INDEX THEOREM FOR CONSTRUCTIBLE SHEAVES

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§0 - INTRODUCTION

0.1. Let X be a complex manifold of dimension n and let $\mathcal{M}$ be a holonomic module over the ring $\underline{\mathcal{D}}_X$ of differential operators on X. Then the Rham complex DR($\mathcal{M}$) of $\mathcal{M}$ has constructible sheaves as its cohomology groups, and its local index $\sum(-1)^i \dim \underline{H}^i(\mathrm{DR}(\mathcal{M}))_x$ at a point x can be expressed in terms of the characteristic cycle $\underline{\mathrm{Ch}}(\mathcal{M})$ of $\mathcal{M}$ (Kashiwara [3], Brylinski-Dubson-Kashiwara [1]). Recently Dubson [2] found a beautiful formula to describe this.

THEOREM - *If* X *is a compact complex manifold, we have*

$$\sum(-1)^i \dim H^i(X;\mathrm{DR}(\mathcal{M})) = (-1)^n \underline{\mathrm{Ch}}(\mathcal{M}).T^*_X X .$$

Here the last term means the intersection number of two n-cycles in T^*X.

0.2. The purpose of this lecture is to generalize his result to the real case.

Let X be a real analytic manifold of dimension n and F a constructible sheaf on X. First we shall define the characteristic cycle $\widetilde{SS}(F)$ of F as a $\pi^{-1}\omega_X$-valued n-cycle in T^*X. Here ω_X denotes the orientation sheaf of X and $\pi : T^*X \to X$ is the cotangent bundle to X. In order to define this, we use the micro-local theory of sheaves developped in Kashiwara-Schapira [4].

Secondly we prove the index theorem.

THEOREM - *Let* F *be a constructible sheaf, and* $\varphi : X \to \mathbb{R}$ *a* C^2-*function. Set* $Y_\varphi = \{d\varphi(x) ; x \in X\} \subset T^*X$. *We assume that* $\{x \in \mathrm{supp}\, F; \varphi(x) \leq t\}$ *is compact for any* t *and that* $SSF \cap Y_\varphi$ *is compact. Then* $\dim H^j(X;F) < \infty$ *for any* j *and we have*

$$\sum(-1)^j \dim H^j(X;F) = (-1)^{n(n+1)/2}\, \widetilde{SS}(F).Y_\varphi .$$

The proof uses the micro-local version of Morse's theory. Similarly to the Morse function, we deform φ a little in a generic position so that Y_φ intersects SSF transversally. Then we consider $H^j(\{x ; \varphi(x) < t\} ; F)$ and vary t. Then the cohomology groups change at points $t \in \varphi(\pi(Y_\varphi \cap SSF))$, and the obstruction can be calculated locally and coincides with the intersection number of Y_φ

and $\widetilde{SS}(F)$ at $p \in SSF \cap Y_\varphi$ with $t = \varphi\pi(p)$.

§1 - SUBANALYTIC CHAINS

1.1. For a topological manifold X, let us denote by ω_X the orientation sheaf of X. If X is oriented then $\omega_X \cong \mathbb{Z}_X$ and this isomorphism changes the signature when we take the opposite orientation of X.

1.2. If X is a differentiable manifold of dimension n and if θ is a nowhere vanishing n-form on X, then we shall denote by $\operatorname{sgn}\theta$ the section of ω_X given by the orientation that θ determines. Hence we have

(1.2.1) $$\operatorname{sgn}\varphi\theta = \operatorname{sgn}\varphi \ \operatorname{sgn}\theta$$

where $\operatorname{sgn}\varphi = \pm 1$ if $\pm\varphi > 0$.

1.3. From now on, we assume that X is a real analytic manifold. For an integer r, let us denote by $E_r(X)$ the set of pairs (Y,s) of a subanalytic locally closed r-dimensional real analytic submanifold Y of X and a section s of ω_Y. We define the equivalence relation $\sim$ on $E_r(X)$ as follows : $(Y_1,s_1) \sim (Y_2,s_2)$ if and only if there exists a subanalytic locally closed r-dimensional real analytic submanifold Y such that $Y \subset Y_1 \cap Y_2$, $s_1|_Y = s_2|_Y$ and $\overline{\operatorname{supp} s_1} = \overline{\operatorname{supp} s_2} = \overline{\operatorname{supp} s_1 \cap Y}$.

We denote by $C_r(X)$ the set of equivalence classes in $E_r(X)$ and an equivalence class is called *subanalytic* r-*chain*. Remark that its support is not assumed to be compact.

We can define the boundary operator

$$\partial : C_r(X) \longrightarrow C_{r-1}(X),$$

so that $\partial\partial = 0$.

1.4. One can see easily that $C_r : U \longmapsto C_r(U)$ is a fine sheaf on X and we have the exact sequence

(1.4.1) $$0 \longrightarrow \omega_X \rightarrow C_n \xrightarrow{\partial} C_{n-1} \rightarrow \cdots \rightarrow C_0 \rightarrow 0$$

This follows for example from the fact that any subanalytic set admits a subanalytic triangulation.

1.5. For a sheaf F on X, we set $C_r(F) = C_r \otimes F$. By (1.4.1), $\omega_X \otimes F$ is quasi-isomorphic to the complex of soft sheaves

(1.5.1) $$C_n(F) \longrightarrow C_{n-1}(F) \longrightarrow \cdots \longrightarrow C_0(F).$$

We set

(1.5.2) $$C_r(X;F) = \Gamma(X;C_r(F))$$

and call its elements *F-valued subanalytic r-chains*. We have isomorphisms

(1.5.3) $$H_r^{\inf}(X;F) \underset{\text{def}}{=} H_r(C.(X;F)) = H^{n-r}(X;F \otimes \omega_X).$$

(1.5.4) $$H_r(X;F) \underset{\text{def}}{=} H_r(\Gamma_c(X;C.(F))) = H_c^{n-r}(X;F \otimes \omega_X).$$

1.6. Assume further that F is locally constant. For a subanalytic r-dimensional real analytic submanifold Y of X and for a section s of $F \otimes \omega_Y$ over Y, the pair (Y,s) determines an F-valued subanalytic r-chain.

1.7. The following criterion for a chain to be a cycle is evident.

LEMMA 1.1 - *Let* α *be a subanalytic r-chain,* $\varphi : X \longrightarrow \mathbb{R}^r$ *be a real analytic map. We assume that*

(i) Supp $\alpha \longrightarrow \mathbb{R}^r$ *is a finite map,*

(ii) Supp $\partial\alpha \longrightarrow \mathbb{R}^r$ *is an immersion,*

(iii) the intersection number of α *and* $\varphi^{-1}(t)$ *is constant in* $t \in \mathbb{R}^r \setminus \varphi(\text{Supp } \partial\alpha)$.

Then α *is a cycle, i.e.* $\partial\alpha = 0$.

§2 - SYMPLECTIC GEOMETRY

2.1. Let X be an n-dimensional real analytic manifold of dimension n and $\pi : T^*X \longrightarrow X$ the cotangent bundle to X. Let θ_X denote the canonical 1-form on T^*X. Then $(d\theta_X)^n$ is nowhere vanishing and this gives the orientation of T^*X.

2.2. Now, let Y be a real analytic submanifold of X. Let T_Y^*X be the conormal bundle to Y. Then we have the canonical isomorphism

(2.2.1) $$\omega_{T_Y^*X} \otimes \pi^{-1}\omega_X \cong \mathbb{Z}_{T_Y^*X} .$$

Since the choice of signature is important in the future arguments, we shall write this explicitely. Let $(x_1, \ldots, x_n)$ be a local coordinate system of X such that Y is given by $x_1 = \ldots = x_r = 0$, and let $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ be the coordinates of T^*X such that $\theta_X = \sum \xi_j\, dx_j$. Then the section $(-1)^r$ sgn $(d\xi_1 \ldots d\xi_r\, dx_{r+1} \ldots dx_n) \otimes$ sgn $(dx_1 \ldots dx_n)$ of $\omega_{T_Y^*X} \otimes \pi^{-1}\omega_X$ does not depend on the choice of coordinates and it determines the isomorphism (2.2.1).

2.3. Let Λ be a subanalytic conic locally closed Lagrangian subvariety of T^*X such that the projection $\Lambda \to X$ has a constant rank. Then we have $\omega_\Lambda \otimes \pi^{-1}\omega_X \cong \mathbb{Z}_\Lambda$. In fact, locally, Λ is an open subset of T_Y^*X for a real analytic submanifold Y of X and we can apply 2.2. Therefore Λ defines the $\pi^{-1}\omega_X$-valued n-chain in T^*X (see 1.6), which we shall denote by $[\Lambda]$.

§3 - CHARACTERISTIC CYCLE

3.1. Let us fix a commutative field k once for all, and vector spaces mean vector spaces over k. Let X be a real analytic manifold of dimension n. Let D(X) be the derived category of the abelian category of sheaves of vector spaces on X.

An object F of D(X) is called *constructible* if the following conditions are satisfied.

(3.1.1) $H^j(F) = 0$ except for finitely many j's.

(3.1.2) There exists a subanalytic locally finite decomposition $X = \bigcup X_\alpha$ of X such that $H^j(F)|_{X_\alpha}$ is a locally constant sheaf of finite rank for any j and any α.

We denote by $D_c^b(X)$ the full subcategory of D(X) consisting of constructible complexes.

3.2. For the notion of micro-support and its properties, we refer to [4]. We just mention the following properties.

For $F \in \mathrm{Ob}(D^+(X))$, we can define the micro-support $SS(F)$ of F as a closed conic subset of T^*X.

PROPOSITION 3.1 - *Let* $F \in \mathrm{Ob}(D^+(X))$, φ *a* C^1- *function on* X *and let* $t_1 \leqq t_2$ *be two real numbers. Assume that* φ $\mathrm{Supp}\, F \to \mathbb{R}$ *is proper and that* $d\varphi(x) \notin SSF$ *for any* $x \in X$ *with* $t_1 \leqq \varphi(x) < t_2$. *Then the restriction homomorphism*

$$H^j(\{x;\varphi(x) < t_2\};F) \to H^j(\{x;\varphi(x) < t_1\};F)$$

is an isomorphism for any j.

PROPOSITION 3.2. - *If* $F \in \mathrm{Ob}(D^b_c(X))$, *then* SSF *is a closed subanalytic Lagrangian subset of* T^*X.

3.3. A morphism $u : F \to F'$ in $D^+(X)$ is called an isomorphism at $p \in T^*X$, if, for a distinguished triangle $F \xrightarrow{u} F' \to F'' \to F[1]$, we have $p \notin SSF''$. We denote by $D^+(X;p)$ the category obtained by localizing $D^+(X)$ by the isomorphisms at p (see [4]).

In particular, if φ is a C^1-function such that $d\varphi(\pi(p)) = p$ $\varphi(\pi(p)) = 0$, then $F \mapsto \mathbb{R}\Gamma_{\varphi^{-1}(\mathbb{R}^+)}(F)_{\pi(p)}$ is a functor from $D^+(X;p)$. Here $\mathbb{R}^+$ signifies the set of non-negative numbers.

PROPOSITION 3.3 - *Let* $F \in \mathrm{Ob}(D^b_c(X))$ *and* Y *a real analytic submanifold. If* $SSF \subset T^*_YX$ *on a neighborhood of* $p \in T^*_YX$, *then we have*

$$F \cong \underline{V}_Y \quad \text{in } D^+(X;p)$$

where V *is a bounded complex of finite-dimensional vector spaces and* V_Y *is the constant sheaf on* Y *with* V *as fiber.*

3.4 Let F be an object of $D^b_c(X)$. Then $\Lambda = SSF$ is a subanalytic Lagrangian subvariety. Hence there exists a locally finite family $\{\Lambda_\alpha\}$ of real analytic subsets of T^*X satisfying the following conditions.

(3.3.1) Λ_α is subanalytic and connected.

(3.3.2) There exists a real analytic submanifold Y_α of X such that Λ_α is an open subset of $T^*_{Y_\alpha}X$.

(3.3.3) $\Lambda \subset \bigcup_\alpha \bar{\Lambda}_\alpha$.

(3.3.4) $\Lambda_\alpha \cap \bar{\Lambda}_\beta = \emptyset$ if $\alpha \neq \beta$.

Then by proposition 3.3, for $p \in \Lambda_\alpha$ there exists a bounded complex V_α of finite-dimensional vector spaces such that $F \cong V_{\alpha Y_\alpha}$ in $D^+(X;p)$. Then $\chi(V_\alpha) = \sum (-1)^j \dim H^j(V_\alpha)$ is locally constant in p and hence determined by α. We set $m_\alpha = \chi(V_\alpha)$.

DEFINITION 3.4 - *We define the* $\pi^{-1}\omega_X$*-valued n-chain* $\widetilde{SS}(F)$ *by*

(3.3.5) $$\widetilde{SS}(F) = \sum_\alpha m_\alpha[\Lambda_\alpha]$$

It is almost obvious that this chain does not depend on the choice of $\{\Lambda_\alpha\}$. We shall call this the *characteristic cycle* of F. Later we shall show that $\widetilde{SS}(F)$ is in fact an n-cycle.

§4. INDEX THEOREM

4.1. Let X be a real analytic manifold of dimension n. For a real valued C^2-function φ on X we set

(4.1.1) $Y_\varphi = \{ d\varphi(x) \ ; \ x \in X \} \subset T^*X$ and

(4.1.2) $Y_\varphi^a = \{ -d\varphi(x) \ ; \ x \in X \} \subset T^*X$.

Then Y_φ and Y_φ^a are isomorphic to X and hence we can regard them as $\pi^{-1}\omega_X$-valued n-cycles in T^*X.

4.2. Now, we state the following three main theorems, whose proof is given in the next three sections.

THEOREM 4.1 - *For* $F \in Ob(D_c^b(X))$, $\widetilde{SS}(F)$ *is an* n-*cycle, i.e.*, $\partial\widetilde{SS}(F) = 0$.

THEOREM 4.2 - *Let* φ *be a* C^2*-function and* $F \in Ob(D_c^b(X))$*. We assume*

(4.2.1) *For any* $t \in \mathbb{R}$, $\{x \in \mathrm{Supp}\, F; \varphi(x) \leq t\}$ *is compact.*

(4.2.2) $Y_\varphi \cap SSF$ *is compact.*

Then, $\dim H^j(X;F) < \infty$ *for any* j *and we have*

$$\chi(X;F) \underset{\mathrm{def}}{=} \sum (-1)^j \dim H^j(X;F) = (-1)^{n(n+1)/2}\, \widetilde{SS}(F).Y_\varphi.$$

THEOREM 4.3 - *Let* φ *and* F *be as in the preceding. We assume* (4.2.1) *and the following condition.*

(4.2.3) $Y_\varphi^a \cap SSF$ *is compact.*

Then $\dim H^j_c(X;F) < \infty$ *for any* j *and we have*

$$\chi_c(X;F) \underset{\text{def}}{=} \sum (-1)^j \dim H^j_c(X;F) = (-1)^{n(n+1)/2}\, \widetilde{SS}(F).Y^a_\varphi .$$

Remark that Theorem 4.1, $\pi^{-1}(\omega_X) \otimes \pi^{-1}(\omega_X) \cong \mathbb{Z}_{T^*X}$ and the condition (4.2.2) or (4.2.3) permit us to define the intersection number $\widetilde{SS}(F).Y_\varphi$ or $\widetilde{SS}(F).Y^a_\varphi$.

§5 - PROOF OF MAIN THEOREMS (I)

5.1 We shall prove first the local version of Theorem 4.2 in a generic case. Let F be an object of $D^b_c(X)$, and we choose $\{\Lambda_\alpha\}$ and $\{Y_\alpha\}$ as in 3.4. Let $x_\circ$ be a point of X and φ a C^2-function on X such that

(5.1.1) $\varphi(x_\circ) = 0$,

(5.1.2) $d\varphi(x_\circ) \in \Lambda_\alpha$ and Y_φ intersects transversally Λ_α at $p = d\varphi(x_\circ)$.

PROPOSITION 5.1 - *Under these conditions we have*

$$\chi(\mathbb{R}\Gamma_{\varphi^{-1}(\mathbb{R}^+)}(F)_{x_\circ}) = (-1)^{n(n+1)/2}\, (\widetilde{SS}(F).Y_\varphi)_p .$$

Here the last term means the intersection number of $\widetilde{SS}(F)$ *and* Y_φ *at* $p = d\varphi(x_\circ)$.

PROOF - We shall take a local coordinate system $(x_1 , \dots , x_n)$ of X such that Y_α is given by $x_1 = \dots = x_r = 0$ and $x_\circ = 0$. Then we have

$$T_p(T^*_{Y_\alpha}X) = \{ (x,\xi) \ ; \ x_1 = \dots = x_r = \xi_{r+1} = \dots = \xi_n \}$$

and

$$T_p(Y_\varphi) = \{ (x,\xi) \ ; \ \xi_j = \sum_k \frac{\partial^2\varphi}{\partial x_j \partial x_k}(0)\, x_k \} .$$

The transversality condition (5.1.2) implies that the Hessian matrix $\left(\frac{\partial^2\varphi}{\partial x_j \partial x_k}(0)\right)_{r<j,k\leqq n}$ is non-degenerate. Hence by Morse's lemma, after a change of local coordinates, we may assume that

$$\varphi|_{Y_\alpha} = \sum_{j>r} a_j x_j^2 \quad \text{for} \quad a_j \in \mathbb{R}\setminus\{0\} .$$

Let V be a bounded complex of vector spaces such that $F \cong \underline{V}_{Y_\alpha}$ in $D^+(X;p)$. Then as stated in 3.3, we have

$$(5.1.3) \qquad \mathbb{R}\Gamma_{\varphi^{-1}\mathbb{R}^+}(F)_{x_\circ} \cong \mathbb{R}\Gamma_{\varphi^{-1}\mathbb{R}^+}(\underline{V}_{Y_\alpha})_{x_\circ} .$$

Let us note the following lemma.

LEMMA 5.2 - Let $Q(x)$ be a non-degenerate quadratic form on $\mathbb{R}^n$, q the number of negative eigenvalues of Q . Then for any vector spaces V , we have

$$H^j_{Q^{-1}(\mathbb{R}^+)}(\mathbb{R}^n;V_{\mathbb{R}^n})$$

$$= \underline{H}^j_{Q^{-1}(\mathbb{R}^+)}(V_{\mathbb{R}^n})_\circ = \begin{cases} V & \text{for } j=q \\ 0 & \text{for } j\neq q \end{cases} .$$

Hence we have, by denoting $q = \#\{j;\ a_j < 0\}$,

$$H^k(\mathbb{R}\Gamma_{\varphi^{-1}\mathbb{R}^+}(F)_{x_\circ}) \cong \underline{H}^k_{\varphi^{-1}\mathbb{R}^+}(\underline{V}_{Y_\alpha})_{x_\circ} = H^{k-q}(V) .$$

Therefore we obtain

$$(5.1.4) \qquad \chi(\mathbb{R}\Gamma_{\varphi^{-1}\mathbb{R}^+}(F)_{x_\circ}) = (-1)^q\chi(V) = (-1)^q\, m_\alpha .$$

On the other hand, we have

$$(\widetilde{SS}(F).Y_\varphi)_p = m_\alpha([T^*_{Y_\alpha}X].Y_\varphi)_p \quad ,$$

and we can easily verify

$$([T^*_{Y_\alpha}X].Y_\varphi)_p = (-1)^{n(n+1)/2 + q}$$

This completes the proof of Proposition 5.1. Q.E.D.

5.2. Now we assume the condition (4.2.1) and the following conditions :

(5.2.1) $SSF \cap Y_\varphi \subset \bigsqcup_\alpha \Lambda_\alpha$

(5.2.2) SSF and Y_φ intersect transversally.

(5.2.3) $\#(SSF \cap Y_\varphi) < \infty$

PROPOSITION 5.3 - *Under these conditions we have* $\dim H^k(X;F) < \infty$

and

$$\chi(X;F) = (-1)^{n(n+1)/2}\,\widetilde{SSF}.Y_\varphi \quad .$$

PROOF - Set $\Omega_t = \{x;\varphi(x) < t\}$ and $Z_t = \{x;\varphi(x) \leq t\}$, and $\varphi\pi(Y_\varphi \cap SSF) = \{t_1, \ldots, t_N\}$ with $t_1 < \ldots < t_N$. We also set $t_0 = -\infty$, $t_{N+1} = \infty$, $\Omega_j = \Omega_{t_j}$ and $Z_j = Z_{t_j}$. Then by Proposition 3.1, we have

$$H^k(\Omega_{j+1}\,;\,F) \cong H^k(\Omega_t\,;\,F) \text{ for } t_{j+1} \geqslant t > t_j \text{ and } 0 \leqq j \leqq N.$$

Taking the inductive limit with respect to t we obtain

$$H^k(\Omega_{j+1}\,;\,F) \xrightarrow{\sim} H^k(Z_j\,;\,F) \tag{5.2.4}$$

Then by the following well-known lemma, we have

$$\dim H^k(\Omega_{j+1}\,;\,F) = \dim H^k(Z_j\,;\,F) < \infty$$

LEMMA - If K is a compact set and if U is an open neighborhood of K, then the image of $H^k(U;F) \longrightarrow H^k(K;F)$ is finite-dimensional.

Since $\Omega_{N+1} = X$ and $Z_0 = \emptyset$, (5.2.4) implies

$$\chi(X;F) = \sum_{j=1}^{N}(\chi(Z_j\,;F) - \chi(\Omega_j\,;F)) \quad . \tag{5.2.5}$$

Now we have a distinguished triangle

$$\mathbb{R}\Gamma(Z_j \setminus \Omega_j\,;\mathbb{R}\Gamma_{X\setminus\Omega_j}(F)) \to \mathbb{R}\Gamma(Z_j\,;F) \to \mathbb{R}\Gamma(\Omega_j\,;F)$$

Hence we obtain

$$\chi(Z_j\,;F) - \chi(\Omega_j\,;F) = \chi(\mathbb{R}\Gamma(Z_j \setminus \Omega_j\,;\mathbb{R}\Gamma_{X\setminus\Omega_j}(F))) \quad . \tag{5.2.6}$$

By the definition of the micro-support, we have

$$\operatorname{supp} \mathbb{R}\Gamma_{X\setminus\Omega_j}(F)|_{\varphi^{-1}(t_j)} \subset \pi(Y_\varphi \cap SSF) \quad .$$

Hence we obtain

$$\mathbb{R}\Gamma(Z_j \setminus \Omega_j\,;\,\mathbb{R}\Gamma_{X\setminus\Omega_j}(F)) = \bigoplus_{x \in \pi(Y_\varphi \cap SSF) \cap \varphi^{-1}(t_j)} \mathbb{R}\Gamma_{X\setminus\Omega_j}(F)_x \,. \tag{5.2.7}$$

The identities (5.2.5), (5.2.6) and (5.2.7) imply

$$\chi(X;F) = \sum_{\substack{x \in \pi(Y_\varphi \cap SSF) \\ \varphi(x)=t_j}} \chi(\mathbb{R}T_{X\setminus\Omega_j}(F)x) .$$

Thus Proposition 5.3 follows from Proposition 5.1. Q.E.D.

§6 - PROOF OF MAIN THEOREMS (II)

6.1. We shall prove Theorem 4.1. We give only an outline of the proof.

Since $\widetilde{SS}(F \otimes k_{\{0\}}) = \widetilde{SS}(F) \times T^*_{\{0\}}\mathbb{R}$, it is sufficient to show that $\widetilde{SS}(F)$ is a cycle outside the zero section.

The support of $\beta = \partial\widetilde{SS}(F)$ is an $(n-1)$-dimensional subanalytic subset contained in $\bigsqcup_\alpha \partial\Lambda_\alpha$. Taking a smooth point p of $\mathrm{supp}\ \beta \setminus T^*_X X$, we shall derive the contradiction by the use of Lemma 1.1 and Proposition 5.3 .

6.2. Let us take a local coordinate system $(x_1, \ldots, x_n)$ of X such that $p = (0,\xi_0)$ and that the map $(x,\xi) \mapsto \xi$ from T^*X to $\mathbb{R}^n$ gives a local embedding from supp β into $\mathbb{R}^n$ and a finite map from SSF into $\mathbb{R}^n$.

Set $\varphi(x,y) = \frac{1}{2} x^2 + xy$ and $\varphi_y(x) = \varphi(x,y)$.

Then we have

$$SSF \cap Y_{\varphi_y} \cap \{x;\ |x| = \varepsilon\} = \emptyset \text{ for } |y| \leqq \varepsilon \text{ and } 0 < \varepsilon \ll 1 .$$

Therefore, if $|y| \ll \varepsilon$ and if Y_{φ_y} satisfies the conditions (5.2.1) - (5.2.3), then we have, by Proposition 5.3

$$\chi(\{x;\ |x| < \varepsilon\};\ F) = (-1)^{n(n+1)/2}\ \widetilde{SS}(F).Y_{\varphi_y} .$$

In particular, $\widetilde{SS}(F).Y_{\varphi_y}$ does not depend on y .

The relation $\xi = \mathrm{grad}_x \varphi_y = x + y$ gives the projection $g : T^*X \to \mathbb{R}^n$ by $g(x,\xi) = \xi - x$. Since $g^{-1}(y) = Y_{\varphi_y}$, $g^{-1}(y).\widetilde{SS}(F)$ is constant in y .

Therefore we can apply Lemma 1.1 to see $\partial\widetilde{SS}(F) = 0$.

§7 - PROOF OF MAIN THEOREMS (III)

7.1. In order to prove Theorem 4.2, we shall note the following

LEMMA 7.1. (i) *Let* Λ *be an n-dimensional subanalytic conic real analytic submanifold of* T^*X. *Then* $\{\varphi\ ;\ Y_\varphi$ *and* Λ *intersect transversally* $\}$ *is dense in the space* $C^\infty(X)$ *of* C^∞*-functions on* X *with respect to the* C^2*-topology.*

(ii) *Let* Z *be an* (n-1)*-dimensional subanalytic conic subset of* T^*X. *Then* $\{\varphi\ ;\ Y_\varphi \cap Z = \emptyset\}$ *is a dense subset of* $C^\infty(X)$.

They can be shown by using Baire's category theorem similarly to the proof of the existence theorem of Morse's function.

Let φ and F satisfy the conditions in Theorem 4.2. Then there exists a function φ' close to φ which satisfies the conditions (5.2.1) - (5.2.3). Hence Proposition 5.3 can be applied to see $\chi(X;F) = (-1)^{n(n+1)/2}\, \widetilde{SS}(F).Y_{\varphi'}$.

Since Y_φ and $Y_{\varphi'}$ are homotopic, we have

$$\widetilde{SS}(F).Y_\varphi = \widetilde{SS}(F).Y_{\varphi'} \ .$$

This shows Theorem 4.2.

7.2. Theorem 4.3 can be proven in a similar argument or by reducing to Theorem 4.2 by the use of the Poincaré duality and the following proposition, which can be shown easily.

PROPOSITION 7.2 - *For* $F \in Ob(D^b_c(X))$ *,we have*

$$\widetilde{SS}(\mathbb{R}\underline{\mathcal{H}om}_k(F,k_X)) = a^*(\widetilde{SS}(F)) \ ,$$

where a *is the antipodal map of* T^*X .

§8 - APPLICATIONS

8.1. The following theorem follows immediately from Theorem 4.2.

THEOREM 8.1 - *Let* X *be a compact complex manifold, and* $F \in Ob(D^b_c(X))$. *Then*

$$\chi(X;F) = (-1)^{n(n+1)/2}\ \widetilde{SS}(F).T^*_X X .$$

8.2. When X is a complex manifold and $\mathcal{M}$ is a holonomic module over the ring $\mathcal{D}_X$ of differential operators. Then $SS(DR(\mathcal{M}))$ coincides with the characteristic variety $Ch(\mathcal{M})$ of $\mathcal{M}$ and $\widetilde{SS}(DR(\mathcal{M}))$ coincides with the characteristic cycle $\underline{Ch}(\mathcal{M})$ of $\mathcal{M}$. Hence the results in this paper can be easily applied to holonomic modules.

8.3. Let φ be a real -valued real analytic function defined on X and $x_o \in X$.

(8.3.1) $\qquad \varphi(x) > 0$ for $x \in X \setminus \{x_o\}$.

LEMMA 8.2. *For any subanalytic closed conic Lagrangian set* Λ, $d\varphi(x_o)$ *is an isolated point of* $\Lambda \cap Y_\varphi$.

PROOF - Otherwise there exists a real analytic path $x = x(t)$ such that $x(0) = x_o$; $x(t) \neq x_o$ for $t \neq 0$ and $d\varphi(x(t)) \in \Lambda$. Since Λ is Lagrangian, $\theta = d\varphi(x(t)) = 0$. Hence $\varphi(x(t))$ is a constant function, which is a contradiction. Q.E.D.

Along with this lemma, the following theorem follows immediately from Theorems 8.2 and 8.3.

THEOREM 8.3 - *Let* $F \in Ob(D^b_c(X))$ *and let* φ *satisfy* (8.3.1). *Then we have*

(8.3.1) $$\chi(F_{x_o}) = (-1)^{n(n+1)/2}\ (\widetilde{SS}(F).Y_\varphi)_{x_o} ,$$

(8.3.2) $$\chi(\mathbb{R}\Gamma_{\{x_o\}}(X;F)) = (-1)^{n(n+1)/2}\ (\widetilde{SS}(F).Y^a_\varphi)_{x_o} .$$

Here (.) means the intersection number of two cycles at $x_o \in T^*_X X \cong X \subset T^*X$.

8.4. A $\mathbb{Z}$-valued function φ on X is called *constructible* if there exists a subanalytic stratification $X = \bigsqcup X_\alpha$ of X such that $\varphi|_{X_\alpha}$ is constant. We define the $\pi^{-1}\omega_X$-valued n-cycle

(8.4.1) $$c(\varphi) = \sum_\alpha \varphi(X_\alpha)\widetilde{SS}(Q_{X_\alpha}) .$$

Then it is immediate that this does not depend on the choice of stratification.

Let us denote by $C(X)$ the space of $\mathbb{Z}$-valued constructible functions on X. Let $K(D_c^b(X))$ be the additive group generated by $Ob(D_c^b(X))$ with the relation

$$[F] = [F'] + [F'']$$

for distinguished triangles $F' \to F \to F'' \to F'[1]$.

For $F \in Ob(D_c^b(X))$ we define the constructible function $\chi(F)$ by $X \ni x \mapsto \chi(F_x)$. Then this passes through the quotient and we obtain the commutative diagram

(8.4.2)
$$\begin{array}{ccc} K(D_c^b(X)) & \xrightarrow{\chi} & C(X) \\ & \underset{\sim}{SS} \searrow \quad \swarrow c & \\ & Z_n(T^*X\ ;\ \pi^{-1}\omega_X) & \end{array}$$

Here $Z_n(T^*X\ ;\ \pi^{-1}\omega_X)$ denotes the space of $\pi^{-1}\omega_X$-valued subanalytic n-cycles.

EXAMPLE 8.5.

(i) Let Y be a closed r-codimensional submanifold of X and χ_Y the characteristic function of Y. Then

$$c(\chi_Y) = [T_Y^*X]$$

(ii) Set $X = \mathbb{R}$, $Z_\pm = \{x\ ;\ \pm x > 0\}$, $Z_\circ = \{0\}$.
We define the 1-cycles $\alpha_\pm$ and $\beta_\pm$ by

$$\alpha_\pm = \{(x,\xi)\ ;\ \xi = 0,\ \pm x > 0\} \text{ with } \operatorname{sgn} dx \otimes \operatorname{sgn} dx\ ,$$

$$\beta_\pm = \{(x,\xi)\ ;\ x = 0,\ \pm\xi > 0\} \text{ with } \operatorname{sgn} d\xi \otimes \operatorname{sgn} dx\ .$$

Then we have

$$c(\chi_{Z_+}) = \alpha_+ + \beta_-\ ,$$
$$c(\chi_{Z_-}) = \alpha_- + \beta_+ \text{ and}$$
$$c(\chi_{Z_\circ}) = -\beta_+ - \beta_-\ .$$

(iii) Set $X = \mathbb{R}^n$, $q(x) = x_1^2 - x_2^2 - \dots - x_n^2$ $(n \geqslant 2)$,

$$dx' = dx_2 \wedge \ldots \wedge dx_n \ , \ dx = dx_1 \wedge dx' \ ,$$
$$Z_\pm = \{x \in X \ ; \ q(x) \geqq 0, \ \pm x_1 \geqslant 0\} \ ,$$
$$Z_\circ = \{x \in X \ ; \ q(x) \leqq 0\} \ ,$$
and $U_\varepsilon = \mathrm{Int}\, Z_\varepsilon \quad (\varepsilon = \pm \ , \ 0)$.

We define the n-cycles in T^*X by

$$\alpha_\varepsilon = \{(x,\xi) \ ; \ x \in U_\varepsilon , \xi = 0\} \text{ with } \operatorname{sgn} dx \otimes \operatorname{sgn} dx \ ,$$
$$\beta_\varepsilon = \{(x,\xi) \ ; \ x = 0, \xi \in U_\varepsilon\} \text{ with } \operatorname{sgn} d\xi \otimes \operatorname{sgn} dx \ ,$$
for $\varepsilon = \pm \ , \ 0$, and
$$\gamma_{\varepsilon_1, \varepsilon_2} = \{(x,\xi) \ ; \ \varepsilon_1 x_1 > 0, \ \varepsilon_2 \xi_1 > 0, \ \xi_j / x_j = -\xi_1 / x_1 \text{ for } j \geqslant 2, \ q(x) = 0\}$$
with $\operatorname{sgn}(d\xi_1 \wedge dx') \otimes \operatorname{sgn} dx$, for $\varepsilon_1 , \varepsilon_2 = \pm 1$.

Then we have

$$c(\chi_{Z_\pm}) = \alpha_\pm - \gamma_{\pm,\pm} + (-)^n \beta_\pm \ ,$$
$$c(\chi_{U_\pm}) = \alpha_\pm + \gamma_{\pm,\mp} + \beta_\mp \ ,$$
$$c(\chi_{Z_\circ}) = \alpha_\circ - \gamma_{+,-} - \gamma_{-,+} - \beta_+ - \beta_- \text{ and}$$
$$c(\chi_{U_\circ}) = \alpha_\circ + \gamma_{+,+} + \gamma_{-,-} - (-)^n \beta_+ - (-)^n \beta_- \ .$$

§9 - VARIATIONS OF MAIN THEOREMS

9.1. Let f be a real analytic function on X . We define, for $F \in \mathrm{Ob}(D(X))$,

(9.1.1) $$\mu_f(F) = \mathbb{R}\Gamma_{f^{-1}(\mathbb{R}^+)}(F)|_{f^{-1}(0)} \ .$$

Let $F \in \mathrm{Ob}(D^b_c(X))$ and Ω an open subset of $f^{-1}(0)$. We assume

(9.1.2) $\Omega \cap \operatorname{supp} F$ is relatively compact.

(9.1.3) $SSF \cap Y_f \cap \pi^{-1}(\partial\Omega) = \emptyset$.

Then we have the following

THEOREM 9.1 - *Under these conditions we have* $\dim H^k(\Omega \ ; \ \mu_f(F)) < \infty$

and

$$\chi(\Omega\ ;\ \mu_f(F)) = (-1)^{n(n+1)/2}\ (\widetilde{SS}F \cap \Omega).(Y_f \cap \Omega) \quad .$$

This theorem can be shown by deforming f to a generic position with respect to SSF .

9.2. Let F and F' be two objects of $D^b_c(X)$ and φ a C^1-function on T^*X . We assume the following

(9.2.1) $\Omega = \{p \in T^*X\ ;\ \varphi(p) < 0\}$ is relatively compact in T^*X .

(9.2.2) $C_p(SS(F'),SS(F)) \not\ni -H_\varphi(p)$ for any $p \in \varphi^{-1}(0)$.

Here C_p means the normal cone (see [4]), and H_φ means the Hamiltonian vector field of φ. We set

$$SS(F)^\varepsilon = e^{-\varepsilon H_\varphi}(SSF)$$

$$\text{and}\quad \widetilde{SS}(F)^\varepsilon = e^{-\varepsilon H_\varphi}(\widetilde{SS}F)$$

Then (8.6.2) implies for $0 < \varepsilon \ll 1$

$$(SS(F)^\varepsilon \cap \Omega) \cap (SS(F') \cap \Omega) = \emptyset \ .$$

THEOREM 9.2 - *Under these conditions we have*

$$\dim H^k(\Omega\ ;\ \mu hom(F,F')) < \infty$$

and

$$\chi(\Omega\ ;\ \mu hom(F,F')) = (-1)^{n(n+1)/2}\ (\widetilde{SS}(F') \cap \Omega).(\widetilde{SS}(F)^\varepsilon \cap \Omega) \ .$$

For the definition of μhom , we refer to [4] . This theorem can be shown by reducing to Theorem 9.1 with the aid of contact transformations.

If we assume instead of (9.2.2)

(9.2.3) $C_p(SS(F'),SSF) \not\ni H_\varphi(p)$ for any $p \in \varphi^{-1}(0)$.

Then we have

THEOREM 9.3 - *Under* (9.2.1) *and* (9.2.3) *we have*

$$\dim H^k_c(\Omega\ ;\ \mu hom(F,F')) < \infty$$

and

$$\chi_c(\Omega\ ;\ \mu hom\ (F,F')) = (-1)^{n(n+1)/2}\ (\widetilde{SS}(F') \cap \Omega).(\widetilde{SS}(F)^{-\varepsilon} \cap \Omega).$$

Remark that if we take as F the constant sheaf k_X , then we can recover Theorems 4.2 and 4.3.

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Corrections to "Microlocal study of Sheaves", M.Kashiwara, P.Schapira. Astérisque 128, 1985.

1) p.48, l.-6 ; p.85, l.-8, -9 ; p.86, l.-2 ; p.191, l.-8, -5 :
read "... $\overset{L}{\otimes} \omega^{!} \underline{\mathbb{Z}}_{T^*_M X}$"

2) p.40, l.-3 : p.47, l.-9 :
read "... convex proper cone of..."

3) p.40, l.-2 : read "...$\mathbb{R}\Gamma(\mathrm{Int}(A^{oa}), \underline{F})$..."

4) p.47, l.-6 : read "... $\cap\ \mathrm{Int}\ Z^{oa}$..."

5) p.189, l.4 : read "... is punctually endowed..."

6) p.119, l. 4, l.6 : read " $\alpha \geqslant 3$ " , " a C^2-function"