Vol. 69, No. 2 DUKE MATHEMATICAL JOURNAL © February 1993

GLOBAL CRYSTAL BASES OF QUANTUM GROUPS

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0. Introduction.

0.1. In [K_2], we constructed the global crystal bases of $U_q^-(\mathfrak{g})$ and of the irreducible $U_q(\mathfrak{g})$-modules with highest weight. The purpose of this article is to construct the global crystal basis of the q-analogue $A_q(\mathfrak{g})$ of the coordinate ring of the reductive algebraic group associated with the Lie algebra $\mathfrak{g}$. The idea of construction is similar to [K_2]. By the q-analogue of the Peter-Weyl theorem, $A_q(\mathfrak{g})$ has a decomposition $\oplus_\lambda V(\lambda)^* \otimes V(\lambda)$ as a bi-$U_q(\mathfrak{g})$-module, where $V(\lambda)$ is the irreducible $U_q(\mathfrak{g})$-module with a dominant integral weight λ as highest weight. Hence $A_q(\mathfrak{g})$ has a (upper) crystal base $(L(A_q(\mathfrak{g})), B(A_q(\mathfrak{g}))) = \oplus(L(\lambda)^*, B(\lambda)^*) \otimes (L(\lambda), B(\lambda))$ at $q = 0$ and similarly a crystal base $(\bar{L}(A_q(\mathfrak{g})), \bar{B}(A_q(\mathfrak{g})))$ at $q = \infty$ (see §7 for their normalization). We denote by $U_q^{\mathbf{Q}}(\mathfrak{g})$ the sub-$\mathbf{Q}[q, q^{-1}]$-algebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$, $f_i^{(n)}$, q^h, and $\begin{Bmatrix} q^h \\ n \end{Bmatrix}$. We denote by $\langle\ ,\ \rangle: A_q(\mathfrak{g}) \times U_q(\mathfrak{g}) \to \mathbf{Q}(q)$ the canonical pairing, and we define

$$A_q^{\mathbf{Q}}(\mathfrak{g}) = \{u \in A_q(\mathfrak{g}); \langle u, U_q^{\mathbf{Q}}(\mathfrak{g})\rangle \subset \mathbf{Q}[q, q^{-1}]\}.$$

Then $A_q^{\mathbf{Q}}(\mathfrak{g})$ is a subalgebra of $A_q(\mathfrak{g})$ satisfying $A_q(\mathfrak{g}) = \mathbf{Q}(q) \otimes_{\mathbf{Q}[q,q^{-1}]} A_q^{\mathbf{Q}}(\mathfrak{g})$.

Now the main result of this article is the following.

THEOREM 1. (i) *Set* $E = A_q^{\mathbf{Q}}(\mathfrak{g}) \cap L(A_q(\mathfrak{g})) \cap \bar{L}(A_q(\mathfrak{g}))$. *Then* $E \to L(A_q(\mathfrak{g}))/qL(A_q(\mathfrak{g}))$ *is an isomorphism, and* $A_q^{\mathbf{Q}}(\mathfrak{g}) = \mathbf{Q}[q, q^{-1}] \otimes_{\mathbf{Q}} E$.

(ii) *Letting G be the inverse of the isomorphism above, we have*

$$A_q^{\mathbf{Q}}(\mathfrak{g}) = \bigoplus_{b \in B(A_q(\mathfrak{g}))} \mathbf{Q}[q, q^{-1}]G(b).$$

0.2. Theorem 1 is a consequence of the following theorem, Theorem 2.

Let M be an integrable $U_q(\mathfrak{g})$-module with highest weights and $M_{\mathbf{Q}}$ a sub-$U_q^{\mathbf{Q}}(\mathfrak{g})$-module of M such that $Q(q) \otimes_{\mathbf{Q}[q,q^{-1}]} M_{\mathbf{Q}} \cong M$. Let (L_0, B_0) and (L_∞, B_∞) be an upper crystal base of M at $q = 0$ and $q = \infty$, respectively. Let $H = \{u \in M; e_i u = 0$ for any $i\}$ be the set of highest-weight vectors.

THEOREM 2. *Assume the following conditions*:

(i) $\{u \in M; e_i^{(n)}u \in M_{\mathbf{Q}}$ *for any i and* $n \geqslant 1\} = M_{\mathbf{Q}} + H$;

(ii) $H \cap M_{\mathbf{Q}} \cap L_0 \cap L_\infty \to (H \cap L_0)/(H \cap qL_0)$ *is an isomorphism.*

Received 24 March 1992. Revision received 31 August 1992.

Then $M_{\mathbf{Q}} \cap L_0 \cap L_\infty \to L_0/qL_0$ *is an isomorphism, and*

$$M_{\mathbf{Q}} \cong \mathbf{Q}[q, q^{-1}] \underset{\mathbf{Q}}{\otimes} (M_{\mathbf{Q}} \cap L_0 \cap L_\infty).$$

0.3. The following conjecture is communicated by A. Zelevinsky (in a weaker form).

Conjecture (A. D. Berenstein and A. Zelevinsky). Assume that $\mathfrak{g}$ is finite-dimensional. Then there exists a finite subset F of $B(A_q(\mathfrak{g}))$ satisfying the following condition:

(0.1) For any $b \in B(A_q(\mathfrak{g}))$, there exist $b_1, \dots, b_m \in F$ and $n \in \mathbf{Z}$ satisfying the following two properties:
(i) $G(b) = q^n G(b_1) \cdots G(b_m)$ for some n;
(ii) there exist integers m_{ij} such that

$$G(b_i)G(b_j) = q^{m_{ij}} G(b_j)G(b_i).$$

1. Notation.

1.1. *Definition of* $U_q(\mathfrak{g})$. We shall review the definition of $U_q(\mathfrak{g})$. We shall follow the notation in [K]. Let us consider the following data:

(1.1.1) a finite-dimensional $\mathbf{Q}$-vector space $\mathfrak{t}$,
(1.1.2) an index set I (of simple roots),
(1.1.3) a linearly independent subset $\{\alpha_i; i \in I\}$ of $\mathfrak{t}^*$ and a subset $\{h_i; i \in I\}$ of $\mathfrak{t}$,
(1.1.4) an inner product $(\ ,\ )$ on $\mathfrak{t}^*$, and
(1.1.5) a lattice P of $\mathfrak{t}^*$.

We assume that they satisfy the following properties:

(1.1.6) $$\{\langle h_i, \alpha_j\rangle\} \text{ is a generalized Cartan matrix}$$

(i.e., $\langle h_i, \alpha_i\rangle = 2$, $\langle h_i, \alpha_j\rangle \in \mathbf{Z}_{\leqslant 0}$ for $i \neq j$ and $\langle h_i, \alpha_j\rangle = 0 \Leftrightarrow \langle h_j, \alpha_i\rangle = 0$);

(1.1.7) $$(\alpha_i, \alpha_i) \in 2\mathbf{Z}_{>0};$$

(1.1.8) $$\langle h_i, \lambda\rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \text{ for any } i \in I \quad \text{and} \quad \lambda \in \mathfrak{t}^*;$$

(1.1.9) $$\alpha_i \in P \quad \text{and} \quad h_i \in P^* = \{h \in \mathfrak{t}; \langle h, P\rangle \subset \mathbf{Z}\}.$$

Then the $\mathbf{Q}(q)$-algebra $U_q(\mathfrak{g})$ is the algebra generated by e_i, $f_i (i \in I)$ and $q^h (h \in P^*)$

with the following defining relations:

(1.1.10) $q^h = 1$ for $h = 0$ and $q^{h+h'} = q^h q^{h'}$;

(1.1.11) $q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i$ and $q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i$;

(1.1.12) $[e_i, f_j] = \delta_{ij}(t_i - t_i^{-1})/(q_i - q_i^{-1})$ where $q_i = q^{(\alpha_i,\alpha_i)/2}$ and $t_i = q^{(\alpha_i,\alpha_i)h_i/2}$;

(1.1.13) $\sum(-1)^n e_i^{(n)} e_j e_i^{(b-n)} = \sum(-1)^n f_i^{(n)} f_j f_i^{(b-n)} = 0$ for $i \neq j$ and $b = 1 - \langle h_i, \alpha_j\rangle$.

Here we used the notation $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{k=1}^n [k]_i$, $\begin{bmatrix} n \\ m \end{bmatrix}_i = [n]_i!/([n-m]_i![m]_i!)$, $e_i^{(n)} = e_i^n/[n]_i!$, and $f_i^{(n)} = f_i^n/[n]_i!$. We understand $e_i^{(n)} = f_i^{(n)} = 0$ for $n < 0$. We set

(1.1.14) $$Q = \sum \mathbf{Z}\alpha_i, \qquad Q_+ = \sum \mathbf{Z}_{\geq 0}\, \alpha_i, \text{ and } Q_- = -Q_+.$$

1.2. *Automorphisms of* $U_q(\mathfrak{g})$. As in [K_2], we define the **Q**-ring automorphism $-$ of $U_q(\mathfrak{g})$ by

(1.2.1) $$\bar{e}_i = e_i, \qquad \bar{f}_i = f_i, \qquad \bar{q} = q^{-1}, \text{ and } (q^h)^- = q^{-h}.$$

We define the $\mathbf{Q}(q)$-linear antiautomorphisms $*$ and φ of $U_q(\mathfrak{g})$ by

(1.2.2) $$e_i^* = e_i, \qquad f_i^* = f_i, \qquad (q^h)^* = q^{-h},$$

(1.2.3) $$\varphi(e_i) = f_i, \qquad \varphi(f_i) = e_i, \text{ and } \varphi(q^h) = q^h.$$

Note that φ, $-$, and $*$ commute to each other and $*^2 = -^2 = \varphi^2 = 1$.

1.3. *Integrable* $U_q(\mathfrak{g})$*-module.* We say that a $U_q(\mathfrak{g})$-module M is *integrable* if

(1.3.1) $$M = \bigoplus_{\lambda \in P} M_\lambda;$$

(1.3.2) for any i, M is a union of finite-dimensional $U_q(\mathfrak{g}_i)$-modules.

Here $U_q(\mathfrak{g}_i)$ is the $\mathbf{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, and $q^h (h \in P^*)$.

Let $O_{\text{int}}(\mathfrak{g})$ be the category of integrable $U_q(\mathfrak{g})$-modules M such that, for any $u \in M$, there exists $l \geq 0$ satisfying $e_{i_1} \cdots e_{i_l} u = 0$ for any $i_1, \ldots, i_l \in I$.

Then $O_{\text{int}}(\mathfrak{g})$ is semisimple and any simple object is isomorphic to the irreducible module $V(\lambda)$ with highest weight λ. These terminologies are slightly different from those in [K_2] where we added finiteness conditions, but almost all results there hold with suitable modifications.

For an object M in $O_{\text{int}}(\mathfrak{g})$ and $\lambda \in P_+$, we set

(1.3.3) $$I_\lambda(M) = \operatorname{Hom}_{U_q(\mathfrak{g})}(V(\lambda), M) \underset{\mathbf{Q}(q)}{\otimes} V(\lambda).$$

Hence $I_\lambda(M)$ is the isotypical component of M with type $V(\lambda)$, and we have

(1.3.4) $$M \cong \bigoplus_{\lambda \in P_+} I_\lambda(M) \quad \text{as } U_q(\mathfrak{g})\text{-module.}$$

We set, for $\lambda \in P_+$,

(1.3.5) $$W_\lambda(M) = \bigoplus_{\mu \in P_+ \cap (\lambda + Q_-)} I_\mu(M),$$
$$W_{<\lambda}(M) = \bigoplus_{\mu \in P_+ \cap (\lambda + Q_-)\setminus\{\lambda\}} I_\mu(M).$$

Here Q_- is as in (1.1.14). More generally, for a subset F of P_+, we set

(1.3.6) $$W_F(M) = \sum_{\lambda \in F} W_\lambda(M) = \sum_{\lambda \in P_+ \cap (F + Q_-)} I_\lambda(M).$$

Thus W forms a filtration of M. We set

(1.3.7) $$Gr_\lambda^W M = W_\lambda(M)/W_{<\lambda}(M).$$

Hence $Gr_\lambda^W(M)$ is isomorphic to $I_\lambda(M)$.

For a $\mathbf{Q}$-vector subspace S of M, we set

(1.3.8) $$W_\lambda(S) = W_\lambda(M) \cap S, \qquad W_{<\lambda}(S) = W_\lambda(M) \cap S,$$
$$Gr_\lambda^W(S) = W_\lambda(S)/W_{<\lambda}(S), \text{ etc.}$$

2. Balanced triple.

2.1. *Definition of balanced triple.* Let A (resp. $\bar{A}$) denote the ring of rational functions in q which are regular at $q = 0$ (resp. $q = \infty$). Hence $A/qA \cong \mathbf{Q}$ and $\bar{A}/q^{-1}\bar{A} \cong \mathbf{Q}$ by the evaluation at $q = 0$ and $q = \infty$, respectively.

Let V be a $\mathbf{Q}(q)$-vector space, L_0 a sub-A-module of V, L_∞ a sub-$\bar{A}$-module of V, and $V_\mathbf{Q}$ a sub-$\mathbf{Q}[q, q^{-1}]$-module of V satisfying the conditions

(2.1.1) $$V \cong \mathbf{Q}(q) \underset{\mathbf{Q}[q,q^{-1}]}{\otimes} V_\mathbf{Q} \cong \mathbf{Q}(q) \underset{A}{\otimes} L_0 \cong \mathbf{Q}(q) \underset{\bar{A}}{\otimes} L_\infty.$$

Note that we do not assume in the beginning that either $V_\mathbf{Q}$, L_0, or L_∞ are free. As a consequence of (2.1.1), we have

(2.1.2) $$A \underset{\mathbf{Q}[q]}{\otimes} (V_\mathbf{Q} \cap L_0) \cong L_0 \quad \text{and} \quad \bar{A} \underset{\mathbf{Q}[q^{-1}]}{\otimes} (V_\mathbf{Q} \cap L_\infty) \cong L_\infty.$$

In fact, for any $u \in L_0$, there exists a nonzero $f(q) \in \mathbf{Q}[q]$ such that $f(q)u \in V_\mathbf{Q}$. Since $V_\mathbf{Q}$ is a $\mathbf{Q}[q, q^{-1}]$-module, we may assume $f(0) \neq 0$, and hence $A \otimes_{\mathbf{Q}[q]} (V_\mathbf{Q} \cap L_0) \to$

L_0 is surjective. Injectivity follows from

$$A \underset{\mathbf{Q}[q]}{\otimes} (V_{\mathbf{Q}} \cap L_0) \subset A \underset{\mathbf{Q}[q]}{\otimes} L_0 \cong L_0 .$$

The proof is similar for L_∞.

LEMMA 2.1.1. *Under the condition* (2.1.1), *set* $E = V_{\mathbf{Q}} \cap L_0 \cap L_\infty$. *Then the following three conditions are equivalent.*

(2.1.3) $$E \to L_0/qL_0 \text{ is an isomorphism.}$$

(2.1.4) $$E \to L_\infty/q^{-1}L_\infty \text{ is an isomorphism.}$$

(2.1.5) $$\mathbf{Q}[q, q^{-1}] \underset{\mathbf{Q}}{\otimes} E \to V_{\mathbf{Q}}, \quad A \underset{\mathbf{Q}}{\otimes} E \to L_0 \quad \text{and} \quad \bar{A} \underset{\mathbf{Q}}{\otimes} E \to L_\infty \text{ are isomorphisms.}$$

Note that (2.1.5) implies $\mathbf{Q}(q) \otimes_{\mathbf{Q}} E \to V$ is an isomorphism.

Proof. It is obvious that (2.1.5) implies (2.1.3) and (2.1.4). Hence it is enough to show that (2.1.3) implies (2.1.5). Under the condition (2.1.3), we shall prove

(2.1.6) $$\left(\bigoplus_{0 \leqslant k \leqslant n} \mathbf{Q}q^k\right) \underset{\mathbf{Q}}{\otimes} E \to V_{\mathbf{Q}} \cap L_0 \cap q^n L_\infty \text{ is an isomorphism}$$

by induction on n. If $n \leqslant 0$, it is obvious. Assume $n > 0$ and consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \left(\bigoplus_{k=1}^{n} \mathbf{Q}q^k\right) \otimes E & \longrightarrow & \left(\bigoplus_{k=0}^{n} \mathbf{Q}q^k\right) \otimes E & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \wr & & \\ 0 & \longrightarrow & V_{\mathbf{Q}} \cap qL_0 \cap q^n L_\infty & \longrightarrow & V_{\mathbf{Q}} \cap L_0 \cap q^n L_\infty & \longrightarrow & L_0/qL_0 . & & \end{array}$$

Since α is an isomorphism by the hypothesis of the induction, β is an isomorphism. Hence we obtain (2.1.6), and we have an isomorphism, for any $a \leqslant b$,

$$\left(\bigoplus_{a \leqslant k \leqslant b} \mathbf{Q}q^k\right) \underset{\mathbf{Q}}{\otimes} E \xrightarrow{\sim} V_{\mathbf{Q}} \cap q^a L_0 \cap q^b L_\infty .$$

Letting $-a$ and (or) b tend to the infinity, we obtain

(2.1.7) $$\mathbf{Q}[q, q^{-1}] \underset{\mathbf{Q}}{\otimes} E \xrightarrow{\sim} V_{\mathbf{Q}} .$$

$$\mathbf{Q}[q] \underset{\mathbf{Q}}{\otimes} E \tilde{\rightarrow} V_{\mathbf{Q}} \cap L_0 .$$

$$\mathbf{Q}[q^{-1}] \underset{\mathbf{Q}}{\otimes} E \tilde{\rightarrow} V_{\mathbf{Q}} \cap L_\infty .$$

Then the desired results follow from (2.1.2). Q.E.D.

Definition 2.1.2. We call $(V_{\mathbf{Q}}, L_0, L_\infty)$ a balanced triple if it satisfies (2.1.1) and the equivalent conditions in Lemma 2.1.2 are satisfied.

Note that if $(V_{\mathbf{Q}}, L_0, L_\infty)$ is balanced, then $V_{\mathbf{Q}}$, L_0, and L_∞ are free modules over $\mathbf{Q}[q, q^{-1}]$, A, and $\bar{A}$, respectively.

2.2. *Translation.* We shall translate the notion above by the language of sheaves on $\mathbf{P}^1$. For $V, V_{\mathbf{Q}}, L_0, L_\infty$ with (2.1.1), we shall associate a quasi-coherent $O_{\mathbf{P}^1}$-module $\mathscr{F}$ on $\mathbf{P}^1 = Spec(\mathbf{Q}[q]) \cup Spec(\mathbf{Q}[q^{-1}])$ as follows.

(2.2.1)
$$\Gamma(Spec(\mathbf{Q}[q]), \mathscr{F}) = V_{\mathbf{Q}} \cap L_0 ,$$
$$\Gamma(Spec(\mathbf{Q}[q^{-1}]), \mathscr{F}) = V_{\mathbf{Q}} \cap L_\infty , \quad \text{and}$$
$$\Gamma(Spec(\mathbf{Q}[q, q^{-1}]), \mathscr{F}) = V_{\mathbf{Q}} ,$$

with the obvious restriction map. The existence of such an $\mathscr{F}$ follows from

$$V_{\mathbf{Q}} \simeq \mathbf{Q}[q, q^{-1}] \otimes_{\mathbf{Q}[q]} (V_{\mathbf{Q}} \cap L_0)$$
$$\simeq \mathbf{Q}[q, q^{-1}] \otimes_{\mathbf{Q}[q^{-1}]} (V_{\mathbf{Q}} \cap L_\infty).$$

Thus, $\mathscr{F}$ is a torsion-free quasi-coherent $O_{\mathbf{P}^1}$-module. Let 0 and ∞ be the closed point of $\mathbf{P}^1$ corresponding to $q = 0$, $q^{-1} = 0$, respectively. Then

$$\mathscr{F}_0 = O_{\mathbf{P}^1, 0} \otimes \Gamma(Spec(\mathbf{Q}[q]), \mathscr{F}) \cong A \underset{\mathbf{Q}[q]}{\otimes} (V_{\mathbf{Q}} \cap L_0) \cong L_0 .$$

Thus we obtain

(2.2.2)
$$\mathscr{F}_0 \cong L_0 \quad \text{and} \quad \mathscr{F}_\infty \cong L_\infty .$$

In this way, the set of data $(V, V_{\mathbf{Q}}, L_0, L_\infty)$ with (2.1.1) is equivalent to the set of quasi-coherent torsion-free $O_{\mathbf{P}^1}$-modules. Let $O_{\mathbf{P}^1}(-1)$ be the invertible sheaf of regular functions that vanishes at $q = 0$.

LEMMA 2.2.1. *$(V_{\mathbf{Q}}, L_0, L_\infty)$ is a balanced triple if and only if $H^0(\mathbf{P}^1; \mathscr{F}(-1)) = H^1(\mathbf{P}^1; \mathscr{F}(-1)) = 0$. Here $\mathscr{F}(-1) = \mathscr{F} \otimes O_{\mathbf{P}^1}(-1)$.*

Proof. The exact sequence

$$0 \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{F}/\mathscr{F}(-1) \to 0 \tag{2.2.3}$$

gives the long exact sequence

$$\begin{array}{ccccccc} H^0(\mathbf{P}^1; \mathscr{F}(-1)) & \longrightarrow & H^0(\mathbf{P}^1; \mathscr{F}) & \longrightarrow & H^0(\mathbf{P}^1; \mathscr{F}/\mathscr{F}(-1)) & \longrightarrow & H^1(\mathbf{P}^1; \mathscr{F}(-1)). \\ & & \wr\| & & \wr\| & & \\ & & V_{\mathbf{Q}} \cap L_0 \cap L_\infty & & L_0/qL_0 & & \end{array}$$

Hence $H^0(\mathbf{P}^1; \mathscr{F}(-1)) = H^1(\mathbf{P}^1; \mathscr{F}(-1)) = 0$ implies that $(V_{\mathbf{Q}}, L_0, L_\infty)$ is balanced. Conversely, if $(V_{\mathbf{Q}}, L_0, L_\infty)$ is balanced, then $\mathscr{F}$ is a free O-module and $H^0(\mathbf{P}^1; \mathscr{F}(-1)) = H^1(\mathbf{P}^1; \mathscr{F}(-1)) = 0$ is a well-known result of algebraic geometry. Q.E.D.

LEMMA 2.2.2. *Let*

$$0 \to V^1 \xrightarrow{f} V^2 \xrightarrow{g} V^3 \to 0 \tag{2.2.4}$$

be an exact sequence of $\mathbf{Q}(q)$*-vector spaces. Let* $V^j_{\mathbf{Q}}$, L^j_0, *and* L^j_∞ *be a sub-*$\mathbf{Q}[q, q^{-1}]$*-module, a sub-*A*-module, and a sub-*$\bar{A}$*-module of* V_j*, respectively, and we assume*

$$V_j \cong \mathbf{Q}(q) \otimes_{\mathbf{Q}[q,q^{-1}]} V^j_{\mathbf{Q}} \cong \mathbf{Q}(q) \otimes_A L^j_0 \cong \mathbf{Q}(q) \otimes_{\bar{A}} L^j_\infty \qquad (j = 1, 2, 3).$$

Assume furthermore that (2.2.4) *induces exact sequences*

$$\begin{aligned} &0 \to V^1_{\mathbf{Q}} \to V^2_{\mathbf{Q}} \to V^3_{\mathbf{Q}} \to 0, \\ &0 \to L^1_0 \to L^2_0 \to L^3_0 \to 0, \quad \textit{and} \\ &0 \to L^1_\infty \to L^2_\infty \to L^3_\infty \to 0. \end{aligned} \tag{2.2.5}$$

Then if two of $(V^j_{\mathbf{Q}}, L^j_0, L^j_\infty)$ *are balanced, then so is the third.*

Proof. Let $\mathscr{F}^j$ be the quasi-coherent $O_{\mathbf{P}^1}$-module constructed above from the data $(V^j, V^j_{\mathbf{Q}}, L^j_0, L^j_\infty)$. Then the exactitude of (2.2.5) implies that of $0 \to \mathscr{F}^1 \to \mathscr{F}^2 \to \mathscr{F}^3 \to 0$. Then the preceding lemma implies the desired result. Q.E.D.

LEMMA 2.2.3. *Let* V *be a finite-dimensional* $\mathbf{Q}(q)$*-vector space and let* $(V_{\mathbf{Q}}, L_0, L_\infty)$ *be a balanced triple in* V*. Set*

$$\begin{aligned} V^\perp_{\mathbf{Q}} &= \{u \in V^*; \langle u, V_{\mathbf{Q}} \rangle \subset \mathbf{Q}[q, q^{-1}]\}, \\ L^\perp_0 &= \{u \in V^*; \langle u, L_0 \rangle \subset A\}, \\ L^\perp_\infty &= \{u \in V^*; \langle u, L_\infty \rangle \subset \bar{A}\}. \end{aligned}$$

Then $(V^\perp_{\mathbf{Q}}, L^\perp_0, L^\perp_\infty)$ *is a balanced triple.*

The proof is straightforward.

LEMMA 2.2.4. *Let V be a $\mathbf{Q}(q)$-vector space and let $(V_{\mathbf{Q}}, L_0, L_\infty)$ and $(V'_{\mathbf{Q}}, L'_0, L'_\infty)$ be a pair of balanced triples in V such that $V_{\mathbf{Q}} \supset V'_{\mathbf{Q}}$, $L_0 \supset L'_0$, $L_\infty \supset L'_\infty$. Then they coincide.*

Proof. If we set $E = V_{\mathbf{Q}} \cap L_0 \cap L_\infty$ and $E' = V'_{\mathbf{Q}} \cap L'_0 \cap L'_\infty$, then $E \supset E'$ and $\mathbf{Q}(q) \otimes E = \mathbf{Q}(q) \otimes E'$. Hence $E = E'$. Thus the desired result follows from Lemma 2.1.1. Q.E.D.

3. Upper and lower crystal bases.

3.1. *Definition.* We shall recall the definition of upper and lower crystal bases (see $[K_1]$ and $[K_2]$).

For an integrable $U_q(\mathfrak{g})$-module M, we define $\tilde{e}_i^{up}$, $\tilde{f}_i^{up}$, $\tilde{e}_i^{low}$, $\tilde{f}_i^{low}$ as follows: for $u \in \operatorname{Ker} e_i \cap M_\lambda$ and $0 \leqslant n \leqslant \langle h_i, \lambda \rangle$,

$$\tilde{e}_i^{low}(f_i^{(n)}u) = f_i^{(n-1)}u \quad \text{and} \quad \tilde{f}_i^{low}(f_i^{(n)}u) = f_i^{(n+1)}u, \tag{3.1.1}$$

$$\tilde{e}_i^{up}(f_i^{(n)}u) = \frac{[\langle h_i \lambda \rangle - n + 1]_i}{[n]_i} f_i^{(n-1)}u \quad \text{and} \quad \tilde{f}_i^{up}(f_i^{(n)}u) = \frac{[n+1]_i}{[\langle h_i, \lambda \rangle - n]_i} f_i^{(n+1)}u. \tag{3.1.2}$$

Hence we have

$$e_i((\tilde{f}_i^{up})^n u) = [n]_i (\tilde{f}_i^{up})^{n-1}u \quad \text{and} \quad f_i((\tilde{f}_i^{up})^n u) = [\langle h_i, \lambda \rangle - n]_i (\tilde{f}_i^{up})^{n+1}u. \tag{3.1.3}$$

We have also

$$e_i^{(n)} f_i^{(n)} u = (\tilde{e}_i^{up})^n f_i^{(n)} u = u \qquad \text{for any } u \in M \text{ with } e_i u = 0. \tag{3.1.4}$$

LEMMA 3.1.1. *For $n \geqslant 0$ and $u \in M$, if $e_i^{1+n}u = 0$, then $(\tilde{e}_i^{up})^n u = e_i^{(n)}u$.*

Proof. We can write

$$u = \sum_{k=0}^{n} f_i^{(k)} v_k \qquad \text{with } e_i v_k = 0.$$

Then the desired result follows from (3.1.4). Q.E.D.

Definition 3.1.2. An upper (resp. lower) crystal lattice at $q = 0$ of M is a free sub-A-module L such that

$$M \cong \mathbf{Q}(q) \underset{A}{\otimes} L, \tag{3.1.5}$$

(3.1.6) $$L = \bigoplus_{\lambda \in P} L_\lambda \qquad \text{where } L_\lambda = L \cap M_\lambda,$$

(3.1.7) $$\tilde{e}_i^{up} L \subset L \quad \text{and} \quad \tilde{f}_i^{up} L \subset L \text{ (resp. } \tilde{e}_i^{low} L \subset L \quad \text{and} \quad \tilde{f}_i^{low} L \subset L).$$

By replacing A with $\bar{A}$ in this definition, we define *crystal lattices at* $q = \infty$.

Definition 3.1.3. An upper (resp. lower) crystal base at $q = 0$ is a pair (L, B) where L is an upper (resp. lower) crystal lattice and B is a base of the $\mathbf{Q}$-vector space L/qL satisfying

(3.1.8) $$B = \bigcup_{\lambda \in P} B_\lambda \qquad \text{where } B_\lambda = B \cap (L/qL),$$

(3.1.9) $$\tilde{e}_i^{up} B \subset B \sqcup \{0\} \quad \text{and} \quad \tilde{f}_i^{up} B \subset B \sqcup \{0\},$$

$$\text{(resp. } \tilde{e}_i^{low} B \subset B \sqcup \{0\} \quad \text{and} \quad \tilde{f}_i^{low} B \subset B \sqcup \{0\}),$$

(3.1.10) $$\text{For } b, b' \in B, \qquad b' = \tilde{f}_i^{up} b \text{ (resp. } b' = \tilde{f}_i^{low} b)$$

$$\text{is equivalent to } b = \tilde{e}_i^{up} b' \text{ (resp. } b = \tilde{e}_i^{low} b').$$

Similarly, we can define *upper crystal bases at* $q = \infty$.

3.2. *Relations of upper and lower crystal bases.* We shall recall the relation between upper and lower crystal bases (see [K$_1$] and [K$_2$]). For an integrable $U_q(\mathfrak{g})$-module M, we define the $\mathbf{Q}(q)$-linear automorphism ψ_M: $M \to M$ by $\psi_M(u) = q^{-(\lambda, \lambda)/2} u$ for $\lambda \in P$ and $u \in M_\lambda$.

LEMMA 3.2.1. *(L, B) is a lower crystal base at $q = 0$ if and only if $\psi_M(L, B)$ is an upper crystal base at $q = 0$.*

Another relation is duality. Let M and N be integrable $U_q(\mathfrak{g})$-modules such that $\dim M_\lambda = \dim N_\lambda < \infty$ for any $\lambda \in P$ and let $(\ , \ )$ be a nondegenerate pairing between M and N such that $(\varphi(P)u, v) = (u, Pv)$ for any $u \in M$, $v \in N$ and $P \in U_q(\mathfrak{g})$. Then one can easily see

(3.2.1) $$(\tilde{e}_i^{up} u, v) = (u, \tilde{f}_i^{low} v)$$

and

(3.2.2) $$(\tilde{f}_i^{up} u, v) = (u, \tilde{e}_i^{low} v).$$

For a sub-A-module L of N, we set

(3.2.3) $$L^\perp = \{u \in M; \langle u, L \rangle \subset A\}.$$

PROPOSITION 3.2.2. (i) *$L^{\perp}$ is an upper crystal lattice of M if and only if L is a lower crystal base of N.*

(ii) *If (L, B) is a lower (resp. upper) crystal base of N, let $B^{\perp}$ be the base of $L^{\perp}/qL^{\perp}$ dual to B by the pairing $(L^{\perp}/qL^{\perp}) \times (L/qL) \to \mathbf{Q}$. Then $(L^{\perp}, B^{\perp})$ is an upper (resp. lower) crystal base of M.*

This proposition follows immediately from (3.2.1) and (3.2.2).

3.3. *The crystal base of $V(\lambda)$.* In [K_2], we proved the existence of lower global crystal bases of $V(\lambda)$. Let us recall it. We fix a highest-weight vector u_λ of $V(\lambda)$, and we set

(3.3.1) $L^{low}(\lambda)$ is the smallest A-module that contains u_λ and that is stable by $\tilde{f}_i^{low}$;

$$B^{low}(\lambda) = \{\tilde{f}_{i_1}^{low} \cdots \tilde{f}_{i_1}^{low} u_\lambda \bmod qL^{low}(\lambda)\} \backslash \{0\}. \tag{3.3.2}$$

By using duality, $V(\lambda)$ has a unique upper crystal base $(L^{up}(\lambda), B^{up}(\lambda))$ such that

$$L^{up}(\lambda)_\lambda = Au_\lambda, \tag{3.3.3}$$

$$B^{up}(\lambda)_\lambda = \{u_\lambda \bmod qL(\lambda)\}. \tag{3.3.4}$$

The following theorem can be proven similarly to Theorem 3 in [K_2].

THEOREM 3.3.1. *Let M be an integrable $U_q(\mathfrak{g})$-module in $O_{int}(\mathfrak{g})$.*

(i) *If L is an upper crystal lattice of M, then there is an isomorphism $M \cong \oplus_j V(\lambda_j)$ by which L is isomorphic to $\oplus L(\lambda_j)$.*

(ii) *If (L, B) is an upper crystal base of M, then there is an isomorphism $M \cong \oplus_j V(\lambda_j)$ by which (L, B) is isomorphic to $\oplus_j(L^{up}(\lambda_j), B^{up}(\lambda_j))$.*

Hence, if L is an upper crystal lattice, setting $I_\lambda(L) = L \cap I_\lambda(M)$, we have

$$L = \bigoplus_{\lambda \in P_+} I_\lambda(L) \tag{3.3.5}$$

and

$$I_\lambda(L) \cong (L \cap I_\lambda(M)_\lambda) \underset{A}{\otimes} L^{up}(\lambda). \tag{3.3.6}$$

If (L, B) is an upper crystal base, then B decomposes into $I_\lambda(B) = B \cap (I_\lambda(L)/qI_\lambda(L))$ and $I_\lambda(B)$ is isomorphic to $I_\lambda(B)_\lambda \times B^{up}(\lambda)$.

4. Global crystal base of $V(\lambda)$.

4.1. *$\mathbf{Q}$-form.* Let $U_q^+(\mathfrak{g})_{\mathbf{Z}}$ be the sub-$\mathbf{Z}[q, q^{-1}]$-algebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$, $U_q^-(\mathfrak{g})_{\mathbf{Z}}$ the one generated by $f_i^{(n)}$, and $U_q^{\mathbf{Z}}(\mathfrak{g})$ the one generated by $e_i^{(n)}$, $f_i^{(n)}$, q^h, and $\begin{Bmatrix} q^h \\ n \end{Bmatrix}$, where i, n, h range over I, $\mathbf{Z}_{\geqslant 0}$, and P^*, respectively. Here, for $n \geqslant 0$ we set

$$\begin{Bmatrix} x \\ n \end{Bmatrix} = \prod_{k=1}^{n} \frac{q^{1-k}x - q^{k-1}x^{-1}}{q^k - q^{-k}}.$$

We set $U_q^+(\mathfrak{g})_{\mathbf{Q}} = \mathbf{Q} \otimes U_q^+(\mathfrak{g})_{\mathbf{Z}}$, $U_q^-(\mathfrak{g})_{\mathbf{Q}} = \mathbf{Q} \otimes U_q^-(\mathfrak{g})_{\mathbf{Z}}$, and $U_q^{\mathbf{Q}}(\mathfrak{g}) = \mathbf{Q} \otimes U_q^{\mathbf{Z}}(\mathfrak{g})$.

PROPOSITION 4.1.1. *Let M be a $U_q(\mathfrak{g})$-module such that $M = \oplus_{\lambda \in P} M_\lambda$ and let M be a sub-$U_q^{\mathbf{Z}}(\mathfrak{g})$-module of M. Then $M_{\mathbf{Z}} = \oplus_{\lambda \in P} (M_{\mathbf{Z}} \cap M_\lambda)$.*

Proof. For $h \in P^*$ and $u = \sum_{k \in \mathbf{Z}} u_k \in M$ with $q^h u_k = q^k u_k$, it is enough to show that

$$\text{if } u \in M_{\mathbf{Z}}, \quad \text{then all } u_k \text{ belong to } M_{\mathbf{Z}}. \tag{4.1.1}$$

Writing $u = \sum_{a \leqslant k \leqslant b} u_k$, we shall prove (4.1.1) by the induction on $b - a$. If $b - a \leqslant 0$, then it is obvious. Assume that $b - a > 0$. Then setting $c = b - a$, we have (see §1.1)

$$\begin{Bmatrix} q^{-a}q^h \\ c \end{Bmatrix} u = \sum_{a \leqslant k \leqslant b} \begin{bmatrix} k-a \\ c \end{bmatrix} u_k = u_b$$

where the last equality follows from

$$\begin{bmatrix} k-a \\ c \end{bmatrix} = 0 \qquad \text{for } c > k - a \geqslant 0.$$

Hence $u_b \in M$, and induction proceeds. Q.E.D.

PROPOSITION 4.1.2. *Let M be an integrable $U_q(\mathfrak{g})$-module and $M_{\mathbf{Z}}$ a sub-$U_q^{\mathbf{Z}}(\mathfrak{g})$-module. For $\lambda \in P$ and $i \in I$, assume $n = -\langle h_i, \lambda \rangle > 0$. Then $\{u \in M_\lambda;\, e_i^{(k)} u \in M_{\mathbf{Z}}$ for $k \geqslant n\} = M_{\mathbf{Z}\lambda}$.*

This follows immediately from the identity (Lemma 6.1.4. in [K$_2$])

$$u = \sum_{k \geqslant n} (-1)^{k-n} \begin{bmatrix} k-1 \\ k-n \end{bmatrix}_i f_i^{(k)} e_i^{(k)} u \qquad \text{for } u \in M_\lambda. \tag{4.1.2}$$

PROPOSITION 4.1.3. *Let M be an integrable $U_q(\mathfrak{g})$-module and L an upper crystal lattice of M. Let $M_{\mathbf{Z}}$ be a sub-$U_q(\mathfrak{g})$-module. Then $L \cap M_{\mathbf{Z}}/qL \cap M_{\mathbf{Z}} \subset L/qL$ is stable by $\tilde{e}_i^{up}$ and $\tilde{f}_i^{up}$.*

The statement for $\tilde{f}_i^{up}$ follows immediately from the following lemma, and the one for $\tilde{e}_i^{up}$ can be proven similarly.

LEMMA 4.1.4. *We set $B_n^{(i)}(x) = \prod_{j=0}^{n-1}(1 - q_i^{2j}x)$. We define an endomorphism F_i of M by*

$$F_i(u) = \sum (-1)^k q_i^{-(3/2)k(k+1)+k} (q_i^{-1} - q_i)^{k+1} [k+1]_i! B_k^{(i)}(q_i) f_i^{(k+1)} t_i^{k+1} e_i^{(k)} u. \tag{4.1.3}$$

Then $F_i L \subset L$, for any crystal lattice of M and the induced action of F_i on L/qL, coincides with that of f_i^{up}.

Proof. It is enough to show that for $u \in M$ such that $e_i u = 0$ and $t_i u = q_i^l u$

(4.1.4)
$F_i((f_i^{up})^n u) \in (1 + qA)(\tilde{f}_i^{up})^{n+1} u$ for $0 \leqslant n < l$ (see [K_2], Proposition 2.3.2).

We have

$$f_i^{(k+1)} t_i^{k+1} e_i^{(k)} (f_i^{up})^n u = f_i^{(k+1)} t_i^{(k+1)} \begin{bmatrix} n \\ k \end{bmatrix}_i (\tilde{f}_i^{up})^{n-k} u$$

$$= q_i^{(k+1)(l-2n+2k)} \begin{bmatrix} n \\ k \end{bmatrix}_i \begin{bmatrix} l-n+k \\ k+1 \end{bmatrix}_i (\tilde{f}_i^{up})^{n+1} u.$$

Hence it is enough to show

$$S = \sum_k (-1)^k q_i^{(k+1)(l-2n+2k)-(3/2)k(k+1)+k} (q_i^{-1} - q_i)^{k+1} \tag{4.1.5}$$

$$[k+1]_i! \begin{bmatrix} n \\ k \end{bmatrix}_i \begin{bmatrix} l-n+k \\ k+1 \end{bmatrix}_i B_k^{(i)}(q_i)$$

belongs to $1 + q_i \mathbf{Z}[q_i]$. In the sequel, we omit the index i (we write q, B_n for q_i, $B_n^{(i)}$, etc.). Setting $l = n + j$ ($j > 0$), we have

$$(q^{-1} - q)^{k+1} [k+1]! \begin{bmatrix} l-n+k \\ k+1 \end{bmatrix} = \prod_{\nu=0}^{k} (q^{\nu-j-k} - q^{j+k-\nu})$$

$$= q^{-(k+1)(j+k)+k(k+1)/2} \prod_{\nu=0}^{k} (1 - q^{2(j+k-\nu)})$$

$$= (1 - q^{2j}) q^{-(k+1)(j+k)+k(k+1)/2} B_k(q^{2j+2}).$$

Hence we obtain

$$S = (1 - q^{2j}) \sum (-1)^k q^{(k+1)(j-n+2k)-k(k+1)+k-(k+1)(j+k)} \begin{bmatrix} n \\ k \end{bmatrix} B_k(q) B_k(q^{2j+2})$$

$$= (1 - q^{2j}) \sum (-1)^k q^{-n(k+1)+k} \begin{bmatrix} n \\ k \end{bmatrix} B_k(q) B_k(q^{2j+2}).$$

Hence the statement follows from the following sublemma.

SUBLEMMA 4.1.5. *We set* $B_n(x) = \prod_{k=0}^{n-1}(1 - q^{2k}x)$.

(i) $\sum_{k=0}^{n}(-1)^k q^{-n(k+1)+k}\begin{bmatrix} n \\ k \end{bmatrix} B_{k+m}(q)B_k(qx)$ *belongs to* $\mathbf{Z}[q, x]$ *for any* $n, m \geqslant 0$.

(ii) $\sum_{k=0}^{n}(-1)^k q^{-n(k+1)+k}\begin{bmatrix} n \\ k \end{bmatrix} B_k(q) = 1$.

Proof. The second statement follows from the well-known formula

$$\sum (-1)^k q^{(1-n)k} B_k(x) \begin{bmatrix} n \\ k \end{bmatrix} = x^n. \tag{4.1.6}$$

Let us prove the first statement by the induction on n. If $n = 0$, then this is obvious. Let us set $f_{n,m}$ equal to the left-hand side of the equality in (i). Then using $\begin{bmatrix} n \\ k \end{bmatrix} = q^{-k}\begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$, we have

$$\begin{aligned} f_{n,m} &= \sum (-1)^k q^{-n(k+1)} \begin{bmatrix} n-1 \\ k \end{bmatrix} B_{k+m}(q)B_k(qx) \\ &\quad + \sum (-1)^k q^{-nk} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} B_{k+m}(q)B_k(qx) \\ &= \sum (-1)^k q^{-n(k+1)} \begin{bmatrix} n-1 \\ k \end{bmatrix} B_{k+m}(q)B_k(qx)(1 - (1 - q^{2k+2m+1})(1 - q^{2k+1}x)) \\ &= \sum (-1)^k q^{-n(k+1)} \begin{bmatrix} n-1 \\ k \end{bmatrix} B_{k+m}(q)B_k(qx)(q^{2k+1}x + q^{2k+2m+1} - q^{4k+2m+2}x) \\ &= \sum (-1)^k q^{-(n-1)(k+1)+k} \begin{bmatrix} n-1 \\ k \end{bmatrix} B_{k+m}(q)B_k(qx)(q^{2m} + (1 - q^{2k+2m+1})x) \\ &= q^{2m} f_{n-1,m} + x f_{n-1,m+1}. \end{aligned}$$

Thus we obtain the desired result. Q.E.D.

4.2. *Global crystal base of* $V(\lambda)$. We set

$$V_{\mathbf{Q}}^{low}(\lambda) = U_q^{\mathbf{Q}}(\mathfrak{g})u_\lambda. \tag{4.2.1}$$

Let $-$ be the automorphism of $V(\lambda)$ defined by

$$\overline{Pu_\lambda} = \overline{P}u_\lambda \qquad \text{for } P \in U_q(\mathfrak{g}). \tag{4.2.2}$$

Let $\bar{L}^{low}(\lambda)$ be the image of $L^{low}(\lambda)$ by $-$. Then $(V_{\mathbf{Q}}^{low}(\lambda), L^{low}(\lambda), \bar{L}^{low}(\lambda))$ is a balanced triple. Now let (,) be the symmetric form of $V(\lambda)$ such that

$$(Pu, v) = (u, \varphi(P)v) \qquad \text{for } u, v \in V(\lambda) \text{ and } P \in U_q(\mathfrak{g}); \tag{4.2.4}$$

$$(u_\lambda, u_\lambda) = 1. \tag{4.2.5}$$

We set

$$V_{\mathbf{Q}}^{up}(\lambda) = \{u \in V(\lambda); (u, V_{\mathbf{Q}}^{low}(\lambda)) \subset \mathbf{Q}[q, q^{-1}]\}, \tag{4.2.6}$$

$$L^{up}(\lambda) = \{u \in V(\lambda); (u, L^{low}(\lambda)) \subset A\}, \quad \text{and} \tag{4.2.7}$$

$$\bar{L}^{up}(\lambda) = \{u \in V(\lambda); (u, \bar{L}^{low}(\lambda)) \subset \bar{A}\}. \tag{4.2.8}$$

Then by Lemma 2.3.3 we have the following lemma.

LEMMA 4.2.1. *$(V_{\mathbf{Q}}^{up}(\lambda), L^{up}(\lambda), \bar{L}^{up}(\lambda))$ is balanced.*

By Proposition 3.2.2, $L^{up}(\lambda)$ and $\bar{L}^{up}(\lambda)$ are upper crystal lattices at $q = 0$ and $q = \infty$, respectively, and we have

$$L^{up}(\lambda)_\mu = q^{(\lambda,\lambda)/2-(\mu,\mu)/2} L^{low}(\lambda)_\mu \qquad \text{for } \mu \in P. \tag{4.2.9}$$

Now if we set $B^{up}(\lambda)_\mu = q^{(\lambda,\lambda)/2-(\mu,\mu)/2} B^{low}(\lambda)_\mu$, then $(L^{up}(\lambda), B^{up}(\lambda))$ is an upper crystal base of $V(\lambda)$ at $q = 0$. Similarly, we can define the upper crystal base $(\bar{L}^{up}(\lambda), \bar{B}^{up}(\lambda))$ at $q = \infty$.

The vector space $V_{\mathbf{Q}}^{up}(\lambda)$ is a $U_q^{\mathbf{Q}}(\mathfrak{g})$-module, and it is characterized by the following two properties:

$$V_{\mathbf{Q}}^{up}(\lambda)_\lambda = \mathbf{Q}[q, q^{-1}]u_\lambda; \tag{4.2.10}$$

$$\{u \in V(\lambda); e_i^{(n)}u \in V_{\mathbf{Q}}^{up}(\lambda) \text{ for any } i \in I \text{ and } n \geqslant 1\} = V_{\mathbf{Q}}(\lambda) + \mathbf{Q}(q)u_\lambda. \tag{4.2.11}$$

Note that (4.2.11) may be replaced by the following property:

$$V_{\mathbf{Q}}^{up}(\lambda)_\mu = \{u \in V(\lambda)_\mu; U_q^+(\mathfrak{g})_{\mathbf{Q}\lambda-\mu}u \in \mathbf{Q}[q, q^{-1}]u_\lambda\}. \tag{4.2.12}$$

4.3. *Global crystal base of $U_q^-(\mathfrak{g})$.* In [K$_2$], we defined the crystal base $(L(U_q^-(\mathfrak{g})), B(U_q^-(\mathfrak{g})))$ of $U_q^-(\mathfrak{g})$, and we have

$$U_q^-(\mathfrak{g})_{\mathbf{Q}} = \bigoplus_{b \in B(\infty)} \mathbf{Q}[q, q^{-1}]G_\infty(b). \tag{4.3.1}$$

Here G_∞ is the inverse of the isomorphism $U_q^-(\mathfrak{g})_{\mathbf{Q}} \cap L(U_q^-(\mathfrak{g})) \cap \bar{L}(U_q^-(\mathfrak{g})) \xrightarrow{\sim} L(U_q^-(\mathfrak{g}))/qL(U_q^-(\mathfrak{g}))$.

If we define π_λ: $U_q^-(\mathfrak{g}) \to V(\lambda)$ by $P \mapsto Pu_\lambda$, then π_λ induces the surjective map $L(U_q^-(\mathfrak{g})) \to L^{low}(\lambda)$ and $\bar{\pi}_\lambda$: $L(U_q^-(\mathfrak{g}))/qL(U_q^-(\mathfrak{g})) \to L^{low}(\lambda)/qL^{low}(\lambda)$. Then we have

$$G_\infty(b)u_\lambda = G_\lambda(\bar{\pi}_\lambda b) \qquad \text{for any } b \in B(\infty) \tag{4.3.2}$$

and

$$\{b \in B(\infty); \bar{\pi}_\lambda b \neq 0\} \xrightarrow{\sim} B(\lambda). \tag{4.3.3}$$

Here G_λ denotes the inverse of the isomorphism $V_{\mathbf{Q}}^{low}(\lambda) \cap L^{low}(\lambda) \cap \bar{L}^{low}(\lambda) \xrightarrow{\sim} L^{low}(\lambda)/qL^{low}(\lambda)$.

PROPOSITION 4.3.1. *Let $\xi \in Q_-$ and $K = \{b \in B(\infty)_\xi;\ \bar{\pi}_\lambda(b) \neq 0\}$. Let u be an element of $V(\lambda)_{\lambda+\xi}$.*

(i) *If $b \in B(\infty)_\xi$ satisfies $\bar{\pi}_\lambda(b) = 0$, then $\varphi(G_\infty(b))u = 0$.*
(ii) *$u \in V_{\mathbf{Q}}^{up}(\lambda)_{\lambda+\xi}$ if and only if $\varphi(G_\infty(b))u \in \mathbf{Q}[q, q^{-1}]u_\lambda$ for any $b \in K$.*
(iii) *$u \in L^{up}(\lambda)_{\lambda+\xi}$ if and only if $\varphi(G_\infty(b))u \in Au_\lambda$ for any $b \in K$.*
(iv) *If $\varphi(G_\infty(b))u = 0$ for any $b \in K$, then $u = 0$.*
(v) *$U_q^-(\mathfrak{g})_{\mathbf{Q}} = \oplus_{b\in K} \mathbf{Q}[q, q^{-1}]G_\infty(b) \oplus (\sum_{n>\langle h_i,\lambda\rangle} U_q^-(\mathfrak{g})_{\mathbf{Q}} f_i^{(n)})$.*

Proof. Note that

$$(G_\infty(b)u_\lambda, u) = (u_\lambda, \varphi(G_\infty(b))u),$$

$$L^{low}(\lambda)_{\lambda+\xi} = \bigoplus_{b\in K} AG(b)u_\lambda, \quad \text{and}$$

$$V_{\mathbf{Q}}^{low} = \bigoplus_{b\in K} \mathbf{Q}[q, q^{-1}]G_\infty(b)u_\lambda.$$

They imply immediately (i)–(iv).

(v) follows from the fact that

$$V(\lambda) = U_q^-(\mathfrak{g})\Big/\left(\sum_i U_q^-(\mathfrak{g})f_i^{\langle h_i,\lambda\rangle+1}\right) \quad \text{and}$$

$$U_q^-(\mathfrak{g})_{\mathbf{Q}} \cap \left(\sum_i U_q^-(\mathfrak{g})f_i^{\langle h_i,\lambda\rangle+1}\right) = \sum_{n>\langle h_i,\lambda\rangle} U_q^-(\mathfrak{g})_{\mathbf{Q}} f_i^{(n)}. \qquad \text{Q.E.D.}$$

5. Properties of global crystal bases. *Hereafter, a crystal lattice (base) means an upper crystal lattice (base). We denote $V_{\mathbf{Q}}(\lambda)$, $L(\lambda)$, for $V_{\mathbf{Q}}^{up}(\lambda)$, $L^{up}(\lambda)$, etc.*

5.1. *Elementary property.* Let M be an integrable $U_q(\mathfrak{g})$-module and $M_{\mathbf{Q}}$ a sub-$U_q^{\mathbf{Q}}(\mathfrak{g})$-module of M such that $M \cong \mathbf{Q}(q) \otimes_{\mathbf{Q}[q,q^{-1}]} M_{\mathbf{Q}}$. We do not assume that M is in $O_{\text{int}}(\mathfrak{g})$.

Let L_0 and L_∞ be upper crystal lattices of M at $q = 0$ and at $q = \infty$, respectively.

In this section we assume

(5.1.1) $\qquad (M_{\mathbf{Q}}, L_0, L_\infty)$ is a balanced triple.

We set

(5.1.2) $\qquad E = M_{\mathbf{Q}} \cap L_0 \cap L_\infty,$

and we denote by G the inverse of the isomorphism $E \xrightarrow{\sim} L_0/qL_0$.

By the definition of balanced triple, we have

(5.1.3) $$M_{\mathbf{Q}} \simeq \mathbf{Q}[q, q^{-1}] \otimes E, \qquad L_0 \simeq A \otimes E \quad \text{and} \quad L_\infty \simeq \bar{A} \otimes E.$$

LEMMA 5.1.1. *Let $b \in L_0/qL_0$ and $n \in \mathbf{Z}_{\geqslant 0}$.*
(i) *$e_i^{1+n}G(b) = 0$ if and only if $\tilde{e}_i^{n+1}b = 0$.*
(ii) *If $\tilde{e}_i^{1+n}b = 0$, then $e_i^{(n)}G(b) = G(\tilde{e}_i^n b)$.*

Proof. Assume $e_i^{1+n}G(b) = 0$. Then $\tilde{e}_i^{n+1}b = 0$ is evident and Lemma 3.1.1 implies $\tilde{e}_i^n G(b) = e_i^{(n)}G(b) \in M_{\mathbf{Q}} \cap L_0 \cap L_\infty$. Hence $e_i^{(n)}G(b) = G(\tilde{e}_i^n b)$. It remains to prove $\tilde{e}_i^{n+1}b = 0$ implies $\tilde{e}_i^{1+n}G(b) = 0$. Take the smallest $m \geqslant n$ such that $e_i^{1+m}G(b) = 0$. Then $e_i^{(m)}G(b) = G(\tilde{e}_i^m b)$. Hence if $m > n$, then $e_i^{(m)}G(b) = 0$, which is a contradiction. Q.E.D.

COROLLARY 5.1.2 *For any i and $n \in \mathbf{Z}_{\geqslant 0}$,*

$$\{u \in M; e_i^{1+n}u = 0\} = \mathbf{Q}(q) \otimes G(\{b \in L_0/qL_0; \tilde{e}_i^{1+n}b = 0\}).$$

Proof. Any $u \in M$ can be written

(5.1.4) $$u = \sum_{k=0}^{r} a_k G(b_k) \qquad \text{where } a_k \in \mathbf{Q}(q) \text{ and } b_k \in L_0/qL_0$$

satisfies $\tilde{e}_i^{k+1}b_k = 0$ and $b_k = \tilde{f}_i^k \tilde{e}_i^k b_k$.

It is enough to show that, if $e_i^{1+n}u = 0$ and if $r > n$, then $a_r = 0$ or $b_r = 0$. We have $0 = e_i^{(r)}u = a_r G(\tilde{e}_i^r b_r)$. Hence $a_r = 0$ or $b_r = 0$. Q.E.D.

LEMMA 5.1.3. *For any $i \in I$ and $n \in \mathbf{Z}_{\geqslant 0}$,*

$$\{u \in M; e_i^{(m)}u \in M_{\mathbf{Q}} \text{ for any } m \geqslant n\} = M_{\mathbf{Q}} + \operatorname{Ker} e_i^n.$$

Proof. We write $u = \sum_{k=0}^{r} a_k G(b_k)$ as in (5.1.4). Assuming $e_i^{(m)}u \in M_{\mathbf{Q}}$ for $m \geqslant n$, we shall prove $u \in M_{\mathbf{Q}} + \operatorname{Ker} e_i^n$ by induction on r. If $r < n$, then $u \in \operatorname{Ker} e_i^n$. If $r \geqslant n$, then $e_i^{(r)}u = a_r G(\tilde{e}_i^r b_r) \in M_{\mathbf{Q}}$, and hence $b_r = 0$ or $a_r \in \mathbf{Q}[q, q^{-1}]$. Hence $a_r G(b_r) \in M_{\mathbf{Q}}$. Thus induction proceeds. Q.E.D.

LEMMA 5.1.4. *Let J_+, J_- be subsets of I and for $i \in J_\pm$ let $a_\pm(i)$ be a nonnegative integer. Then we have*

(5.1.5)
$$\{u \in M; e_i^{(n)}u \in M_{\mathbf{Q}} \text{ for } i \in J_1 \text{ and } n \geqslant a_+(i) \text{ and } f_i^{(n)}u \in M_{\mathbf{Q}} \text{ for } i \in J_2 \text{ and } n \geqslant a_-(i)\}$$

$$= M_{\mathbf{Q}} + \{u \in M : e_i^{a_+(i)}u = 0 \text{ for } i \in J_1 \text{ and } f_i^{a_-(i)}u = 0 \text{ for } i \in J_2\}.$$

Proof. We set $(x_i^{\pm})^{(n)} = e_i^{(n)}$ or $f_i^{(n)}$, $\tilde{x}_i^{\pm} = \tilde{e}_i$ or $\tilde{f}_i$ according to $\pm$. Let $K_i^{\pm}$ be the $\mathbf{Q}[q, q^{-1}]$-module generated by $\{G(b); b \in L_0/qL_0, (\tilde{x}_i^{\pm})^{a_{\pm}(i)}b = 0\}$, and $S_i^{\pm} = \{u \in M; (x_i^{\pm})^{(n)}u \in M_{\mathbf{Q}}$ for $n \geqslant a_{\pm}(i)\}$. Then by the preceding lemma

$$S_i^{\pm} \subset M_{\mathbf{Q}} + L_i^{\pm}.$$

Let $\varphi: M \to M/M_{\mathbf{Q}} = (\mathbf{Q}(q)/\mathbf{Q}[q, q^{-1}]) \otimes E$ be the projection. Then $\varphi(S_i^{\pm}) \subset \mathbf{Q}(q)/\mathbf{Q}[q, q^{-1}] \otimes L_i^{\pm}$. Hence

$$\varphi(\textstyle\bigcap_{i \in J_+} S_i^+ \cap \bigcap_{i \in J_-} S_i^-) \subset (\mathbf{Q}(q)/\mathbf{Q}[q, q^{-1}]) \cap (\bigcap_{i \in J_+} L_i \cap \bigcap_{i \in J_-} L_i^-),$$

which gives the desired result. Q.E.D.

Remark. We have the isomorphism $L_0/qL_0 \xleftarrow{\sim} E \xrightarrow{\sim} L_\infty/q^{-1}L_\infty$. In general, the thus-obtained isomorphism $L_0/qL_0 \xrightarrow{\sim} L_\infty/q^{-1}L_\infty$ does not commute with $\tilde{e}_i$ or $\tilde{f}_i$.

5.2. *Global crystal bases and W.* Let us investigate the relation between the global crystal bases and the filtration W (see §1.3). In this subsection, we assume that $(M_{\mathbf{Q}}, L_0, L_\infty)$ is balanced and that M is in $O_{\text{int}}(\mathfrak{g})$.

PROPOSITON 5.2.1. *For any subset F of P_+, $(W_F(M_{\mathbf{Q}}), W_F(L_0), W_F(L_\infty))$ is balanced, where $W_F(M_{\mathbf{Q}}) = M_{\mathbf{Q}} \cap W_F(M)$, etc.*

Proof. We may assume F finite. Set $\mathscr{W} = \{\lambda \in P_+; I_\lambda(M) \neq 0\}$. We shall show, for $\mu \in P$,

$$W_F(M_{\mathbf{Q}}) \cap L_{0\mu} \cap L_{\infty\mu} \xrightarrow{\sim} W_F(L_0)_\mu / qW_F(L_0)_\mu \tag{5.2.1}$$

by the induction of $\#(F + Q_-) \cap (\mu + Q_+) \cap \mathscr{W}$. Note that $(F + Q_-) \cap (\mu + Q_+)$ is a finite set for any $\mu \in P$.

If $\#(F + Q_-) \cap (\mu + Q_+) \cap \mathscr{W} = \phi$, then $W_F(M)_\mu = 0$, and there is nothing to prove. Otherwise, there exist $F' \subset F$ and $\lambda \in P_+$ such that

$$(F + Q_-) \cap (\mu + Q_+) \cap \mathscr{W} = \{\lambda\} \sqcup (F' + Q_-) \cap (\mu + Q_+) \cap \mathscr{W}.$$

By the hypothesis of induction, $(W_{F'}(M_{\mathbf{Q}})_\mu, W_{F'}(L_0)_\mu, W_{F'}(L_\infty)_\mu)$ is balanced. Hence $((M_{\mathbf{Q}}/W_{F'}(M_{\mathbf{Q}}))_\mu, (L_0/W_{F'}(L_0))_\mu, (L_\infty/W_{F'}(L_\infty))_\mu)$ is balanced. Since it is enough to show $((W_F(M_{\mathbf{Q}})/W_{F'}(M_{\mathbf{Q}}))_\mu, (W_F(L_0)/W_{F'}(L_0))_\mu, (W_F(L_\infty)/W_{F'}(L_\infty))_\mu)$ is balanced by Lemma 2.2.1, we may assume $F' = \phi$ by replacing M with $M/W_{F'}(M)$. Hence it is enough to show

(5.2.2) for $\lambda \in P_+$, $\mu \in P$, such that $(\lambda + Q_-) \cap (\mu + Q_+) \cap \mathscr{W} = \{\lambda\}$,

$(W_\lambda(M_{\mathbf{Q}})_\mu, W_\lambda(L_0)_\mu, W_\lambda(L_\infty)_\mu)$ is balanced.

Since the injectivity of $W_\lambda(M_{\mathbf{Q}})_\mu \cap W_\lambda(L_0)_\mu \cap W_\lambda(L_\infty)_\mu \to W_\lambda(L_0)_\mu/qW_\lambda(L_0)_\mu$ is evident, it is enough to show

$$\text{(5.2.3)} \qquad \text{if } b \in W_\lambda(L_0)_\mu/qW_\lambda(L_0)_\mu \subset L_0/qL_0, \quad \text{then } G(b) \in I_\lambda(M).$$

Let us take the smallest subset F of P_+ such that $G(b) \in \sum_{\zeta \in F} I_\zeta(M)$. If $F \not\subset \{\lambda\}$, then take an element $\lambda_0 \in F$ such that $\lambda_0 \neq \lambda$ and $(\lambda_0 + Q_+) \cap F = \{\lambda_0\}$. We have $\xi = \mu - \lambda_0 \in Q_-$. Take any element P of $U_q^-(\mathfrak{g})_{\mathbf{Q}\xi} \cap L(U_q^-(\mathfrak{g})) \cap \bar{L}(U_q^-(\mathfrak{g}))$. Then writing $G(b) = \sum v_\zeta$ with $v_\zeta \in I_\zeta(M)$, $\varphi(P)v_\zeta = 0$ for $\zeta \neq \lambda_0$ because $\zeta + Q_- \not\ni \lambda_0$. On the other hand $v_{\lambda_0} \in qL_0$. Hence $\varphi(P)G(b) = \varphi(P)v_{\lambda_0} \in M_{\mathbf{Q}} \cap qL_0 \cap L_\infty$ by Proposition 4.3.1. Thus $\varphi(P)v_{\lambda_0} = 0$. This implies $v_{\lambda_0} = 0$ and hence $G(b) \in \sum_{\mu \neq \lambda_0} I_\mu(M)$, which is a contradiction. Q.E.D.

COROLLARY 5.2.2. *Set*

$$\text{(5.2.4)} \qquad H = \{u \in M;\, e_i u = 0 \text{ for } i \in I\}.$$

Then $Gr_\lambda^W H \cap Gr_\lambda^W E = Gr_\lambda^W(H \cap E)$, *where* $Gr_\lambda^W H = (W_\lambda(M) \cap H)/(W_{<\lambda}(M) \cap H)$, *etc.*

Proof. We may assume $M = W_\lambda(M)$. Hence it is enough to show

$$E \cap (W_{<\lambda}(M) + H) = E \cap W_{<\lambda}(H) + E \cap H.$$

This follows from $E = G(L_0/qL_0)$, $W_{<\lambda}(M) = \mathbf{Q}(q) \otimes_{\mathbf{Q}} G(W_{<\lambda}(L_0/qL_0))$ and $H = \mathbf{Q}(q) \otimes_{\mathbf{Q}} G(\{b \in L_0/qL_0;\, \tilde{e}_i b = 0\})$. Q.E.D.

PROPOSITION 5.2.2. *Define H and E as in* (5.1.2) *and* (5.2.4). *Then for any $\lambda \in P_+$,* $(Gr_\lambda^W(M_{\mathbf{Q}}), Gr_\lambda(L_0), Gr_\lambda(L_\infty))$ *is isomorphic to* $(Gr_\lambda^W(H \cap E) \otimes V_{\mathbf{Q}}(\lambda), Gr_\lambda^W(H \cap E) \otimes L(\lambda), Gr_\lambda^W(H \cap E) \otimes \bar{L}(\lambda))$.

Proof. By Proposition 5.2.1 and Corollary 5.2.2, we may assume that $M = I_\lambda(M)$. Then $M = (H \cap E) \otimes V(\lambda)$, $L_0 = (H \cap E) \otimes L(\lambda)$, $L_\infty = (H \cap E) \otimes \bar{L}(\lambda)$. Now, $M_{\mathbf{Q}} = (H \cap E) \otimes V_{\mathbf{Q}}(\lambda)$ follows from Lemma 5.1.4 and (4.2.11).

5.3. *Matrix coefficients of e_i and f_i.* Let M, $M_{\mathbf{Q}}$, L_0, and L_∞ be as in §5.1. In particular, $(M_{\mathbf{Q}}, L_0, L_\infty)$ is assumed to be balanced. We do not assume in this section that M is in $O_{\mathrm{int}}(\mathfrak{g})$. Let us assume that B is given so that (L_0, B) is a crystal base of M.

For $b \in B$, we set

$$\text{(5.3.1)} \quad \varepsilon_i(b) = \max\{n \geqslant 0;\, \tilde{e}_i^n b \in B\} \quad \text{and} \quad \varphi_i(b) = \max\{n \geqslant 0;\, \tilde{f}_i^n b \in B\}.$$

Let G be the inverse of the isomorphism $M_{\mathbf{Q}} \cap L_0 \cap L_\infty \xrightarrow{\sim} L_0/qL_0$.

PROPOSITION 5.3.1. *For any $b \in B$, and $i \in I$, there exist $F_{bb'}^i \in \mathbf{Q}[q, q^{-1}]$ such that*

$$\text{(5.3.2)} \qquad f_i G(b) = [\varphi_i(b)]_i G(\tilde{f}_i b) + \sum_{b'} F_{bb'}^i G(b'),$$

(5.3.3) $$F^i_{bb'} = 0 \quad \textit{unless } \varepsilon_j(b') \leqslant \varepsilon_j(b) \textit{ for any } j \in I,$$

(5.3.4) $$F^i_{bb'} \in qq_i^{1-\varphi_i(b)}\mathbf{Q}[q].$$

Proof. (a) *Case* $I = \{i\}$. Then M belongs to $O_{\mathrm{int}}(\mathfrak{g})$. If $b \in I_\lambda(B)_\mu$ with $\lambda \in P_+$ and $\mu \in P$, then $\mu = \lambda - \varepsilon_i(b)\alpha_i$. Hence $G(b)$ belongs to $W_\lambda(M)$ by Proposition 5.2.1 and $f_i G(b) \equiv [\varphi_i(b)]_i G(\tilde{f}_i b) \bmod W_{<\lambda}(M)$ by Proposition 5.2.3.

Thus we obtain (5.3.2) and (5.3.3). It remains to prove (5.3.4). Let us write

$$G(b) = \sum_{k=0}^{n} \tilde{e}_i^k u_k$$

with $u_k \in \operatorname{Ker} f_i \cap M_{\mu-(n-k)\alpha_i}$ and $n = \varphi_i(b)$. Then all u_k belong to L_0. Moreover, $u_n = \tilde{f}_i^n G(b)$ by Lemma 3.1.1, and hence

(5.3.5) $$G(\tilde{f}_i b) \equiv \tilde{e}_i^{n-1} u_n \bmod qL_0.$$

By (3.1.3), we have

(5.3.6) $$f_i G(b) = \sum_{k=1}^{n} [k]_i \tilde{e}_i^{k-1} u_k$$

$$\equiv [n]_i G(\tilde{f}_i b) + \sum_{k=1}^{n-1} [k]_i \tilde{e}_i^{k-1} u_k \bmod qq_i^{1-n} L_0.$$

Thus we obtain

(5.3.7) $$f_i G(b) - [\varphi_i(b)]_i G(\tilde{f}_i b) \in q \cdot q_i^{1-n}(L_0 \cap M_{\mathbf{Q}}).$$

Hence we obtain (5.3.4).

(b) *General case.* By (a), we can write

$$f_i G(b) = [\varphi_i(b)]_i G(\tilde{f}_i b) + \sum_{\varepsilon_i(b') \leqslant \varepsilon_i(b)} F^i_{bb'} G(b').$$

Then, for any $j \neq i$, setting $a = \varepsilon_j(b) + 1$, we have $e_j^a f_i G(b) = f_i e_j^a G(b) = 0$ by Lemma 5.1.1. Hence Corollary 5.1.2 implies

$$f_i G(b) \in \sum_{\varepsilon_j(b') \leqslant \varepsilon_j(b)} \mathbf{Q}[q, q^{-1}] G(b'),$$

which gives (5.3.3). Q.E.D.

Remark. (1) By replacing e_i and f_i, we obtain

(5.3.8) $$e_i G(b) = [\varepsilon_i(b)] G(\tilde{e}_i b) + \sum_{b'} E^i_{bb'} G(b') \text{ with}$$

(5.3.9) $$E^i_{bb'} = 0 \quad \text{unless } \varphi_j(b') \leqslant \varphi_j(b) \text{ for any } j \in J,$$

(5.3.10) $$E^i_{bb'} \in qq_i^{1-\varepsilon_i(b)}\mathbf{Q}[q].$$

Note also that (5.3.3) is equivalent to

(5.3.3)′ $$F^i_{bb'} = 0 \quad \text{unless } \varphi_j(b') \leqslant \varphi_j(b) - \langle h_j, \alpha_i \rangle \text{ for any } j \in J.$$

(2) The proof of Proposition 5.3.1 shows that for $b \in B$ and $i \neq j \in I$, if $\tilde{f}_i b \neq 0$, then $\varepsilon_j(\tilde{f}_i b) \leqslant \varepsilon_j(b)$. More precise arguments show the following. For $b \in B(\lambda)$, set $\psi(n) = \varepsilon_j(\tilde{f}_i^n b) - \varepsilon_j(\tilde{f}_i^{n+1} b)$ for $0 \leqslant n < \varphi_i(b)$. Then there exist integers c_0, c_1 $(0 \leqslant c_0 \leqslant c_1 \leqslant \varphi_i(b))$ such that

(i) $\psi(n) = -\langle h_i, \alpha_j \rangle$ for $0 \leqslant n < c_0$,

(ii) $\psi(n) > \psi(n+1)$ for $c_0 \leqslant n < c_1$, and

(iii) $\psi(n) = 0$ for $c_1 \leqslant n < \varphi_i(b)$.

(3) For a subset J of I, let W^J_λ be the filtration of M by regarding M as modules over the subalgebra $U_q(\mathfrak{g}_J)$ generated by e_i, f_i $(i \in J)$ and q^h $(h \in P^*)$. Assume $i \notin J$ and $\lambda \in P$. If $b \in W^J_\lambda(B)$, then $f_i G(b) \in W^J_{\lambda-\alpha_i}(M)$. In fact, setting $\mu = wt(b)$, for $\xi \in Q_+ \setminus ((\lambda + Q_-) - \mu) = Q_+ \setminus ((\lambda - \alpha_i) + Q_-) - (\mu - \alpha_i))$ and $P \in U_q^+(\mathfrak{g}_J)_\xi$, we have $Pf_i G(b) = f_i PG(b) = 0$. Hence we obtain

(5.3.11) $$\tilde{f}_i W^J_\lambda(B) \subset W^J_{\lambda-\alpha_i}(B) \cup \{0\},$$

(5.3.12) $$\text{for } b \in W^J_\lambda(B), \qquad F^i_{bb'} = 0 \qquad \text{unless } b' \in W^J_{\lambda-\alpha_i}(B).$$

6. Sufficient condition for the existence of global bases.

6.1. *Statement.* In the preceding section, we discussed necessary conditions for the balancedness of $(M_{\mathbf{Q}}, L_0, L_\infty)$. In this section, we shall study the sufficient condition.

Let M be an integrable $U_q(\mathfrak{g})$-module in $O_{\mathrm{int}}(\mathfrak{g})$ and $M_{\mathbf{Q}}$ a sub-$U_q^{\mathbf{Q}}(\mathfrak{g})$-module such that $M = \mathbf{Q}(q) \otimes M_{\mathbf{Q}}$. Let L_0 and L_∞ be crystal lattices of M at $q = 0$ and $q = \infty$. As in the preceding section, we set

(6.1.1) $$H = \{u \in M;\, e_i u = 0 \text{ for any } i\}.$$

THEOREM 2. *Assume the following two conditions.*

(a) $(H \cap M_{\mathbf{Q}}, H \cap L_0, H \cap L_\infty)$ *is a balanced triple in* H.

(b) $\{u \in M;\, e_i^{(n)} u \in M_{\mathbf{Q}}$ *for any* $i \in I$ *and any* $n \geqslant 1\} = M_{\mathbf{Q}} + H$.

Then $(M_{\mathbf{Q}}, L_0, L_\infty)$ *is a balanced triple.*

The converse was proven in the preceding section.

6.2. *Preliminary reduction.* For any finite set F of P_+, $W_F(M)$ satisfies the similar condition. Hence we may assume $M = W_F(M)$. Hence, for any $\mu \in P$, $Wt(M) \cap (\mu + Q_+)$ is a finite set. Here we set

(6.2.1) $$Wt(M) = \{\lambda \in P;\, M_\lambda \neq 0\}.$$

Hence we can modify the statement in the following form suitable for induction.

(S) *Let G be a subset of P such that $G = G + Q_+$. Assume*

$$\text{(S.1)} \qquad Wt(M) \cap G \text{ is finite};$$

$$\text{(S.2)} \qquad (H_\lambda \cap M_{\mathbf{Q}}, H_\lambda \cap L_0, H_\lambda \cap L_\infty) \text{ is balanced for any } \lambda \in G;$$

$$\text{(S.3)} \qquad \{u \in M_\mu;\ e_i^{(n)}u \in M_{\mathbf{Q}} \text{ for any } i \in I \text{ and } n \geqslant 1\} = M_{\mathbf{Q}\mu} + H_\mu \text{ for any } \mu \in G.$$

Then $(M_{\mathbf{Q}\mu}, L_{0\mu}, L_{\infty\mu})$ is balanced for any $\mu \in G$.

6.3. *Lowest highest-weight parts.* We shall start by proving the following lemma.

LEMMA 6.3.1. *Under the conditions* (S.1) *and* (S.3), *we have*

$$\{u \in M_\mu;\ U_q^+(\mathfrak{g})_{\mathbf{Q}\lambda-\mu}u \in M_{\mathbf{Q}} \text{ for any } \lambda \in Wt(H) \cap G\} = M_{\mathbf{Q}\mu} \qquad \text{for any } \mu \in G.$$

Proof. We shall prove this by induction on $\sharp(Wt(M) \cap (\mu + Q_+))$. If $\mu \in Wt(H)$, then this is obvious. Otherwise, we have $U_q^+(\mathfrak{g})_{\mathbf{Q}\lambda-\mu-n\alpha_i}e_i^{(n)}u \in M_{\mathbf{Q}}$ for $n \geqslant 1$ and $\lambda \in Wt(H) \cap G$. Hence $e_i^{(n)}u \in M_{\mathbf{Q}}$ by the hypothesis of the induction, and (S.3) implies $u \in M_{\mathbf{Q}} + H_\mu = M_{\mathbf{Q}}$. Q.E.D.

Now we shall prove (S) by induction on $\sharp(Wt(H) \cap G)$. If $Wt(H) \cap G = \phi$, then (S) is obvious because $M_\mu = 0$ for any $\mu \in G$. If $Wt(H) \cap G$ is nonempty, we shall take $\lambda \in Wt(H) \cap G$ such that $Wt(H) \cap G \cap (\lambda + Q_-) = \{\lambda\}$. Set $I_\lambda(M_{\mathbf{Q}}) = I_\lambda(M) \cap M_{\mathbf{Q}}$, etc. Set $E = M_{\mathbf{Q}} \cap L_0 \cap L_\infty$. Then by (S.2)

$$\text{(6.3.1)} \qquad H_\lambda = \mathbf{Q}(q) \underset{\mathbf{Q}}{\otimes} (H_\lambda \cap E).$$

Hence (3.3.4) implies

$$\text{(6.3.2)} \qquad I_\lambda(M) = H_\lambda \underset{\mathbf{Q}(q)}{\otimes} V(\lambda) = (H_\lambda \cap E) \underset{\mathbf{Q}}{\otimes} V(\lambda),$$

$$I_\lambda(L_0) = (H \cap L_0)_\lambda \underset{A}{\otimes} L(\lambda) = (H_\lambda \cap E) \underset{\mathbf{Q}}{\otimes} L(\lambda),$$

$$I_\lambda(L_\infty) = (H \cap L_\infty)_\lambda \underset{\bar{A}}{\otimes} \bar{L}(\lambda) = (H_\lambda \cap E) \underset{\mathbf{Q}}{\otimes} \bar{L}(\lambda).$$

Since $V_{\mathbf{Q}}(\lambda)_\mu = \{u \in V(\lambda);\ U_q^+(\mathfrak{g})_{\mathbf{Q}\lambda-\mu}u \in \mathbf{Q}[q, q^{-1}]u_\lambda\}$ by (4.2.12), Lemma 6.3.1 implies

$$\text{(6.3.3)} \qquad I_\lambda(M_{\mathbf{Q}})_\mu = (H_\lambda \cap E) \underset{\mathbf{Q}}{\otimes} V_{\mathbf{Q}}(\lambda)_\mu \qquad \text{for any } \mu \in G.$$

Thus Lemma 4.2.1 implies

$$\text{(6.3.4)} \qquad (I_\lambda(M_{\mathbf{Q}})_\mu, I_\lambda(L_0)_\mu, I_\lambda(L_\infty)_\mu) \text{ is a balanced triple}.$$

6.4. *End of Proof.* Now set $N = M/I_\lambda(M)$, $N_{\mathbf{Q}} = M_{\mathbf{Q}}/I_\lambda(M_{\mathbf{Q}})$, $L_0(N) = L_0/I_\lambda(L_0)$, $L_\infty(N) = L_\infty/I_\lambda(L_\infty)$. By the hypothesis of the induction, if N, $N_{\mathbf{Q}}$, $L_0(N)$, and $L_\infty(N)$ satisfy (S.1), (S.2), and (S.3), then $(N_{\mathbf{Q}\mu}, L_0(N)_\mu, L_\infty(N)_\mu)$ is balanced and hence $(M_{\mathbf{Q}\mu}, L_{0\mu}, L_{\infty\mu})$ is balanced by Lemma 2.2.2. Since (S.1) and (S.2) for N are obvious, it remains to prove (S.3) for $N_{\mathbf{Q}}$, or equivalently

(6.4.1) for any $\mu \in G$ and $u \in M_\mu$, if $e_i^{(n)}u \in M_{\mathbf{Q}} + I_\lambda(M)$ for any i and $n \geqslant 1$,

then $u \in M_{\mathbf{Q}} + H + I_\lambda(M)$.

If $(\mu + (Q_+\backslash\{0\})) \cap (\lambda + Q_-) = \phi$, (6.4.1) follows immediately from (S.3). Hence we may assume

$$\mu \in (\lambda + Q_-)\backslash\{\lambda\} \tag{6.4.2}$$

from the beginning. We shall divide the proof of (6.4.1) into two cases.

(a) $\mu = \lambda - n\alpha_i$ with $n > \langle h_i, \lambda\rangle$. In this case, $\langle h_i, \mu\rangle < -n$ and $e_i^{(k)}u \in M_{\mathbf{Q}}$ for $k \geqslant -\langle h_i, \mu\rangle$. Hence Lemma 4.1.2 implies $u \in M_{\mathbf{Q}}$.

(b) $\mu \notin \bigcup_i \{\lambda - n\alpha_i;\, n > \langle h_i, \lambda\rangle\}$. Set $K = \{b \in B(\infty)_{\mu-\lambda};\, \bar{\pi}_\lambda(b) \neq 0\}$. Then we have (see Proposition 4.3.1(v))

$$U_q^+(\mathfrak{g})_{\mathbf{Q}\lambda-\mu} = \left(\sum_{b\in K} Q[q, q^{-1}]\varphi(G_\infty(b))\right) + \sum_{\substack{i\in I\\ n>\langle h_i,\lambda\rangle}} e_i^{(n)}U_q^+(\mathfrak{g})_{\mathbf{Q}\lambda-\mu-n\alpha_i}. \tag{6.4.3}$$

For any $b \in K$, there exists $v_b \in H_\lambda$ such that $\varphi(G_\infty(b))u - v_b \in M_{\mathbf{Q}}$. There exists $v \in I_\lambda(M)$ such that $\varphi(G(b))v = v_b$ for any $b \in K$. Hence replacing u with $u - v$, we may assume from the beginning

$$\varphi(G(b))u \in M_{\mathbf{Q}} \qquad \text{for any } b \in K. \tag{6.4.4}$$

Now we shall show $u \in M_{\mathbf{Q}}$ by applying Lemma 6.3.1. It is obvious that $U_q^+(\mathfrak{g})_{\mathbf{Q}\lambda'-\mu}u \in M_{\mathbf{Q}}$ for any $\lambda' \in Wt(H) \cap G\backslash\{\lambda\}$. Hence by (6.4.3) and (6.4.4), it is enough to show

$$e_i^{(n)}Pu \in M_{\mathbf{Q}} \qquad \text{for any } i \in I, \qquad n > \langle h_i, \lambda\rangle \quad \text{and} \quad P \in U_q^+(\mathfrak{g})_{\mathbf{Q}\lambda-\mu-n\alpha_i}. \tag{6.4.5}$$

Since $\lambda - \mu - n\alpha_i \neq 0$ by the assumption, we have

$$Pu \in M_{\mathbf{Q}} + I_\lambda(M). \tag{6.4.6}$$

Since the weight $\lambda - n\alpha_i$ of Pu is not a weight of $V(\lambda)$, (6.4.6) implies $Pu \in M_{\mathbf{Q}}$, and hence we obtain (6.4.5). This completes the proof of Theorem 2.

Remark. In this paper we discussed upper crystal bases. By using duality (see §3.2), we can derive similar results for lower crystal bases. For example, Theorem 2 can be reformulated as follows.

THEOREM 6.4.1. *Let M be an integrable $U_q(\mathfrak{g})$-module in $O_{\mathrm{int}}(\mathfrak{g})$ such that $I_\lambda(M) = 0$ except finitely many $\lambda \in P_+$. Let $M_{\mathbf{Q}}$ be as in §6.1 and let L_0 and L_∞ be a lower crystal lattice at $q = 0$ and $q = \infty$, respectively. We assume*

(6.4.7) $(M_{\mathbf{Q}}/S \cap M_{\mathbf{Q}}, L_0/S \cap L_0, L_\infty/S \cap L_\infty)$ *is a balanced triple in M/S,*

where $S = \sum_i f_i M$.

(6.4.8) $$M_{\mathbf{Q}} \cap S = \sum_{n>0, i \in I} f_i^{(n)} M_{\mathbf{Q}}.$$

Then $(M_{\mathbf{Q}}, L_0, L_\infty)$ is a balanced triple.

7. The q-analogue of the coordinate ring.

7.1. *Right module.* A right $U_q(\mathfrak{g})$-module M can be considered as a left $U_q(\mathfrak{g})$-module via the antiautomorphism φ. By this we define the notion of integrable right $U_q(\mathfrak{g})$-module and the category $O_{\mathrm{int}}(\mathfrak{g}^{\mathrm{opp}})$ of right integrable $U_q(\mathfrak{g})$-modules M such that, for any $u \in M$, there are $l \geqslant 0$ satisfying $uf_{i_1} \cdots f_{i_l} = 0$ for any $i_1, \ldots, i_l \in I$. We set for a right $U_q(\mathfrak{g})$-module M

(7.1.1) $$M_\lambda = \{u \in M; uq^h = q^{\langle h, \lambda \rangle} u\}.$$

Let $V^r(\lambda)$ be the irreducible integrable right $U_q(\mathfrak{g})$-module with highest weight $\lambda \in P_+$; i.e., $V^r(\lambda) \cong U_q(\mathfrak{g})/(\sum_{h \in P^*} (q^h - q^{\langle h, \lambda \rangle}) U_q(\mathfrak{g}) + \sum_i f_i U_q(\mathfrak{g}) + \sum_i e_i^{1+\langle h_i, \lambda \rangle} U_q(\mathfrak{g}))$. By the antiautomorphism φ of $U_q(\mathfrak{g})$, $V^r(\lambda)$ is isomorphic to $V(\lambda)$. Let v_λ be the highest-weight vector of $V^r(\lambda)$. Then there is a unique pairing $\langle \ , \ \rangle$

(7.1.2) $$V^r(\lambda) \otimes V(\lambda) \to \mathbf{Q}(q)$$

such that

(7.1.3) $$\langle v_\lambda, u_\lambda \rangle = 1,$$

(7.1.4) $\langle vP, u \rangle = \langle v, Pu \rangle$ for $v \in V^r(\lambda)$, $u \in V(\lambda)$, and $P \in U_q(\mathfrak{g})$.

We define similarly crystal lattice and crystal base for the integrable right $U_q(\mathfrak{g})$-module. We denote by $(L^r(\lambda), B^r(\lambda))$ the canonical upper crystal base of $V^r(\lambda)$. Thus, $L(\lambda)$ and $L^r(\lambda)$ are so normalized that $L(\lambda)_\lambda = Au_\lambda$ and $L^r(\lambda)_\lambda = \bar{A}v_\lambda$.

7.2. *The coordinate ring $A_q(\mathfrak{g})$.* Let $U_q(\mathfrak{g})^*$ be $\mathrm{Hom}_{\mathbf{Q}(q)}(U_q(\mathfrak{g}), \mathbf{Q}(q))$. We denote by

$$\langle\ ,\ \rangle\colon U_q(\mathfrak{g})^* \otimes U_q(\mathfrak{g}) \to \mathbf{Q}(q)$$

the canonical pairing.

We define the comultiplication

$$\Delta_+\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \tag{7.2.1}$$

by

(7.2.2)

$$\Delta_+(q^h) = q^h \otimes q^h, \qquad \Delta_+(e_i) = e_i \otimes 1 + t_i \otimes e_i, \qquad \Delta_+(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i.$$

Then Δ_+ induces the multiplication

$$\mu\colon U_q(\mathfrak{g})^* \otimes U_q(\mathfrak{g})^* \to (U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}))^* \to U_q(\mathfrak{g})^*, \tag{7.2.2}$$

by which $U_q(\mathfrak{g})^*$ has a structure of ring with unit.

Since $U_q(\mathfrak{g})$ has the structure of a bi-$U_q(\mathfrak{g})$-module, $U_q(\mathfrak{g})^*$ has the structure of a bi-$U_q(\mathfrak{g})$-module. Then the multiplication $\mu\colon U_q(\mathfrak{g})^* \otimes U_q(\mathfrak{g})^* \to U_q(\mathfrak{g})^*$ is a morphism of a bi-$U_q(\mathfrak{g})$-module, where $U_q(\mathfrak{g})^* \otimes U_q(\mathfrak{g})^*$ has the structure of a bi-$U_q(\mathfrak{g})$-module via the comultiplication Δ_+.

Definition 7.2.1. We set

$$A_q(\mathfrak{g}) = \{u \in U_q(\mathfrak{g})^*;\ U_q(\mathfrak{g})u \text{ belongs to } O_{\text{int}}(\mathfrak{g}) \text{ and } uU_q(\mathfrak{g}) \text{ belongs to } O_{\text{int}}(\mathfrak{g}^{\text{opp}})\}.$$

Since the multiplication map of $U_q(\mathfrak{g})^*$ is bi-$U_q(\mathfrak{g})$-linear and $O_{\text{int}}(\mathfrak{g})$ and $O_{\text{int}}(\mathfrak{g}^{\text{opp}})$ are closed by tensor product, $A_q(\mathfrak{g})$ is a subring of $U_q(\mathfrak{g})^*$.

The following theorem is the q-analogue of the Peter-Weyl theorem, and the proof follows easily from the semisimplicity of the category $O_{\text{int}}(\mathfrak{g})$.

PROPOSITION 7.2.2. *As a bi-$U_q(\mathfrak{g})$-module, $A_q(\mathfrak{g})$ is isomorphic to $\oplus_\lambda V^r(\lambda) \otimes V(\lambda)$ by the homomorphisms*

$$\Phi_\lambda\colon V^r(\lambda) \otimes V(\lambda) \to A_q(\mathfrak{g})$$

given by

$$\langle \Phi_\lambda(v \otimes u), P\rangle = \langle vP, u\rangle$$

for $v \in V^r(\lambda)$, $u \in V(\lambda)$, and $P \in U_q(\mathfrak{g})$.

7.3. *Crystal base and the* $\mathbf{Q}$*-form.* We define the automorphism $-$ of the $\mathbf{Q}$-ring $A_q(\mathfrak{g})$ by

$$(7.3.1) \qquad \langle \bar{u}, P \rangle = \langle u, \bar{P} \rangle^- \quad \text{for } u \in A_q(\mathfrak{g}) \text{ and } P \in U_q(\mathfrak{g}).$$

Here $-: \mathbf{Q}(q) \to \mathbf{Q}(q)$ is the ring automorphism sending q to q^{-1}.
We denote also by $-$ the automorphisms of $V(\lambda)$ and $V^r(\lambda)$ by

$$(7.3.2) \qquad \bar{u}_\lambda = u_\lambda, \qquad \bar{v}_\lambda = v_\lambda$$

and

$$(7.3.3) \quad \overline{Pu} = \bar{P}\bar{u}, \qquad \overline{vP} = \bar{v}\bar{P} \qquad \text{for } u \in V(\lambda), \qquad v \in V^r(\lambda) \text{ and } P \in U_q(\mathfrak{g}).$$

Then we have the commutative diagram

$$(7.3.4) \qquad \begin{array}{ccc} V^r(\lambda) \underset{\mathbf{Q}(q)}{\otimes} V(\lambda) & \xrightarrow{\Phi_\lambda} & A_q(\mathfrak{g}) \\ {\scriptstyle -\otimes -}\downarrow & & \downarrow{\scriptstyle -} \\ V^r(\lambda) \underset{\mathbf{Q}(q)}{\otimes} V(\lambda) & \xrightarrow{\Phi_\lambda} & A_q(\mathfrak{g}). \end{array}$$

We define

$$L(A_q(\mathfrak{g})) = \bigoplus_\lambda L^r(\lambda) \underset{A}{\otimes} L(\lambda) \subset A_q(\mathfrak{g}),$$

$$\bar{L}(A_q(\mathfrak{g})) = \bigoplus_\lambda \bar{L}^r(\lambda) \underset{\bar{A}}{\otimes} \bar{L}(\lambda) \subset A_q(\mathfrak{g}), \quad \text{and}$$

$$B(A_q(\mathfrak{g})) = \bigcup_\lambda B^r(\lambda) \otimes B(\lambda), \qquad \bar{B}(A_q(\mathfrak{g})) = \bigcup_\lambda \bar{B}^r(\lambda) \otimes \bar{B}(\lambda).$$

Then the automorphism $-$ of $A_q(\mathfrak{g})$ sends $L(A_q(\mathfrak{g}))$ and $\bar{L}(A_q(\mathfrak{g}))$ to each other.
We define the $\mathbf{Q}$-form of $A_q(\mathfrak{g})$ as follows:

$$(7.3.5) \qquad A_q^{\mathbf{Q}}(\mathfrak{g}) = \{u \in A_q(\mathfrak{g}); \langle u, U_q^{\mathbf{Q}}(\mathfrak{g}) \rangle \subset \mathbf{Q}[q, q^{-1}]\}.$$

Then $A_q^{\mathbf{Q}}(\mathfrak{g})$ is a sub-bi-$U_q^{\mathbf{Q}}(\mathfrak{g})$-module of $A_q(\mathfrak{g})$ and also a subring of $A_q(\mathfrak{g})$ because Δ_+ sends $U_q^{\mathbf{Q}}(\mathfrak{g})$ into $U_q^{\mathbf{Q}}(\mathfrak{g}) \otimes U_q^{\mathbf{Q}}(\mathfrak{g})$.

LEMMA 7.3.1. $A_q(\mathfrak{g}) \cong \mathbf{Q}(q) \otimes_{\mathbf{Q}[q,q^{-1}]} A_q^{\mathbf{Q}}(\mathfrak{g})$.

Proof. For $u \in A_q(\mathfrak{g})$, if we set $J_1 = \{P \in U_q(\mathfrak{g}), Pu = 0\}$ and $J_2 = \{P \in U_q(\mathfrak{g}), uP = 0\}$, then u can be considered as an element of $\mathrm{Hom}_{\mathbf{Q}(q)}(U_q(\mathfrak{g})/(J_2 U_q(\mathfrak{g}) + U_q(\mathfrak{g})J_1), \mathbf{Q}(q))$. Since $\dim(U_q(\mathfrak{g})/(J_2 U_q(\mathfrak{g}) + U_q(\mathfrak{g})J_1)) < \infty$, there exists a nonzero $\varphi \in \mathbf{Q}(q)$ such that φu belongs to $A_q^{\mathbf{Q}}(\mathfrak{g})$. Q.E.D.

7.4. *Main theorem.* Now we shall prove the following theorem.

THEOREM 1. (i) $A_q^{\mathbf{Q}}(\mathfrak{g}) \cap L(A_q(\mathfrak{g})) \cap \bar{L}(A_q(\mathfrak{g})) \to L(A_q(\mathfrak{g}))/qL(A_q(\mathfrak{g}))$ *is an isomorphism.*

(ii) *Letting* G *be the inverse of the isomorphism above, we have*

$$A_q^{\mathbf{Q}}(\mathfrak{g}) = \bigoplus_{b \in B(A_q(\mathfrak{g}))} \mathbf{Q}[q, q^{-1}]G(b).$$

Proof. Regarding $A_q(\mathfrak{g})$ as a left $U_q(\mathfrak{g} \oplus \mathfrak{g})$-module belonging to $O_{\text{int}}(\mathfrak{g} \oplus \mathfrak{g})$, we shall apply Theorem 2 (in §6.2). Hence setting

$$H = \{u \in A_q(\mathfrak{g});\ e_i u = u f_i = 0 \text{ for any } i\}, \tag{7.4.1}$$

it is enough to show

$$A_q^{\mathbf{Q}}(\mathfrak{g}) + H = \{u \in A_q(\mathfrak{g});\ e_i^{(n)} u \text{ and } u f_i^{(n)} \text{ belong to } A_q^{\mathbf{Q}}(\mathfrak{g}) \text{ for } n \geqslant 1 \text{ and } i \in I\} \tag{7.4.2}$$

and

$$H \cap A_q^{\mathbf{Q}}(\mathfrak{g}) \cap L(A_q(\mathfrak{g})) \cap \bar{L}(A_q(\mathfrak{g})) \xrightarrow{\sim} (H \cap L(A_q(\mathfrak{g})))/(H \cap qL(A_q(\mathfrak{g}))).$$

By the identification

$$A_q(\mathfrak{g}) \cong \bigoplus_{\lambda \in P_+} V^r(\lambda) \otimes V(\lambda),$$

we have

$$H \cong \bigoplus_{\lambda \in P_+} \mathbf{Q}(q)(v_\lambda \otimes u_\lambda). \tag{7.4.4}$$

Also, it is obvious that

$$H \cap A_q^{\mathbf{Q}}(\mathfrak{g}) \cong \bigoplus_{\lambda} \mathbf{Q}[q, q^{-1}](v_\lambda \otimes u_\lambda), \tag{7.4.5}$$

$$H \cap L(A_q^{\mathbf{Q}}(\mathfrak{g})) \cong \bigoplus_{\lambda} A(v_\lambda \otimes u_\lambda), \tag{7.4.6}$$

and

$$H \cap \bar{L}(A_q^{\mathbf{Q}}(\mathfrak{g})) \cong \bigoplus_{\lambda} \bar{A}(v_\lambda \otimes u_\lambda). \tag{7.4.7}$$

Then (7.4.3) follows by these three. Hence it remains to prove

(7.4.8) if $u \in A_q(\mathfrak{g})$ satisfies $e_i^{(n)} u,\ u f_i^{(n)} \in A_q^{\mathbf{Q}}(\mathfrak{g})$ for $n \geqslant 1$ and any $i \in I$, then u belongs to $A_q^{\mathbf{Q}}(\mathfrak{g}) + H$.

By Proposition 4.1.1, we may assume that u is a weight vector, i.e., that there are $\lambda_r, \lambda_l \in P$, such that $q^h u = q^{\langle h, \lambda_l \rangle} u$ and $uq^h = uq^{\langle h, \lambda_r \rangle}$.

If $\langle h_i, \lambda_l \rangle < 0$ for some i, then u belongs to $A_q^{\mathbf{Q}}(\mathfrak{g})$ by Proposition 4.1.2. Therefore, we may assume $\lambda_l \in P_+$ without loss of generality. Now if $\lambda_l \neq \lambda_r$, then $\langle u, 1 \rangle = 0$. If $\lambda_l = \lambda_r$, then setting $c = \langle u, 1 \rangle$, $u' = u - c\Phi_{\lambda_l}(v_{\lambda_l} \otimes u_{\lambda_l})$ satisfies $\langle u', 1 \rangle = 0$ and $e_i^{(n)} u'$, $u' f_i^{(n)}$ belong to $A_q^{\mathbf{Q}}(\mathfrak{g})$. Hence replacing u with u' we may assume $\langle u, 1 \rangle = 0$. Thus in both cases, we may assume $\langle u, 1 \rangle = 0$. Let $\mathscr{C}$ be the commutative $\mathbf{Q}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by $\left\{ \begin{matrix} q^h \\ n \end{matrix} \right\}$ and q^h. Then $\langle u, \mathscr{C} \rangle = 0$. Finally, $\langle u, U_q^{\mathbf{Q}}(\mathfrak{g}) \rangle \subset \mathbf{Q}[q, q^{-1}]$ follows from

$$\begin{aligned} U_q^{\mathbf{Q}}(\mathfrak{g}) &= U_q^-(\mathfrak{g})_{\mathbf{Q}} \otimes \mathscr{C} \otimes U_q^+(\mathfrak{g})_{\mathbf{Q}} \\ &= \sum_{\substack{n \geq 1 \\ i \in I}} (f_i^{(n)} U_q^{\mathbf{Q}}(\mathfrak{g}) + U_q^{\mathbf{Q}}(\mathfrak{g}) e_i^{(n)}) + \mathscr{C}. \end{aligned}$$

This completes the proof of Theorem 1. Q.E.D.

Remark. (1) The isomorphism $L(A_q(\mathfrak{g}))/qL(A_q(\mathfrak{g})) \xrightarrow{\sim} \bar{L}(A_q(\mathfrak{g}))/q^{-1}\bar{L}(A_q(\mathfrak{g}))$ obtained through $A_q^{\mathbf{Q}}(\mathfrak{g}) \cap L(A_q(\mathfrak{g})) \cap \bar{L}(A_q(\mathfrak{g}))$ coincides with $-$. Hence it sends $B(A_q(\mathfrak{g}))$ onto $\overline{B}(A_q(\mathfrak{g}))$ and commutes with $\tilde{e}_i$ and $\tilde{f}_i$.

(2) Set $A_q^{\mathbf{Z}}(\mathfrak{g}) = \{u \in A_q(\mathfrak{g}); \langle u, U_q^{\mathbf{Z}}(\mathfrak{g}) \rangle \subset \mathbf{Z}[q, q^{-1}]\}$. Then, more precise arguments show

$$A_q^{\mathbf{Z}}(\mathfrak{g}) = \bigoplus_{b \in B(A_q(\mathfrak{g}))} \mathbf{Z}[q, q^{-1}] G(b).$$

7.5. *Conjugate of the product.* We shall investigate the relation between $\overline{uv}$ and $\bar{v}\,\bar{u}$ for $u, v \in A_q(\mathfrak{g})$. In order to do this, let ψ be the automorphism of the $\mathbf{Q}(q)$-algebra $U_q(\mathfrak{g})$ defined by

$$\begin{aligned} &\psi(q^h) = q^h, \\ &\psi(e_i) = q_i^{-1} t_i e_i = q_i e_i t_i, \\ &\psi(f_i) = q_i^{-1} t_i^{-1} f_i = q_i f_i t_i^{-1}. \end{aligned} \tag{7.5.1}$$

We have

$$\overline{\psi(P)} = \psi^{-1}(\overline{P}). \tag{7.5.2}$$

LEMMA 7.5.1. *For any $P \in U_q(\mathfrak{g})_\xi$, we have*

$$\psi(P) = q^{(\xi, \xi)/2} q^\xi P.$$

Here, for $\xi = \sum n_i \alpha_i \in Q$, we set $q^\xi = \prod t_i^{n_i}$. (Hence $q^\xi u = q^{(\xi, \lambda)} u$ for a weight vector u of weight λ).

The proof is straightforward.

Let us define the automorphism ψ^* of $A_q(\mathfrak{g})$ by $\langle \psi^*(u), P\rangle = \langle u, \psi(P)\rangle$.

LEMMA 7.5.2. *If $u \in A_q(\mathfrak{g})$ has weight (λ_l, λ_r) (i.e., $q^h u = q^{\langle h, \lambda_l\rangle} u$, $uq^h = q^{\langle h, \lambda_r\rangle} u$), then*

$$\psi^*(u) = q^{(\lambda_r, \lambda_r)/2 - (\lambda_l, \lambda_l)/2} u .$$

Proof. We have

$$\langle u, U_q(\mathfrak{g})_\xi\rangle = 0 \qquad \text{for } \xi \neq \lambda_r - \lambda_l . \tag{7.5.3}$$

If $P \in U_q(\mathfrak{g})_{\lambda_r - \lambda_l}$, then

$$\begin{aligned}\langle u, \psi(P)\rangle &= \langle u, q^{-(\lambda_r - \lambda_l, \lambda_r - \lambda_l)/2} q^{\lambda_r - \lambda_l} P\rangle \\ &= q^{-(\lambda_r - \lambda_l, \lambda_r - \lambda_l)/2 + (\lambda_r - \lambda_l, \lambda_r)} \langle u, P\rangle \\ &= q^{(\lambda_r, \lambda_r)/2 - (\lambda_l, \lambda_l)/2} \langle u, P\rangle .\end{aligned}$$

Q.E.D.

Let us denote also by $-$ the automorphism of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ given by $P \otimes Q \mapsto \overline{P} \otimes \overline{Q}$.

Let σ be the automorphism of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ given by $P \otimes Q \mapsto Q \otimes P$. We have the following lemma.

LEMMA 7.5.3. $\Delta(\overline{P}) = \sigma \circ (\psi \otimes \psi) \circ \overline{\Delta(\psi P)}$ *for any $P \in U_q(\mathfrak{g})$.*

Proof. It is enough to show that, if it holds for P, then it does for $q^h P$, $e_i P$, *and* $f_i P$. This can be checked easily. Q.E.D.

PROPOSITION 7.5.4. *Let u and v be elements of weight (λ_r, λ_l) and (μ_r, μ_l), respectively (i.e., $q^h u = q^{\langle h, \lambda_l\rangle} u$, $uq^h = q^{\langle h, \lambda_r\rangle} u$, etc.). Then we have $\overline{uv} = q^{(\lambda_r, \mu_r) - (\lambda_l, \mu_l)} \bar{v}\, \bar{u}$.*

Proof. For $P \in U_q(\mathfrak{g})$

$$\begin{aligned}\langle \overline{uv}, P\rangle &= \langle uv, \overline{P}\rangle^- = \langle u \otimes v, \Delta\overline{P}\rangle^- \\ &= \langle u \otimes v, \sigma \circ (\psi \otimes \psi)\overline{\Delta\psi P}\rangle^- \\ &= \langle \psi^*(v) \otimes \psi^*(u), \overline{\Delta\psi P}\rangle^- \\ &= q^{((\lambda_l, \lambda_l) - (\lambda_r, \lambda_r) + (\mu_l, \mu_l) - (\mu_r, \mu_r))/2} \langle v \otimes u, \overline{\Delta\psi P}\rangle^- .\end{aligned}$$

The last term is

$$\begin{aligned}\langle v \otimes u, \overline{\Delta\psi P}\rangle^- &= \langle \bar{v} \otimes \bar{u}, \Delta(\psi P)\rangle \\ &= \langle \bar{v}\, \bar{u}, \psi P\rangle = \langle \psi^*(\bar{v}\, \bar{u}), P\rangle \\ &= q^{((\lambda_r + \mu_r, \lambda_r + \mu_r) - (\lambda_l + \mu_l, \lambda_l + \mu_l))/2} \langle \bar{v}\, \bar{u}, P\rangle .\end{aligned}$$

Hence we obtain

$$\langle \overline{uv}, P\rangle = q^{(\lambda_r, \mu_r)-(\lambda_l, \mu_l)}\langle \bar{v}\,\bar{u}, P\rangle .$$

Thus we obtain the desired result. Q.E.D.

8. Comments.

8.1. *The ring* $\tilde{U}_q(\mathfrak{g})$. Let Mod $(\mathfrak{g}, P)$ be the category of $U_q(\mathfrak{g})$-modules M with the weight decomposition $\oplus_{\lambda\in P} M_\lambda$.

For $\lambda \in P$, we set $N(\lambda) = U_q(\mathfrak{g})/\sum U_q(\mathfrak{g})(q^h - q^{\langle h,\lambda\rangle})$ and let w_λ denote the image 1 in $N(\lambda)$. Let 1 be the identity functor of Mod$(\mathfrak{g}, P)$ and $R = End(1)$. Hence, an element φ of R associates an endomorphism $\varphi(M)$ of M with any M in Mod$(\mathfrak{g}, P)$, such that

$$\begin{array}{ccc} M & \xrightarrow{\varphi(M)} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ N & \xrightarrow{\varphi(N)} & N \end{array}$$

is commutative for any morphism $f\colon M \to N$ in Mod$(\mathfrak{g}, P)$.

For $\lambda \in P$, the projector $a_\lambda\colon M \to M_\lambda$ gives an element of R.

LEMMA 8.1.1. (i) *R contains $U_q(\mathfrak{g})$ as a subring.*
(ii) *$R \to \prod_{\lambda\in P} N(\lambda)$ given by $\varphi \mapsto (\varphi(N(\lambda))w_\lambda)_\lambda$ is an isomorphism.*

Proof. (i) is obvious. For $\varphi \in R$, set $\varphi(N(\lambda))(w_\lambda) = P_\lambda w_\lambda$ with $P_\lambda \in U_q(\mathfrak{g})$. Then for M in Mod$(\mathfrak{g}, P)$ and any element $u \in M_\mu$, let $f\colon N(\mu) \to M$ be a morphism given by $f(w_\mu) = u$. Then $\varphi(M)(u) = \varphi(M)f(w_\mu) = f\varphi(N(\lambda))(w_\lambda) = f(P_\mu w_\mu) = P_\mu u$. Conversely if $P_\lambda \in U_q(\mathfrak{g})$, then $\varphi(M)(u) = \sum P_\lambda a_\lambda u$ is well defined and gives an element φ in R. Q.E.D.

Definition 8.1.2. Let $\tilde{U}_q(\mathfrak{g})$ denote the direct sum of $U_q(\mathfrak{g})a_\lambda$ $(\lambda \in P)$.

Note that

$$a_\lambda a_\mu = \delta_{\lambda,\mu} a_\lambda , \tag{8.1.1}$$

$$a_\lambda P = P a_\mu \qquad \text{for } P \in U_q(\mathfrak{g})_{\lambda-\mu} . \tag{8.1.2}$$

Hence $\tilde{U}_q(\mathfrak{g})$ is a ring (without unit in general). We denote by $\tilde{U}_q^{\mathbf{Q}}(\mathfrak{g})$ the subring of $\tilde{U}_q(\mathfrak{g})$ generated by $U_q^{\mathbf{Q}}(\mathfrak{g})a_\lambda$. We have

$$\tilde{U}_q(\mathfrak{g}) \cong U_q^+(\mathfrak{g}) \otimes \Big(\bigoplus_\lambda \mathbf{Q}(q)a_\lambda\Big) \otimes U_q^-(\mathfrak{g}) . \tag{8.1.3}$$

Formally, we have

$$q^h = \sum_{\lambda} q^{\langle h, \lambda \rangle} a_\lambda . \tag{8.1.4}$$

8.2. *Conjectural base of* $\tilde{U}_q(\mathfrak{g})$. We can also define the coupling

$$\langle \ , \ \rangle : A_q(\mathfrak{g}) \otimes R \to \mathbf{Q}(q)$$

by $\langle \Phi_\lambda(v \otimes u), \varphi \rangle = (v, \varphi(V(\lambda))u)$ for $u \in V(\lambda)$, $v \in V^r(\lambda)$ and $\varphi \in R$.

Conjecture 2. Assume $\mathfrak{g}$ is finite dimensional. For any $b \in B(A_q(\mathfrak{g}))$, there exists a unique $P(b) \in \tilde{U}_q^{\mathbf{Q}}(\mathfrak{g})$ such that $(G(b'), P(b)) = \delta_{b,b'}$, and we have $\tilde{U}_q^{\mathbf{Q}}(\mathfrak{g}) = \oplus_b \mathbf{Q}[q, q^{-1}]P(b)$.

It is expected that, when $\mathfrak{g} = sl(n)$, this base coincides with the base constructed by Beilinson-Lusztig-MacPherson [BLM] (via intersection cohomologies). After writing up this paper, I learned that G. Lusztig has constructed bases of $\tilde{U}_q^{\mathbf{Q}}(\mathfrak{g})$ ([L]). It turns out that this gives an affirmative answer to Conjecture 2.

9. Example (sl_2 case).

9.1. We shall give an explicit form of global crystal base of $A_q(sl_2)$ and examine the conjecture of A. D. Berenstein and A. Zelevinsky. We leave the proof to the reader.

Let $I = \{1\}$, $(\alpha_1, \alpha_1) = 1$, $\rho = \alpha_1/2$, $P = \mathbf{Z}\rho$. We write e, f, t for e_1, f_1, t_1, etc. Set $x = \Phi_\rho(v_\rho \otimes u_\rho)$, $u = fx$, $v = xe$ and $y = fxe = fv = ue$. Then it is well known that $A_q(\mathfrak{g})$ is the $\mathbf{Q}(q)$-algebra generated by x, y, u, and v with the defining relations

(9.1.1)
$$xu = qux, \qquad xv = qvx,$$
$$uy = qyu, \qquad vy = qyv, \quad \text{and}$$
$$uv = vu \quad \text{and} \quad xy - quv = yx - q^{-1}uv = 1 .$$

PROPOSITION 9.1.1. (i) *$L(A_q(\mathfrak{g})) = \sum Au^n x^m v^l + \sum Au^n y^m v^l$. Here n, m, and l range over $\mathbf{Z}_{\geqslant 0}$.*

(ii) *$G(B(A_q(\mathfrak{g}))) = \{u^n x^m v^l; n, m, l \geqslant 0\} \cup \{u^n y^m v^l; n, l \geqslant 0, m > 0\}$.*

Hence in this case, the global crystal bases are monomes of x, y, u, and v, and the conjecture of A. D. Berenstein and A. Zelevinsky is true.

PROPOSITION 9.1.2. *Conjecture 2 is true for $\mathfrak{g} = sl_2$, and the $P(b)$'s are of the form*

$$f^{(m)} e^{(n)} a_\lambda \qquad \textit{where } \langle h, \lambda \rangle \geqslant m - n$$

and

$$e^{(n)}f^{(m)}a_\lambda \qquad \text{where } \langle h, \lambda\rangle < m - n.$$

Notice that $f^{(m)}e^{(n)}a_\lambda = e^{(n)}f^{(m)}a_\lambda$ when $\langle h, \lambda\rangle = m - n$.

References

[BLM] A. A. Beilinson, G. Lusztig, and R. MacPherson, *A geometric setting for the quantum deformation of GL_n*, Duke Math. J. **61** (1990), 655–677.

[K1] M. Kashiwara, *Crystalizing the q-analogue of universal enveloping algebras*, Comm. Math. Phys. **133** (1990), 249–260.

[K2] ———, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.

[K3] ———, "Crystallizing the q-analogue of universal enveloping algebras" in *Proceedings of the International Congress of Mathematicians, Kyoto, 1990*, Math. Soc. of Japan, Tokyo, 1991, 791–797.

[L] G. Lusztig, *Canonical bases in tensor product*, Proc. Nat. Acad. Sci. U.S.A. **89** (1992), 8177–8179.

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