar{n}}$ with u_-, and others with u_0 in Remark* 2.1.2.

EXAMPLE 4.5.2. For $\mathfrak{g}=C_3$ and $\lambda=2\Lambda_1+\Lambda_2+\Lambda_3$, we consider the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on

$$v=\begin{array}{|c|c|c|c|}\hline 1&2&3&\bar{3}\\ \hline 3&3\\ \cline{1-2} \bar{3}\\ \cline{1-1}\end{array}\in B(2\Lambda_1+\Lambda_2+\Lambda_3).$$

By the construction of crystal graphs, it can be expressed

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&\bar{3}\\ \hline 3&3\\ \cline{1-2} \bar{3}\\ \cline{1-1}\end{array}=\boxed{\bar{3}}\otimes\boxed{3}\otimes\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline \bar{3}\\ \hline\end{array}=\boxed{\bar{3}}\otimes\boxed{3}\otimes\boxed{2}\otimes\boxed{3}\otimes\boxed{1}\otimes\boxed{3}\otimes\boxed{\bar{3}}.$$

(1) If we consider the actions of $\tilde{e}_2$ and $\tilde{f}_2$, the vector v can be identified with $u_+\otimes u_-\otimes u_+\otimes u_-\otimes u_0\otimes u_-\otimes u_+$ and by Remark 2.1.2,

$$\begin{aligned}&\tilde{e}(u_+\otimes u_-\otimes u_+\otimes u_-\otimes u_0\otimes u_-\otimes u_+)\\&\qquad=u_+\otimes u_-\otimes u_+\otimes u_-\otimes u_0\otimes u_+\otimes u_+,\\&\tilde{f}(u_+\otimes u_-\otimes u_+\otimes u_-\otimes u_0\otimes u_-\otimes u_+)\\&\qquad=u_+\otimes u_-\otimes u_+\otimes u_-\otimes u_0\otimes u_-\otimes u_-.\end{aligned}$$

Thus,

$$\begin{aligned}&\tilde{e}_2(\boxed{\bar{3}}\otimes\boxed{3}\otimes\boxed{2}\otimes\boxed{3}\otimes\boxed{1}\otimes\boxed{3}\otimes\boxed{\bar{3}})\\&\qquad=\boxed{\bar{3}}\otimes\boxed{3}\otimes\boxed{2}\otimes\boxed{3}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{\bar{3}},\\&\tilde{f}_2(\boxed{\bar{3}}\otimes\boxed{3}\otimes\boxed{2}\otimes\boxed{3}\otimes\boxed{1}\otimes\boxed{3}\otimes\boxed{\bar{3}})\\&\qquad=\boxed{\bar{3}}\otimes\boxed{3}\otimes\boxed{2}\otimes\boxed{3}\otimes\boxed{1}\otimes\boxed{3}\otimes\boxed{\bar{2}}.\end{aligned}$$

Hence, we obtain

$$\tilde{e}_2\begin{array}{|c|c|c|c|}\hline 1&2&3&\bar{3}\\ \hline 3&3\\ \cline{1-2} \bar{3}\\ \cline{1-1}\end{array}=\begin{array}{|c|c|c|c|}\hline 1&2&3&\bar{3}\\ \hline 2&3\\ \cline{1-2} \bar{3}\\ \cline{1-1}\end{array},\qquad \tilde{f}_2\begin{array}{|c|c|c|c|}\hline 1&2&3&\bar{3}\\ \hline 3&3\\ \cline{1-2} \bar{3}\\ \cline{1-1}\end{array}=\begin{array}{|c|c|c|c|}\hline 1&2&3&\bar{3}\\ \hline 3&3\\ \cline{1-2} \bar{2}\\ \cline{1-1}\end{array}.$$

(2) If we consider the actions of $\tilde{e}_3$ and $\tilde{f}_3$, the vector v can be identified with $u_-\otimes u_+\otimes u_0\otimes u_+\otimes u_0\otimes u_+\otimes u_-$ and by Remark 2.1.2,

$$\begin{aligned}&\tilde{e}(u_-\otimes u_+\otimes u_0\otimes u_+\otimes u_0\otimes u_+\otimes u_-)\\&\qquad=u_+\otimes u_+\otimes u_0\otimes u_+\otimes u_0\otimes u_+\otimes u_-,\\&\tilde{f}(u_-\otimes u_+\otimes u_0\otimes u_+\otimes u_0\otimes u_+\otimes u_-)\\&\qquad=u_-\otimes u_-\otimes u_0\otimes u_+\otimes u_0\otimes u_+\otimes u_-.\end{aligned}$$

Thus,

$$\begin{aligned}&\tilde{e}_3(\boxed{\bar{3}}\otimes\boxed{3}\otimes\boxed{2}\otimes\boxed{3}\otimes\boxed{1}\otimes\boxed{3}\otimes\boxed{\bar{3}})\\&\qquad=\boxed{3}\otimes\boxed{3}\otimes\boxed{2}\otimes\boxed{3}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{\bar{3}},\\&\tilde{f}_3(\boxed{\bar{3}}\otimes\boxed{3}\otimes\boxed{2}\otimes\boxed{3}\otimes\boxed{1}\otimes\boxed{3}\otimes\boxed{\bar{3}})\\&\qquad=\boxed{\bar{3}}\otimes\boxed{\bar{3}}\otimes\boxed{2}\otimes\boxed{3}\otimes\boxed{1}\otimes\boxed{3}\otimes\boxed{\bar{2}}.\end{aligned}$$

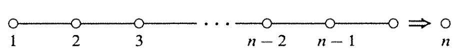
Hence, we obtain

$$\tilde{e}_3\ \begin{array}{|c|c|c|c|}\hline 1&2&3&\bar{3}\\\hline 3&3\\\cline{1-2}\bar{3}\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&3&3\\\hline 3&3\\\cline{1-2}\bar{3}\\\cline{1-1}\end{array}\,, \qquad \tilde{f}_3\ \begin{array}{|c|c|c|c|}\hline 1&2&3&\bar{3}\\\hline 3&3\\\cline{1-2}\bar{3}\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&\bar{3}&\bar{3}\\\hline 3&3\\\cline{1-2}\bar{3}\\\cline{1-1}\end{array}\,.$$

5. Crystal Graphs for $U_q(B_n)$-Modules

5.1. *Notation*

We treat the B_n-case in this section. Let $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ be the orthonormal base of the dual of the Cartan subalgebra of B_n such that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leqslant i < n)$ and $\alpha_n = \varepsilon_n$ are simple roots. The Dynkin diagram is



Hence, $\alpha_1, ..., \alpha_{n-1}$ are long roots and α_n is the short root. Let $\{\Lambda_i\}_{1 \leqslant i \leqslant n}$ be the dual base of $\{h_i\}_{1 \leqslant i \leqslant n}$. Hence $\Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i$ $(i = 1, ..., n-1)$ and $\Lambda_n = (\varepsilon_1 + \cdots + \varepsilon_n)/2$.

5.2. *The Crystal Graph of the Vector Representation*

Let us construct the vector representation $V_\square$. Letting $\boxed{i}$ $(0 \leqslant i \leqslant n)$, $\boxed{\bar{i}}$ $(1 \leqslant i \leqslant n)$ be the base of $\mathbf{Q}(q)^{\oplus 2n+1}$, the vector representation $V_\square$ of $U_q(B_n)$ is explicitly given as

$$q^h\boxed{j} = q^{\langle h, \varepsilon_j\rangle}\boxed{j}, \qquad q^h\boxed{\bar{j}} = q^{\langle h, -\varepsilon_j\rangle}\boxed{\bar{j}}, \qquad q^h\boxed{0} = \boxed{0}, \tag{5.2.1}$$

$$\left.\begin{aligned}&e_i\boxed{j} = \delta_{i+1,j}\boxed{j-1}\,, \quad e_i\boxed{\bar{j}} = \delta_{i,j}\boxed{\overline{j+1}}\,, \quad e_i\boxed{0} = 0\\&f_i\boxed{j} = \delta_{i,j}\boxed{j+1}\,, \quad f_i\boxed{\bar{j}} = \delta_{i+1,j}\boxed{\overline{j-1}}\,, \quad f_i\boxed{0} = 0\end{aligned}\right\}\ (1 \leqslant i < n,\ 1 \leqslant j \leqslant n) \tag{5.2.2}$$

$$\left.\begin{aligned}&e_n\boxed{j} = 0, \quad e_n\boxed{0} = [2]_n\boxed{n}, \quad e_n\boxed{\bar{j}} = \delta_{n,j}\boxed{0}\\&f_n\boxed{j} = \delta_{n,j}\boxed{0}, \quad f_n\boxed{0} = [2]_n\boxed{\bar{n}}, \quad f_n\boxed{\bar{j}} = 0\end{aligned}\right\}\ (1 \leqslant j \leqslant n) \tag{5.2.3}$$

In (5.2.2), we understand $\boxed{j} = \boxed{\bar{j}} = 0$ unless $j = 0, 1, 2, \cdots, n$. Then the crystal base $(L(V_{\square}), B(V_{\square}))$ is given by

$$\begin{aligned} L(V_{\square}) &= \bigoplus_{i=1}^{n} \left(A\boxed{i} \oplus A\boxed{\bar{i}}\right) \oplus A\boxed{0}, \\ B(V_{\square}) &= \left\{\boxed{i}, \boxed{\bar{i}};\ 1 \leqslant i \leqslant n\right\} \cup \left(\boxed{0}\right). \end{aligned} \tag{5.2.4}$$

From (5.2.2) and (5.2.3) we obtain the actions of $\tilde{e}_i$ and $\tilde{f}_i$ by replacing e_i and f_i with $\tilde{e}_i$ and $\tilde{f}_i$ except the actions of $\tilde{e}_n$ and $\tilde{f}_n$ on $\boxed{0}$. Those are given as

$$\tilde{e}_n \boxed{0} = \boxed{n} \qquad \text{and} \qquad \tilde{f}_n \boxed{0} = \boxed{\bar{n}}. \tag{5.2.5}$$

Then the crystal graph of $V_{\square}$ is given by

$$\begin{aligned} &\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\bar{n}} \\ &\xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}. \end{aligned} \tag{5.2.6}$$

5.3. *The Crystal Graph of Anti-symmetric Tensor-Representations*

We set $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ $(1 \leqslant i \leqslant n)$ and call the irreducible representation with the highest weight ω_i "the anti-symmetric tensor representation with the highest weight ω_i." Remark that $\omega_i = \Lambda_i$ $(1 \leqslant i < n)$ and $\omega_n = 2\Lambda_n$.

The anti-symmetric tensor representation $V(\omega_N)$ with the highest weight ω_N can be embedded into $V_{\square}^{\otimes N}$ $(1 \leqslant N \leqslant n)$. Similarly to the A_n and C_n cases, the connected component of the crystal graph of $B(V_{\square}^{\otimes N}) = B(V_{\square})^{\otimes N}$ containing $\boxed{1} \otimes \boxed{2} \otimes \cdots \otimes \boxed{N}$ is the crystal graph of $B(\omega_N)$. We denote by u_{ω_N} the highest weight vector $\boxed{1} \otimes \boxed{2} \otimes \cdots \otimes \boxed{N}$. We give the linear order on $\{i, \bar{i};\ 1 \leqslant i \leqslant n\} \cup \{0\}$ by

$$1 \prec 2 \prec \cdots \prec n \prec 0 \prec \bar{n} \prec \cdots \prec \bar{2} \prec \bar{1}. \tag{5.3.1}$$

This ordering is derived from the crystal graph (5.2.6). Now using the same expression as in Subsection 4.3, we set

$$I_N^{(B)} = \left\{ \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \in B(V_{\square})^{\otimes N};\ \begin{array}{l} (1)\ 1 \preccurlyeq i_1 \preccurlyeq \cdots \preccurlyeq i_N \preccurlyeq \bar{1}, \\ \text{but any element other than 0} \\ \text{cannot appear more than once.} \\ (2)\ \text{if } i_k = p \text{ and } i_l = \bar{p}\ (1 \leqslant p \leqslant n), \\ \text{then } k + (N - l + 1) \leqslant p. \end{array} \right\}. \tag{5.3.2}$$

PROPOSITION 5.3.1. *$B(\omega_N)$ coincides with $I_N^{(B)}$.*

The arguments are similar to Proposition 4.3.2 and we omit the proof.

We remark that the actions of $\tilde{e}_i$ and $\tilde{f}_i$ $(1 \leqslant i < n)$ on $I_N^{(B)}$ are given by identifying $\boxed{i}$ and $\boxed{\overline{i+1}}$ with u_+, $\boxed{i+1}$ and $\boxed{\bar{i}}$ with u_-, and others with u_0 in Remark 2.1.2 and the actions of $\tilde{e}_n$ and $\tilde{f}_n$ are given by identifying $\boxed{n}$ with u_1, $\boxed{0}$ with u_2, $\boxed{\bar{n}}$ with u_3, and others with u_0 in Remark 2.1.3.

Remark 5.3.2. By Remark 2.1.2 and Remark 2.1.3, we obtain

(i) $\tilde{e}_i^3 = \tilde{f}_i^3 = 0$ on $I_N^{(B)}$.

(ii) If $u \in I_N^{(B)}$ satisfies $\tilde{f}_i^2 u \neq 0$ $(1 \leqslant i < n)$, then u contains i and $\overline{i+1}$ but neither $i+1$ nor $\bar{i}$. If $u \in I_N^{(B)}$ satisfies $\tilde{e}_i^2 u \neq 0$ $(1 \leqslant i < n)$, then u contains $i+1$ and $\bar{i}$ but neither i nor $\overline{i+1}$.

(iii) If u satisfies $\tilde{f}_i u \neq 0$ and $\tilde{e}_i u \neq 0$ $(1 \leqslant i < n)$, then u contains $i+1$ and $\overline{i+1}$ but neither i nor $\bar{i}$.

(iv) If $u \in I_N^{(B)}$ satisfies $\tilde{f}_n^2 u \neq 0$, then u contains n but no $\bar{n}$. If $u \in I_N^{(B)}$ satisfies $\tilde{e}_n^2 u \neq 0$, then u contains $\bar{n}$ but no n.

(v) If u satisfies $\tilde{f}_n u \neq 0$ and $\tilde{e}_n u \neq 0$, then u contains 0 but neither n nor $\bar{n}$.

Remark 5.3.3. We can define $I_N^{(B)}$ by the formula (5.3.2) for an arbitrary N. Even if $N > n$, $I_N^{(B)}$ is a crystal subgraph of $B(V_\square)^{\otimes N}$ (i.e., stable by $\tilde{e}_i$ and $\tilde{f}_i$). The proof is similar to Propositions 4.3.2 and 5.3.1.

5.4. *The Crystal Graph of the Spin Representation*

The finite-dimensional irreducible representation with the highest weight Λ_n is called the spin representation and denoted by V_{sp}. It has the explicit description as follows (cf. [Re]); we set

$$B_{sp} = \{v = (i_1, i_2, \cdots, i_n);\ i_j = \pm\},$$

$$V_{sp} = \bigoplus_{v \in B_{sp}} \mathbf{Q}(q)\, v.$$

We define the actions of generators of $U_q(B_n)$ as

$$q^h v = q^{\langle h,\, wt(v)\rangle} v, \qquad \text{where } wt(v) = \frac{1}{2} \sum_{j=1}^{n} i_j \varepsilon_j \qquad \text{for } v = (i_1, \ldots, i_n); \tag{5.4.1}$$

$$e_j(i_1, \ldots, i_n) = \begin{cases} (i_1, \ldots, \overset{j}{+}, \overset{j+1}{-}, \ldots, i_n) & i_j = - \text{ and } i_{j+1} = +, \\ 0, & \text{otherwise;} \end{cases} \tag{5.4.2}$$

$$f_j(i_1, \ldots, i_n) = \begin{cases} (i_1, \ldots, \overset{j}{-}, \overset{j+1}{+}, \ldots, i_n), & i_j = + \text{ and } i_{j+1} = -, \\ 0, & \text{otherwise,} \end{cases} \tag{5.4.3}$$

for $j = 1, \ldots, n-1$;

$$e_n(i_1, ..., i_n) = \begin{cases} (i_1, ..., i_{n-1}, \overset{n}{+}), & i_n = -, \\ 0, & \text{otherwise}; \end{cases} \tag{5.4.4}$$

$$f_n(i_1, ..., i_n) = \begin{cases} (i_1, ..., i_{n-1}, \overset{n}{-}), & i_n = +, \\ 0, & \text{otherwise}. \end{cases} \tag{5.4.5}$$

If we set $L_{sp} = \bigoplus_{v \in B_{sp}} Av$, then (L_{sp}, B_{sp}) is the crystal base of V_{sp}. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are given by the same formula (5.4.2)–(5.4.5).

We give another expression for B_{sp}.

First we give the linear order on $\{1, 2, ..., n, \bar{n}, ..., \bar{2}, \bar{1}\}$ as in (5.3.1), then

$$B_{sp} \cong \left\{ \begin{array}{|c|} \hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_n \\ \hline \end{array} \; ; \; \begin{array}{l} (1)\ i_j \in \{1, 2, ..., n, \bar{n}, ..., \bar{2}, \bar{1}\}, \\ (2)\ i_1 \prec \cdots \prec i_n, \\ (3)\ i \text{ and } \bar{i} \text{ do not appear simultaneously} \end{array} \right\}. \tag{5.4.6}$$

In this description, i corresponds to the ith $+$ and $\bar{i}$ corresponds to the ith $-$ in the former description. The actions of f_i, e_i, $\tilde{f}_i$, and $\tilde{e}_i$ are given in the following remark.

Remark 5.4.1. For $u \in B_{sp}$ in the latter description,

(1) $q^h u = q^{\langle h, wt(u) \rangle} u$ where $wt(u) = (\sum_{i_k \preccurlyeq n} \varepsilon_{i_k} - \sum_{i_k \succcurlyeq \bar{n}} \varepsilon_{\bar{i_k}})/2$. Remark that $\bar{\bar{j}} = j$.

(2) For $j = 1, ..., n-1$, if u contains j and $\overline{j+1}$, then $f_j u = \tilde{f}_j u$ is obtained by replacing j with $j+1$ and replacing $\overline{j+1}$ with $\bar{j}$. Otherwise, $f_j u = \tilde{f}_j u = 0$.

(3) For $j = 1, ..., n-1$ if u contains $j+1$ and $\bar{j}$, then $e_j u = \tilde{e}_j u$ is obtained by replacing $j+1$ with j and replacing $\bar{j}$ with $\overline{j+1}$. Otherwise, $e_j u = \tilde{e}_j u = 0$.

(4) If u contains n, then $f_n u = \tilde{f}_n u$ is obtained by replacing n with $\bar{n}$. Otherwise, $f_n u = \tilde{f}_n u = 0$.

(5) If u contains $\bar{n}$, then $e_n u = \tilde{e}_n u$ is obtained by replacing $\bar{n}$ with n. Otherwise, $e_n u = \tilde{e}_n u = 0$.

(6) From (2)–(5), $\tilde{e}_i^2 = \tilde{f}_i^2 = 0$ on B_{sp} for any i.

In the rest of this paper, we use the latter description.

5.5. *The Crystal Graph of* $V(\omega_M + \omega_N)$

Now, we investigate the crystal graph of $V(\omega_M + \omega_N)$ $(1 \leqslant M \leqslant N \leqslant n)$. Most of the arguments are similar to the C_n-case; $V(\omega_M + \omega_N)$ can be embedded into $V(\omega_M) \otimes V(\omega_N)$ uniquely and $B(\omega_M + \omega_N)$ is identified with the connected component of $B(\omega_M) \otimes B(\omega_N)$ containing the highest weight vector $u_{\omega_M}\otimes u_{\omega_N}$. We express $u\otimes v$ ($u\in B(\omega_M)$ and $v\in B(\omega_N)$) as in Subsection 4.4.

DEFINITION 5.5.1. (1) For $1\leqslant a\leqslant b<n$, we say that $w=u\otimes v\in I_M^{(B)}\otimes I_N^{(B)}$ is in the (a, b)-configuration if $w=u\otimes v$ satisfies the same condition as Definition 4.4.1.

(2) For $1\leqslant a<n$, we say that $w=u\otimes v\in I_M^{(B)}\otimes I_N^{(B)}$ is in the (a, n)-configuration if there exist $1\leqslant p\leqslant q<r=q+1\leqslant s\leqslant M$ such that $i_p=a$, $j_s=\bar{a}$ and one of the following conditions is satisfied:

(i) i_q and i_r $(=i_{q+1})$ are n, 0, or $\bar{n}$.

(ii) j_q and j_r $(=j_{q+1})$ are n, 0, or $\bar{n}$.

(3) We say that $w=u\otimes v\in I_M^{(B)}\otimes I_N^{(B)}$ is in the (n, n)-configuration if there are $1\leqslant p<q\leqslant M$ such that $i_p=n$ or 0 and $j_q=0$ or $\bar{n}$.

This definition includes the case $a=b$, $p=q$, and $r=s$.

Now, for w in the (a, b)-configuration we define

$$p(a, b; w)=(q-p)+(s-r). \tag{5.5.1}$$

If $a=b=n$, we set $p(a, b; w)=0$.

Let us set

$$I_{(M,N)}^{(B)}=\left\{ w=u\otimes v=\begin{array}{|c|c|}\hline i_1 & j_1 \\ \hline \vdots & \vdots \\ \cline{2-2} \vdots & j_M \\ \hline i_N & \\ \cline{1-1}\end{array}\in I_M^{(B)}\otimes I_N^{(B)};\quad \begin{array}{l} w \text{ satisfies} \\ \text{(M.N.1) and (M.N.2)}\end{array}\right\}. \tag{5.5.2}$$

Here, (M.N.1) and (M.N.2) are the following conditions.

(M.N.1) $i_k\preccurlyeq j_k$ for $1\leqslant k\leqslant M$, and i_k and j_k cannot be 0 simultaneously.

(M.N.2) If w is in the (a, b)-configuration, then $p(a, b; w)<b-a$.

(5.5.3)

When $a=b=n$, (M.N.2) means that there is no $1\leqslant p<q\leqslant M$ such that $i_p=n$ or 0 and that $j_q=0$ or $\bar{n}$.

PROPOSITION 5.5.2. *$B(\omega_M+\omega_N)$ coincides with $I_{(M,N)}^{(B)}$.*

Proof. First, we show that $I_{(M,N)}^{(B)}\cup\{0\}$ is closed by the actions of $\tilde{f}_i$ and $\tilde{e}_i$. In order to see this, we remark the following.

If $w = u \otimes v \in I_M^{(B)} \otimes I_N^{(B)}$ and $\tilde{f}_i(u \otimes v) = u \otimes \tilde{f}_i v \neq 0$, then by Theorem 1.1.4 and Remark 5.3.2 there are two possible cases,

$$\tilde{f}_i u = 0, \tag{5.5.4a}$$

$$\tilde{f}_i^2 u = 0, \qquad \tilde{f}_i u \neq 0, \qquad \text{and} \qquad \tilde{e}_i v \neq 0. \tag{5.5.4b}$$

Now, assuming that $w = u \otimes v \in I_{(M,N)}^{(B)}$ and $\tilde{f}_i w \neq 0$, we prove that (M.N.1) holds for $\tilde{f}_i w$. Otherwise, there are the following cases:

(i) and (ii) These are the same as (1-i) and (1-ii) in the proof of Proposition 4.4.3.

(iii) $i = n$ and there exists $1 \leqslant k \leqslant M$ such that $i_k = j_k = n$ and $i_k = n$ changes to 0 by the action of $\tilde{f}_n$.

(iv) $i = n$ and there exists $1 \leqslant k \leqslant M$ such that $i_k = n$, $j_k = 0$, and $i_k = n$ changes to 0 by the action of $\tilde{f}_n$.

In these cases, $\tilde{f}_i(u \otimes v) = u \otimes \tilde{f}_i v$ and hence (5.5.4a) or (5.5.4b) holds. We can easily obtain that (i) and (ii) cannot occur by similar arguments to the proof of Proposition 4.4.3. In the case (iii), if (5.5.4a) occurs, u contains $\bar{n}$. This contradicts (M.N.2). If (5.5.4b) occurs, we have that $\tilde{e}_n v \neq 0$, but $i_k = n$ implies $\tilde{e}_n v = 0$, this is a contradiction. Hence, (iii) cannot occur. Similarly the case (iv) cannot occur. Thus (M.N.1) holds for $w \in I_{(M,N)}^{(B)}$ with $\tilde{f}_i w \neq 0$. By a similar argument, (M.N.1) holds for $w \in I_{(M,N)}^{(B)}$ with $\tilde{e}_i w \neq 0$.

By a similar argument to the C_n-case and Remark 5.3.3 instead of Lemma 4.4.2, the condition (M.N.2) is preserved by $\tilde{e}_i$ and $\tilde{f}_i$. Thus, we obtain that $I_{(M,N)}^{(B)} \cup \{0\}$ is closed under the action of $\tilde{f}_i$ and $\tilde{e}_i$.

It remains to prove that if $w = u \otimes v \in I_{(M,N)}^{(B)}$ satisfies $\tilde{e}_i w = 0$ $(1 \leqslant i \leqslant n)$ then $w = u_{\omega_M} \otimes u_{\omega_N}$. This is easily obtained by the following lemma.

LEMMA 5.5.3. *If* $u_{\omega_M} \otimes v \in I_{(M,N)}^{(B)}$ *satisfies* $\tilde{e}_i v = 0$ *for* $M < i \leqslant n$, *then* $v = u_{\omega_M}$.

The proof being similar to the proof of Lemma 4.4.4, we omit it.

Thus, we complete the proof of Proposition 5.5.2. Q.E.D.

5.6. *The Crystal Graph of* $V(\omega_M + \Lambda_n)$

We consider the crystal graph of $V(\omega_M + \Lambda_n)$ $(1 \leqslant M \leqslant n)$. The crystal graph $B(\omega_M + \Lambda_n)$ is the connected component of $B(\omega_M) \otimes B(\Lambda_n)$ containing $u_{\omega_M} \otimes u_{\Lambda_n}$.

For

$$u=\begin{array}{|c|}\hline j_1\\ \hline \vdots\\ \hline j_M\\ \hline\end{array}\in I_M^{(B)} \text{ and } v=\begin{array}{|c|}\hline i_1\\ \hline \vdots\\ \hline i_n\\ \hline\end{array}\in B_{sp},\ u\otimes v\in I_M^{(B)}\otimes B_{sp} \text{ will be denoted by } \begin{array}{|c|c|}\hline i_1 & j_1\\ \hline \vdots & \vdots\\ \cline{2-2} & j_M\\ \hline i_n & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array}.$$

Here, v is expressed by (5.4.6).

DEFINITION 5.6.1. Let $u\in I_M^{(B)}$ and $v\in B_{sp}$ be as above. For $1\leqslant a\leqslant b\leqslant n$ we say that $w=u\otimes v$ is in the (a, b)-configuration if w satisfies the same condition as Definition 5.5.1. We define $p(a, b; w)$ by the formula (5.5.1).

Let us set

$$I_{M,sp}^{(B)}=\left\{ w=u\otimes v=\begin{array}{|c|c|}\hline i_1 & j_1\\ \hline \vdots & \vdots\\ \cline{2-2} & j_M\\ \hline i_n & \multicolumn{1}{c}{}\\ \cline{1-1}\end{array}\in I_M^{(B)}\otimes B_{sp};\quad \begin{array}{l} w \text{ satisfies the conditions}\\ \text{(M.sp.1) and (M.sp.2).}\end{array}\right\}. \tag{5.6.1}$$

Here, the conditions (M.sp.1) and (M.sp.2) are

(M.sp.1) $i_k \preccurlyeq j_k$ for $1\leqslant k\leqslant M$,

(M.sp.2) If w is in the (a, b)-configuration, then $p(a, b; w)<b-a$.

Remark 5.6.2. If w is in the (a, b)-configuration, a pair $(b, \bar{b})$ or 0 can appear only in $u\in I_M^{(B)}$.

PROPOSITION 5.6.3. *$B(\omega_M+\Lambda_n)$ coincides with $I_{M,sp}^{(B)}$.*

Proof. Similarly to the previous arguments, we first show that $I_{M,sp}^{(B)}\cup\{0\}$ is closed under the action of $\tilde{e}_i$ and $\tilde{f}_i$. In order to see this, remark the following.

Remark 5.6.4. If $w=u\otimes v\in I_M^{(B)}\otimes B_{sp}$ and $\tilde{f}_i(u\otimes v)=u\otimes\tilde{f}_iv\neq 0$, then by Theorem 1.1.4 and Remark 5.4.1, there is only one possible case, $\tilde{f}_iu=0$.

Assuming that $w=u\otimes v\in I_{M,sp}^{(B)}$ and $\tilde{f}_iw\neq 0$, we prove that (M.sp.1) holds for $\tilde{f}_iw$. Otherwise, there are the following cases:

(i) $i<n$ and there exists $1\leqslant k\leqslant M$ such that $i_k=j_k=i$ and $i_k=i$ changes to $i+1$ by the action of $\tilde{f}_i$.

(ii) $i<n$ and there exists $1\leqslant k\leqslant M$ such that $i_k=j_k=\overline{i+1}$ and $i_k=\overline{i+1}$ changes to $\bar{i}$ by the action of $\tilde{f}_i$.

(iii) $i=n$ and there exists $1\leqslant k\leqslant M$ such that $i_k=j_k=n$ and $i_k=n$ changes to $\bar{n}$ by the action of $\tilde{f}_n$.

(iv) $i=n$ and there exists $1 \leqslant k \leqslant M$ such that $i_k=n$, $j_k=0$, $i_k=n$ changes to $\bar{n}$ by the action of $\tilde{f}_n$.

In these cases, not u but v changes by the action of $\tilde{f}_i$. Therefore we can apply Remark 5.6.4. It is easy to see that (i) and (ii) cannot occur by a similar argument to that in the proof of Proposition 4.4.3.

In the case (iii), v contains n, and then v does not contain $\bar{n}$ by the definition of B_{sp}. Hence $i_{k+1} \geqslant \overline{n-1}$, and (M.sp.1) implies $j_{k+1} \geqslant \overline{n-1}$. Therefore $\bar{n}$ does not appear in u. Then $\tilde{f}_n u \neq 0$ and $\tilde{e}_n v = 0$, which contradicts Remark 5.6.4. Thus, the case (iii) cannot occur. The case (iv) cannot occur by the same arguments as in the case (iii). Thus, we have (M.sp.1) for $\tilde{f}_i w$.

By Remark 5.6.2, we can prove that (M.sp.2) is stable by $\tilde{f}_i$ as in the C_n-case. Thus we have shown that $I^{(B)}_{M,sp}$ is stable by $\tilde{f}_i$. Similarly, we can obtain the stability of $I^{(B)}_{M,sp}$ by $\tilde{e}_i$. Now, it remains to see that if $w = u \otimes v \in I^{(B)}_{M,sp}$ satisfies $\tilde{e}_i w = 0$ $(1 \leqslant i \leqslant n)$, then $w = u_{\omega_M} \otimes u_{\Lambda_n}$. Since $\tilde{e}_i w = 0$ $(1 \leqslant i \leqslant n)$ implies $\tilde{e}_i u = 0$ for any i, u must be u_{ω_M}. Hence, this follows from the following lemma.

LEMMA 5.6.5. *If* $u_{\omega_M} \otimes v \in I^{(B)}_{M,sp}$ *satisfies* $\tilde{e}_i v = 0$ *for* $M < i \leqslant n$, *then* $v = u_{\Lambda_n}$.

Proof. We set

$$v = \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_n \\ \hline \end{array}, \qquad \text{where } i_j \in \{1, 2, \cdots, n, \bar{n}, \cdots, \bar{2}, \bar{1}\}.$$

Then by (M.sp.1), we have $i_\nu = \nu$ $(1 \leqslant \nu \leqslant M)$. If $M = n$, there is nothing to prove. If $M < n$, $\tilde{e}_n v = 0$ implies that v does not contain $\bar{n}$. Hence, v contains n. Since $\tilde{e}_{n-1} v = 0$, v does not contain $\overline{n-1}$. Hence, v contains $n-1$. Repeating this argument we have that $v = u_{\Lambda_n}$. Q.E.D.

Thus we complete the proof of Proposition 5.6.2.

5.7. *The Crystal Graph of* $V(\lambda)$

Let λ be a dominant integral weight. Let us consider the crystal graph of $V(\lambda)$. We can classify λ into following two types:

(E) $\lambda = \sum_{i=1}^{p} \omega_{l_i}$ $(1 \leqslant l_1 \leqslant \cdots \leqslant l_p \leqslant n)$

(O) $\lambda = \sum_{i=1}^{p-1} \omega_{l_i} + \Lambda_n$ $(1 \leqslant l_1 \leqslant \cdots \leqslant l_{p-1} \leqslant n)$.

When λ is of type (E), we define

$$I_\lambda^{(B)} = \left\{ u_1 \otimes \cdots \otimes u_p = \begin{array}{|c|c|c|c|c|} \hline & & & \vdots & t^1_l \\ \cline{5-5} & \vdots & \vdots & & \\ \cline{4-4} t^p_{l'} & & & & \\ \cline{3-3} & & & & \\ \cline{2-2} & & & & \\ \cline{1-1} \end{array} \in I^{(B)}_{l_1} \otimes \cdots \otimes I^{(B)}_{l_p}; \quad \begin{array}{l} u_k \otimes u_{k+1} \in I^{(B)}_{(l_k, l_{k+1})} \\ \text{for } k = 1, \ldots, p-1 \end{array} \right\}.$$

Here, for

$$u_k = \begin{array}{|c|} \hline t^k_1 \\ \hline \vdots \\ \hline t^k_{l_k} \\ \hline \end{array} \in I^{(B)}_{l_k},$$

we denote

$$u_1 \otimes \cdots \otimes u_p \text{ by } \begin{array}{|c|c|c|c|c|} \hline & & & \vdots & t^1_l \\ \cline{5-5} & \vdots & \vdots & & \\ \cline{4-4} t^p_{l'} & & & & \\ \cline{3-3} & & & & \\ \cline{2-2} & & & & \\ \cline{1-1} \end{array} \in I^{(B)}_{l_1} \otimes \cdots \otimes I^{(B)}_{l_p}.$$

When λ is of type (O), we define

$$I_\lambda^{(B)} = \left\{ u_1 \otimes \cdots \otimes u_p = \begin{array}{|c|c|c|c|c|} \hline & & & \vdots & t^1_l \\ \cline{5-5} & \vdots & \vdots & & \\ \cline{4-4} t^p_{l'} & & & & \\ \cline{3-3} & & & & \\ \cline{2-2} & & & & \\ \cline{1-1} \end{array} \in I^{(B)}_{l_1} \otimes \cdots \otimes I^{(B)}_{l_{p-1}} \otimes B_{sp}; \quad \begin{array}{l} u_k \otimes u_{k+1} \in I^{(B)}_{(l_k, l_{k+1})} \\ u_{p-1} \otimes u_p \in I^{(B)}_{l_{p-1}, sp} \\ \text{for } k = 1, \ldots, p-1 \end{array} \right\}.$$

Here, for $u_k \in I^{(B)}_{l_k}$ $(1 \leqslant k < p)$ (with the same expression as above) and,

$$u_p = \begin{array}{|c|}\hline t^p_1 \\ \hline \vdots \\ \hline t^p_n \\ \hline \end{array} \in B_{sp},$$

we denote

$$u_1 \otimes \cdots \otimes u_p \text{ by } \begin{array}{|c|c|c|c|c|}\hline & & & & t^1_l \\ t^p_{l'} & \vdots & \vdots & \vdots & \\ \hline \end{array} \in I^{(B)}_{l_1} \otimes \cdots \otimes I^{(B)}_{l_{p-1}} \otimes B_{sp}.$$

An element of $I^{(B)}_\lambda$ is called a *semi-standard B-tableau* of shape λ.

By the proof of Proposition 5.5.2 and Proposition 5.6.2, we get that $\omega_1, ..., \omega_n$ and Λ_n satisfy the hypothesis of Proposition 2.2.1. Along with Proposition 5.5.2 and Proposition 5.6.2, we obtain the following theorem.

THEOREM 5.7.1. *Let $\lambda \in P_+$.*

(i) *$B(\lambda)$ coincides with the set of semi-standard B-tableaux of shape λ.*

(ii) *The actions of $\tilde{e}_i$ and $\tilde{f}_i$ $(1 \leqslant i < n)$ are given by identifying $\boxed{i}$, $\boxed{\overline{i+1}}$ and the pair $(i, \overline{i+1})$ in an element of B_{sp} with u_+, and $\boxed{i+1}$, $\boxed{\bar{i}}$ and the pair $(i+1, \bar{i})$ in an element of B_{sp} with u_-, and others with u_0 in Remark* 2.1.2 *and the actions of $\tilde{e}_n$ and $\tilde{f}_n$ are given by identifying $\boxed{n}$ with u_1, $\boxed{0}$ with u_2, $\boxed{\bar{n}}$ with u_3 in Remark* 2.1.3 *and n in an element of B_{sp} with u_+, $\bar{n}$ in an element of B_{sp} with u_-, and others with u_0 in Remark* 2.1.2.

EXAMPLE 5.7.2. For $\mathfrak{g} = B_3$ and $\lambda = \Lambda_1 + \Lambda_2 + 3\Lambda_3 = \omega_1 + \omega_2 + \omega_3 + \Lambda_3$, we consider the actions of $\tilde{e}_3$ and $\tilde{f}_3$ on

$$v = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 0 \\ \hline 2 & \bar{3} & \bar{3} \\ \cline{1-3} 3 & \bar{2} \\ \cline{1-2} \end{array} \in B(\lambda).$$

By the construction of crystal graphs, it can be expressed

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 0 \\ \hline 2 & \bar{3} & \bar{3} \\ \cline{1-3} 3 & \bar{2} \\ \cline{1-2} \end{array} = \boxed{0} \otimes \begin{array}{|c|}\hline 3 \\ \hline \bar{3} \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 2 \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} = \boxed{0} \otimes \boxed{3} \otimes \boxed{\bar{3}} \otimes \boxed{2} \otimes \boxed{\bar{3}} \otimes \boxed{\bar{2}} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}.$$

The vector v can be identified with $u_2 \otimes u_1 \otimes u_3 \otimes u_0 \otimes u_3 \otimes u_0 \otimes u_+$, which is the same one in the example in Remark 2.1.3 and then

$$\tilde{e}(u_2 \otimes u_1 \otimes u_3 \otimes u_0 \otimes u_3 \otimes u_0 \otimes u_+) = u_2 \otimes u_1 \otimes u_3 \otimes u_0 \otimes u_2 \otimes u_0 \otimes u_+,$$

$$\tilde{f}(u_2 \otimes u_1 \otimes u_3 \otimes u_0 \otimes u_3 \otimes u_0 \otimes u_+) = u_2 \otimes u_1 \otimes u_3 \otimes u_0 \otimes u_3 \otimes u_0 \otimes u_-.$$

Thus,

$$\tilde{e}_3\left(\boxed{0}\otimes\boxed{3}\otimes\boxed{\bar{3}}\otimes\boxed{2}\otimes\boxed{\bar{3}}\otimes\boxed{\bar{2}}\otimes\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right)$$

$$=\boxed{0}\otimes\boxed{3}\otimes\boxed{\bar{3}}\otimes\boxed{2}\otimes\boxed{0}\otimes\boxed{\bar{2}}\otimes\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array},$$

$$\tilde{f}_3\left(\boxed{0}\otimes\boxed{3}\otimes\boxed{\bar{3}}\otimes\boxed{2}\otimes\boxed{\bar{3}}\otimes\boxed{\bar{2}}\otimes\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right)$$

$$=\boxed{0}\otimes\boxed{3}\otimes\boxed{\bar{3}}\otimes\boxed{2}\otimes\boxed{\bar{3}}\otimes\boxed{\bar{2}}\otimes\begin{array}{|c|}\hline 1\\\hline 2\\\hline \bar{3}\\\hline\end{array}.$$

Hence, we obtain

$$\tilde{e}_3\begin{array}{|c|c|c|c|}\hline 1&2&3&0\\\hline 2&\bar{3}&\bar{3}\\\cline{1-3} 3&\bar{2}\\\cline{1-2}\end{array}=\begin{array}{|c|c|c|c|}\hline 1&2&3&0\\\hline 2&0&\bar{3}\\\cline{1-3} 3&\bar{2}\\\cline{1-2}\end{array},\qquad \tilde{f}_3\begin{array}{|c|c|c|c|}\hline 1&2&3&0\\\hline 2&\bar{3}&\bar{3}\\\cline{1-3} 3&\bar{2}\\\cline{1-2}\end{array}=\begin{array}{|c|c|c|c|}\hline 1&2&3&0\\\hline 2&\bar{3}&\bar{3}\\\cline{1-3} \bar{3}&\bar{2}\\\cline{1-2}\end{array}.$$

6. Crystal Graphs for $U_q(D_n)$-Modules

6.1. *Notation*

We treat the D_n-case in this section. Let $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ be the orthonormal base of the dual of the Cartan subalgebra of D_n such that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leqslant i < n)$ and $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$ are the simple roots. Let $\{\Lambda_i\}_{1 \leqslant i \leqslant n}$ be the dual base of $\{h_i\}_{1 \leqslant i \leqslant n}$. Hence $\Lambda_i = \varepsilon_1 + \cdots + \varepsilon_i$ $(i = 1, ..., n-2)$ and $\Lambda_{n-1} = (\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n)/2$, $\Lambda_n = (\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n)/2$.

6.2. *The Crystal Graph of the Vector Representation*

Let us construct the vector representation $V_\square$. Letting $\boxed{i}$, $\boxed{\bar{i}}$ $(1 \leqslant i \leqslant n)$ be the base of $\mathbf{Q}(q)^{\oplus 2n}$, the vector representation $V_\square$ of $U_q(D_n)$ is explicitly given as

$$q^h \boxed{j} = q^{\langle h, \varepsilon_j \rangle} \boxed{j}, \qquad q^h \boxed{\bar{j}} = q^{\langle h, -\varepsilon_j \rangle} \boxed{\bar{j}}, \tag{6.2.1}$$

$$\left.\begin{array}{ll} e_i \boxed{j} = \delta_{i+1,j} \boxed{j-1}, & e_i \boxed{\bar{j}} = \delta_{i,j} \boxed{\overline{j+1}} \\ f_i \boxed{j} = \delta_{i,j} \boxed{j+1}, & f_i \boxed{\bar{j}} = \delta_{i+1,j} \boxed{\overline{j-1}} \end{array}\right. \quad (1 \leqslant i < n,\ 1 \leqslant j \leqslant n), \tag{6.2.2}$$

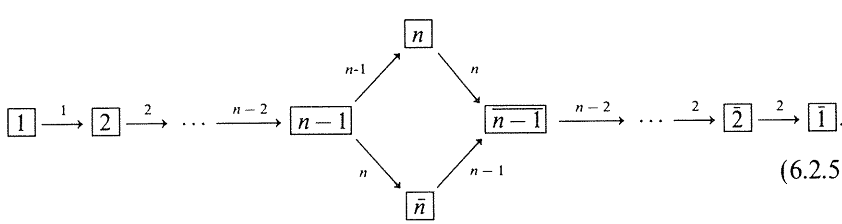
$$\left.\begin{array}{ll} e_n \boxed{j} = 0, & e_n \boxed{\bar{j}} = \begin{cases} \boxed{n} & \text{if } j = n-1, \\ \boxed{n-1} & \text{if } j = n, \\ 0 & \text{otherwise,} \end{cases} \\ f_n \boxed{\bar{j}} = 0, & f_n \boxed{j} = \begin{cases} \boxed{\bar{n}} & \text{if } j = n-1, \\ \boxed{\overline{n-1}} & \text{if } j = n, \\ 0 & \text{otherwise,} \end{cases} \end{array}\right. \quad (1 \leqslant j \leqslant n). \tag{6.2.3}$$

Here, we understand $\boxed{j} = \boxed{\bar{j}} = 0$ unless $j = 1, \ldots, n$.

Now, the crystal base $(L(V_\square), B(V_\square))$ is given by

$$\begin{aligned} L(V_\square) &= \bigoplus_{i=1}^{n} (A \boxed{i} \oplus A \boxed{\bar{i}}), \\ B(V_\square) &= \left\{ \boxed{i}, \boxed{\bar{i}};\ 1 \leqslant i \leqslant n \right\}. \end{aligned} \tag{6.2.4}$$

From (6.2.2) and (6.2.3) we obtain the actions of $\tilde{e}_i$ and $\tilde{f}_i$ by replacing e_i and f_i with $\tilde{e}_i$ and $\tilde{f}_i$. Then the crystal graph of $V_\square$ is described as



(6.2.5)

6.3. *The Crystal Graph of the Anti-symmetric Tensor Representations*

We set $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ $(1 \leqslant i \leqslant n)$ and $\overline{\omega_n} = \varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n$. We call the irreducible representation with highest weight ω_i (resp. $\overline{\omega_n}$) "anti-symmetric tensor representation with highest weight ω_i (resp. $\overline{\omega_n}$)" and denote $V(\omega_i)$ (resp. $V(\overline{\omega_n})$). Here, remark that $\omega_i = \Lambda_i$ $(1 \leqslant i \leqslant n-2)$, $\omega_{n-1} = \Lambda_{n-1} + \Lambda_n$, $\omega_n = 2\Lambda_n$, and $\overline{\omega_n} = 2\Lambda_{n-1}$.

$V(\omega_N)$ $(1 \leqslant N \leqslant n)$ (resp. $V(\overline{\omega_n})$) can be embedded into $V_{\square}^{\otimes N}$ (resp. $V_{\square}^{\otimes n}$). Similarly to the previous cases, the connected component of $B(V_{\square}^{\otimes N}) = B(V_{\square})^{\otimes N}$ (resp. $B(V_{\square}^{\otimes N}) = B(V_{\square})^{\otimes n}$) containing $\boxed{1} \otimes \cdots \otimes \boxed{N}$ (resp. $\boxed{1} \otimes \cdots \otimes \boxed{n-1} \otimes \boxed{\bar{n}}$) is $B(\omega_N)$ (resp. $V(\overline{\omega_n})$).

We denote by u_{ω_N} (resp. $u_{\overline{\omega_n}}$) the highest weight vector $\boxed{1} \otimes \cdots \otimes \boxed{N}$ (resp. $\boxed{1} \otimes \cdots \otimes \boxed{n-1} \otimes \boxed{\bar{n}}$). We give the ordering on $\{i, \bar{i};\ 1 \leqslant i \leqslant n\}$ by

$$1 \prec 2 \prec \cdots \prec n-1 \prec \begin{matrix} n \\ \bar{n} \end{matrix} \prec \overline{n-1} \prec \cdots \prec \bar{2} \prec \bar{1}. \tag{6.3.1}$$

Remark that there is no order between n and $\bar{n}$. This ordering is derived from the crystal graph of $V_{\square}$ (6.2.5). Using the same expression as Subsections 4.3 and 5.3, for $1 \leqslant N < n$ we set

$$I_N^{(D)} = \left\{ \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_N \\ \hline \end{array} \in B(V_{\square})^{\otimes N};\quad \begin{array}{l} (1)\ i_\nu \not\succeq i_{\nu+1} \text{ for } 1 \leqslant \nu < N, \\ (2)\ \text{if } i_k = p \text{ and } i_l = \bar{p}\ (1 \leqslant p \leqslant n) \\ \quad \text{then } k + (N - l + 1) \leqslant p \end{array} \right\}. \tag{6.3.2}$$

Remark 6.3.1. (i) The condition (1) in (6.3.2) is equivalent to saying that for any ν either $i_\nu \prec i_{\nu+1}$, or $(i_\nu, i_{\nu+1}) = (n, \bar{n})$ or $(\bar{n}, n)$.

(ii) The actions of $\tilde{e}_i$ and $\tilde{f}_i$ $(1 \leqslant i < n)$ are given by identifying $\boxed{i}$ and $\boxed{\overline{i+1}}$ with u_+, $\boxed{i+1}$ and $\boxed{\bar{i}}$ with u_-, and others with u_0 in Remark 2.1.2. The actions of $\tilde{e}_n$ and $\tilde{f}_n$ are given by identifying $\boxed{\overline{n-1}}$ and $\boxed{n}$ with u_+, $\boxed{n-1}$ and $\boxed{\bar{n}}$ with u_-, and others with u_0 in Remark 2.1.2.

PROPOSITION 6.3.2. *$B(\omega_N)$ coincides with $I_N^{(D)}$ $(1 \leqslant N < n)$.*

Proof. In order to see this, we have to show the following two statements:

(a) $I_N^{(D)} \cup \{0\}$ is stable by $\tilde{e}_i$ and $\tilde{f}_i$.

(b) If $u \in I_N^{(D)}$ satisfies $\tilde{e}_i u = 0$ for any i, then $u = \boxed{1} \otimes \cdots \otimes \boxed{N}$.

Here, (a) can be easily obtained by Remark 2.1.2, Remark 6.3.1, and a similar argument to the proof of Proposition 4.3.2.

Let us show (b). Assuming that $u = \boxed{i_1} \otimes \cdots \otimes \boxed{i_N} \in I_N^{(D)}$ satisfy $\tilde{e}_i u = 0$ for any i and $u \neq \boxed{1} \otimes \cdots \otimes \boxed{N}$, we take k $(1 \leqslant k < N)$ such that $i_\nu = \nu$ for $\nu < k$ and $i_k \succ k$. If $i_k \in \{k+1, ..., n-1\}$, then $\tilde{e}_{i_k - 1} u \neq 0$ and if $i_k \in \{\bar{k}, \cdots, \overline{n-1}\}$, then there exist some p such that $i_k = \bar{p}$ and $\tilde{e}_p u \neq 0$ by Remark 2.1.2 and Remark 6.3.1. If $i_k = n$ or $\bar{n}$, then $i_{k-1} \neq n-1$

by $k \leqslant N < n$ and then $\tilde{e}_{n-1}u \neq 0$ or $\tilde{e}_n u \neq 0$, respectively. If $i_k = \bar{p} \in \{\bar{1}, \cdots, \overline{k-1}\}$ and $k > 1$, then $p + (N-k+1) > p$, which contradicts the condition (2) of (6.3.2). Thus we get $i_k = k$, which contradicts the definition of k. Hence, we obtain $u = \boxed{1} \otimes \cdots \otimes \boxed{N}$. Q.E.D.

Next, we set

$$I_n^{(D)} \text{ (resp. } \bar{I}_n^{(D)}) = \left\{ \begin{array}{|c|} \hline i_1 \\ \hline \vdots \\ \hline i_n \\ \hline \end{array} \in B(V_{\llcorner})^{\otimes n}; \quad \begin{array}{l} (1) \text{ and } (2) \text{ are the same conditions as in } (6.3.2) \\ (3) \text{ If } i_k = n, \text{ then } n-k \text{ is even (resp. odd)} \\ \quad \text{and if } i_k = \bar{n}, \text{ then } n-k \text{ is odd (resp. even)} \end{array} \right\}. \tag{6.3.3}$$

Remark 6.3.3. (i) Any element of $I_n^{(D)}$ or $\bar{I}_n^{(D)}$ contains either n or $\bar{n}$. This can be proved similarly to Lemma 4.3.1.

(ii) Under the conditions (1) and (2) in (6.3.3), if (3) is satisfied for some k such that $i_k = n$ or $\bar{n}$, then it is satisfied for all k since n and $\bar{n}$ appear in u alternatively and successively.

PROPOSITION 6.3.4. *$B(\omega_n)$ (resp. $B(\overline{\omega_n})$) coincides with $I_n^{(D)}$ (resp. $\bar{I}_n^{(D)}$).*

Proof. Similarly to the previous proof, we have to show the following two statements:

(a) $I_n^{(D)} \cup \{0\}$ (resp. $\bar{I}_n^{(D)} \cup \{0\}$) is stable by $\tilde{f}_i$ and $\tilde{e}_i$.

(b) If $u \in I_n^{(D)}$ (resp. $\bar{I}_n^{(D)}$) satisfies $\tilde{e}_i u = 0$ for any i, then $u = \boxed{1} \otimes \cdots \otimes \boxed{n}$ (resp. $\boxed{1} \otimes \cdots \otimes \boxed{n-1} \otimes \boxed{\bar{n}}$).

We can easily obtain (b) by a similar argument to the proof of Proposition 6.3.2.

For $u \in I_n^{(D)}$ (resp. $\bar{I}_n^{(D)}$), (a) follows as before if $\tilde{f}_i u$ and $\tilde{e}_i u$ satisfy the condition (3) or are zero. Now, we assume that $u = \boxed{i_1} \otimes \cdots \otimes \boxed{i_n} \in I_n^{(D)}$ (resp. $\bar{I}_n^{(D)}$) and $\tilde{f}_i u \neq 0$ for some i, and does not satisfy the condition (3). Under this assumption, there are the following two possibilities:

(i) There exists k such that $n-k$ is odd (resp. even) and $i_k = n-1$ changes to n by $\tilde{f}_{n-1}$.

(ii) There exists k such that $n-k$ is even (resp. odd) and $i_k = n-1$ changes to $\bar{n}$ by $\tilde{f}_n$.

In both the cases, u contains n or $\bar{n}$ by Remark 6.3.3(i), hence we know that both the cases contradict Remark 6.3.3(ii). Then we obtain that for $u \in I_n^{(D)}$ (resp. $\bar{I}_n^{(D)}$), $\tilde{f}_i u$ satisfies the condition (3) or vanishes. We obtain the same conclusion for $\tilde{e}_i u$ by similar arguments. Q.E.D.

Remark 6.3.5. By Remark 2.1.2, we obtain

(i) $\tilde{e}_i^3 = \tilde{f}_i^3 = 0$ on $I_N^{(D)}$ and $\bar{I}_n^{(D)}$.

(ii) If $u \in I_N^{(D)}$ or $\bar{I}_n^{(D)}$ satisfies $\tilde{f}_i^2 u \neq 0$ $(1 \leqslant i < n)$, then u contains i and $\overline{i+1}$ but neither $i+1$ nor $\bar{i}$. If $u \in I_N^{(D)}$ or $\bar{I}_n^{(D)}$ satisfies $\tilde{e}_i^2 u \neq 0$ $(1 \leqslant i < n)$, then u contains $i+1$ and $\bar{i}$ but neither i nor $\overline{i+1}$.

(iii) If $u \in I_N^{(D)}$ or $\bar{I}_n^{(D)}$ satisfies $\tilde{f}_i u \neq 0$ and $\tilde{e}_i u \neq 0$ $(1 \leqslant i < n)$, then u contains $i+1$ and $\overline{i+1}$ but neither i nor $\bar{i}$.

(iv) If $u \in I_N^{(D)}$ or $\bar{I}_n^{(D)}$ satisfies $\tilde{f}_n^2 u \neq 0$, then u contains $n-1$ and n. If $u \in I_N^{(D)}$ or $\bar{I}_n^{(D)}$ satisfies $\tilde{e}_n^2 u \neq 0$, then u contains $\overline{n-1}$ and $\bar{n}$ but neither $n-1$ nor n.

(v) If $u \in I_N^{(D)}$ or $\bar{I}_n^{(D)}$ satisfies $\tilde{f}_n u \neq 0$ and $\tilde{e}_n u \neq 0$, then u contains n and $\bar{n}$ but neither $n-1$ nor $\overline{n-1}$.

6.4. *The Crystal Graph of the Spin Representations $V_{sp}^{(+)}$ and $V_{sp}^{(-)}$*

For D_n, there are two spin representations $V_{sp}^{(+)}$ and $V_{sp}^{(-)}$. They are the finite-dimensional irreducible representations with highest weight Λ_n and Λ_{n-1}, respectively. They have the explicit description as follows (cf. [Re]); we set

$$B_{sp}^{(+)} \text{ (resp. } B_{sp}^{(-)}) = \{v = (i_1, ..., i_n);\ i_j = \pm,\ i_1 \cdots i_n = + \text{ (resp. } -)\},$$

$$V_{sp}^{(\pm)} = \bigoplus_{v \in B_{sp}^{(\pm)}} \mathbf{Q}(q)\, v.$$

We define the actions of $U_q(\mathfrak{g})$ on $V_{sp}^{(\pm)}$ as

$$q^h v = q^{\langle h, wt(v) \rangle} v, \qquad \text{where} \quad wt(v) = \frac{1}{2} \sum_{j=1}^{n} i_j \varepsilon_j \qquad \text{for} \quad v = (i_1, \cdots, i_n); \tag{6.4.1}$$

$$e_j(i_1, ..., i_n) = \begin{cases} (i_1, ..., \overset{j}{+}, \overset{j+1}{-}, ..., i_n), & i_j = - \text{ and } i_{j+1} = +, \\ 0, & \text{otherwise,} \end{cases} \tag{6.4.2}$$

$$f_j(i_1, ..., i_n) = \begin{cases} (i_1, ..., \overset{j}{-}, \overset{j+1}{+}, ..., i_n), & i_j = + \text{ and } i_{j+1} = -, \\ 0, & \text{otherwise} \end{cases} \tag{6.4.3}$$

$(j = 1, \cdots, n-1);$

$$e_n(i_1, ..., i_n) = \begin{cases} (i_1, ..., \overset{n-1}{+}, \overset{n}{+}), & i_{n-1} = - \text{ and } i_n = -, \\ 0, & \text{otherwise;} \end{cases} \tag{6.4.4}$$

$$f_n(i_1, ..., i_n) = \begin{cases} (i_1, ..., \overset{n-1}{-}, \overset{n}{-}), & i_{n-1} = + \text{ and } i_n = +, \\ 0, & \text{otherwise.} \end{cases} \tag{6.4.5}$$

If we set $L_{sp}^{(\pm)} = \bigoplus_{v \in B_{sp}^{(\pm)}} Av$, then $(L_{sp}^{(\pm)}, B_{sp}^{(\pm)})$ is the crystal base of $V_{sp}^{(\pm)}$ with the same actions of $\tilde{e}_i$ and $\tilde{f}_i$ as those of e_i and f_i in (6.4.2)–(6.4.5).

Now, similar to the B_n-case we introduce another expression of $B_{sp}^{(+)}$ and $B_{sp}^{(-)}$. We give the order on $P = \{1, 2, ..., n, \bar{n}, ..., \bar{2}, \bar{1}\}$ as the same as (6.3.1). Then we set

$$B_{sp}^{(+)} \text{ (resp. } B_{sp}^{(-)}) = \left\{ v = \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_n \\ \hline \end{array}\ ; \quad \begin{array}{ll} (1) & i_j \in P \\ (2) & i_1 \prec i_2 \prec \cdots \prec i_{n-1} \prec i_n \\ (3) & i \text{ and } \bar{i} \text{ cannot appear simultaneously} \\ (4) & \text{If } i_k = n, \text{ then } n-k \text{ is even (resp. odd)} \\ (5) & \text{If } i_k = \bar{n}, \text{ then } n-k \text{ is odd (resp. even)} \end{array} \right\}. \tag{6.4.6}$$

In this expression, i corresponds to the ith $+$ in the previous expression and $\bar{i}$ corresponds to the ith $-$. This gives the actions of $\tilde{e}_i$ and $\tilde{f}_i$.

Remark 6.4.1. Let $u \in B_{sp}^{(\pm)}$ be with the notation as in (6.4.6).

(1) For $j = 1, ..., n-1$ if u contains j and $\overline{j+1}$, then $f_j u = \tilde{f}_j u$ is the element of $B_{sp}^{(\pm)}$ obtained by replacing j with $j+1$ and replacing $\overline{j+1}$ with $\bar{j}$. Otherwise, $f_j u = \tilde{f}_j u = 0$.

(2) For $j = 1, ..., n-1$ if u contains $j+1$ and $\bar{j}$, then $e_j u = \tilde{e}_j u$ is the element of $B_{sp}^{(\pm)}$ obtained by replacing $j+1$ with j and replacing $\bar{j}$ with $\overline{j+1}$. Otherwise, $e_j u = \tilde{e}_j u = 0$.

(3) If u contains $n-1$ and n, then $f_n u = \tilde{f}_n u$ is the element of $B_{sp}^{(\pm)}$ obtained by replacing $n-1$ with $\bar{n}$ and replacing n with $\overline{n-1}$. Otherwise, $f_n u = \tilde{f}_n u = 0$.

(4) If u contains $\bar{n}$ and $\overline{n-1}$, then $e_n u = \tilde{e}_n u$ is the element of $B_{sp}^{(\pm)}$ obtained by replacing $\bar{n}$ with $n-1$ and replacing $\overline{n-1}$ with n. Otherwise, $e_n u = \tilde{e}_n u = 0$.

(5) From (1)–(4), $\tilde{e}_i^2 = \tilde{f}_i^2 = 0$ on $B_{sp}^{(\pm)}$ for any i.

6.5. *The Crystal Graph of* $V(\omega_M+\omega_N)$

We investigate the crystal graph of $V(\omega_M+\omega_N)$ $(1\leqslant M\leqslant N\leqslant n)$. The representation $V(\omega_M+\omega_N)$ can be embedded into $V(\omega_M)\otimes V(\omega_N)$ uniquely and $B(\omega_M+\omega_N)$ is the connected component of $B(\omega_M+\omega_N)$ containing the highest weight vector $u_{\omega_M}\otimes u_{\omega_N}$. Similarly to Definition 4.4.1 and with the same notations there, we give the definition of the (a,b)-configuration in the D_n-case.

DEFINITION 6.5.1. (1) For $1\leqslant a\leqslant b<n$, we say that $u\otimes v\in I_M^{(D)}\otimes I_N^{(D)}$ is in the (a,b)-configuration if $u\otimes v$ satisfies the same condition as in Definition 4.4.1.

(2) For $1\leqslant a<n$, we say that $u\otimes v\in I_M^{(D)}\otimes I_N^{(D)}$ is in the (a,n)-configuration if there exists $1\leqslant p\leqslant q<r=q+1\leqslant s\leqslant M$ such that $i_p=a$, $j_s=\bar{a}$, and one of the following conditions is satisfied:

(i) i_q and $i_r(=i_{q+1})$ are n or $\bar{n}$.

(ii) j_q and $j_r(=j_{q+1})$ are n or $\bar{n}$.

(3) We say that $u\otimes v\in I_M^{(D)}\otimes I_N^{(D)}$ is in the (n,n)-configuration if there exist $1\leqslant p<q\leqslant M$ such that $i_p=n$ or $\bar{n}$ and that $j_q=n$ or $\bar{n}$.

(4) For $1\leqslant a<n$, we say that $u\otimes v\in I_M^{(D)}\otimes I_N^{(D)}$ is in the a-odd-configuration if $u\otimes v$ satisfies the following conditions; there exist $1\leqslant p\leqslant q<r\leqslant s\leqslant M$ such that

(a) $r-q+1$ is odd.

(b) $i_p=a$ and $j_s=\bar{a}$.

(c) $j_q=n$, $i_r=\bar{n}$ or $j_q=\bar{n}$, $i_r=n$.

(5) For $1\leqslant a<n$, we say that $u\otimes v\in I_M^{(D)}\otimes I_N^{(D)}$ is in the a-even-configuration if $u\otimes v$ satisfies the following conditions; there exist $1\leqslant p\leqslant q<r\leqslant s\leqslant M$ such that

(a) $r-q+1$ is even.

(b) $i_p=a$ and $j_s=\bar{a}$.

(c) $j_q=n$, $i_r=n$ or $j_q=\bar{n}$, $i_r=\bar{n}$.

If w is in the (a,b)-configuration for $1\leqslant a\leqslant b\leqslant n$, we define $p(a,b;w)=(q-p)+(s-r)$. When $a=b=n$, we set $p(a,b;w)=0$.

If $w\in I_M^{(D)}\otimes I_N^{(D)}$ is in the a-odd-configuration or the a-even-configuration, we define

$$q(a;w)=s-p. \tag{6.5.1}$$

For $1 \leqslant M \leqslant N \leqslant n$, let us set

$$I^{(D)}_{(M,N)} = \left\{ w = u \otimes v = \begin{array}{|c|c|} \hline i_1 & j_1 \\ \hline \vdots & \vdots \\ \cline{2-2} & j_M \\ \hline i_N \\ \cline{1-1} \end{array} \in I^{(D)}_M \otimes I^{(D)}_N; \quad \begin{array}{l} w \text{ satisfies the following} \\ \text{(M.N.1)–(M.N.3)} \end{array} \right\}. \tag{6.5.2}$$

Here, the conditions (M.N.1)–(M.N.3) are as follows.

(M.N.1) $i_k \preccurlyeq i_j$ for $1 \leqslant k \leqslant M$.

(M.N.2) If w is in the (a, b)-configuration, then $p(a, b; w) < b - a$.

(M.N.3) If w is in the a-odd-configuration or the a-even-configuration, then $q(a; w) < n - a$.

When $a = b = n$, (M.N.2) means that there is no $1 \leqslant p < q \leqslant M$ such that $i_p = n$ or $\bar{n}$ and that $j_q = n$ or $\bar{n}$.

PROPOSITION 6.5.2. *$B(\omega_M + \omega_N)$ coincides with $I^{(D)}_{(M,N)}$ for $1 \leqslant M \leqslant N \leqslant n$.*

Proof. We have to show

(1) $I^{(D)}_{(M,N)} \cup \{0\}$ is stable by $\tilde{e}_i$ and $\tilde{f}_i$.

(2) If $w \in I^{(D)}_{(M,N)}$ satisfies that $\tilde{e}_i w = 0$ for any i, then $w = u_{\omega_M} \otimes u_{\omega_N}$.

First we show (1). In order to see this, we remark that if $w = u \otimes v \in I^{(D)}_M \otimes I^{(D)}_N$ and $\tilde{f}_i(u \otimes v) = u \otimes \tilde{f}_i v \neq 0$, then by Theorem 1.1.4 and Remark 6.3.5 there are two possible cases:

$$\tilde{f}_i u = 0 \tag{6.5.3a}$$

$$\tilde{f}_i^2 u = 0, \qquad \tilde{f}_i u \neq 0, \qquad \text{and} \qquad \tilde{e}_i v \neq 0. \tag{6.5.3b}$$

Assuming that $u \otimes v \in I^{(D)}_{(M,N)}$ and $\tilde{f}_i(u \otimes v) \neq 0$, we prove that (M.N.1) holds for $\tilde{f}_i(u \otimes v)$. Otherwise, there are the following cases:

(1-i) $i < n$ and there exists $1 \leqslant k \leqslant M$ such that $i_k = j_k = i$ and $i_k = i$ changes to $i+1$ by $\tilde{f}_i$.

(1-ii) $i < n$ and there exists $1 \leqslant k \leqslant M$ such that $i_k = j_k = \overline{i+1}$ and $i_k = \overline{i+1}$ changes to $\bar{i}$ by $\tilde{f}_i$.

(1-iii) $i = n$ and there exists $1 \leqslant k \leqslant M$ such that $i_k = j_k = n-1$ and $i_k = n-1$ changes to $\bar{n}$ by $\tilde{f}_n$.

(1-iv) $i = n$ and there exists $1 \leqslant k \leqslant M$ such that $i_k = j_k = n$ and $i_k = n$ changes to $\overline{n-1}$ by $\tilde{f}_n$.

(1-v) $i = n$ and there exists $1 \leqslant k \leqslant M$ such that $i_k = n-1$, $j_k = n$, and $i_k = n-1$ changes to $\bar{n}$ by $\tilde{f}_n$.

(1-vi) $i = n-1$ and there exists $1 \leqslant k \leqslant M$ such that $i_k = n-1, j_k = \bar{n}$, and $i_k = n-1$ changes to n by $\tilde{f}_{n-1}$.

In the case (1-i), not u but v changes by $\tilde{f}_i$. Then we may consider (6.5.3a) and (6.5.3b). If $\tilde{f}_i u = 0$, then u contains $i+1$ or $\bar{i}$. In any case, we can easily derive a contradiction similar to the C_n-case. If (6.5.3b) occurs, v contains $i+1$ and $\bar{i}$ because $\tilde{e}_i v \neq 0$. This is a contradiction to the fact that i in v changes to $i+1$. Thus (1-i) cannot occur.

In the case (1-ii), not u but v changes by $\tilde{f}_i$, hence (6.5.3a) or (6.5.3b) occurs. If (6.5.3a) occurs, u satisfies $j_{k+1} = \bar{i}$ by applying Remark 2.1.2 and Remark 6.3.1 to (6.5.3a). Hence, we get $\overline{i+1} = i_k \prec i_{k+1} \preccurlyeq j_{k+1} = \bar{i}$. This implies $i_{k+1} = \bar{i}$, which contradicts the fact that $i_k = \overline{i+1}$ changes to $\bar{i}$. If (6.5.4b) occurs, v contains $i+1$ by Remark 6.3.5(iii). Thus $u \otimes v$ is in the $(i+1, i+1)$-configuration, which contradicts (M.N.2). Hence, (1-ii) cannot occur.

In the case (1-iii), $i_k = n-1$ in v changes to $\bar{n}$. Then (6.5.3a) or (6.5.3b) occurs. If (6.5.3a) occurs, then u contains $\overline{n-1}$ or $\bar{n}$. If u contains $\overline{n-1}$, $u \otimes v$ is in the $(n-1, n-1)$-configuration, then this cannot occur by (M.N.2). If u contains $\bar{n}$ and no $\overline{n-1}$, u satisfies $j_{k+1} = \bar{n}$. Then we have $n-1 = i_k \prec i_{k+1} \preccurlyeq j_{k+1} = \bar{n}$, this implies $i_{k+1} = \bar{n}$. This contradicts the fact that $i_k = n-1$ changes to $\bar{n}$. Hence (6.5.3a) cannot occur. If (6.5.3b) occurs, $\tilde{f}_n v \neq 0$, $\tilde{e}_n v \neq 0$, and $i_k = n-1$. This contradicts Remark 6.3.5(v). Thus (1-iii) cannot occur.

In the case (1-iv), $i_k = n$ in v changes to $\overline{n-1}$. This implies that $i_{k+1} \succ \overline{n-1}$, then $j_{k+1} \succ \overline{n-1}$. Hence, $\tilde{f}_n u \neq 0$. We may consider only (6.5.3b). By $\tilde{e}_n v \neq 0$, we have $i_{k-1} \succ \bar{n}$, which implies that $u \otimes v$ is in the (n, n)-configuration. This contradicts (M.N.2). Thus (1-iv) cannot occur.

In the case (1-v), since v changes, we may consider (6.5.3a) and (6.5.3b). If (6.5.3a) occurs, u contains $\bar{n}$ or $\overline{n-1}$. If u contains $\bar{n}$, then $j_{k+1} = \bar{n}$ and therefore we have $n-1 = i_k \prec i_{k+1} \preccurlyeq j_{k+1} = \bar{n}$. This implies $i_{k+1} = \bar{n}$, and this contradicts the fact that $i_k = n-1$ changes to $\bar{n}$. If u contains $\overline{n-1}$ and no $\bar{n}$, then $i_{k+1} = \overline{n-1}$ and we have $n-1 = i_k \prec i_{k+1} \preccurlyeq j_{k+1} = \overline{n-1}$. This implies $i_{k+1} = n, \bar{n}$, or $\overline{n-1}$. First $i_{k+1} = \bar{n}$ never occurs because $i_k = n-1$ changes to $\bar{n}$. If $i_{k+1} = \overline{n-1}$, then $i_k = n-1$ does not change by $\tilde{f}_n$ by Remark 2.1.2. If $i_{k+1} = n$, then $u \otimes v$ is in the $(n-1)$-even-configuration ($p = q = k$ and $r = s = k+1$ in Definition 6.5.1(4)) and $q(n-1; u \otimes v) = (k+1) - k = n - (n-1)$, which contradicts (M.N.3). Thus (6.5.3a) cannot occur. If (6.5.3b) occurs, $\tilde{e}_n u \neq 0$ and $\tilde{f}_n u \neq 0$ contradicts the fact that v contains $n-1$ by Remark 6.3.5(v). Thus (1-v) cannot occur.

In the case (1-vi), v changes. Then (6.5.3a) or (6.5.3b) holds. If (6.5.3a) occurs, then $j_{k+1} = n$ or $\overline{n-1}$ because $\tilde{f}_{n-1} u = 0$. If $j_{k+1} = n$, we have $n-1 = i_k \prec i_{k+1} \preccurlyeq j_{k+1} = n$. This implies $i_{k+1} = n$, which contradicts the fact that $i_k = n-1$ changes to n. If $j_{k+1} = \overline{n-1}$, then we have $n-1 =$

$i_k \prec i_{k+1} \preccurlyeq j_{k+1} = \overline{n-1}$. This implies $i_{k+1} = n$, $\bar{n}$, or $\overline{n-1}$. If $i_{k+1} = n$ or $\overline{n-1}$, this contradicts the fact that $i_k = n-1$ changes by $\tilde{f}_{n-1}$ by Remark 2.1.2. If $i_{k+1} = \bar{n}$, $u \otimes v$ is in the $(n-1)$-even-configuration and $q(n-1; u \otimes v) = (k+1) - k = n - (n-1)$, which contradicts (M.N.3). Thus (6.5.3a) cannot occur. If (6.5.3b) occurs, $\tilde{f}_{n-1} v \neq 0$, $\tilde{e}_{n-1} v \neq 0$, and v contains $n-1$. This contradicts Remark 6.3.5(iii). Hence, (1-vi) cannot occur. Thus (M.N.1) is preserved by $\tilde{f}_i$. By a similar argument, (M.N.1) is preserved by $\tilde{e}_i$.

We can obtain that (M.N.2) is stable by $\tilde{e}_i$ and $\tilde{f}_i$ by similar arguments to the cases of B_n and C_n.

Finally, assuming that $w = u \otimes v \in I^{(D)}_{(M,N)}$ and $\tilde{f}_i w \neq 0$, we prove that $\tilde{f}_i w$ satisfies (M.N.3). Otherwise, there are the following cases.

(2-i) There exist $1 \leqslant p \leqslant q < r \leqslant s \leqslant M$ and $1 \leqslant a < n-1$ such that $i_p = a$, $j_q = n$ (resp. $\bar{n}$), $i_r = \bar{n}$ (resp. n), $j_s = \overline{a+1}$, and $i_p = a$ changes to $a+1$ by $\tilde{f}_a$, then $\tilde{f}_a w$ is in the $(a+1)$-odd-configuration and $q(a+1; \tilde{f}_a w) \geqslant n-(a+1)$.

(2-ii) There exist $1 \leqslant p \leqslant q < r \leqslant s \leqslant M$ and $1 \leqslant a < n-1$ such that $i_p = a$, $j_q = n$ (resp. $\bar{n}$), $i_r = \bar{n}$ (resp. n), $j_s = \overline{a+1}$, and $j_s = \overline{a+1}$ changes to $\bar{a}$ by $\tilde{f}_a$, then $\tilde{f}_a w$ is in the a-odd-configuration and $q(a; \tilde{f}_a w) \geqslant n-a$.

(2-iii) There exist $1 \leqslant p \leqslant q < r \leqslant s \leqslant M$ and $1 \leqslant a < n$ such that $i_p = a$, $j_q = \bar{n}$, $i_r = n-1$, $j_s = \bar{a}$, and $i_r = n-1$ changes to n by $\tilde{f}_{n-1}$, then $\tilde{f}_{n-1} w$ is in the a-odd-configuration and $q(a; \tilde{f}_{n-1} w) \geqslant n-a$.

(2-iv) There exist $1 \leqslant p \leqslant q < r \leqslant s \leqslant M$ and $1 \leqslant a < n$ such that $i_p = a$, $j_q = n-1$, $i_r = n$, $j_s = \bar{a}$, and $j_q = n-1$ changes to $\bar{n}$ by $\tilde{f}_n$, then $\tilde{f}_n w$ is in the a-odd-configuration and $q(a; \tilde{f}_n w) \geqslant n-a$.

(2-v) There exist $1 \leqslant p \leqslant q < r \leqslant s \leqslant M$ and $1 \leqslant a < n$ such that $i_p = a$, $j_q = n$, $i_r = n-1$, $j_s = \bar{a}$, and $i_r = n-1$ changes to $\bar{n}$ by $\tilde{f}_n$, then $\tilde{f}_n w$ is in the a-odd-configuration and $q(a; \tilde{f}_n w) \geqslant n-a$.

(2-vi) There exist $1 \leqslant p \leqslant q < r \leqslant s \leqslant M$ and $1 \leqslant a < n$ such that $i_p = a$, $j_q = n-1$, $i_r = \bar{n}$, $j_s = \bar{a}$, and $j_q = n-1$ changes to n by $\tilde{f}_{n-1}$, then $\tilde{f}_{n-1} w$ is in the a-odd-configuration and $q(a; \tilde{f}_{n-1} w) \geqslant n-a$.

Cases (2-vii)–(2-xii) are the cases where a-even-configuration appears.

In the case (2-i), i_p in v changes. Then (6.5.3a) or (6.5.3b) occurs. If (6.5.3a) occurs, then $j_{s+1} = \bar{a}$ because $\tilde{f}_a u = 0$. This implies that w is in the a-odd-configuration and

$$q(a; w) = q(a+1; \tilde{f}_a w) + 1 \geqslant n - (a+1) + 1 = n - a.$$

This contradicts (M.N.3). If (6.5.3b) occurs, we get that $\tilde{f}_a v \neq 0$ and $\tilde{e}_a v \neq 0$. This contradicts the fact that v contains a by Remark 6.3.5(iii). Thus (2-i) cannot occur.

In order to derive a contradiction from (2-ii), we prove the following lemma.

LEMMA 6.5.3. *Let* $u \otimes v \in I^{(D)}_{(M,N)}$ *(where* u *contains* $j_1, ..., j_M$ *and* v *contains* $i_1, ..., i_N$*) and* $r-q+1$ *is odd for* $1 \leqslant q < r \leqslant M$. *If either* $j_q = n$ *and* $i_r = \bar{n}$ *or* $j_q = \bar{n}$ *and* $i_r = n$, *then there exists some* t *such that* $q \leqslant t \leqslant r$ *and*

$$i_t \preccurlyeq n-1 \quad \text{and} \quad \overline{n-1} \preccurlyeq j_{t+1}. \tag{6.5.4}$$

Proof. Assuming that there is no such t, by (1) in (6.3.2) there exists some $s \in \{q, ..., r\}$ such that $i_s = j_s = n$ or $\bar{n}$. In both cases, $s-q$ and $r-s+1$ are both even or both odd by the condition (1) or (6.3.2). This implies that $r-q+1 = (s-q)+(r-s+1)$ is even, which is a contradiction. Q.E.D.

In the case (2-ii), by Lemma 6.5.3, there exists $q \leqslant t \leqslant r$ with (6.5.4). Let t be the smallest one which satisfies (6.5.4) and t' the largest one. Hence $q \leqslant t \leqslant t' \leqslant r$, moreover $j_t = n$ or $\bar{n}$ and $i_{t'+1} = n$ or $\bar{n}$. Now, we set

$$\alpha = \{i_{p+1}, ..., i_t\}, \qquad \beta = \{j_{t'+1}, ..., j_s\}.$$

Since $i_t \prec \cdots \prec i_{t'} \preccurlyeq n-1$, we have $i_t \preccurlyeq n-1-(t'-t)$. Similarly we have $j_{t'+1} \succcurlyeq \overline{n-1-(t'-t)}$. Hence,

$$\alpha \subset \{a+1, ..., n-1-(t'-t)\}$$

and

$$\beta \subset \varphi = \{\overline{n-1-(t'-t)}, \cdots, \overline{a+1}\}. \tag{6.5.5}$$

By the definition of α and β,

$$\begin{aligned} \#\alpha + \#\beta &= (t-p)+(s-t') = s-p-(t'-t) \\ &= q(a; \tilde{f}_a w) - (t'-t) \geqslant n-a-(t'-t). \end{aligned} \tag{6.5.6}$$

By (6.5.5), $\bar{\alpha}$ and β are both contained in φ. By the definition of φ in (6.5.5),

$$\#\varphi = n-a-(t'-t)-1. \tag{6.5.7}$$

By (6.5.6) and (6.5.7), $\#\alpha + \#\beta > \#\varphi$, this implies that $\bar{\alpha} \cap \beta \neq \emptyset$. Hence, there exist m and m' such that $p+1 \leqslant m \leqslant t$, $t'+1 \leqslant m' \leqslant s$, $i_m = k$, and $j_{m'} = \bar{k}$. Let k be the smallest one among such k's. Then w is in the k-odd or even-configuration according to whether $j_t = i_{t'+1}$ ($= n$ or $\bar{n}$) or $j_t = \overline{i_{t'+1}}$ ($= n$ or $\bar{n}$). We set

$$\begin{aligned} \gamma &= \{i_{p+1}, ..., i_{m-1}\} \subset \{a+1, ..., k-1\}, \\ \delta &= \{j_{m'+1}, ..., j_s\} \subset \{\overline{k-1}, ..., \overline{a+1}\} \end{aligned}$$

(where if $m-1 < p+1$, $\gamma = \emptyset$ and if $m'+1 > s$, $\delta = \emptyset$.)

By the choice of k, $\bar{\gamma} \cap \delta = \emptyset$ and hence $\#\gamma + \#\delta \leqslant k - a - 1$. Therefore, we have

$$(m - p - 1) + (s - m') \leqslant k - a - 1. \tag{6.5.8}$$

Hence we obtain

$$\begin{aligned} q(k; w) &= m' - m = s - p - (s - m') - (m - p) \\ &\geqslant q(a; \tilde{f}_a w) - (k - a) \geqslant n - k. \end{aligned} \tag{6.5.9}$$

This contradicts (M.N.3). Thus (2-ii) cannot occur.

In the case (2-iii), we set

$$\alpha = \{i_p, ..., i_r\} \subset \{a, ..., n-1\} \qquad \text{and} \qquad \beta = \{i_{r+1}, ..., i_s\}.$$

Since $i_r = n - 1$ changes to n by $\tilde{f}_{n-1}$, we get $i_{r+1} \neq n$. If $r < s$ and $i_{r+1} = \bar{n}$, w is in the a-even-configuration. Hence $q(a; w) = q(a; \tilde{f}_{n-1} w) < n - a$. Then we may assume $r = s$ or $i_{r+1} \succcurlyeq \overline{n-1}$. In both cases, we have

$$\begin{aligned} \#\alpha + \#\beta &= (r - p + 1) + (s - r) = s - p + 1 \\ &= q(a; \tilde{f}_{n-1} w) + 1 > n - a. \end{aligned} \tag{6.5.10}$$

Since $\bar{\alpha} \cup \beta$ is contained in the set $\{\overline{n-1}, ..., \bar{a}\}$ with $n - a$ elements, (6.5.10) implies $\bar{\alpha} \cap \beta \neq \emptyset$. The rest of the argument is similar to the previous case. Thus (2-iii) cannot occur. By a similar argument we get that (2-iv)–(2-vi) and (2-vii)–(2-xii) cannot occur. Hence, we obtain that (M.N.3) is preserved by $\tilde{f}_i$. By a similar argument, (M.N.3) is preserved by $\tilde{e}_i$. Therefore, we have completed the proof of (1).

It remains to prove that if $w = u \otimes v \in I^{(D)}_{(M,N)}$ satisfies $\tilde{e}_i w = 0$ $(1 \leqslant i \leqslant n)$ then $w = u_{\omega_M} \otimes u_{\omega_N}$. Since $\tilde{e}_i w = 0$ implies $\tilde{e}_i u = 0$, u must be u_{ω_M}. Therefore it reduced to the following lemma.

LEMMA 6.5.4. *If* $u_{\omega_M} \otimes v \in I^{(D)}_{(M,N)}$ *satisfies* $\tilde{e}_i v = 0$ *for* $M < i \leqslant n$, *then* $v = u_{\omega_N}$.

Proof. We set $v = \boxed{i_1} \otimes \cdots \otimes \boxed{i_N} \in I^{(D)}_N$. Then by (M.N.1) we have $i_\nu = \nu$ for $\nu \leqslant M$. Hence, if $M = N$, then $v = u_{\omega_N}$. In the case $M < N$, assuming $i_\nu = \nu$ for $\nu < k \leqslant N$, let us show $i_k = k$. Assume first $k < n$. If $i_k \in \{k+1, ..., n\}$, $\tilde{e}_{i_k - 1} v \neq 0$ and if $i_k \in \{\bar{k}, ..., \bar{n}\}$, $\tilde{e}_{\bar{i}_k} v \neq 0$, which contradicts the hypothesis. By the condition (2) in (6.3.2), v does not contain $\bar{1}, ..., \bar{k}$. Then we get $i_k = k$. If $k = N = n$, the condition (3) in (6.3.3) implies $i_n = n$. Q.E.D.

Thus we have completed the proof of Proposition 6.5.1. Q.E.D.

Next, we consider $V(\omega_M+\overline{\omega_n})$ $(1\leqslant M<n)$ and $V(2\overline{\omega_n})$. We set

$$I^{(D)}_{(M,\bar{n})}\ (\text{resp. } I^{(D)}_{(\bar{n},\bar{n})})=\left\{ w=\begin{array}{|c|c|}\hline i_1 & j_1\\ \hline \vdots & \vdots\\ \cline{2-2} & i_M \\ \hline j_n & \\ \hline\end{array}\in I^{(D)}_M\otimes \bar{I}^{(D)}_n\ (\text{resp. } \bar{I}^{(D)\otimes 2}_n);\right.$$

$$\left. w \text{ satisfies (M.N.1)–(M.N.3)}\right\}. \quad (6.5.11)$$

PROPOSITION 6.5.5. *$I^{(D)}_{(M,\bar{n})}$ (resp. $I^{(D)}_{(\bar{n},\bar{n})}$) coincides with $B(\omega_M+\overline{\omega_n})$ (resp. $B(2\overline{\omega_n})$).*

The proof is similar to the one of Proposition 6.5.1.

6.6. *The Crystal Graph of $V(\omega_M+\Lambda_n)$, $V(\omega_N+\Lambda_{n-1})$, and $V(\overline{\omega_n}+\Lambda_{n-1})$*

We treat the crystal graph of $V(\omega_M+\Lambda_n)$ $(1\leqslant M\leqslant n)$, $V(\omega_N+\Lambda_{n-1})$ $(1\leqslant N<n)$, and $V(\overline{\omega_n}+\Lambda_{n-1})$. They can be embedded into $V(\omega_M)\otimes V^{(+)}_{sp}$, $V(\omega_N)\otimes V^{(-)}_{sp}$, and $V(\overline{\omega_n})\otimes V^{(-)}_{sp}$, respectively, with multiplicity free. Similar to the previous cases $B(\omega_M+\Lambda_n)$, $B(\omega_N+\Lambda_{n-1})$, and $B(\overline{\omega_n}+\Lambda_{n-1})$ are respectively the connected component of $I^{(D)}_M\otimes B^{(+)}_{sp}$, $I^{(D)}_N\otimes B^{(-)}_{sp}$, and $\bar{I}^{(D)}_n\otimes B^{(-)}_{sp}$ containing the highest weight vector $u_{\omega_M}\otimes u_{\Lambda_n}$, $u_{\omega_N}\otimes u_{\Lambda_{n-1}}$, and $u_{\overline{\omega_n}}\otimes u_{\Lambda_{n-1}}$. For $u\otimes v\in I^{(D)}_M\otimes B^{(+)}_{sp}$, $I^{(D)}_N\otimes B^{(-)}_{sp}$, $\bar{I}^{(D)}_n\otimes B^{(-)}_{sp}$ we use the same expression as in Subsection 5.6 (where the expression of v is the latter one in Subsection 6.4).

DEFINITION 6.6.1. (1) $u\otimes v\in I^{(D)}_M\otimes B^{(+)}_{sp}$ $(1\leqslant M\leqslant n)$, $I^{(D)}_N\otimes B^{(-)}_{sp}$ $(1\leqslant N<n)$, $\bar{I}^{(D)}_n\otimes B^{(-)}_{sp}$ is in the (a,b)-configuration $(1\leqslant a\leqslant b\leqslant n)$ if $u\otimes v$ satisfies the same condition as Definition 6.5.1(1), (2), or (3).

(2) $u\otimes v\in I^{(D)}_M\otimes B^{(+)}_{sp}$ $(1\leqslant M\leqslant n)$, $I^{(D)}_N\otimes B^{(-)}_{sp}$ $(1\leqslant N<n)$, $\bar{I}^{(D)}_n\otimes B^{(-)}_{sp}$ is in the a-odd (resp. even)-configuration $(1\leqslant a<n)$ if $u\otimes v$ satisfies the same condition as Definition 6.5.1(4) (resp. (5)).

$p(a,b;w)$ and $q(a;w)$ are the same ones defined in Definition 6.5.1.

We set

$$I^{(D)}_{M,sp+}\left(\text{resp. } \begin{matrix} I^{(D)}_{N,sp-} \\ I^{(D)}_{\bar{n},sp-}\end{matrix}\right) = \left\{ w = \begin{array}{|c|c|}\hline i_1 & j_1 \\ \hline \vdots & \vdots \\ \hline \vdots & j_M \\ \hline i_n & \\ \hline \end{array} \in I^{(D)}_M \otimes B^{(+)}_{sp} \left(\text{resp. } \begin{matrix} I^{(D)}_N \otimes B^{(-)}_{sp} \\ \bar{I}^{(D)}_n \otimes B^{(-)}_{sp}\end{matrix}\right); \; w \text{ holds (M.N.1)–(M.N.3) in (6.5.2)} \right\}. \tag{6.6.1}$$

PROPOSITION 6.6.2. (1) *For* $1 \leqslant M \leqslant n$, $I^{(D)}_{M,sp+}$ *coincides with* $B(\omega_M + \Lambda_n)$.

(2) *For* $1 \leqslant N < n$, $I^{(D)}_{N,sp-}$ *coincides with* $B(\omega_N + \Lambda_{n-1})$.

(3) $I^{(D)}_{\bar{n},sp-}$ *coincides with* $B(\overline{\omega_n} + \Lambda_{n-1})$.

As the proof is similar to Proposition 6.5.2, we omit it.

Remark 6.6.3. (1) For $u \otimes v \in I^{(D)}_M \otimes B^{(+)}_{sp}$, $I^{(D)}_N \otimes B^{(-)}_{sp}$, $\bar{I}^{(D)}_n \otimes B^{(-)}_{sp}$, the actions of $\tilde{e}_i$ and $\tilde{f}_i$ $(1 \leqslant i < n)$ are given by identifying $\boxed{i}$, $\boxed{\overline{i+1}}$ in u and the pair $(i, \overline{i+1})$ in v with u_+, identifying $\boxed{i+1}$, $\boxed{\bar{i}}$ in u and the pair $(i+1, \bar{i})$ in v with u_- and identifying others with u_0 in Remark 2.1.2.

(2) For $u \otimes v \in I^{(D)}_M \otimes B^{(+)}_{sp}$, $I^{(D)}_N \otimes B^{(-)}_{sp}$, $\bar{I}^{(D)}_n \otimes B^{(-)}_{sp}$, the actions of $\tilde{e}_n$ and $\tilde{f}_n$ are given by identifying $\boxed{n-1}$, $\boxed{n}$ in u and the pair $(n-1, n)$ in v with u_+, identifyingh $\boxed{\bar{n}}$, $\boxed{\overline{n-1}}$ in u and the pair $(\bar{n}, \overline{n-1})$ in v with u_- and identifying others with u_0 in Remark 2.1.2.

6.7. *The Crystal Graph of* $V(\lambda)$

Let $\lambda = \sum_{i=1}^n m_i \Lambda_i$ $(m_i \mathbf{Z}_{\geqslant 0})$ be a dominant integral weight of D_n. Now, we rewrite λ by use of ω_M, $\overline{\omega_n}$, Λ_n, and Λ_{n-1}. By the definition of ω_M and $\overline{\omega_n}$, we have $\Lambda_i = \omega_i$ $(1 \leqslant i \leqslant n-2)$, $\Lambda_{n-1} + \Lambda_n = \omega_{n-1}$, $2\Lambda_n = \omega_n$, and $2\Lambda_{n-1} = \overline{\omega_n}$. Hence, any $\lambda \in P_+$ can be written as

(W1) $\lambda = \sum_{i=1}^n m_i \omega_i$.

(W2) $\lambda = \sum_{i=1}^n m_i \omega_i + \Lambda_n$.

(W3) $\lambda = \sum_{i=1}^{n-1} m_i \omega_i + m_n \overline{\omega_n}$.

(W4) $\lambda = \sum_{i=1}^{n-1} m_i \omega_i + m_n \overline{\omega_n} + \Lambda_{n-1}$.

Here $m_i \in \mathbf{Z}_{\geqslant 0}$.

If λ is of type (W4), we can write $\lambda = \omega_{l_1} + \cdots + \omega_{l_p} + \overline{\omega_{l_{p+1}}} + \cdots + \overline{\omega_{l_{q-1}}} + \Lambda_{n-1}$ with $1 \leqslant l_1 \leqslant \cdots \leqslant l_p < n = l_{p+1} = \cdots = l_{q-1}$. By Proposition 6.5.2, Lemma 6.5.4, Proposition 6.5.5, and Proposition 6.6.2, we obtain that $\omega_{l_1}, \ldots, \omega_{l_p}, \overline{\omega_{l_{p+1}}}, \ldots, \overline{\omega_{l_{q-1}}}$ and Λ_{n-1} satisfy the hypothesis of Proposition 2.2.1.

For

$$u_k = \begin{array}{|c|}\hline t_1^k \\ \hline \vdots \\ \hline t_{l_k}^k \\ \hline \end{array} \in I_{l_k}^{(D)} \text{ or } \bar{I}_{l_k}^{(D)} \ (1 \leqslant k < q-1) \qquad \text{and} \qquad u_q = \begin{array}{|c|}\hline t_1^q \\ \hline \vdots \\ \hline t_n^q \\ \hline \end{array} \in B_{sp}^{(-)},$$

we denote

$$u_1 \otimes \cdots \otimes u_q = \begin{array}{|c|c|c|c|c|}\hline & & & \vdots & t_l^1 \\ \hline t_{l'}^q & \vdots & \vdots & & \\ \hline \end{array} \in \left(\bigotimes_{k=1}^{p} I_{l_k}^{(D)}\right) \otimes \left(\bigotimes_{k=p+1}^{q-1} \bar{I}_{l_k}^{(D)}\right) \otimes B_{sp}^{(-)}.$$

We set

$$I_\lambda^{(D)} = \left\{ w = \begin{array}{|c|c|c|c|c|}\hline & & & \vdots & t_l^1 \\ \hline t_{l'}^q & \vdots & \vdots & & \\ \hline \end{array} \in \left(\bigotimes_{k=1}^{p} I_{l_k}^{(D)}\right) \otimes \left(\bigotimes_{k=p+1}^{q-1} \bar{I}_{l_k}^{(D)}\right) \otimes B_{sp}^{(-)}; \right.$$

$$\left. w \text{ satisfies the following (1)–(4)} \right\}, \qquad (6.7.1)$$

(1) $u_k \otimes u_{k+1} \in I_{(l_k, l_{k+1})}^{(D)}$ for any $k = 1, \ldots, p-1$,

(2) $u_p \otimes u_{p+1} \in I_{(l_p, \overline{l_{p+1}})}^{(D)}$,

(3) $u_k \otimes u_{k+1} \in I_{(\overline{l_k}, \overline{l_{k+1}})}^{(D)}$, for any $k = p+1, \ldots, q-2$,

(4) $u_{q-1} \otimes u_q \in I_{l_{q-1}, sp-}^{(D)}$.

Also, for λ of type (W1)–(W3), we define $I_\lambda^{(D)}$ similarly. An element of $I_\lambda^{(D)}$ is called a *semi-standard D-tableau* of shape λ.

By Proposition 2.2.1, Proposition 6.5.2, Proposition 6.5.5, and Proposition 6.6.2, we obtain the following result;

THEOREM 6.7.1. *Let* $\lambda \in P_+$.

(i) *$B(\lambda)$ coincides with the set of semi-standard D-tableaux of shape λ.*

(ii) *The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are given by the same rule as Remark* 6.6.3.

EXAMPLE 6.7.2. For D_4 and $\lambda = \Lambda_1 + 2\Lambda_2 + \Lambda_4$, we consider the actions of $\tilde{e}_4$ and $\tilde{f}_4$ on

$$v = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & \bar{3} \\ \hline 2 & \bar{4} & \bar{4} \\ \cline{1-3} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \in B(\Lambda_1 + 2\Lambda_2 + \Lambda_4).$$

By the constructions of crystal graphs, it can be expressed,

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & \bar{3} \\ \hline 2 & \bar{4} & \bar{4} \\ \cline{1-3} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} = \boxed{\bar{3}} \otimes \begin{array}{|c|}\hline 4 \\ \hline \bar{4} \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 3 \\ \hline \bar{4} \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array} = \boxed{\bar{3}} \otimes \boxed{4} \otimes \boxed{\bar{4}} \otimes \boxed{3} \otimes \boxed{\bar{4}} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}.$$

The vector v can be identified with $u_- \otimes u_+ \otimes u_- \otimes u_+ \otimes u_- \otimes u_+$ and by Remark 2.1.2,

$$\tilde{e}(u_- \otimes u_+ \otimes u_- \otimes u_+ \otimes u_- \otimes u_+) = u_+ \otimes u_+ \otimes u_- \otimes u_+ \otimes u_- \otimes u_+,$$

$$\tilde{f}(u_- \otimes u_+ \otimes u_- \otimes u_+ \otimes u_- \otimes u_+) = u_- \otimes u_+ \otimes u_- \otimes u_+ \otimes u_- \otimes u_-.$$

Thus,

$$\tilde{e}_4\left(\boxed{\bar{3}} \otimes \boxed{4} \otimes \boxed{\bar{4}} \otimes \boxed{3} \otimes \boxed{\bar{4}} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}\right) = \boxed{4} \otimes \boxed{4} \otimes \boxed{\bar{4}} \otimes \boxed{3} \otimes \boxed{\bar{4}} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array},$$

$$\tilde{f}_4\left(\boxed{\bar{3}} \otimes \boxed{4} \otimes \boxed{\bar{4}} \otimes \boxed{3} \otimes \boxed{\bar{4}} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}\right) = \boxed{\bar{3}} \otimes \boxed{4} \otimes \boxed{\bar{4}} \otimes \boxed{3} \otimes \boxed{\bar{4}} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \bar{4} \\ \hline \bar{3} \\ \hline\end{array}.$$

Hence, we obtain

$$\tilde{e}_4 \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & \bar{3} \\ \hline 2 & \bar{4} & \bar{4} \\ \cline{1-3} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 4 \\ \hline 2 & \bar{4} & \bar{4} \\ \cline{1-3} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}, \qquad \tilde{f}_4 \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & \bar{3} \\ \hline 2 & \bar{4} & \bar{4} \\ \cline{1-3} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & \bar{3} \\ \hline 2 & \bar{4} & \bar{4} \\ \cline{1-3} \bar{4} \\ \cline{1-1} \bar{3} \\ \cline{1-1}\end{array}.$$

REFERENCES

[D-J-M] E. DATE, M. JIMBO, AND T. MIWA, Representations of $U_q(\mathfrak{gl}(n, \mathbf{C}))$ at $q = 0$ and the Robinson–Schensted correspondence, *in* "Memorial Volume for Vadim Kniznik, Physics and Mathematics of Strings" (L. Brink, D. Friedan, and A.M. Polyakov, Eds.), pp. 185–211, World Scientific, Singapore, 1990.

[D] V.G. DRINFELD, "Quantum Groups," pp. 798–820, ICM Proceedings, Berkeley, 1986.

[J] M. JIMBO, A q-difference analogue of $U(\mathfrak{g})$ and the Yang–Baxter equation, *Lett. Math. Phys.* **11** (1986), 247–252.

[K1] M. KASHIWARA, Crystallizing the q-analogue of universal enveloping algebra, *Comm. Math. Phys.* **133** (1990), 249–260.

[K2] M. KASHIWARA, On crystal base of q-analogue of universal enveloping algebras, *Duke Math. J.* **63** (1991), 456–516.

[K3] M. KASHIWARA, Crystallizing the q-analogue of universal enveloping algebra, *in* "Proceedings of the International Congress of Mathematicians, Kyoto, Japan, 1990," pp. 791–797.

[K-E] R.C. KING AND N.G.I. EL-SHARKAWAY, Standard Young tableaux and weight multiplicities of classical Lie groups, *J. Phys. A* **16** (1983), 3153–3177.

[L1] G. LUSZTIG, Canonical bases arising from quantized enveloping algebra, *J. Amer. Math. Soc.* **3** (1990), 447–498.

[L2] G. LUSZTIG, Canonical bases arising from quantized enveloping algebra, II, *Prog. Theoret. Phys.* **102** (1990), 175–202.

[N] T. NAKASHIMA, A basis of symmetric tensor representation for the quantum analogue of the Lie algebra B_n, C_n, and D_n, *Publ. Res. Inst. Math. Sci.* **26** (1990), 723–733.

[O] M. OKADO, Quantum R matrices related to the spin representations of B_n and D_n, *Comm. Math. Phys.* **134** (1990), 467–486.

[Re] N. YU. RESHETIKHIN, Quantized universal enveloping algebras, Yang–Baxter equation and invariants of links, I, LOMI preprint, E-4-48, 1990.

[Ro] M. ROSSO, Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra, *Comm. Math. Phys.* **117** (1988), 581–593.

[S] S. SUNDARAM, Orthogonal tableaux and an insertion algorighm for $SO(2n+1)$, *J. Combin. Theory Ser. A* **53** (1990), 239–256.

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