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Algebraic study of systems of partial differential equations. (Master's thesis, Tokyo University, December 1970. Translated by Andrea D'Agnolo and Pierre Schneiders. With a foreword by Pierre Schapira)

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Algebraic Study of Systems of Partial Differential Equations

(Master's Thesis, Tokyo University, December 1970)

Masaki KASHIWARA

Translated by Andrea D'AGNOLO and Jean-Pierre SCHNEIDERS

Abstract - This Mémoire is a translation of M. Kashiwara's thesis. In this pioneering work, the author initiates the study of systems of linear partial differential equations with analytic coefficients from the point of view of modules over the ring $\mathcal{D}$ of differential operators. Following some preliminaries on good filtrations and non-commutative localization, the author introduces the notion of characteristic variety and of multiplicity of a $\mathcal{D}$-module. Then he shows that the classical Cauchy-Kovalevskaya theorem may be generalized as a formula for the solutions of non-characteristic inverse images of $\mathcal{D}$-modules. Among the applications of this result, we find a solvability criterion in the complex domain and a study of the Cauchy problem for hyperfunctions. The author also investigates the homological properties of $\mathcal{D}$-modules linking, in particular, their homological dimension to the codimension of their characteristic variety. The thesis concludes with an index formula for holonomic systems on smooth complex curves.

Résumé - Ce mémoire est une traduction de la thèse de M. Kashiwara. Dans ce travail de pionnier, l'auteur entreprend l'étude des systèmes d'équations aux dérivées partielles linéaires à coefficients analytiques du point de vue des modules sur l'anneau $\mathcal{D}$ des opérateurs différentiels. Après quelques préliminaires sur les bonnes filtrations et la localisation non-commutative, l'auteur introduit la notion de variété caractéristique et de multiplicité d'un $\mathcal{D}$-module. Ensuite, il montre que le théorème classique de Cauchy-Kovalevskaya peut être généralisé en une formule pour les solutions des images inverses non-caractéristiques des $\mathcal{D}$-modules. Parmi les applications de ce résultat, nous trouvons un critère de résolubilité dans le domaine complexe et une étude du problème de Cauchy pour les hyperfonctions. L'auteur examine également les propriétés homologiques des $\mathcal{D}$-modules, reliant en particulier leur dimension homologique à la codimension de leur variété caractéristique. La thèse se conclut avec une formule d'indice pour les systèmes holonomes sur les courbes complexes lisses.

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To the memory of our friend Emmanuel Andronikof

Contents

Translators' Foreword

The study of analytic linear partial differential equations using the powerful tools of homological algebra and sheaf theory began in the seventies. It has proved to be a very successful approach for a broad range of mathematical questions (microlocal analysis, index formulas, representation theory, etc.). The set of results obtained using these methods forms what is often called "Algebraic Analysis".

One of the important components of algebraic analysis is the study of analytic $\mathcal{D}$-modules (i.e., modules over the ring of linear partial differential operators with analytic coefficients). In his master's thesis (Tokyo University, December 1970), M. Kashiwara did a very important pioneering work on this subject. Until now, this text has been unavailable to a large public (it only appeared in a local handwritten publication in Japanese).

Although various expository texts on this subject are now available, we feel that Kashiwara's thesis is still interesting in its own right; not only as an important landmark in the historical development of algebraic analysis, but also as an illuminating introduction to analytic $\mathcal{D}$-modules.

In this volume, we present an almost faithful translation of this work. The only differences with the original consist in a few minor corrections and the use of up-to-date notations. We felt it unnecessary to mention these small changes explicitly.

We hope that in reading Kashiwara's thesis the reader's interest in our field of research will be stimulated. Of course, since 1970, the field has evolved much. To get a better perspective, one may refer to the list of suggested further readings which can be found hereafter. Although far from exhaustive, this list may nevertheless serve as a source of pointers to the extensive research literature.

We wish to thank Tae Morikawa for her help in preparing a typewritten Japanese version of Kashiwara's handwritten notes, and Kimberly De Haan for her attempts to clean up our franco-italian English.

Paris, February 1995

Andrea D'Agnolo — Jean-Pierre Schneiders

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Foreword

The idea of regarding a system of linear equations as a module over a ring is basic to algebraic geometry. However, it only appeared in the '70s for systems of partial differential equations with analytic coefficients, after pioneering talks by Sato in the '60s and Quillen thesis in '64. The two seminal papers on this subject are certainly Kashiwara's thesis (December '70) and Bernstein's papers of '71 and '72 published in "Functional Analysis" [3, 4]. Unfortunately, Kashiwara's thesis has never been translated nor published, and only exists in the form of handwritten mimeographed notes. Nevertheless, it has been distributed, inside and outside Japan, and some people have found the material to their inspiration at its reading.

However, the aim of this publication is by no means historical. As it will become evident to the reader, this thesis could have been written last week (modulo minor modifications): it contains a great deal of little known or even unknown results and could be used almost without any changes as a textbook for post-graduate courses.

Twenty-five years have passed since this thesis was written, and $\mathcal{D}$-module theory is now a basic tool in many branches of Mathematics: linear partial differential equations, harmonic analysis and representation theory, algebraic geometry, etc. Without being exhaustive, let us describe a few directions in which $\mathcal{D}$-modules techniques and Kashiwara's contribution have been decisive.

Before $\mathcal{D}$-modules were studied for themselves, mathematicians were interested in their solutions (distributions, hyperfunctions, etc.). In this perspective, the theory takes its full strength with the microlocal point of view, introduced by M. Sato in '69 and developed with Kawai and Kashiwara in [26]. This paper is at the origin of what is now called "microlocal analysis", and gave rise to enormous literature in the '80s.

Among $\mathcal{D}$-modules, there is a class of particular importance — holonomic $\mathcal{D}$-modules — which generalizes the notion of ordinary differential equations. In his 1975 paper [9], Kashiwara proved that if $\mathcal{M}$ is holonomic, and if one calls Sol the functor which to a $\mathcal{D}$-module associates the complex of its holomorphic solutions, then $Sol(\mathcal{M})$ is constructible, and is even perverse (although the theory of perverse sheaves didn't exist at the time). That same year (see [23], p. 287), he gave a precise statement for the Riemann-Hilbert correspondence. Then with Oshima [18] and Kawai [16], he developed the theory of regular holonomic $\mathcal{D}$-modules (a generalization in higher dimension of the Picard-Fuchs theory) and in 1980 he gave a proof of the Riemann-Hilbert correspondence by constructing an inverse to the functor Sol, the functor of temperate cohomology [12, 14]. As is well known, holonomic $\mathcal{D}$-modules together with the work of Goresky-MacPherson [6] have been a main theme to the theory of perverse sheaves. This last theory, introduced by Gabber-Beilinson-Bernstein-Deligne [2], plays an important role in algebraic geometry via Hodge theory, which in turn, enlarges $\mathcal{D}$-module theory with the introduction by Saito [24, 25] of the category of Hodge modules.

The complex of holomorphic solutions of a holonomic $\mathcal{D}$-module is constructible. It is then natural to try to calculate its local Euler-Poincaré index in terms of the characteristic cycle of the $\mathcal{D}$-module. It is remarkable that the answer already appeared in '73 [8]. The proof (which can be found in detail in [13]) introduces important topological constructions discovered independently by other people (in particular MacPherson [21]), such as Lagrangian cycles and Euler obstruction. This local index formula was generalized later by Kashiwara to a local Lefschetz formula [15] which has important applications to group representation.

Many properties of holonomic $\mathcal{D}$-modules are closely related to the so-called b-function or Bernstein-Sato polynomial, itself related to deep topological notions (see [22]). In 1978, Kashiwara proved the rationality of the zeroes of the b-functions [11]. For that purpose he first obtained a theorem of Grauert type on direct images of $\mathcal{D}$-modules which is important in its own right.

Notice that the link between $\mathcal{D}$-modules and sheaves is not confined to holonomic $\mathcal{D}$-module and constructible sheaves. In 1982, Kashiwara and the author introduced the notion of the "micro-support" of a sheaf on a real manifold, a notion which appears as very similar to that of the characteristic variety of a $\mathcal{D}$-module in the complex case (both will be shown to be involutive). This is the starting point for a microlocal theory of sheaves, partly inspired by $\mathcal{D}$-module theory (see [19]).

Other fundamental fields of application of $\mathcal{D}$-modules are harmonic analysis, Lie groups and representation theory. In '78, Kashiwara et al. [17] solved the Helgason conjecture and in the '90s, in a series of papers, he reinterpreted the Harish-Chandra theory in terms of $\mathcal{D}$-modules (see in particular [7] and [20]). But the most spectacular application of $\mathcal{D}$-modules to representation theory may be Beilinson-Bernstein's theorem [1] on the equivalence of (twisted) $\mathcal{D}$-modules on flag manifolds and modules over enveloping algebras and their proof of the Kazhdan-Lusztig conjecture, simultaneously obtained by Brylinski-Kashiwara [5].

Thanks to these successes, $\mathcal{D}$-modules have obtained an importance comparable to that of distributions in real analysis or coherent analytic sheaves ($\mathcal{O}$-modules) in complex geometry.

To paraphrase Grothendieck (in "Récoltes et Semailles"): a new philosophy of coefficients was born.

Paris, January 1996

Pierre Schapira

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Notations

X : complex manifold

T^*X : cotangent vector bundle of X

P^*X : cotangent projective bundle of X (given by $P^*X \simeq (T^*X \setminus X)/\mathbb{C}^\times$)

$\mathcal{O}_X$: sheaf of holomorphic functions on X

$\mathcal{D}_X$: sheaf of rings of linear partial differential operators with holomorphic function coefficients on X

$\mathcal{D}_{X,k}$: subsheaf of $\mathcal{D}_X$ of operators of degree less than k

$\overline{\mathcal{D}}_{X,k}$: $= \mathcal{D}_{X,k}/\mathcal{D}_{X,k-1}$

$\overline{\mathcal{D}}_X$: $= \bigoplus_k \overline{\mathcal{D}}_{X,k}$

Θ_X : sheaf of holomorphic tangent vector fields on X

$\bigodot \Theta_X$: symmetric algebra of Θ_X (note that $\bigodot \Theta_X \simeq \overline{\mathcal{D}}_X$)

M : real analytic manifold

$\mathcal{A}_M$: sheaf of real analytic functions on M

$\mathcal{B}_M$: sheaf of hyperfunctions on M

Γ_Z : local cohomology with support in Z

Chapter 0

Introduction

Elie Cartan performed the first systematic study of systems of partial differential equations. Here, we will approach the study of systems of partial differential equations from a different angle. Namely, we will consider systems of partial differential operators as modules over the sheaf of rings $\mathcal{D}$. In this new direction, there have been some successes for constant coefficient partial differential operators, while many problems in the variable coefficient case have not been solved.

Let us start with a simple example. Let:

$$\underset{(i=1,\dots,N')}{\textstyle\sum_{j=1}^{N} P_{ij}u_j = 0,} \qquad \underset{(\lambda=1,\dots R')}{\textstyle\sum_{\mu=1}^{R} Q_{\lambda\mu}v_\mu = 0,}$$

be two systems. We will say that:

$$\mathcal{M} = \mathcal{D}^N/\mathcal{D}^{N'}P \quad \text{and} \quad \mathcal{N} = \mathcal{D}^R/\mathcal{D}^{R'}Q$$

are of the same type, if there is a transformation interchanging the solutions of these systems of differential operators. Namely, what is essential is the $\mathcal{D}$-module $\mathcal{M}$, and not its presentation $\sum_j P_{ij}u_j = 0$.

Let:

$$\mathcal{E} \xrightarrow{P} \mathcal{F}$$

be a system of partial differential operators from the vector bundle $\mathcal{E}$ to $\mathcal{F}$, and set $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{O})$. Since $\mathcal{D} \otimes \mathcal{E}^\vee$ is identified to:

$$\mathcal{H}omdiff(\mathcal{E}, \mathcal{O}) = \{\text{differential operators from } \mathcal{E} \text{ to } \mathcal{O}\},$$

we get the $\mathcal{D}$-homomorphism:

$$\mathcal{D} \otimes \mathcal{E}^\vee \xleftarrow{P} \mathcal{D} \otimes \mathcal{F}^\vee.$$

Let $\mathcal{M}$ be its cokernel. Then, $\mathcal{M}$ is the $\mathcal{D}$-module which represents $\mathcal{E} \xrightarrow{P} \mathcal{F}$. Since we have:

$$\mathcal{H}om_{\mathcal{D}}(\mathcal{D} \otimes \mathcal{E}^\vee, \mathcal{O}) = \mathcal{H}om_{\mathcal{O}}(\mathcal{E}^\vee, \mathcal{O}) = \mathcal{E},$$

we get the exact sequence:

$$0 \to \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O}) \to \mathcal{E} \xrightarrow{P} \mathcal{F},$$

and $\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$ appears as the sheaf of solutions of $P : \mathcal{E} \to \mathcal{F}$. In the same way, given:

$$\mathcal{E} \xrightarrow{P} \mathcal{F} \xrightarrow{Q} \mathcal{G}, \qquad (QP = 0),$$

such that:

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D} \otimes \mathcal{E}^{\vee} \xleftarrow{P} \mathcal{D} \otimes \mathcal{F}^{\vee} \xleftarrow{Q} \mathcal{D} \otimes \mathcal{G}^{\vee} \tag{0.0.1}$$

is exact, the cohomology of:

$$\mathcal{E} \to \mathcal{F} \to \mathcal{G}$$

is isomorphic to $\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$. Therefore, $\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$ represents the obstruction for an $f \in \mathcal{F}$ satisfying the compatibility condition $Qf = 0$ to be written as $f = Pg$, $g \in \mathcal{E}$. Since (0.0.1) is exact, $Qf = 0$ is the best possible compatibility condition (among those expressed by a system of partial differential operators). Thus, the computation of $\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$ becomes an essential problem.

In order to obtain further results through this neat approach, we need to perform a detailed investigation of the nature of $\mathcal{D}$. In these short notes, we will lay the foundations for such a study.

We wish to thank Prof. Mikio Sato for his constant help during the preparation of this work. The root ideas of these notes are his. The Sato-Komatsu semester in algebraic analysis gave rise to much emulation among the participants. In particular, the frequent discussions we had with Prof. Aomoto were very helpful. We wish to thank him here.

Chapter 1

Algebraic preliminaries

1.1 Coherence

In this section, we will present several facts on finite type $\mathcal{D}$-modules needed for the further development of the theory. In Chapter 0, we saw the correspondence between a left $\mathcal{D}$-module $\mathcal{M}$ and a system of partial differential equations. The fact that $\mathcal{M}$ is of finite type, that is, locally admits a resolution:

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}_X^r,$$

means that it corresponds to a system of equations with a finite number of unknown functions. We say that $\mathcal{M}$ is of finite presentation, if it admits a resolution:

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}_X^r \leftarrow \mathcal{D}_X^s,$$

corresponding to a system with a finite number of equations, in a finite number of unknown functions. Hence, the meaning of a left $\mathcal{D}$-module of finite presentation is clear.

Definition 1.1.1. A left $\mathcal{D}$-module of finite presentation is simply called a system.

If we want to develop the theory further, we need to study the finiteness properties of $\mathcal{D}$. In this chapter, we will consider results which depend only on the filtered structure of $\mathcal{D}$.

Below, $\mathcal{S}$ will be a sheaf of (not necessarily commutative) algebras on X endowed with a filtration $\mathcal{S} = \bigcup_{k=0}^{\infty} \mathcal{S}_k$ satisfying the following conditions:

(1) $\mathcal{S}_0 \ni 1$ (the unit), and $\mathcal{S}_0$ is an $\mathcal{O}_X$-algebra,

(2) $\mathcal{S}_k\mathcal{S}_l = \mathcal{S}_{k+l}$ for $k, l \geq 0$,

(3) every $\mathcal{S}_k$ is a (left and right) coherent $\mathcal{O}_X$-module, and $\mathcal{S}_k \subset \mathcal{S}_{k+1}$,

(4) given $\mathcal{S}_k \ni u$, $\mathcal{S}_l \ni v$, $[u, v] = uv - vu \in \mathcal{S}_{k+l-1}$.

Assuming the above axioms, and setting:

$$\overline{\mathcal{S}}_k = \operatorname{gr}_k \mathcal{S} = \mathcal{S}_k/\mathcal{S}_{k-1},$$

we see that

$$\overline{\mathcal{S}} = \operatorname{gr} \mathcal{S} = \bigoplus_{k=0}^{\infty} \overline{\mathcal{S}}_k$$

is a commutative $\mathcal{O}_X$-algebra, and that $\overline{\mathcal{S}}$ is generated by $\overline{\mathcal{S}}_0$ and $\overline{\mathcal{S}}_1$. For instance, $\mathcal{D}_X$ satisfies the axioms for $\mathcal{S}$, the corresponding $\overline{\mathcal{S}}$ being $\bigodot \Theta_X$.

Definition 1.1.2. Let K be a compact subset of X. If, for every coherent $(\mathcal{O}_X|_K)$-module $\mathcal{F}$

$$H^i(K;\mathcal{F}) = 0 \qquad \text{for } i > 0,$$

we say that K is *Stein.* If, moreover, $\Gamma(K;\mathcal{O}_X)$ is a Noetherian ring, we say that K is *Noetherian.*

In Harvey [7], it is shown that, in a Stein manifold X, a compact subset K is Stein if and only if it is an intersection of Stein open subsets of X. Frish [5] shows that, if K can be defined near its boundary by finitely many inequalities involving real analytic functions, then K is Noetherian. This result is also discussed in Hironaka [8]. Therefore, X has sufficiently many Noetherian compact subsets.

Proposition 1.1.3. *If K is a Noetherian compact subset of X, then $\Gamma(K,\mathcal{S})$ is both left and right Noetherian.*

Proof. Let us only show that $\Gamma(K;\mathcal{S})$ is left Noetherian. We have

$$\Gamma(K;\mathcal{S}) = \bigcup_k \Gamma(K;\mathcal{S}_k).$$

Moreover,

$$\Gamma(K;\mathcal{S}_k)/\Gamma(K;\mathcal{S}_{k-1}) = \Gamma(K;\mathcal{S}_k/\mathcal{S}_{k-1}) = \Gamma(K;\overline{\mathcal{S}}_k).$$

Hence, $\Gamma(K;\overline{\mathcal{S}}_0)$ and $\Gamma(K;\overline{\mathcal{S}}_1)$ generate $\Gamma(K;\overline{\mathcal{S}})$. Therefore, $\Gamma(K;\overline{\mathcal{S}})$ is finitely generated over $\Gamma(K;\mathcal{O}_X)$, and is a Noetherian ring. Now, let $\mathcal{I} \subset \Gamma(K;\mathcal{S})$ be a left ideal. Set

$$\mathcal{I}_k = \Gamma(K;\mathcal{S}_k) \cap \mathcal{I}, \quad \overline{\mathcal{I}}_k = \mathcal{I}_k/\mathcal{I}_{k-1} \subset \Gamma(K;\overline{\mathcal{S}}_k), \quad \overline{\mathcal{I}} = \bigoplus_k \overline{\mathcal{I}}_k \subset \Gamma(K;\overline{\mathcal{S}}).$$

Since $\overline{\mathcal{I}}$ is of finite type, there is N, such that $\overline{\mathcal{I}}$ is generated by $\overline{\mathcal{I}}_0 \oplus \cdots \oplus \overline{\mathcal{I}}_N$. Now, it is sufficient to prove that $\mathcal{I}$ is generated by $\mathcal{I}_N$. For $k \geq N$

$$\overline{\mathcal{I}}_k \subset \sum_{i=0}^{N} \Gamma(K;\overline{\mathcal{S}}_{k-i})\overline{\mathcal{I}}_i$$

therefore,

$$\begin{aligned}\mathcal{I}_k &\subset \mathcal{I}_{k-1} + \sum_{i=0}^{N} \Gamma(K;\mathcal{S}_{k-i})\mathcal{I}_i \\ &\subset \mathcal{I}_{k-1} + \Gamma(K;\mathcal{S})\mathcal{I}_N\end{aligned}$$

Consequently, by induction on k, we get

$$\mathcal{I}_k \subset \Gamma(K;\mathcal{S})\mathcal{I}_N.$$

So,

$$\mathcal{I} = \Gamma(K;\mathcal{S})\mathcal{I}_N.$$

□

In the following, by module (resp. ideal) we mean left module (resp. left ideal).

Proposition 1.1.4. *Let $\mathcal{I}$ be a sub $\mathcal{S}$-module of $\mathcal{S}^l$. If, for any k, $\mathcal{I} \cap \mathcal{S}_k^l$ is a coherent $\mathcal{O}_X$-module, then $\mathcal{I}$ is an $\mathcal{S}$-module of finite type.*

Proof. Let $x \in X$, and let K be a Noetherian compact neighborhood of x. Set $I = \Gamma(K;\mathcal{I})$, $S = \Gamma(K;\mathcal{S})$, $S_k = \Gamma(K;\mathcal{S}_k)$, $I_k = I \cap S_k^l = \Gamma(K;\mathcal{I} \cap \mathcal{S}_k^l)$. Since $I \subset S^l$, and S is a Noetherian ring, I is a finite type S-module. Therefore, there is a k_0 such that I is generated by I_{k_0}. Hence, the exact sequence

$$S \otimes I_{k_0} \to I \to 0.$$

On the other hand, since every $\mathcal{I}_k = \mathcal{I} \cap \mathcal{S}_k^l$ is a coherent $\mathcal{O}_X$-module, the map

$$\mathcal{O}_X \otimes I_k \to \mathcal{I}_k$$

is surjective on K. Therefore, the map

$$\mathcal{S} \otimes I_{k_0} \to \mathcal{I}$$

is also surjective. Hence, $\mathcal{I}$ is of finite type. □

Proposition 1.1.5. *$\mathcal{S}$ is coherent.*

Proof. We must show that if $0 \to \mathcal{M} \to \mathcal{S}^l \xrightarrow{f} \mathcal{S}$ is an exact sequence, then $\mathcal{M}$ is of finite type. By the preceding proposition, it is sufficient to prove that $\mathcal{M}_k = \mathcal{M} \cap \mathcal{S}_k^l$ is a coherent $\mathcal{O}_X$-module. For any k, there is k' such that $f(\mathcal{S}_k^l) \subset \mathcal{S}_{k'}$. For such a k',

$$0 \to \mathcal{M}_k \to \mathcal{S}_k^l \to \mathcal{S}_{k'}$$

is an exact sequence of $\mathcal{O}_X$-modules. Therefore, $\mathcal{M}_k$ is a coherent $\mathcal{O}_X$-module. □

Therefore, the category of coherent $\mathcal{S}$-modules is an abelian category associated to $\mathcal{S}$. Hereafter, we will investigate coherent $\mathcal{S}$-modules.

1.2 Good filtrations

In the proofs of § 1.1, the properties of the non-commutative ring $\mathcal{S}$ were deduced from those of the commutative ring $\overline{\mathcal{S}}$. This was made possible by the use of filtrations.

Definition 1.2.1. Let $\mathcal{M}$ be an $\mathcal{S}$-module. A filtration of $\mathcal{M}$ is an increasing sequence $\{\mathcal{M}_k\}_{k\in\mathbb{Z}}$ of sub $\mathcal{O}_X$-modules $\mathcal{M}_k$ of $\mathcal{M}$, such that

(o) $\mathcal{S}_l\mathcal{M}_k \subset \mathcal{M}_{k+l}$

A *filtered $\mathcal{S}$-module* is an $\mathcal{S}$-module endowed with a filtration. Let $\mathcal{M}$ be a filtered $\mathcal{S}$-module. For any integer l, we denote by $\mathcal{M}(l)$ the $\mathcal{S}$-module $\mathcal{M}$ endowed with the filtration

$$\mathcal{M}(l)_k = \mathcal{M}_{k+l}.$$

A *homomorphism of filtered $\mathcal{S}$-modules* is an $\mathcal{S}$-linear homomorphism $f : \mathcal{M} \to \mathcal{N}$ such that $f(\mathcal{M}_k) \subset \mathcal{N}_k$. A homomorphism of filtered $\mathcal{S}$-modules $f : \mathcal{M} \to \mathcal{N}$ is strict (resp. essentially strict) if $f(\mathcal{M}_k) = \mathcal{N}_k \cap f(\mathcal{M})$ for any k (resp. any sufficiently large k). We call a filtered $\mathcal{S}$-module of the type $\mathcal{L} = \bigoplus_{j=1}^{r} \mathcal{S}(l_j)$ a *quasi-free filtered $\mathcal{S}$-module.* Let $\mathcal{M}$ be a filtered $\mathcal{S}$-module. Then, $\operatorname{gr}\mathcal{M} = \bigoplus_k(\mathcal{M}_k/\mathcal{M}_{k-1})$ has a structure of $\overline{\mathcal{S}} = \operatorname{gr}\mathcal{S}$-module. Since $\overline{\mathcal{S}}$ is a sheaf of commutative rings, we can use the classical results of commutative algebra and algebraic geometry. To any homomorphism of filtered $\mathcal{S}$-modules $\mathcal{M} \to \mathcal{N}$, we associate the $\overline{\mathcal{S}}$-linear homomorphism

$$\operatorname{gr}\mathcal{M} \to \operatorname{gr}\mathcal{N}.$$

We will skip the proof of the next statement, since it is obvious.

Lemma 1.2.2. *Let $\mathcal{L} \xrightarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{N}$ be a sequence of filtered $\mathcal{S}$-modules such that $g \circ f = 0$. If f and g are strict, and if $\mathcal{L} \to \mathcal{M} \to \mathcal{N}$ is exact, then so is*

$$\operatorname{gr}\mathcal{L} \to \operatorname{gr}\mathcal{M} \to \operatorname{gr}\mathcal{N}. \tag{1.2.1}$$

Conversely, if (1.2.1) is exact, and $\mathcal{M} = \bigcup_k \mathcal{M}_k$, $\mathcal{M}_k = 0$ for $k \ll 0$, then f and g are strict, and $\mathcal{L} \to \mathcal{M} \to \mathcal{N}$ is exact.

Definition 1.2.3. A filtration of an $\mathcal{S}$-module $\mathcal{M}$ is *good* if conditions (i)–(iii) below hold:

(i) $\mathcal{M}_k$ is a coherent $\mathcal{O}_X$-module,

(ii) $\mathcal{M} = \bigcup_k \mathcal{M}_k$, $\mathcal{M}_k = 0$ for $k \ll 0$,

(iii) $\operatorname{gr}\mathcal{M}$ is a coherent $\overline{\mathcal{S}}$-module.

The next statement is clear.

Translators' note: In this section, most of the results are to be understood to hold semi-locallly (i.e. in a neighborhood of a Noetherian compact subset of X).

Lemma 1.2.4. *Let $\mathcal{M}$ be an $\mathcal{S}$-module endowed with a good filtration. Then, there is a surjective strict homomorphism $\mathcal{L} \to \mathcal{M}$, where $\mathcal{L}$ is a quasi-free filtered $\mathcal{S}$-module.*

Proposition 1.2.5. *Let $\mathcal{M}$ be an $\mathcal{S}$-module endowed with a good filtration $\{\mathcal{M}_k\}$. Then, $\mathcal{M}$ is a coherent $\mathcal{S}$-module, and*

(iv) $\mathcal{S}_1\mathcal{M}_k = \mathcal{M}_{k+1}$ *for* $k \gg 0$.

Proof. Since $\operatorname{gr}\mathcal{M}$ is an $\overline{\mathcal{S}}$-module of finite type, it follows that $\operatorname{gr}\mathcal{M}$ is generated by $\bigoplus_{k_1 \leq k \leq k_0} \operatorname{gr}_k \mathcal{M}$. Then, clearly, for $k \geq k_0$

$$\mathcal{M}_k = \mathcal{S}_{k-k_0}\mathcal{M}_{k_0} + \mathcal{M}_{k-1}.$$

Therefore,

$$\begin{aligned}
\mathcal{M}_{k+1} &= \mathcal{S}_{k+1-k_0}\mathcal{M}_{k_0} + \mathcal{M}_k \\
&= \mathcal{S}_1\mathcal{S}_{k-k_0}\mathcal{M}_{k_0} + \mathcal{M}_k \\
&= \mathcal{S}_1(\mathcal{S}_{k-k_0}\mathcal{M}_{k_0} + \mathcal{M}_{k-1}) + \mathcal{M}_k \\
&= \mathcal{S}_1\mathcal{M}_k + \mathcal{M}_k \\
&= \mathcal{S}_1\mathcal{M}_k.
\end{aligned}$$

Hence, we get (iv). Furthermore, this shows that $\mathcal{M}$ is of finite type. Let

$$\mathcal{L} \xrightarrow{f} \mathcal{M} \to 0$$

be a strict homomorphism, where $\mathcal{L}$ is a quasi-free filtered $\mathcal{S}$-module. Let $\mathcal{N}$ be the kernel of f, and let $\mathcal{N}_k = \mathcal{N} \cap \mathcal{L}_k$ be the induced filtration. Hence,

$$0 \to \mathcal{N}_k \to \mathcal{L}_k \to \mathcal{M}_k \to 0$$

being an exact sequence, $\mathcal{N}_k$ is a coherent $\mathcal{O}_X$-module. Therefore, according to Proposition 1.1.4, $\mathcal{N}$ is a coherent $\mathcal{S}$-module. It follows that $\mathcal{M}$ is also coherent. □

Proposition 1.2.6. *Let $\mathcal{M}$ be a coherent $\mathcal{S}$-module. A filtration $\{\mathcal{M}_k\}$ is good if (i), (ii), and (iv) hold.*

Proof. Clearly $\operatorname{gr}\mathcal{M}$ is of finite type. It is possible to choose a quasi-free filtered $\mathcal{S}$-module $\mathcal{L}$, and a surjective strict homomorphism

$$\mathcal{L} \xrightarrow{f} \mathcal{M} \to 0.$$

Let $\mathcal{N}$ be its kernel. Set $\mathcal{N}_k = \mathcal{N} \cap \mathcal{L}_k$. Since

$$0 \to \operatorname{gr}\mathcal{N} \to \operatorname{gr}\mathcal{L} \to \operatorname{gr}\mathcal{M} \to 0$$

is exact, it is sufficient to prove that $\operatorname{gr}\mathcal{N}$ is of finite type. However, since the $\mathcal{O}_X$-modules $\operatorname{gr}_k \mathcal{L}$, $\operatorname{gr}_k \mathcal{M}$ are clearly coherent, $\operatorname{gr}_k \mathcal{N}$ is also a coherent $\mathcal{O}_X$-module. Because of Proposition 1.1.4, $\operatorname{gr}\mathcal{N}$ is a coherent $\overline{\mathcal{S}}$-module. □

A reason to introduce the notion of good filtration is the fact that the following proposition holds.

Proposition 1.2.7. *Let*

$$0 \to \mathcal{M}' \xrightarrow{f} \mathcal{M} \xrightarrow{g} \mathcal{M}'' \to 0$$

be an exact sequence of coherent filtered $\mathcal{S}$-modules with strict homomorphisms. Then, the filtration of $\mathcal{M}$ is good if and only if the filtrations of $\mathcal{M}'$ and $\mathcal{M}''$ are also good.

Proof. (1). Assume that $\mathcal{M}'$ and $\mathcal{M}''$ are good filtered modules. Since

$$0 \to \mathcal{M}'_k \to \mathcal{M}_k \to \mathcal{M}''_k \to 0$$

is exact, $\mathcal{M}_k$ is a coherent $\mathcal{O}_X$-module. Furthermore, since

$$0 \to \operatorname{gr} \mathcal{M}' \to \operatorname{gr} \mathcal{M} \to \operatorname{gr} \mathcal{M}'' \to 0$$

is exact, $\operatorname{gr} \mathcal{M}$ is a coherent $\overline{\mathcal{S}}$-module. Since $\mathcal{M}'_k = \mathcal{M}''_k = 0$ for $k \ll 1$, $\mathcal{M}_k = 0$ for $k \ll 0$. Moreover, it is clear that $\mathcal{M} = \bigcup_k \mathcal{M}_k$.

(2). Conversely, assume that $\mathcal{M}$ is a good filtered module. Choose h such that

$$\mathcal{S}^m \xrightarrow{h} \mathcal{M} \to \mathcal{M}'' \to 0$$

is exact. Then, $h^{-1}(\mathcal{M}_k) \cap \mathcal{S}^m_l$ is a coherent $\mathcal{O}_X$-module. Moreover, $h(\mathcal{S}^m_l) \cap \mathcal{M}_k$ is also coherent, and gives an increasing sequence in l. Thanks to the Noetherian property, this sequence is stationary. Hence, $h(\mathcal{S}^m) \cap \mathcal{M}_k$ is also coherent. Therefore, $\mathcal{M}''_k$ is also coherent. Since $\{\mathcal{M}''_k\}$ satisfies (iv), it gives a good filtration. Since

$$0 \to \operatorname{gr} \mathcal{M}' \to \operatorname{gr} \mathcal{M} \to \operatorname{gr} \mathcal{M}'' \to 0$$

is exact, $\{\mathcal{M}'_k\}$ satisfies (iii), and is also a good filtration. □

Corollary 1.2.8. *Locally, any coherent $\mathcal{S}$-module admits good filtrations.*

Proposition 1.2.9. *Let $\mathcal{M} = \bigcup_k \mathcal{M}_k$ be a coherent $\mathcal{S}$-module endowed with a good filtration. Let $\mathcal{N} \subset \mathcal{M}$ be a sub $\mathcal{S}$-module, such that $\mathcal{N}_k = \mathcal{N} \cap \mathcal{M}_k$ is a coherent $\mathcal{O}_X$-module. Then, $\mathcal{N}$ is a coherent $\mathcal{S}$-module.*

Proof. Let $\mathcal{L} = \bigoplus_{j=1}^r \mathcal{S}(l_j) \xrightarrow{f} \mathcal{M} \to 0$ be a strict surjective homomorphism. Set $\mathcal{N}' = f^{-1}(\mathcal{N})$. Clearly, $\mathcal{N}'_k = \mathcal{N}' \cap \mathcal{L}_k$ is a coherent $\mathcal{O}_X$-module. Moreover, thanks to Proposition 1.1.4, $\mathcal{N}'$ is a coherent $\mathcal{S}$-module. Since $\mathcal{N}$ is the image of $\mathcal{N}' \to \mathcal{M}$, it is a coherent $\mathcal{S}$-module. □

Corollary 1.2.10. *Let $\mathcal{M}$ be a coherent $\mathcal{S}$-module, and let $\{\mathcal{M}_\lambda\}$ be a directed family of coherent sub $\mathcal{O}_X$-modules of $\mathcal{M}$, such that $\mathcal{M} = \bigcup_\lambda \mathcal{M}_\lambda$. If $\mathcal{N}$ is a sub $\mathcal{S}$-module of $\mathcal{M}$ such that $\mathcal{N} \cap \mathcal{M}_\lambda$ is a coherent $\mathcal{O}_X$-module, then $\mathcal{N}$ is a coherent $\mathcal{S}$-module. Moreover, if $\mathcal{N}'$ is a quotient $\mathcal{S}$-module of $\mathcal{M}$, and the image $\mathcal{N}'_\lambda$ of $\mathcal{M}_\lambda$ in $\mathcal{N}'$ is a coherent $\mathcal{O}_X$-module, then $\mathcal{N}'$ is a coherent $\mathcal{S}$-module.*

Proof. Let $\mathcal{M} = \bigcup_k \mathcal{M}'_k$ be a good filtration of $\mathcal{M}$. For any $\mathcal{M}'_k$, there exists λ such that $\mathcal{M}'_k \subset \mathcal{M}_\lambda$. Since $\mathcal{N} \cap \mathcal{M}_\lambda$ is a coherent $\mathcal{O}_X$-module, $\mathcal{N} \cap \mathcal{M}'_k$ is also coherent. Therefore, $\mathcal{N}$ is a coherent $\mathcal{S}$-module. As for $\mathcal{N}'$, if we call $\mathcal{N}$ the kernel of $\mathcal{M} \to \mathcal{N}'$, then $\mathcal{N} \cap \mathcal{M}_\lambda$ is coherent. Since $\mathcal{N}$ is a coherent $\mathcal{S}$-module, $\mathcal{N}'$ is also coherent. □

Proposition 1.2.11. *Let $\mathcal{L} \to \mathcal{M} \to \mathcal{N}$ be an exact sequence of coherent $\mathcal{S}$-modules endowed with good filtrations, and assume the morphisms are filtered. If*

$$\mathrm{gr}_k \mathcal{L} \to \mathrm{gr}_k \mathcal{M} \to \mathrm{gr}_k \mathcal{N}$$

is exact for sufficiently large k, then

$$\mathcal{L}_k \to \mathcal{M}_k \to \mathcal{N}_k$$

is also exact for sufficiently large k.

Proof. Let $\mathcal{H}_k$ be the cohomology of $\mathcal{L}_k \to \mathcal{M}_k \to \mathcal{N}_k$. $\mathcal{H}_k$ is a coherent $\mathcal{O}_X$-module. Since the columns in

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{L}_{k-1} & \longrightarrow & \mathcal{M}_{k-1} & \longrightarrow & \mathcal{N}_{k-1} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{L}_k & \longrightarrow & \mathcal{M}_k & \longrightarrow & \mathcal{N}_k \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{gr}_k \mathcal{L} & \longrightarrow & \mathrm{gr}_k \mathcal{M} & \longrightarrow & \mathrm{gr}_k \mathcal{N} \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

are exact,

$$\mathcal{H}_{k-1} \to \mathcal{H}_k \to \mathcal{G}_k$$

is exact, where $\mathcal{G}_k$ denotes the cohomology of

$$\mathrm{gr}_k \mathcal{L} \to \mathrm{gr}_k \mathcal{M} \to \mathrm{gr}_k \mathcal{N}.$$

Therefore, for sufficiently large k, $\mathcal{H}_{k-1} \to \mathcal{H}_k$ is surjective. Hence, due to the Noetherian property, $\mathcal{H}_{k-1} \to \mathcal{H}_k$ is an isomorphism for $k \gg 0$. It follows that for $k \gg 0$,

$$\mathcal{H}_k = \varinjlim_{\nu} \mathcal{H}_\nu = (\text{cohomology of } \mathcal{L} \to \mathcal{M} \to \mathcal{N}) = 0.$$

□

Let us explain E. Cartan's "prolongation" idea in our framework. Let $\mathcal{M}$ be a system, and let

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}_X^{r_0} \overset{P}{\leftarrow} \mathcal{D}_X^{r_1}$$

be one of its presentations. Note that, in general, this presentation is not strict. Then, the idea of "prolongation" is to use the fact that we can replace the preceding sequence by a strict one

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}_X^{l_0} \leftarrow \mathcal{D}_X^{l_1}.$$

1.3 Characteristic variety

Recently, in the framework of the theory of hyperfunctions, both the sheaf of Sato's microfunctions $\mathcal{C}_M$ and the sheaf of pseudo-differential operators were constructed on the cotangent sphere bundle S^*M of a real analytic manifold M. In this section, we will mimic this construction for the case of $\mathcal{D}_X$. The cotangent projective bundle P^*X on X will replace S^*M. Similarly, the role of the sheaf of Sato's hyperfunctions $\mathcal{B}_M$ will be played by $\mathcal{D}_X$, and that of $\mathcal{C}_M$ by $\widetilde{\mathcal{D}}_X$.

In fact, $\widetilde{\mathcal{D}}_X$ should be called the sheaf of algebraic pseudo-differential operators. Holomorphic pseudo-differential operators were introduced in Kashiwara-Kawai [10] under the name of pseudo-differential operators of finite type[1].

Intuitively, the characteristic variety of a system $\mathcal{M}$ will be a generalization of the zero-set of the principal symbol of a differential operator. The multiplicities will give other important invariants.

Let $\mathcal{S}$ be an $\mathcal{O}_X$-algebra satisfying conditions (1), (2), (3), (4) of § 1.1. Then, since $\overline{\mathcal{S}}$ is an $\mathcal{O}_X$-algebra of finite presentation, it is possible to construct $\operatorname{Proj}(\overline{\mathcal{S}})$. Let us set $X^* = \operatorname{Proj}(\overline{\mathcal{S}})$. For a graded $\overline{\mathcal{S}}$-module $\mathcal{G}$, we can construct the $\mathcal{O}_{X^*}$-module $\widetilde{\mathcal{G}}$. Let $\pi : X^* \to X$ be the canonical projection.

Theorem 1.3.1. *Let $\mathcal{M}$ be a coherent $\mathcal{S}$-module. Then, $\operatorname{supp}(\widetilde{\operatorname{gr}\mathcal{M}})$ does not depend on the chosen good filtration.*

Proof. Let $\{\mathcal{M}_k\}$, $\{\mathcal{M}'_k\}$ be two good filtrations. Let $\operatorname{gr}\mathcal{M}$, $\operatorname{gr}\mathcal{M}'$ be the associated graded modules. Since $\operatorname{supp}(\widetilde{\operatorname{gr}\mathcal{M}})$ is invariant by a shift of the degree of the filtration, we may assume that $\mathcal{M}_k \supset \mathcal{M}'_k$ (for $k \gg 0$). Moreover, there is N such that $\mathcal{M}'_{k+N} \supset \mathcal{M}_k$ (for $k \gg 0$). Let us proceed by induction on N. If $N = 1$, $\mathcal{M}_k \supset \mathcal{M}'_k \supset \mathcal{M}_{k-1} \supset \mathcal{M}'_{k-1}$. We get the exact sequences

$$0 \to \mathcal{M}'_k/\mathcal{M}_{k-1} \to \mathcal{M}_k/\mathcal{M}_{k-1} \to \mathcal{M}_k/\mathcal{M}'_k \to 0,$$

$$0 \to \mathcal{M}_{k-1}/\mathcal{M}'_{k-1} \to \mathcal{M}'_k/\mathcal{M}'_{k-1} \to \mathcal{M}'_k/\mathcal{M}_{k-1} \to 0.$$

Set

$$\mathcal{F} = \bigoplus_k (\mathcal{M}_k/\mathcal{M}'_k), \qquad \mathcal{L} = \bigoplus_k (\mathcal{M}'_k/\mathcal{M}_{k-1}).$$

From the two exact sequences:

$$0 \to \mathcal{L} \to \operatorname{gr}\mathcal{M} \to \mathcal{F} \to 0,$$
$$0 \to \mathcal{F}(-1) \to \operatorname{gr}'\mathcal{M} \to \mathcal{L} \to 0,$$

[1] *Translators' note:* See also [L. Boutet de Monvel, P. Kree, *Pseudo-differential operators and Gevrey classes*, Ann. Inst. Fourier (Grenoble) **17** (1967), 295–323].

we see that $\mathcal{F}$, $\mathcal{L}$ are coherent $\overline{\mathcal{S}}$-modules. Moreover, the sequences:

$$0 \to \widetilde{\mathcal{L}} \to \widetilde{\operatorname{gr}\mathcal{M}} \to \widetilde{\mathcal{F}} \to 0,$$
$$0 \to \widetilde{\mathcal{F}} \underset{\mathcal{O}_{X^*}}{\otimes} \mathcal{O}_{X^*}(-1) \to \widetilde{\operatorname{gr}'\mathcal{M}} \to \widetilde{\mathcal{L}},$$

are exact. Therefore,

$$\begin{aligned}\operatorname{supp}(\widetilde{\operatorname{gr}\mathcal{M}}) &= \operatorname{supp}(\widetilde{\mathcal{L}}) \cup \operatorname{supp}(\widetilde{\mathcal{F}}) \\ &= \operatorname{supp}(\widetilde{\mathcal{L}}) \cup \operatorname{supp}(\widetilde{\mathcal{F}} \underset{\mathcal{O}_{X^*}}{\otimes} \mathcal{O}_{X^*}(-1)) \\ &= \operatorname{supp}(\widetilde{\operatorname{gr}'\mathcal{M}}).\end{aligned}$$

Assume now that $N > 1$. Set $\mathcal{M}''_k = \mathcal{M}_{k-1} + \mathcal{M}'_k$. $\mathcal{M}''_k$ is a good filtration of $\mathcal{M}$ (it satisfies (i), (ii), (iv) of § 1.2). Since

$$\mathcal{M}_k \supset \mathcal{M}''_k \supset \mathcal{M}_{k-1},$$
$$\mathcal{M}''_k \supset \mathcal{M}'_k \supset \mathcal{M}''_{k-N+1},$$

the induction hypothesis shows that

$$\operatorname{supp}(\widetilde{\operatorname{gr}\mathcal{M}}) = \operatorname{supp}(\widetilde{\operatorname{gr}''\mathcal{M}}) = \operatorname{supp}(\widetilde{\operatorname{gr}'\mathcal{M}}),$$

where we denote by $\operatorname{gr}''\mathcal{M}$ the graded module associated to $\mathcal{M}''_k$. □

Definition 1.3.2. Let $\mathcal{M}$ be a coherent $\mathcal{S}$-module. Then, for a good filtration of $\mathcal{M}$,

$$\operatorname{supp}(\widetilde{\operatorname{gr}\mathcal{M}}) \subset X^*$$

is the *characteristic variety*[2] of $\mathcal{M}$. We denote it by $\operatorname{char}(\mathcal{M})$.

Lemma 1.3.3. *Let*

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

be an exact sequence of coherent $\mathcal{S}$-modules. Then,

$$\operatorname{char}(\mathcal{M}) = \operatorname{char}(\mathcal{M}') \cup \operatorname{char}(\mathcal{M}'').$$

Proof. Let us choose a good filtration on $\mathcal{M}$, and endow $\mathcal{M}'$, $\mathcal{M}''$ with the induced good filtrations, so that

$$0 \to \operatorname{gr}\mathcal{M}' \to \operatorname{gr}\mathcal{M} \to \operatorname{gr}\mathcal{M}'' \to 0$$

is exact. Then, $0 \to \widetilde{\operatorname{gr}\mathcal{M}'} \to \widetilde{\operatorname{gr}\mathcal{M}} \to \widetilde{\operatorname{gr}\mathcal{M}''} \to 0$ is also exact, and

$$\begin{aligned}\operatorname{char}(\mathcal{M}) &= \operatorname{supp}(\widetilde{\operatorname{gr}\mathcal{M}}) \\ &= \operatorname{supp}(\widetilde{\operatorname{gr}\mathcal{M}'}) \cup \operatorname{supp}(\widetilde{\operatorname{gr}\mathcal{M}''}) \\ &= \operatorname{char}(\mathcal{M}') \cup \operatorname{char}(\mathcal{M}'').\end{aligned}$$

□

[2] *Translators' note:* In today's literature, the characteristic variety of a $\mathcal{D}_X$-module is usually defined in the cotangent bundle instead of the projective cotangent bundle.

Proposition 1.3.4. *Let s be a section of $\mathcal{S}_m$, and denote by $\overline{s}$ the associated section of $\overline{\mathcal{S}}_m$. Assume that $\overline{s}$ is $\mathcal{O}_{X^*}$-regular on $U \subset X^*$ (i.e., $\mathcal{O}_{X^*}|_U \xrightarrow{\overline{s}} \mathcal{O}_{X^*}(m)|_U$ is injective). Then, if we set $\mathcal{M} = \mathcal{S}/\mathcal{S}s$, $U \cap \operatorname{char}(\mathcal{M}) = \{x^* \in U : \overline{s}(x) = 0\}$. Furthermore, if we endow $\mathcal{M}$ with the good filtration $\mathcal{M}_k = \mathcal{S}_k/(\mathcal{S}_k \cap \mathcal{S}s)$, we get*

$$\widetilde{\operatorname{gr}\mathcal{M}}|_U = (\mathcal{O}_{X^*}/\overline{s}\,\mathcal{O}_{X^*}(-m))|_U.$$

Proof. It is sufficient to prove that

$$\begin{array}{ccccc} \widetilde{\operatorname{gr}\mathcal{M}} & \longleftarrow & \widetilde{\operatorname{gr}\mathcal{S}} & \xleftarrow{\;\overline{s}\;} & \widetilde{\operatorname{gr}\mathcal{S}}(-m) \\ & & \wr\downarrow & & \wr\downarrow \\ & & \mathcal{O}_{X^*} & & \mathcal{O}_{X^*}(-m) \end{array}$$

is exact at $x^* \in U$. For $k \gg 0$, if

$$t \in \mathcal{S}_k \quad \text{and} \quad t \in \mathcal{S}_{k-1} + \mathcal{S}s,$$

we must show that there is $\alpha \in \mathcal{S}_\ell$ with $\overline{\alpha}(x^*) \neq 0$ ($\overline{\alpha} \in \overline{\mathcal{S}}_\ell$ is the image of $\alpha \in \mathcal{S}_\ell$), such that

$$\alpha t \in \mathcal{S}_{k+\ell-1} + \mathcal{S}_{k+\ell-m}s.$$

There is $\beta \in \mathcal{S}_{k+\ell}$ such that $t - \beta s \in \mathcal{S}_{k-1}$, for some $\ell \gg 0$. We will show by decreasing induction on ν, that there are $\alpha_\nu \in \mathcal{S}_{k_\nu}$, $\gamma_\nu \in \mathcal{S}_{k+k_\nu-m+\nu}$ ($\nu \geq 0$), such that $\alpha_\nu t - \gamma_\nu s \in \mathcal{S}_{k+k_\nu-1}$, $\overline{\alpha_\nu}(x^*) \neq 0$. This is clear for $\nu \geq \ell$. For $\nu < \ell$, there exists $\alpha_{\nu+1}$, $\gamma_{\nu+1}$ such that

$$\alpha_{\nu+1}t - \gamma_{\nu+1}s \in \mathcal{S}_{k+k_{\nu+1}-1}.$$

Therefore, $\overline{\gamma}_{\nu+1}\overline{s} = 0$ in $\overline{\mathcal{S}}_{k+k_{\nu+1}+\nu+1}$, and since $\overline{s}$ is $\mathcal{O}_{X^*}$-regular, there is $\delta \in \mathcal{S}_n$ such that $\overline{\delta}(x^*) \neq 0$, $\overline{\delta}\,\overline{\gamma}_{\nu+1} = 0$. If we set $\alpha_\nu = \delta\alpha_{\nu+1}$, $\gamma_\nu = \delta\gamma_{\nu+1}$, $k_\nu = k_{\nu+1} + n$, then $\gamma_\nu \in \mathcal{S}_{k+k_\nu-m+\nu}$, and

$$\alpha_\nu t - \gamma_\nu s \in \mathcal{S}_{k+k_\nu-1}.$$

Finally, we get $\alpha_0 \in \mathcal{S}_{k_0}$, $\gamma_0 \in \mathcal{S}_{k+k_0-m}$ such that

$$\alpha_0 t - \gamma_0 s \in \mathcal{S}_{k+k_0-1}, \qquad \overline{\alpha_0}(x^*) \neq 0.$$

□

1.4 Sheaf of pseudo-differential operators

We have defined the characteristic variety of an $\mathcal{S}$-module, but this definition using good filtrations is not direct. Hence, we define a sheaf of rings $\widetilde{\mathcal{S}}$ on X^*, such that the support of $\widetilde{\mathcal{S}} \otimes_{\pi^{-1}\mathcal{S}} \pi^{-1}\mathcal{M}$ is the characteristic variety of $\mathcal{M}$. In the case of $\mathcal{S} = \mathcal{D}_X$, we recover the sheaf of pseudo-differential operators. Before we can give the definition, we need a few lemmas.

Lemma 1.4.1. *Let $x^* \in X^*$, $x = \pi(x^*) \in X$, t be a section of $\mathcal{S}_m$, and $\bar{t}$ the associated section of $\overline{\mathcal{S}}_m$. Assume that $\bar{t}(x^*) \neq 0$. Then, for any section $u \in \mathcal{S}_x$, there exists $v \in \mathcal{S}_x$, and $s \in \mathcal{S}_{\ell,x}$ such that conditions (a), (b) below hold:*

(a) $vt = su$,

(b) if we denote by $\bar{s}$ the image of s in $\overline{\mathcal{S}}_{\ell,x}$, then $\bar{s}(x^*) \neq 0$.

Proof. Set

$$\mathcal{M} = \mathcal{S}/\mathcal{S}t, \qquad \mathcal{N} = (\mathcal{S}u + \mathcal{S}t)/\mathcal{S}t \subset \mathcal{M}.$$

If we endow $\mathcal{M}$ with the good filtration $\mathcal{M}_k = \mathcal{S}_k/(\mathcal{S}_k \cap \mathcal{S}t)$, then, since

$$0 \leftarrow \operatorname{gr} \mathcal{M} \leftarrow \overline{\mathcal{S}}/\overline{\mathcal{S}}\,\bar{t}$$

is an exact sequence,

$$\operatorname{char}(\mathcal{M}) = \operatorname{supp}(\widetilde{\operatorname{gr} \mathcal{M}}) \subset \operatorname{supp}(\widetilde{\overline{\mathcal{S}}/\overline{\mathcal{S}}\,\bar{t}})$$

and $\operatorname{char}(\mathcal{M}) \not\ni x^*$. Hence $\operatorname{char}(\mathcal{N}) \not\ni x^*$. Endow $\mathcal{N}$ with the good filtration

$$\mathcal{N}_k = (\mathcal{S}_k u + \mathcal{S}t)/\mathcal{S}t.$$

Denote by $\bar{u}$ the section of $\operatorname{gr}_0 \mathcal{N}$ associated to u. Since $\operatorname{supp}(\widetilde{\operatorname{gr} \mathcal{N}}) \not\ni x^*$, there is $\bar{s} \in \overline{\mathcal{S}}_{\ell,x}$ such that:

$$\begin{cases} \bar{s}\,\bar{u} = 0 & \text{in } \operatorname{gr}_\ell \mathcal{N}, \\ \bar{s}(x^*) \neq 0. \end{cases}$$

If $s \in \mathcal{S}_{\ell,x}$ is a representative of $\bar{s}$, then

$$su \in \mathcal{S}_{\ell-1}u + \mathcal{S}t.$$

Furthermore, there are $s' \in \mathcal{S}_{\ell-1}$, $v \in \mathcal{S}$ such that

$$(s - s')u = vt.$$

Note that $s - s' \in \mathcal{S}_\ell$ and that its image in $\overline{\mathcal{S}}_\ell$ is $\bar{s}$. Since $\bar{s}(x^*) \neq 0$, the proof is complete. □

Lemma 1.4.2. *Let $x^* \in X^*$, $x = \pi(x^*) \in X$. Assume $u \in \mathcal{S}_x$, $t \in \mathcal{S}_{m,x}$ and $ut = 0$. Denote by $\bar{t}$ the section of $\overline{\mathcal{S}}_m$ associated with t, and assume that $\bar{t}(x^*) \neq 0$. Then, there is $s \in \mathcal{S}_{\ell,x}$ such that $\bar{s}(x^*) \neq 0$ and $su = 0$.*

Proof. Set

$$\mathcal{I} = \mathcal{A}nn(t) = \{v \in \mathcal{S} : vt = 0\} \subset \mathcal{S}.$$

Let us first show that $\operatorname{char}(\mathcal{I}) \not\ni x^*$. Let us endow $\mathcal{I}$ with the good filtration $\mathcal{I}_k = \mathcal{S}_k \cap \mathcal{I}$. Since $\mathcal{I}\,t = 0$,

$$(\operatorname{gr} \mathcal{I})\,\bar{t} = 0,$$

and $\operatorname{supp}\widetilde{\operatorname{gr}\mathcal{I}} \not\ni x^*$. So, $\operatorname{char}(\mathcal{I}) \not\ni x^*$.

Since $\mathcal{S}u \subset \mathcal{I}$,

$$\operatorname{char}(\mathcal{S}u) \not\ni x^*.$$

Set $\mathcal{M} = \mathcal{S}u$ and endow it with the filtration $\mathcal{M}_k = \mathcal{S}_k u$. Let $\overline{u} \in \operatorname{gr}_0 \mathcal{M}$ be the element associated to $u \in \mathcal{M}_0$. Since

$$\operatorname{supp}(\widetilde{\operatorname{gr}\mathcal{M}}) \not\ni x^*,$$

there is $s \in \mathcal{S}_{\ell,x}$ such that

$$\begin{cases} \overline{s}\,\overline{u} = 0 & \text{in } \operatorname{gr}_\ell \mathcal{M}, \\ \overline{s}(x^*) \neq 0. \end{cases}$$

Hence,

$$su \in \mathcal{S}_{\ell-1}u.$$

So, $su = s'u$ with $s' \in \mathcal{S}_{\ell-1}$, and $(s - s')u = 0$. Since the section of $\overline{\mathcal{S}}_\ell$ associated to $s - s' \in \mathcal{S}_\ell$ is $\overline{s}$, and $\overline{s}(x^*) \neq 0$, the proof is complete. □

The two preceding lemmas and the following general considerations will allow us to define the sheaf $\widetilde{\mathcal{S}}$ on X^*.

Definition 1.4.3. Let A be a (not necessarily commutative) ring with a unit element. Let S be a subset of A satisfying conditions (i)–(iv) below.

(i) $S \ni 1$

(ii) $S \ni s, t \Rightarrow S \ni st$

(iii) $S \ni s$, $A \ni a$ implies that there are $b \in A$ and $t \in S$ such that $ta = bs$.

(iv) $a \in A$, $s \in S$, $as = 0$, implies that there is $t \in S$ such that $ta = 0$.

Then, for any left A-module M, we may define the quotient by S, $S^{-1}M$. We obtain $S^{-1}M$ as the quotient of the set of pairs $(s, x) \in S \times M$, by the equivalence relation defined by

$$(s_1, x_1) \sim (s_2, x_2)$$
$$\Leftrightarrow \exists s_1', s_2' \in S \text{ such that } s_1's_1 = s_2's_2 \text{ and } s_1'x_1 = s_2'x_2.$$

The element of $S^{-1}M$ associated to (s, x) is denoted by $s^{-1}x$. Clearly, $S^{-1}M$ has a canonical A-module structure. Moreover, $S^{-1}A$ has a canonical ring structure. For this structure, the map $\varphi : A \to S^{-1}A$ is a ring homomorphism, and the elements of $\varphi(S)$ are invertible in $S^{-1}A$. We have

$$S^{-1}M = (S^{-1}A) \underset{A}{\otimes} M,$$

and $S^{-1}A$ is a flat right A-module. If, furthermore, S satisfies

(iii)' $S \ni s$, $A \ni a$ implies that there is $b \in A$, $t \in S$ such that $at = sb$, (iv)' $a \in A$, $s \in S$, $sa = 0$ implies that there are $t \in S$ such that $at = 0$,

then $S^{-1}A = AS^{-1}$.

Let $(A_\lambda, S_\lambda)_\lambda$ be an inductive system, and assume that $S_\lambda \subset A_\lambda$ satisfies (i), (ii), (iii) and (iv). Then, if we set

$$A = \varinjlim A_\lambda, \qquad S = \varinjlim S_\lambda,$$

we get $\varinjlim S_\lambda^{-1} A_\lambda = S^{-1}A$.

Now, let $\mathcal{A}$ be a sheaf of rings over a topological space X. Assume that $\mathcal{V} \subset \mathcal{A}$, and that the stalk $\mathcal{V}_x \subset \mathcal{A}_x$ satisfies (i), (ii), (iii) and (iv). Then, it is possible to construct a sheaf of rings

$$\mathcal{V}^{-1}\mathcal{A},$$

such that

$$(\mathcal{V}^{-1}\mathcal{A})_x = \mathcal{V}_x^{-1}\mathcal{A}_x.$$

For any $\mathcal{A}$-module $\mathcal{M}$, we may also construct

$$\mathcal{V}^{-1}\mathcal{M} = \mathcal{V}^{-1}\mathcal{A} \underset{\mathcal{A}}{\otimes} \mathcal{M}.$$

Let us go back to the preceding situation. Let us define the subsheaf $\mathcal{V}$ of the sheaf of rings $\pi^{-1}\mathcal{S}$ on X^*, by

$$\mathcal{V} = \{s \in \pi^{-1}\mathcal{S} : \overline{s} \neq 0\}.$$

Thanks to Lemmas 1.4.1, 1.4.2, for any x^*, $\mathcal{V}_{x^*} \subset \mathcal{S}_{x^*}$ satisfy (i), (ii), (iii) and (iv). We set $\widetilde{\mathcal{S}} = \mathcal{V}^{-1}\pi^{-1}\mathcal{S}$. $\widetilde{\mathcal{S}}$ is a sheaf of rings on X^*. For every $\mathcal{S}$-module $\mathcal{M}$, we set

$$\widetilde{\mathcal{M}} = \mathcal{V}^{-1}\pi^{-1}\mathcal{M} = \widetilde{\mathcal{S}} \underset{\pi^{-1}\mathcal{S}}{\otimes} \pi^{-1}\mathcal{M}.$$

Proposition 1.4.4. *Let $\mathcal{M}$ be a coherent $\mathcal{S}$-module. Then,*

$$\mathrm{supp}(\widetilde{\mathcal{M}}) = \mathrm{char}(\mathcal{M}).$$

Proof. Assume $x^* \notin \mathrm{supp}(\widetilde{\mathcal{M}})$. Let us write $\mathcal{M}$ as $\sum_{j=1}^{\ell} \mathcal{S}u_j$. Since $\widetilde{\mathcal{M}}_{x^*} = \mathcal{V}_{x^*}^{-1}\mathcal{M}_x = 0$, there are $s_j \in \mathcal{V}_{x^*}$ such that $s_j u_j = 0$. Therefore, we get the epimorphism $0 \leftarrow \mathcal{M} \leftarrow \bigoplus_j \mathcal{S}/\mathcal{S}s_j$. Hence, $\mathrm{char}(\mathcal{M}) \subset \bigcup_j \mathrm{char}(\mathcal{S}/\mathcal{S}s_j)$. Moreover, following Proposition 1.3.4, $\mathrm{char}(\mathcal{S}/\mathcal{S}s_j) \not\ni x^*$, so $x^* \notin \mathrm{char}(\mathcal{M})$. Conversely, assume $x^* \notin \mathrm{char}(\mathcal{M})$. We have the epimorphism

$$0 \leftarrow \mathcal{M} \leftarrow \bigoplus_j \mathcal{S}u_j,$$

and the inclusions

$$\mathrm{char}(\mathcal{M}) \subset \bigcup_j \mathrm{char}(\mathcal{S}u_j) \subset \mathrm{char}(\mathcal{M}).$$

Since $\operatorname{supp}(\widetilde{\mathcal{M}}) = \bigcup_j \operatorname{supp}(\widetilde{\mathcal{S}u_j})$, we may assume

$$\mathcal{M} = \mathcal{S}u.$$

Endow $\mathcal{M}$ with the good filtration

$$\mathcal{M}_k = \mathcal{S}_k u.$$

We know that $\operatorname{char}(\mathcal{M}) = \operatorname{supp}(\widetilde{\operatorname{gr}\mathcal{M}}) \not\ni x^*$, and if $\overline{u} \in \operatorname{gr}_0 \mathcal{M}$ denotes the image of $u \in \mathcal{M}_0$, there is $\overline{s} \in \overline{\mathcal{S}}_k$ such that

$$\begin{cases} \overline{s}\,\overline{u} = 0 & \text{in } \operatorname{gr}_k \mathcal{M}, \\ \overline{s}(x^*) \neq 0. \end{cases}$$

Therefore, if $s \in \mathcal{S}_k$ is a representative of $\overline{s}$, we have

$$su \in \mathcal{S}_{k-1}u.$$

Hence, $su = s'u$ with $s' \in \mathcal{S}_{k-1}$, and $s - s' \in \mathcal{V}_{x^*}$ is such that $(s - s')u = 0$. Therefore, $\widetilde{\mathcal{M}}_{x^*} = \mathcal{V}_{x^*}^{-1}\mathcal{M}_x = 0$. So, $x^* \notin \operatorname{supp}\widetilde{\mathcal{M}}$. □

Definition 1.4.5. Let X be an analytic space, $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, $S = \operatorname{supp}(\mathcal{F})$, A an irreducible subset of X, and assume $S \subset A$. Take any $x \in A$, set $Y = \operatorname{Spec}(\mathcal{O}_{X,x})$, and denote by ξ the point of Y corresponding to A. Let $\widetilde{\mathcal{F}}_x$ be the coherent $\mathcal{O}_Y$-module corresponding to $\mathcal{F}_x$. Then, $\operatorname{length}_{\mathcal{O}_{Y,\xi}}(\widetilde{\mathcal{F}}_x)_\xi < \infty$ does not depend on x. We call this number the multiplicity of $\mathcal{F}$ along A.

Multiplicity is additive. Namely, let

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be an exact sequence, such that $\operatorname{supp}\mathcal{F} \subset A$. Then, the multiplicity of $\mathcal{F}$ along A is the sum of the corresponding multiplicities of $\mathcal{F}'$ and $\mathcal{F}''$.

Theorem 1.4.6. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, and A be an irreducible analytic subset of X^*, such that $A \supset \operatorname{char}(\mathcal{M})$. Then, the multiplicity along A of $\widetilde{\operatorname{gr}\mathcal{M}}$ does not depend on the good filtration $\mathcal{M} = \bigcup_k \mathcal{M}_k$ of $\mathcal{M}$.*

Proof. Taking advantage of the additivity of the multiplicity, we may work as in Theorem 1.3.1. Therefore, we will omit the details. □

Definition 1.4.7. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, let A be an irreducible analytic subset of X^*, such that $A \supset \operatorname{char}(\mathcal{M})$. Then, we define the multiplicity of $\mathcal{M}$ along A as the multiplicity of $\widetilde{\operatorname{gr}\mathcal{M}}$ along A.

Proposition 1.4.8. *The multiplicity of coherent $\mathcal{D}_X$-modules is additive. Namely, let A be an irreducible analytic subset of X^*, and assume that*

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

is an exact sequence of coherent $\mathcal{D}_X$-modules such that $\operatorname{char}(\mathcal{M}) \subset A$. Then, the multiplicity of $\mathcal{M}$ along A is the sum of the ones of $\mathcal{M}'$ and $\mathcal{M}''$.

Proof. Let $\mathcal{M} = \bigcup_k \mathcal{M}_k$ be a good filtration of $\mathcal{M}$, and endow $\mathcal{M}'$, $\mathcal{M}''$ with the induced good filtrations. Then, the sequence

$$0 \to \operatorname{gr}\mathcal{M}' \to \operatorname{gr}\mathcal{M} \to \operatorname{gr}\mathcal{M}'' \to 0$$

is exact, and the multiplicity of $\widetilde{\operatorname{gr}\mathcal{M}}$ along A is the sum of the ones of $\widetilde{\operatorname{gr}\mathcal{M}'}$ and $\widetilde{\operatorname{gr}\mathcal{M}''}$. □

1.5 Torsion modules

In this section, we set $\mathcal{S} = \mathcal{D} = \mathcal{D}_X$, hence $X^* = P^*X$.

Lemma 1.5.1. *Let $x \in X$. The set $\Sigma = \{P \in \mathcal{D}_x : P \neq 0\}$ satisfies conditions (i)–(iv), (iii)' and (iv)' of Definition 1.4.3.*

Proof. (i), (ii), (iv) and (iv)' are clear. Let us deal with (iii) and (iii)'. It is sufficient to prove (iii). Let $P \in S$, $Q \in \mathcal{D}_x$. We may assume that $Q, P \in \Gamma(X; \mathcal{D})$. Set

$$\mathcal{I} = \{R \in \mathcal{D} : RQ \in \mathcal{D}P\}.$$

$\mathcal{I}$ is a coherent ideal of $\mathcal{D}$. Let P_m denote the principal symbol of P. For any x' such that $P_m(x', \eta') \not\equiv 0$, Lemma 1.4.1 shows that $\mathcal{I}_{x'} \neq 0$. Hence, $\mathcal{I}_x \neq 0$. In fact, if $\mathcal{I}_x = 0$, there is a neighborhood of x for the points x' of which $\mathcal{I}_{x'} = 0$, and we obtain a contradiction. □

Therefore, it is possible to define

$$\widetilde{\widetilde{\mathcal{D}_x}} = \Sigma^{-1}\mathcal{D}_x = \mathcal{D}_x\Sigma^{-1}.$$

So, it is clearly possible to construct a sheaf of rings on X with $\widetilde{\widetilde{\mathcal{D}_x}}$ (which is non-commutative) as stalks.

Definition 1.5.2. Let $\mathcal{M}$ be a system (see Definition 1.1.1). The torsion part of $\mathcal{M}$ is $\{u \in \mathcal{M} : Pu = 0 \text{ for some } P \neq 0 \text{ in } \mathcal{D}\}$. When $\mathcal{M}$ coincides with its torsion part, we call it a torsion module. When the torsion part of $\mathcal{M}$ is 0, we say that $\mathcal{M}$ is a torsion-free module.

Since the torsion part of $\mathcal{M}$ is the kernel of

$$\mathcal{M} \to \widetilde{\widetilde{\mathcal{D}}} \underset{\mathcal{D}}{\otimes} \mathcal{M},$$

$\mathcal{M}$ is torsion-free when $\mathcal{M} \subset \widetilde{\widetilde{\mathcal{D}}} \otimes_{\mathcal{D}} \mathcal{M}$, and $\mathcal{M}$ is a torsion module when $\widetilde{\widetilde{\mathcal{D}}} \otimes_{\mathcal{D}} \mathcal{M} = 0$.

Proposition 1.5.3. *Let $\mathcal{M}$ be a system. Then, the following conditions are equivalent.*

(a) $\mathcal{M}$ is a torsion-free module,

(b) locally, there is an injection $0 \to \mathcal{M} \to \mathcal{D}^{\ell}$ of $\mathcal{M}$ into a free module,

(c) $\mathcal{M} \to \mathcal{H}om_{\mathcal{D}}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}), \mathcal{D})$ is injective.

Proof. (b) $\Rightarrow$ (c). Let $0 \to \mathcal{M} \to \mathcal{L} = \mathcal{D}^{\ell}$. The result follows clearly from the diagram

$$\begin{tikzcd}
0 \arrow[d] & \\
\mathcal{M} \arrow[r] \arrow[d] & \mathcal{H}om_{\mathcal{D}}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}), \mathcal{D}) \arrow[d] \\
\mathcal{L} \arrow[r, "\sim"] & \mathcal{H}om_{\mathcal{D}}(\mathcal{H}om_{\mathcal{D}}(\mathcal{L}, \mathcal{D}), \mathcal{D})
\end{tikzcd}$$

(c) $\Rightarrow$ (b). Let us choose a free $\mathcal{D}$-module $\mathcal{L}$, and an epimorphism

$$0 \leftarrow \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) \leftarrow \mathcal{L}.$$

We get the exact sequences

$$\begin{aligned}
&0 \to \mathcal{H}om_{\mathcal{D}}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}), \mathcal{D}) \to \mathcal{H}om_{\mathcal{D}}(\mathcal{L}, \mathcal{D}), \\
&0 \to \mathcal{M} \to \mathcal{H}om_{\mathcal{D}}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}), \mathcal{D}).
\end{aligned}$$

Hence, we get

$$0 \to \mathcal{M} \to \mathcal{H}om_{\mathcal{D}}(\mathcal{L}, \mathcal{D}).$$

(b) $\Rightarrow$ (a). A submodule of a torsion-free module is also torsion-free.

(a) $\Rightarrow$ (b). Let $\mathcal{M}$ be a torsion-free module. Fix $x \in X$, and consider $\widetilde{\widetilde{\widetilde{\mathcal{D}}}}_x \otimes_{\mathcal{D}_x} \mathcal{M}_x$. This is a vector space over $\widetilde{\widetilde{\widetilde{\mathcal{D}}}}_x$. Therefore, we get

$$\widetilde{\widetilde{\widetilde{\mathcal{D}}}}_x \underset{\mathcal{D}_x}{\otimes} \mathcal{D}_x^{\ell} \simeq \widetilde{\widetilde{\widetilde{\mathcal{D}}}}_x^{\,\ell} \xrightarrow{\sim} \widetilde{\widetilde{\widetilde{\mathcal{D}}}}_x \underset{\mathcal{D}_x}{\otimes} \mathcal{M}_x,$$

and

$$\mathcal{H}om_{\widetilde{\widetilde{\widetilde{\mathcal{D}}}}_x}(\widetilde{\widetilde{\widetilde{\mathcal{D}}}}_x \underset{\mathcal{D}_x}{\otimes} \mathcal{M}_x, \widetilde{\widetilde{\widetilde{\mathcal{D}}}}_x \underset{\mathcal{D}_x}{\otimes} \mathcal{D}_x^{\ell}) = \mathcal{H}om_{\mathcal{D}_x}(\mathcal{M}_x, \mathcal{D}_x^{\ell}) \underset{\mathcal{D}_x}{\otimes} \widetilde{\widetilde{\widetilde{\mathcal{D}}}}_x.$$

It follows that

$$\widetilde{\widetilde{\widetilde{\mathcal{D}}}}_x \underset{\mathcal{D}_x}{\otimes} \mathcal{M}_x \xrightarrow{\sim} \widetilde{\widetilde{\widetilde{\mathcal{D}}}}_x^{\,\ell}$$

comes from a map

$$\mathcal{M}_x \to \mathcal{D}_x^{\ell}.$$

This map is clearly injective. $\square$

The next proposition comes almost directly from the definition.

Proposition 1.5.4. *Let $\mathcal{M}$ be a system. For $\mathcal{M}$ to be a torsion module, it is necessary and sufficient that* $\operatorname{char}(\mathcal{M}) \neq P^*X$.

Proposition 1.5.5. *Let $\mathcal{M}$ be a system. Then, its torsion part $\mathcal{M}'$ is also coherent, and $\mathcal{M}'' = \mathcal{M}/\mathcal{M}'$ is torsion-free.*

Proof. Fix $x \in X$. Then, it is possible to find an exact sequence

$$0 \to M' \to \mathcal{M}_x \to \mathcal{D}_x^{\ell},$$

where M' is a torsion module. Hence, in a neighborhood U of x, there is a coherent torsion module $\mathcal{M}'$ such that

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{D}^{\ell}.$$

Moreover, $\mathcal{M}'$ is the torsion part of $\mathcal{M}$, since $\mathcal{M}'' = \mathcal{M}/\mathcal{M}' \subset \mathcal{D}^{\ell}$ is torsion-free. $\square$

1.6 First Spencer sequence

Let $\mathcal{M}$ be a $\mathcal{D}$-module endowed with a filtration $\mathcal{M} = \bigcup_k \mathcal{M}_k$. Let us define the $\mathcal{D}$-homomorphism

$$\delta : \mathcal{D} \underset{\mathcal{O}}{\otimes} \bigwedge^p \Theta \underset{\mathcal{O}}{\otimes} \mathcal{M}_{k-1} \to \mathcal{D} \underset{\mathcal{O}}{\otimes} \bigwedge^{p-1} \Theta \underset{\mathcal{O}}{\otimes} \mathcal{M}_k$$

by

$$\begin{aligned}
&\delta(P \otimes (\nu_1 \wedge \cdots \wedge \nu_p) \otimes u) \\
&\quad = \sum_{i=1}^{p} (-1)^{i-1} P\nu_i \otimes (\nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \nu_p) \otimes u \\
&\qquad - \sum_{i=1}^{p} (-1)^{i-1} P \otimes (\nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \nu_p) \otimes \nu_i u \\
&\qquad + \sum_{1 \leq i < j \leq p} (-1)^{i+j} P \otimes ([\nu_i, \nu_j] \wedge \nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \widehat{\nu_j} \wedge \cdots \wedge \nu_p) \otimes u.
\end{aligned}$$

It is simple to check that this is a well posed definition for δ. We call

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{M}_k \xleftarrow{\delta} \mathcal{D} \underset{\mathcal{O}}{\otimes} \Theta \underset{\mathcal{O}}{\otimes} \mathcal{M}_{k-1} \xleftarrow{\delta} \cdots \mathcal{D} \underset{\mathcal{O}}{\otimes} \bigwedge^n \Theta \underset{\mathcal{O}}{\otimes} \mathcal{M}_{k-n} \leftarrow 0 \qquad (1.6.1)$$

the first Spencer sequence of degree k of $\mathcal{M}$. Note that $\mathcal{D} \otimes_{\mathcal{O}} \mathcal{M}_k \to \mathcal{M}$ is defined by $P \otimes u \mapsto Pu$.

The following theorem may be found in Quillen [14] and Malgrange [11], but we will give a detailed proof below.

Theorem 1.6.1. *Let $\mathcal{M}$ be a system. Assume $\mathcal{M}$ is endowed with a good filtration $\mathcal{M} = \bigcup_k \mathcal{M}_k$. Locally, the first Spencer sequence of degree k associated to this filtration becomes exact for k sufficiently large.*

Proof. Step one. Let $\mathcal{M} = \mathcal{D}$, and let $\mathcal{D} = \bigcup_k \mathcal{D}_k$ be the canonical filtration. Then, the first Spencer sequence of degree k is exact for $k \geq 0$. The proof will use an induction on k. For $k = 0$, the result is clear. Assume $k > 0$. Consider the commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
0 & \mathcal{D} \arrow[l] \arrow[d, equal] & \mathcal{D}\underset{\mathcal{O}}{\otimes}\mathcal{D}_{k-1} \arrow[l] \arrow[d] & \mathcal{D}\underset{\mathcal{O}}{\otimes}\Theta\underset{\mathcal{O}}{\otimes}\mathcal{D}_{k-2} \arrow[l, "\delta"'] \arrow[d] & \cdots \arrow[l] & \mathcal{D}\underset{\mathcal{O}}{\otimes}\bigwedge^n\Theta\underset{\mathcal{O}}{\otimes}\mathcal{D}_{k-n-1} \arrow[d] & 0 \arrow[l] \\
0 & \mathcal{D} \arrow[l] \arrow[d] & \mathcal{D}\underset{\mathcal{O}}{\otimes}\mathcal{D}_{k} \arrow[l] \arrow[d] & \mathcal{D}\underset{\mathcal{O}}{\otimes}\Theta\underset{\mathcal{O}}{\otimes}\mathcal{D}_{k-1} \arrow[l, "\delta"'] \arrow[d] & \cdots \arrow[l] & \mathcal{D}\underset{\mathcal{O}}{\otimes}\bigwedge^n\Theta\underset{\mathcal{O}}{\otimes}\mathcal{D}_{k-n} \arrow[d] & 0 \arrow[l] \\
 & 0 & \mathcal{D}\underset{\mathcal{O}}{\otimes}\overline{\mathcal{D}}_{k} \arrow[l] \arrow[d] & \mathcal{D}\underset{\mathcal{O}}{\otimes}\Theta\underset{\mathcal{O}}{\otimes}\overline{\mathcal{D}}_{k-1} \arrow[l, "\overline{\delta}"'] \arrow[d] & \cdots \arrow[l] & \mathcal{D}\underset{\mathcal{O}}{\otimes}\bigwedge^n\Theta\underset{\mathcal{O}}{\otimes}\overline{\mathcal{D}}_{k-n} \arrow[d] & 0 \arrow[l] \\
 & & 0 & 0 & & 0 &
\end{tikzcd}$$

Note that

$$\overline{\delta}(P\otimes(\nu_1 \wedge \cdots \wedge \nu_p)\otimes\overline{u}) = \sum_{i=1}^{p}(-1)^i P\otimes(\nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \nu_p)\otimes\nu_i\overline{u}.$$

Since the first row is exact, it is sufficient to show that the last row is exact. Since $\mathcal{D}\otimes_{\mathcal{O}}(\cdot)$ is an exact functor, it is sufficient to show that the sequence

$$0 \leftarrow \overline{\mathcal{D}}_k \leftarrow \Theta\underset{\mathcal{O}}{\otimes}\overline{\mathcal{D}}_{k-1} \leftarrow \cdots \bigwedge^n\Theta\underset{\mathcal{O}}{\otimes}\overline{\mathcal{D}}_{k-n} \leftarrow 0$$

is exact for $k > 0$. This is the Koszul complex, and its exactness is well known (see e.g. EGA IV [4]).

Step two. The general case. Let us denote the i-th homology of

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}\underset{\mathcal{O}}{\otimes}\mathcal{M}_k \xleftarrow{\delta} \mathcal{D}\underset{\mathcal{O}}{\otimes}\Theta\underset{\mathcal{O}}{\otimes}\mathcal{M}_{k-1} \xleftarrow{\delta} \cdots \mathcal{D}\underset{\mathcal{O}}{\otimes}\bigwedge^n\Theta\underset{\mathcal{O}}{\otimes}\mathcal{M}_{k-n} \leftarrow 0$$

by $H^k_i(\mathcal{M})$. ($\mathcal{M}$ is in degree -1, and $\mathcal{D}\otimes_{\mathcal{O}}\mathcal{M}_k$ is in degree 0). We have to show that

$$H^k_i(\mathcal{M}) = 0$$

for $k \gg 0$. Since $\mathcal{D}\mathcal{M}_k = \mathcal{M}$ for $k \gg 0$,

$$H^k_{-1}(\mathcal{M}) = 0.$$

Following Lemma 1.2.4, there is an exact sequence of filtered $\mathcal{D}$-modules with strict homomorphisms
$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{L} \leftarrow \mathcal{N} \leftarrow 0,$$
where $\mathcal{L}$ is quasi-free. Therefore (thanks to step one),
$$H_i^k(\mathcal{L}) = 0 \qquad (k \gg 0).$$
Since $0 \leftarrow \mathcal{M}_k \leftarrow \mathcal{L}_k \leftarrow \mathcal{N}_k \leftarrow 0$ is exact,
$$H_i^k(\mathcal{L}) \to H_i^k(\mathcal{M}) \to H_{i-1}^k(\mathcal{N}) \to \cdots$$
is an exact sequence. Now, let us show that
$$H_i^k(\mathcal{M}) = 0 \qquad (k \gg 0)$$
by induction on i. In fact, since
$$H_i^k(\mathcal{L}) = 0,\ H_{i-1}^k(\mathcal{N}) = 0 \qquad (k \gg 0),$$
we get $H_i^k(\mathcal{M}) = 0$. □

Let us consider a special case of the first Spencer sequence. If $\mathcal{M} = \mathcal{O}$, we get a resolution of $\mathcal{O}$:
$$0 \leftarrow \mathcal{O} \leftarrow \mathcal{D} \xleftarrow{\delta} \mathcal{D} \underset{\mathcal{O}}{\otimes} \Theta \xleftarrow{\delta} \mathcal{D} \underset{\mathcal{O}}{\otimes} \bigwedge^2 \Theta \xleftarrow{\delta} \cdots \mathcal{D} \underset{\mathcal{O}}{\otimes} \bigwedge^n \Theta \leftarrow 0,$$
where the map
$$\delta : \mathcal{D} \underset{\mathcal{O}}{\otimes} \bigwedge^p \Theta \to \mathcal{D} \underset{\mathcal{O}}{\otimes} \bigwedge^{p-1} \Theta$$
is given by
$$\begin{aligned}
&\delta(P \otimes (\nu_1 \wedge \cdots \wedge \nu_p)) \\
&\quad = \sum_{i=1}^{p} (-1)^{i-1} P\nu_i \otimes (\nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \nu_p) \\
&\qquad + \sum_{1 \le i < j \le p} (-1)^{i+j} P \otimes ([\nu_i, \nu_j] \wedge \nu_1 \wedge \cdots \wedge \widehat{\nu_i} \wedge \cdots \wedge \widehat{\nu_j} \wedge \cdots \wedge \nu_p).
\end{aligned}$$
If $\mathcal{M}$ is the right $\mathcal{D}$-module Ω^n, we get a resolution:
$$0 \to \mathcal{D} \xrightarrow{d} \Omega^1 \underset{\mathcal{O}}{\otimes} \mathcal{D} \xrightarrow{d} \Omega^2 \underset{\mathcal{O}}{\otimes} \mathcal{D} \cdots \xrightarrow{d} \Omega^{n-1} \underset{\mathcal{O}}{\otimes} \mathcal{D} \xrightarrow{d} \Omega^n \underset{\mathcal{O}}{\otimes} \mathcal{D} \to \Omega^n \to 0,$$
where d is given by
$$\begin{cases} d(1) = \sum_{j=1}^n dx_j \otimes \partial_{x_j}, \\ d(\omega \wedge \varphi) = d\omega \wedge \varphi + (-1)^p \omega \wedge d\varphi, \qquad \omega \in \Omega^p, \quad \varphi \in \Omega^q \otimes_{\mathcal{O}} \mathcal{D}. \end{cases}$$

Moreover, since

$$\mathcal{D} \xleftarrow{\delta} \mathcal{D} \underset{\mathcal{O}}{\otimes} \Theta \xleftarrow{\delta} \cdots \mathcal{D} \underset{\mathcal{O}}{\otimes} \bigwedge^n \Theta$$

and

$$\mathcal{D} \xrightarrow{d} \Omega^1 \underset{\mathcal{O}}{\otimes} \mathcal{D} \cdots \xrightarrow{d} \Omega^n \underset{\mathcal{O}}{\otimes} \mathcal{D}$$

are interchanged by the functor $\mathcal{H}om_{\mathcal{D}}(\cdot, \mathcal{D})$, we get

$$\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{O}, \mathcal{D}) = \begin{cases} \Omega^n & \text{for } i = n, \\ 0 & \text{for } i \neq n, \end{cases}$$

$$\mathcal{E}xt^i_{\mathcal{D}}(\Omega^n, \mathcal{D}) = \begin{cases} \mathcal{O} & \text{for } i = n, \\ 0 & \text{for } i \neq n. \end{cases}$$

Chapter 2

On the Cauchy-Kovalevskaya theorem

2.1 Another formulation of the Cauchy-Kovalevskaya theorem

Before giving a new formulation of the Cauchy-Kovalevskaya theorem for systems, let us make some preliminary remarks.

Let X be an analytic space, Y one of its analytic subspaces, and $\mathcal{I}$ the coherent $\mathcal{O}_X$-ideal defining Y. For any $\mathcal{O}_X$-module $\mathcal{F}$, let us set

$$H^k_{[Y]}(\mathcal{F}) = \varinjlim_{\nu} \mathcal{E}xt^k_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^{\nu+1}, \mathcal{F}).$$

There is a canonical homomorphism:

$$H^k_{[Y]}(\mathcal{F}) \to H^k_Y(\mathcal{F}).$$

Now, let X and Y be complex manifolds, and let $f : Y \to X$ be a holomorphic map. Assume X is n-dimensional. Let $\mathcal{O}^{(0,n)}_{Y\times X}$ be the sheaf of holomorphic functions on $Y \times X$, which are n-form in the X variables.

Let us set[1]

$$\mathcal{D}_f = \mathcal{D}_{Y\to X} = H^n_{[Y]}(\mathcal{O}^{(0,n)}_{Y\times X}).$$

$\mathcal{D}_{Y\to X}$ is the sheaf of differential operators (of finite order) from Y to X over f. Locally, let y be a point of Y, and x a point of X, and let $\{x_i\}$ be a local coordinate system of X. Then, a section P of $\mathcal{D}_{Y\to X}$ may be written as

$$P(y, \partial_x) = \sum_{|\alpha|\le m} a_\alpha(y)\partial_x^\alpha,$$

[1] *Translators' note:* Here, Y is viewed as a subspace of $Y \times X$ through the graph embedding.

where $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$. Let us define the action

$$\mathcal{D}_{Y \to X} \times f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$$

as follows: for any

$$P = \sum_\alpha a_\alpha(y)\partial_x^\alpha \in \mathcal{D}_{Y \to X}, \qquad \varphi(x) \in \mathcal{O}_X,$$

we define $P\varphi \in \mathcal{O}_Y$ by the formula

$$(P\varphi)(y) = \sum_\alpha a_\alpha(y) \left[\partial_x^\alpha \varphi(x)\right]_{x=f(y)}.$$

In this way, we can consider a section of $\mathcal{D}_{Y \to X}$ as an operator from $\mathcal{O}_X$ to $\mathcal{O}_Y$. The next statements are clear.

Proposition 2.1.1.

$$\mathcal{D}_{Y \to X} \simeq \mathcal{O}_Y \underset{f^{-1}\mathcal{O}_X}{\otimes} f^{-1}\mathcal{D}_X.$$

Proposition 2.1.2. *$\mathcal{D}_{Y \to X}$ is both a left $\mathcal{D}_Y$-module and a right $f^{-1}\mathcal{D}_X$-module.*

Furthermore, let $g : Z \to Y$ be any holomorphic map. Then, there is a canonical bilinear homomorphism

$$\mathcal{D}_{Z \to Y} \times g^{-1}\mathcal{D}_{Y \to X} \to \mathcal{D}_{Z \to X}.$$

We will always consider $\mathcal{D}_{Y \to X}$ as endowed with its $\mathcal{D}_X$ and $\mathcal{D}_Y$ module structures. For any $\mathcal{D}_X$-module $\mathcal{M}$, we define the $\mathcal{D}_Y$-module $f^*\mathcal{M}$ by the formula

$$f^*\mathcal{M} = \mathcal{D}_{Y \to X} \underset{f^{-1}\mathcal{D}_X}{\otimes} f^{-1}\mathcal{M} = \mathcal{O}_Y \underset{f^{-1}\mathcal{O}_X}{\otimes} f^{-1}\mathcal{M}.$$

Then,

$$\begin{aligned} f^*\mathcal{O}_X &= \mathcal{O}_Y, \\ f^*\mathcal{D}_X &= \mathcal{D}_{Y \to X}. \end{aligned}$$

Let $u_1, \ldots, u_\ell$ be a system of generators of the system $\mathcal{M}$ over X, i.e., $\mathcal{M} = \sum_{j=1}^{\ell} \mathcal{D}_X u_j$. Assume $\mathcal{M}$ has a resolution

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}_X^\ell \xleftarrow{(P_{ij})} \mathcal{D}_X^r.$$

Then,

$$\sum_j P_{ij} u_j = 0. \tag{2.1.1}$$

When u_j satisfies (2.1.1), $f^*(u_j)$ and $f^*(\partial_x^\alpha u_j)$ should also satisfy relations over Y. $f^*\mathcal{M}$ is determined by these generators and relations. For example, let $Y = X \times Z$, and let $f : Y \to X$ be the projection. Then,

$$f^*(\partial_x^\alpha u_j) = \partial_x^\alpha f^*(u_j).$$

Furthermore, since $f^*(u_j)$ should be fiber-wise constant, we should get

$$\partial_{z_\nu} f^*(u_j) = 0 \qquad \text{(where } \{z_\nu\} \text{ is a local coordinate system of } Z\text{)}.$$

In fact, we get

$$f^*\mathcal{M} = \mathcal{D}_Y^\ell / (\mathcal{D}_Y^r(P_{ij}) + \sum_\nu \mathcal{D}_Y^\ell \partial_{z_\nu}).$$

In the case $Y \subset X$, $Y = \{x = (x', x'') \in X : x' = 0\}$, we have

$$f^*(\partial_{x''}^\alpha u) = \partial_{x''}^\alpha f^*(u).$$

Moreover, $f^*(\partial_{x'}^\alpha u_i) = \partial_{x'}^\alpha u_i|_Y$ and we have relations deduced from (2.1.1). In writing them down, we obtain $f^*\mathcal{M}$. For example, in the case $\mathcal{M} = \mathcal{D}_X$ itself, we do not have any real relations, and so $\mathcal{M}$ corresponds to any function u. Therefore, $\partial_{x'}^\alpha u$ are arbitrary. Then, we get $f^*\mathcal{M} = \bigoplus_\alpha \mathcal{D}_Y \partial_{x'}^\alpha$.

Now, let $\mathcal{M}$ and $\mathcal{N}$ be two systems over X. Let us consider the canonical sheaf homomorphism

$$\nu : f^{-1}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) \to \mathcal{H}om_{\mathcal{D}_Y}(f^*\mathcal{M}, f^*\mathcal{N}).$$

It corresponds to taking the initial value of the solution of a partial differential equation. Whether this ν is an isomorphism, is a generalization of the Cauchy problem. In this chapter, we will give sufficient conditions to solve this Cauchy problem.

2.2 Cauchy problem – The smooth case

Let $f : Y \to X$ be a smooth holomorphic map. For any $\mathcal{D}_X$-module $\mathcal{M}$, $f^*\mathcal{M}$ corresponds to the system on Y obtained by merging the equations of $\mathcal{M}$ with the equations of the fiber-wise de Rham system. The next theorem should intuitively be clear.

Theorem 2.2.1. *If $f : Y \to X$ is smooth, for any coherent $\mathcal{D}_X$-modules $\mathcal{M}$ and $\mathcal{N}$,*

$$\nu : f^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) \to R\mathcal{H}om_{\mathcal{D}_Y}(f^*\mathcal{M}, f^*\mathcal{N})$$

is an isomorphism. Moreover, for any k,

$$f^{-1}\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) \to \mathcal{E}xt^k_{\mathcal{D}_Y}(f^*\mathcal{M}, f^*\mathcal{N})$$

is also an isomorphism.

Proof. $\mathcal{M} \mapsto f^*\mathcal{M}$ is an exact functor. Notice that it transforms coherent $\mathcal{D}_X$-modules into coherent $\mathcal{D}_Y$-modules. Since $\mathcal{M}$ has a free resolution, we may assume $\mathcal{M} = \mathcal{D}_X$. Moreover, it is sufficient to prove that:

$$\begin{cases} \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, f^*\mathcal{N}) \simeq f^{-1}\mathcal{N}, \\ \mathcal{E}xt^k_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, f^*\mathcal{N}) = 0 \qquad (k>0). \end{cases} \tag{2.2.1}$$

It follows from the next lemma that for $\mathcal{N} = \mathcal{D}_X$, we have

$$\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, \mathcal{D}_{Y\to X}) \simeq f^{-1}\mathcal{D}_X,$$
$$\mathcal{E}xt^k_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, \mathcal{D}_{Y\to X}) = 0 \qquad (k>0).$$

Then, we will work by induction on the length of the free resolution of $\mathcal{N}$. Hence, we have an exact sequence

$$0 \leftarrow \mathcal{N} \leftarrow \mathcal{L} \leftarrow \mathcal{N}' \leftarrow 0,$$

and we may assume that (2.2.1) holds for $\mathcal{L}$ and $\mathcal{N}'$. Since $\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{D}_{Y\to X}, f^*\mathcal{N}') = 0$, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H}om_{\mathcal{D}}(\mathcal{D}_{Y\to X}, f^*\mathcal{N}') \arrow[r] & \mathcal{H}om_{\mathcal{D}}(\mathcal{D}_{Y\to X}, f^*\mathcal{L}) \arrow[r] & \mathcal{H}om_{\mathcal{D}}(\mathcal{D}_{Y\to X}, f^*\mathcal{N}) \arrow[r] & 0 \\
0 \arrow[r] & f^{-1}\mathcal{N}' \arrow[r] \arrow[u, "\wr"] & f^{-1}\mathcal{L} \arrow[r] \arrow[u, "\wr"] & f^{-1}\mathcal{N} \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

It follows that

$$\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, f^*\mathcal{N}) \xleftarrow{\sim} f^{-1}\mathcal{N},$$

and

$$\mathcal{E}xt^k_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, f^*\mathcal{N}) \xrightarrow{\sim} \mathcal{E}xt^{k+1}_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, f^*\mathcal{N}') = 0 \qquad (k>0).$$

□

Lemma 2.2.2. *Let $f : Y \to X$ be a smooth map. Then,*

$$f^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, \mathcal{D}_{Y\to X}) \xleftarrow{\sim} f^{-1}\mathcal{D}_X,$$
$$\mathcal{E}xt^k_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X}, \mathcal{D}_{Y\to X}) \simeq 0 \qquad (k>0).$$

Proof. Take a local coordinate system $(x_1, \dots, x_n)$ on X, and a coordinate system $(x_1, \dots, x_n, y_1, \dots, y_m)$ on Y, such that

$$f(x_1, \dots, x_n, y_1, \dots, y_m) = (x_1, \dots, x_n).$$

Then,

$$\mathcal{D}_{Y\to X} = \mathcal{D}_Y / (\mathcal{D}_Y \partial_{y_1} + \cdots + \mathcal{D}_Y \partial_{y_m}).$$

We may consider the free resolution of $\mathcal{D}_{Y\to X}$:

$$0 \leftarrow \mathcal{D}_{Y\to X} \leftarrow \mathcal{D}_Y \leftarrow \mathcal{D}_Y^m \leftarrow \mathcal{D}_Y^{\binom{m}{2}} \leftarrow \cdots \mathcal{D}_Y^m \leftarrow \mathcal{D}_Y \leftarrow 0.$$

For a section

$$\varphi = \sum_\alpha \varphi_\alpha(x,y)\,\partial_x^\alpha \in \mathcal{D}_{Y\to X},$$

we get

$$\partial_{y_j}\varphi = \sum_\alpha \partial_{y_j}\varphi_\alpha(x,y)\,\partial_x^\alpha.$$

It is then sufficient to compute the cohomology of the following complex:

$$0\to\mathcal{D}_{Y\to X}\to\mathcal{D}_{Y\to X}^m\cdots\to\mathcal{D}_{Y\to X}^m\to\mathcal{D}_{Y\to X}\to 0. \tag{2.2.2}$$

Using the structure formula

$$\mathcal{D}_{Y\to X} = \bigoplus_\alpha \mathcal{O}_Y\,\partial_x^\alpha,$$

we may transform component by component (2.2.2) into the de Rham complex:

$$0\to\mathcal{O}_Y\,\partial_x^\alpha\to\mathcal{O}_Y^m\,\partial_x^\alpha\cdots\to\mathcal{O}_Y^m\,\partial_x^\alpha\to\mathcal{O}_Y\,\partial_x^\alpha\to 0. \tag{2.2.3}$$

The cohomology of (2.2.3) is $f^{-1}\mathcal{O}_X\,\partial_x^\alpha$ in degree zero, and 0 in other degrees. Therefore,

$$\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X},\mathcal{D}_{Y\to X}) = \bigoplus_\alpha f^{-1}\mathcal{O}_X\,\partial_x^\alpha = f^{-1}\mathcal{D}_X,$$

$$\mathcal{E}xt^k_{\mathcal{D}_Y}(\mathcal{D}_{Y\to X},\mathcal{D}_{Y\to X}) = 0 \qquad (k>0).$$

□

2.3 Cauchy problem – The embedding case

Let X be a complex manifold, and Y one of its submanifolds. Let T^*X be the cotangent vector bundle of X, and $P^*X = (T^*X\setminus X)/\mathbb{C}^\times$ the cotangent projective bundle of X. Let T_Y^*X be the kernel of the canonical projection

$$T^*X\times_X Y\to T^*Y.$$

Set

$$P_Y^*X = (T_Y^*X\setminus Y)/\mathbb{C}^\times \subset P^*X.$$

Let $\mathcal{D}=\mathcal{D}_X$ be the sheaf of differential operators on X. In the first chapter, we associated to any $\mathcal{D}$-module $\mathcal{M}$ its characteristic variety

$$\operatorname{char}(\mathcal{M})\subset P^*X.$$

If

$$\operatorname{char}(\mathcal{M})\cap P_Y^*X = \varnothing,$$

we will say that Y is *non-characteristic* for $\mathcal{M}$. The purpose of this section is to prove the following theorem.

Theorem 2.3.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Let Y be a submanifold which is non-characteristic for $\mathcal{M}$. Then, $\mathcal{M}_Y = \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{M} = \mathcal{D}_{Y\to X} \otimes_{\mathcal{D}_X} \mathcal{M}$ is a coherent $\mathcal{D}_Y$-module, and*

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \to \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)$$

is an isomorphism.

Before proving this theorem, let us first note a few general facts. Let $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$, with $P \in \mathcal{D}_X$. As we proved in Chapter 1,

$$\operatorname{char}(\mathcal{M}) = \{x^* \in P^*X : P_m(x^*) = 0\},$$

where P_m is the principal part of P. If Y is a hypersurface of codimension 1 of X, for Y to be non-characteristic for $\mathcal{M}$ it is necessary and sufficient that Y be non-characteristic for P. Moreover, if P is an operator of degree m, in a local system of coordinates $(x_1, \dots, x_n)$ of X where $Y = \{x : x_1 = 0\}$, we may assume that

$$P = \partial_{x_1}^m + P_1(x, \partial_{x'})\partial_{x_1}^{m-1} + \cdots + P_m(x, \partial_{x'}),$$

where $x' = (x_2, \dots, x_n)$, $\partial_{x'} = (\partial_{x_2}, \dots, \partial_{x_n})$, and $P_i(x, \partial_{x'})$ is an operator of degree i which does not depend on ∂_{x_1}. We have

$$P\partial_{x_1}^i = \partial_{x_1}^{m+i} + P_1(x, \partial_{x'})\partial_{x_1}^{m+i-1} + \cdots + P_m(x, \partial_{x'})\partial_{x_1}^i.$$

Therefore, as an $\mathcal{O}_X$-module, $\mathcal{M}$ is generated by $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ $(\alpha_1 < m)$. More precisely,

$$\mathcal{M} \simeq \bigoplus_{\alpha_1 < m} \mathcal{O}_X\, \partial_x^\alpha.$$

In fact, if

$$\sum_{\alpha_1 < m} f_\alpha(x)\, \partial_x^\alpha = 0 \qquad \text{in } \mathcal{M},$$

then

$$\sum_{\alpha_1 < m} f_\alpha(x)\, \partial_x^\alpha = Q(x, \partial_x)P(x, \partial_x) \qquad \text{in } \mathcal{D}_X.$$

Comparing the terms of the same degree in ∂_{x_1}, we get $Q = 0$. Therefore, $f_\alpha = 0$. So,

$$\mathcal{M}_Y = \mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} \mathcal{M} = \bigoplus_{\alpha_1 < m} \mathcal{O}_Y\, \partial_x^\alpha = \bigoplus_{\alpha_1 < m} \mathcal{D}_Y\, \partial_{x_1}^{\alpha_1},$$

and we obtain the map:

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \to \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y) = \mathcal{O}_Y^m$$

by

$$u \mapsto (u|_Y, \partial_{x_1}u|_Y, \dots, \partial_{x_1}^{m-1}u|_Y),$$

where $u \in \mathcal{O}_X|_Y$ is such that $Pu = 0$. Following the classical Cauchy-Kovalevskaya theorem, this is clearly an isomorphism. Since $\mathcal{O}_X \xrightarrow{P} \mathcal{O}_X$ is surjective,

$$\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) = 0.$$

If we sum up the preceding discussion, we get the following lemma.

Lemma 2.3.2. *Let Y be a 1-codimensional submanifold of X, let P be an operator of degree m, and assume that Y is non-characteristic for P. Set $\mathcal{M} = \mathcal{D}/\mathcal{D}P$. Then, Theorem 2.3.1 holds for $\mathcal{M}$. Moreover, as an $\mathcal{O}_X$-module, $\mathcal{M}$ is locally free, and locally,*

$$\mathcal{M}_Y \simeq \mathcal{D}_Y^m.$$

Furthermore, $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) = 0$ for $i > 0$.

Now that we have translated the classical version of the Cauchy-Kovalevskaya theorem (Lemma 2.3.2), let us return to the proof of Theorem 2.3.1. Let us begin by proving the following proposition.

Proposition 2.3.3. *Let Y be a submanifold of X, $\mathcal{M}$ a coherent $\mathcal{D}_X$-module, $\rho : P^*X \times_X Y \setminus P^*_Y X \to P^*Y$ the canonical projection. Assume Y is non-characteristic for $\mathcal{M}$. Then*[2]*,*

(i) $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M}) = \mathcal{T}or_i^{\mathcal{D}_X}(\mathcal{D}_{Y\hookrightarrow X}, \mathcal{M})$ *is a coherent $\mathcal{D}_Y$-module,*

(ii) $\operatorname{char}(\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M})) \subset \rho(\operatorname{char}(\mathcal{M}) \times_X Y)$.

Remark 1. It is clear that $\mathcal{T}or_i^{\mathcal{D}_X}(\mathcal{D}_{Y\hookrightarrow X}, \mathcal{M})$ is a $\mathcal{D}_Y$-module. Recalling that $\mathcal{D}_X$ is $\mathcal{O}_X$-flat, and considering a free resolution $0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}_X^{r_0} \leftarrow \cdots$ of $\mathcal{M}$, we get

$$\begin{aligned} \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M}) &= H_i(\mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X^{r_\bullet}) \\ &= H_i(\mathcal{D}_{Y\hookrightarrow X} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_X^{r_\bullet}) \\ &= \mathcal{T}or_i^{\mathcal{D}_X}(\mathcal{D}_{Y\hookrightarrow X}, \mathcal{M}). \end{aligned}$$

Proof. (A) *The case where Y has codimension* 1.

Let us assume that $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, and $\operatorname{char}(\mathcal{M}) \cap P^*_Y X = \varnothing$. Let $\mathcal{M} = \sum_{j=1}^r \mathcal{D}_X u_j$ for a system of generators $u_j \in \mathcal{M}$. Since $\operatorname{char}(\mathcal{M}) \cap P^*_Y X = \varnothing$, locally there are operators $P_j \in \mathcal{D}_X$ such that $P_j u_j = 0$, Y being non-characteristic for P_j. Set $\mathcal{L} = \bigoplus_{j=1}^r \mathcal{D}_X/\mathcal{D}_X P_j$. We get the exact sequence

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{L} \leftarrow \mathcal{N} \leftarrow 0. \tag{2.3.1}$$

[2] *Translators' note:* In fact, it was later proved by Kashiwara that $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M}) = 0$ for $i > 0$ (see e.g. [P. Schapira, *Microdifferential Systems in the Complex Domain*, Grundlehren der mathematischen Wissenschaften **269**, Springer, 1985]).

We have $\mathrm{char}(\mathcal{L}) \cap P_Y^*X = \mathrm{char}(\mathcal{N}) \cap P_Y^*X = \varnothing$. Moreover, it follows from the preceding discussion that $\mathcal{L}_Y$ is a coherent $\mathcal{D}_Y$-module, and $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{L}) = 0$ $(i \neq 0)$. Because of (2.3.1), the sequence

$$0 \leftarrow \mathcal{M}_Y \leftarrow \mathcal{L}_Y \leftarrow \mathcal{N}_Y \leftarrow \mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M}) \leftarrow 0$$

is exact. Therefore, $\mathcal{M}_Y$ is a $\mathcal{D}_Y$-module of finite type. Since $\mathcal{N}$ satisfies the same conditions as $\mathcal{M}$, $\mathcal{N}_Y$ is a $\mathcal{D}_Y$-module of finite type. Since $\mathcal{L}_Y$ is a coherent $\mathcal{D}_Y$-module, we see that $\mathcal{M}_Y$ is also coherent. Therefore, $\mathcal{N}_Y$ is also a coherent $\mathcal{D}_Y$-module. Then, $\mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M})$ is also a coherent $\mathcal{D}_Y$-module. Using the isomorphism

$$\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M}) \xrightarrow{\sim} \mathcal{T}or_{i-1}^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{N})$$

for $i > 1$, we see by induction on i that $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M})$ is coherent. This proves (i). Let us now prove (ii). Let $\mathcal{M} = \bigcup_k \mathcal{M}_k$ be a good filtration on $\mathcal{M}$. Denote by $\mathcal{F}_k^i$ the cokernel of

$$\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M}_{k-1}) \to \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M}_k).$$

Setting $\mathcal{F}^i = \bigoplus_{k\in\mathbb{Z}} \mathcal{F}_k^i$, $\mathcal{F}^i$ becomes a graded $\mathrm{gr}\,\mathcal{D}_Y$-module. We have an injective homomorphism

$$0 \to \mathcal{F}_k^i \to \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathrm{gr}_k \mathcal{M}),$$

so $\mathcal{F}^i$ is a submodule of $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathrm{gr}\,\mathcal{M})$. Following Grothendieck's results (EGA III [4]), if $\widetilde{\mathcal{T}or}_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathrm{gr}\,\mathcal{M})$ denotes the corresponding $\mathcal{O}_{P^*Y}$-module, then

$$\widetilde{\mathcal{T}or}_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathrm{gr}\,\mathcal{M}) = R\rho_*(\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \widetilde{\mathrm{gr}\,\mathcal{M}})).$$

We have

$$\mathrm{supp}(\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \widetilde{\mathrm{gr}\,\mathcal{M}})) \subset \mathrm{char}(\mathcal{M}).$$

Since $\mathrm{char}(\mathcal{M}) \cap (P^*X \times_X Y) \xrightarrow{\rho} P^*Y$ is finite, $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathrm{gr}\,\mathcal{M})$ is a coherent $\mathrm{gr}\,\mathcal{D}_Y$-module. Furthermore, $\mathcal{F}^i$ is also a coherent $\mathrm{gr}\,\mathcal{D}_Y$-module (see Proposition 1.2.9). Hence,

$$\mathrm{supp}(\widetilde{\mathcal{F}^i}) \subset \rho(\mathrm{supp}(\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \widetilde{\mathrm{gr}\,\mathcal{M}}))) \subset \rho(\mathrm{char}(\mathcal{M}) \times_X Y).$$

Let us set $\mathcal{N} = \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M})$, and denote by $\mathcal{N}_k$ the image of

$$\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M}_k) \to \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M}).$$

$\mathcal{N} = \bigcup_k \mathcal{N}_k$ is a filtration on the coherent $\mathcal{D}_Y$-module $\mathcal{N}$. Let us show that $\{\mathcal{N}_k\}$ is a good filtration on $\mathcal{N}$. Let us denote by $\mathcal{Z}_l$ $(l \geq k)$ the kernel of

$$\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M}_k) \to \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M}_l).$$

Since $\mathcal{Z}_l$ is an increasing sequence of coherent sub $\mathcal{O}_Y$-modules of $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y,\mathcal{M}_k)$, it is stationary. Hence,

$$\mathcal{N}_k = \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y,\mathcal{M}_k)/\mathcal{Z}_l \qquad (l \gg 0)$$

is a coherent $\mathcal{O}_Y$-module. The commutative diagram

$$\begin{tikzcd}
\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y,\mathcal{M}_{k-1}) \arrow[r] \arrow[d] & \mathcal{N}_{k-1} \arrow[r] \arrow[d] & 0 \\
\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y,\mathcal{M}_k) \arrow[r] \arrow[d] & \mathcal{N}_k \arrow[r] \arrow[d] & 0 \\
\mathcal{F}_k \arrow[r] \arrow[d] & \operatorname{gr}_k \mathcal{N} \arrow[r] \arrow[d] & 0 \\
0 & 0 &
\end{tikzcd}$$

gives a surjective $\operatorname{gr}\mathcal{D}_Y$-linear homomorphism

$$\mathcal{F} \to \operatorname{gr}\mathcal{N} \to 0.$$

Following Proposition 1.2.9, $\operatorname{gr}\mathcal{N}$ is a coherent $\operatorname{gr}\mathcal{D}_Y$-module. So, $\mathcal{N} = \bigcup_k \mathcal{N}_k$ is a good filtration. Hence,

$$\operatorname{char}(\mathcal{N}) = \operatorname{supp}(\widetilde{\operatorname{gr}\mathcal{N}}) \subset \operatorname{supp}(\widetilde{\mathcal{F}}) \subset \rho(\operatorname{char}(\mathcal{M}) \times_X Y).$$

(B) *The general case.*

Let us work by induction on $\operatorname{codim} Y$. If Y is non-characteristic for $\mathcal{M}$, then, locally, there is a hypersurface $Y' \supset Y$ which is non-characteristic for $\mathcal{M}$. Set

$$\begin{array}{rcl}
\rho' : P^*X \times_X Y' \setminus P^*_{Y'}X & \to & P^*Y', \\
\rho'' : P^*Y' \times_{Y'} Y \setminus P^*_Y Y' & \to & P^*Y.
\end{array}$$

By the induction hypothesis, $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_{Y'},\mathcal{M})$ is a coherent $\mathcal{D}_{Y'}$-module, and

$$\operatorname{char}(\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_{Y'},\mathcal{M})) \subset \rho'(\operatorname{char}(\mathcal{M}) \times_X Y').$$

Moreover, $Y \subset Y'$ is non-characteristic for every $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_{Y'},\mathcal{M})$. By the induction hypothesis, since $\operatorname{codim}(Y,Y') = \operatorname{codim}(Y,X) - 1$, $\mathcal{T}or_j^{\mathcal{O}_{Y'}}(\mathcal{O}_Y,\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_{Y'},\mathcal{M}))$ is a coherent $\mathcal{D}_{Y'}$-module, and

$$\begin{array}{rcl}
\operatorname{char}(\mathcal{T}or_j^{\mathcal{O}_{Y'}}(\mathcal{O}_Y,\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_{Y'},\mathcal{M}))) & \subset & \rho''(\operatorname{char}(\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_{Y'},\mathcal{M}) \times_{Y'} Y)) \\
& \subset & \rho(\operatorname{char}(\mathcal{M}) \times_X Y).
\end{array}$$

Hence, we have the regular spectral sequence

$$\mathcal{E}^2_{pq} = \mathcal{T}or_p^{\mathcal{O}_{Y'}}(\mathcal{O}_Y,\mathcal{T}or_q^{\mathcal{O}_X}(\mathcal{O}_{Y'},\mathcal{M})) \quad \Rightarrow \quad \mathcal{E}_{p+q} = \mathcal{T}or_{p+q}^{\mathcal{O}_X}(\mathcal{O}_Y,\mathcal{M}),$$

where

$$\mathrm{char}(\mathcal{E}^2_{pq}) \subset \rho(\mathrm{char}(\mathcal{M}) \times_X Y).$$

Since $\mathcal{E}^{r+1}_{pq}$ is the cohomology of the complex $\mathcal{E}^r_{pq}$,

$$\mathrm{char}(\mathcal{E}^r_{pq}) \subset \rho(\mathrm{char}(\mathcal{M}) \times_X Y).$$

Since

$$\mathcal{E}^\infty_{p,q} = \mathcal{E}^r_{p,q} \qquad (r \gg 0),$$

$\mathrm{char}(\mathcal{E}^\infty_{p,q}) \subset \rho(\mathrm{char}(\mathcal{M}) \times_X Y)$. There is a filtration on $\mathcal{E}_k$ such that

$$\mathrm{gr}_p(\mathcal{E}_k) = \mathcal{E}^\infty_{p,k-p}.$$

Therefore,

$$\mathrm{char}(\mathcal{E}_k) \subset \rho(\mathrm{char}(\mathcal{M}) \times_X Y),$$

and $\mathcal{E}_k$ is coherent $\mathcal{D}_Y$-module. □

After this preparation, we can begin proving Theorem 2.3.1. We will work in two steps.

(A) *The case where Y is a 1-codimensional hypersurface of X.*

As in the proof of Proposition 2.3.3, we have the exact sequence

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{L} \leftarrow \mathcal{N} \leftarrow 0,$$

where

$$\mathcal{L} = \bigoplus_{j=1}^{r} \mathcal{D}/\mathcal{D}P_j,$$

and $\mathrm{char}(\mathcal{L}) \cap P^*_Y X = \varnothing$.

By Lemma 2.3.2, the theorem is clear for $\mathcal{L}$. Moreover, we have the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y & \longrightarrow & \mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \mathcal{O}_X)|_Y & \longrightarrow & \mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_X)|_Y \\
 & {\scriptstyle r}\downarrow & & \wr\downarrow & & {\scriptstyle r'}\downarrow \\
0 \longrightarrow & \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y) & \longrightarrow & \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{L}_Y, \mathcal{O}_Y) & \longrightarrow & \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{N}_Y, \mathcal{O}_Y).
\end{array}$$

Hence r is injective. Since $\mathcal{M}$ was general, r' is also injective. Hence, r is an isomorphism.

(B) *The general case.*

Let us proceed by induction on the codimension of Y. Locally, we may find a hypersurface $Y' \supset Y$ which is non-characteristic for $\mathcal{M}$. Then, we get the isomorphism

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_{Y'} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_{Y'}}(\mathcal{M}_{Y'}, \mathcal{O}_{Y'}).$$

Since $\operatorname{codim}(Y, Y') = \operatorname{codim}(Y, X) - 1$, Proposition 2.3.3 shows that Y is non-characteristic for $\mathcal{M}_{Y'}$, and

$$\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_{Y'}, \mathcal{O}_{Y'})|_Y \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_Y}((\mathcal{M}_{Y'})_Y, \mathcal{O}_Y).$$

Since $(\mathcal{M}_{Y'})_Y = \mathcal{M}_Y$, we get

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y).$$

This concludes the proof of Theorem 2.3.1 which we will now generalize somewhat.

Theorem 2.3.4. *Let Y be a submanifold of X, $\mathcal{M}^\bullet$ be a complex of $\mathcal{D}_X$-modules, assume that $H^i(\mathcal{M}^\bullet)$ is 0 except for finitely many i's. Assume that $H^i(\mathcal{M}^\bullet)$ is a coherent $\mathcal{D}_X$-module, for which Y is non-characteristic. Then, $\mathcal{O}_Y \otimes^L_{\mathcal{O}_X} \mathcal{M}^\bullet = \mathcal{D}_{Y \to X} \otimes^L_{\mathcal{D}_X} \mathcal{M}^\bullet$ (where $\otimes^L$ is the left derived functor of $\otimes$) has coherent cohomology, and*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{O}_X)|_Y \to R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{O}_Y)$$

is an isomorphism (in the derived category). (We refer to Hartshorne [6] for the concept of derived categories).

Proof. (A) *Assume Y has codimension* 1. Using the techniques of Hartshorne [6], and Theorem 2.3.1, we may assume from the beginning that $\mathcal{M} = \mathcal{D}/\mathcal{D}P$. Then, since $\mathcal{M}$ is a flat $\mathcal{O}_X$-module, $\mathcal{O}_Y \otimes^L_{\mathcal{O}_X} \mathcal{M} = \mathcal{M}_Y$. We already know that

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y).$$

Thanks to the Cauchy-Kovalevskaya theorem, $\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) = 0$ (i.e., $\mathcal{M}$ is solvable). The fact that $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) = 0$ $(i \geq 2)$ follows directly from the exact sequence

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}_X \xleftarrow{P} \mathcal{D}_X \leftarrow 0.$$

Since $\mathcal{M}_Y$ is a locally free $\mathcal{D}_Y$-module, $\mathcal{E}xt^i_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y) = 0$ for $i \neq 0$. Hence,

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y).$$

(B) *Assume Y has codimension* > 1. Let $Y' \supset Y$ be a non-characteristic hypersurface. Using the spectral sequence

$$\mathcal{T}or_p^{\mathcal{O}_X}(\mathcal{O}_{Y'}, H^q(\mathcal{M})) \quad \Rightarrow \quad H^{q-p}(\mathcal{O}_{Y'} \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet),$$

we see that Y is non-characteristic for $H^p(\mathcal{O}_{Y'} \otimes^L_{\mathcal{O}_X} \mathcal{M}^\bullet)$. By induction on the codimension of Y, we get

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_{Y'}}(\mathcal{O}_{Y'} \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{O}_{Y'})|_Y &\xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_{Y'}}{\otimes}} (\mathcal{O}_{Y'} \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet), \mathcal{O}_Y) \\ &\xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{O}_Y). \end{aligned}$$

Hence,

$$
\begin{aligned}
R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y &\xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_{Y'}}(\mathcal{O}_{Y'} \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{O}_{Y'})|_Y \\
&\xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{O}_Y).
\end{aligned}
$$

□

2.4 Application I – Solvability in the complex domain

Let $\mathcal{M}$ be a system. We call a system solvable if it satisfies the compatibility conditions

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) = 0 \qquad \text{for } i > 0.$$

Quillen [14] and Palamodov [12] gave sufficient conditions for solvability. Using the results of § 2.3 we get an analogous result.

Definition 2.4.1. Let $\mathcal{M}$ be a system, and $\mathcal{M} = \bigcup_k \mathcal{M}_k$ a good filtration. If $\mathrm{gr}_k \mathcal{M}$ is a locally free $\mathcal{O}_X$-module we will say that the filtration is regular.

Remark 2. If $(\mathrm{gr}_k \mathcal{M})_x$ is a free $\mathcal{O}_{X,x}$-module for every x in U, then the filtration is regular on U. (EGA IV [4], Frish [5])

Remark 3. The points of X where $\mathcal{M} = \bigcup_k \mathcal{M}_k$ is not a regular filtration form an analytic subset of X ($\neq X$).

The purpose of this section is to prove the following theorem.

Theorem 2.4.2. *Let $\mathcal{M}$ be a system endowed with a good regular filtration. Then,*

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) = 0 \qquad \text{for } i \neq 0.$$

In preparation, we will first prove the following lemma.

Lemma 2.4.3. *Let $\mathcal{M} = \bigcup_k \mathcal{M}_k$ be a regular filtration of $\mathcal{M}$. Then, locally, it is possible to find an exact sequence*

$$0 \to \mathcal{D}_X^l \to \mathcal{M} \to \mathcal{N} \to 0$$

such that conditions (a) and (b) hold.

(a) $\mathrm{char}(\mathcal{N}) \neq P^*X$,

(b) The filtration $\mathcal{M} = \bigcup_k \mathcal{M}_k$ induces on $\mathcal{N}$ a regular filtration.

Proof. Let x be any point of X. Let us construct an exact sequence of the requested kind in a neighborhood of x. For any $\mathcal{O}_X$-module $\mathcal{F}$, let us set $\mathcal{F}(x) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$. Then, $M = \operatorname{gr}(\mathcal{M}(x)) = (\operatorname{gr}\mathcal{M})(x)$ is a graded $\bigodot\Theta(x)$-module of finite type. If P_x^*X is the fiber of $P^*X \to X$ over x, then P_x^*X is isomorphic to $\mathbb{P}_{n-1}$ as an algebraic variety. If we take a sufficiently large ν_0, there exists a sheaf homomorphism

$$\widetilde{f} : \mathcal{O}^l_{P_x^*X} \to \widetilde{M}(\nu_0)$$

which is an isomorphism at generic points. Let $\mathcal{F}$ be its kernel. Then[3], since $\operatorname{Ass}(\mathcal{F}) \subset \operatorname{Ass}(\mathcal{O}_{P_x^*X}) = \{\text{generic point of } P_x^*X\}$, $\operatorname{Ass}(\mathcal{F}) = \varnothing$. Therefore, $\mathcal{F} = 0$. That is, $\widetilde{f} : \mathcal{O}^l_{P_x^*X} \to \widetilde{M}(\nu_0)$ is injective. For suitable ν_0 and $\widetilde{f}$, we may assume that $\widetilde{f}$ is induced by

$$\overline{f} : \bigodot\Theta(x)^l \to M(\nu_0).$$

Therefore, $\overline{f}$ is also injective. Let

$$f : \mathcal{D}^l_X \to \mathcal{M}(\nu_0)$$

be a homomorphism inducing $\overline{f}$ at the level of graded objects. Since

$$(\operatorname{gr}_k \mathcal{D}^l_X)(x) \to \operatorname{gr}_{k+\nu_0} \mathcal{M}(x)$$

is injective,

$$\operatorname{gr}_k \mathcal{D}^l_X \to \operatorname{gr}_{k+\nu_0} \mathcal{M}$$

is also injective. Therefore, $\mathcal{D}^l_X \to \mathcal{M}$ is injective. Let $\mathcal{N}$ denote its cokernel. Then,

$$0 \to \operatorname{gr}_{k-\nu_0} \mathcal{D}^l_X \to \operatorname{gr}_k \mathcal{M} \to \operatorname{gr}_k \mathcal{N} \to 0$$

is exact. Since $\operatorname{gr}_{k-\nu_0} \mathcal{D}^l_X(x) \to \operatorname{gr}_k \mathcal{M}(x)$ is injective, $\operatorname{gr}\mathcal{N}$ is a flat $\mathcal{O}_X$-module. (EGA IV [4]) □

Proof of Theorem 2.4.2. We will prove it by induction on the dimension of X.

(A) *The case* $\operatorname{char}(\mathcal{M}) \neq P^*X$. Since $\operatorname{char}(\mathcal{M}) \to X$ is an open mapping (EGA IV [4]), we have $\pi^{-1}(x) \cap \operatorname{char}(\mathcal{M}) \neq \pi^{-1}(x)$ for $x \in X$. Thus there is a submanifold Y with $X \supsetneqq Y$, and $P *_Y X \cap \operatorname{char}(\mathcal{M}) = \varnothing$. Since $\mathcal{M}$ is a free $\mathcal{O}_X$-module,

$$\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M} = \mathcal{M}_Y.$$

Therefore, because of Theorem 2.3.4, $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y \xrightarrow{\sim} \mathcal{E}xt^i_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)$. Let $\mathcal{M} = \bigcup_k \mathcal{M}_k$ be a regular filtration of $\mathcal{M}$. As in the proof of Proposition 2.3.3, we see that the filtration $\mathcal{M}_{Y,k} = \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{M}_k$, is a good filtration on $\mathcal{M}_Y$. Since

$$\operatorname{gr}_k(\mathcal{M}_Y) = \mathcal{O}_Y \underset{\mathcal{O}_X}{\otimes} \operatorname{gr}_k \mathcal{M},$$

[3] *Translators' note:* See [EGA IV, Definition 3.1.1], for the precise meaning of $\operatorname{Ass}(\mathcal{F})$.

$\mathcal{M}_Y$ has a regular filtration. By the induction hypothesis,

$$\mathcal{E}xt^i_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y) = 0 \qquad \text{for } i > 0.$$

Hence,

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) = 0 \qquad \text{for } i > 0.$$

(B) *The general case.* Locally, we have the exact sequence

$$0 \to \mathcal{D}^l_X \to \mathcal{M} \to \mathcal{N} \to 0,$$

where $\mathcal{N}$ has a regular filtration, and $\operatorname{char}(\mathcal{N}) \neq P^*X$. Case (A) shows that

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_X) = 0 \qquad \text{for } i > 0.$$

Because of the exact sequence

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{D}^l_X, \mathcal{O}_X) \leftarrow \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \leftarrow \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_X)$$

we get

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) = 0 \qquad \text{for } i > 0.$$

□

Working as in the proof of the preceding theorem, we get the following theorem.

Theorem 2.4.4. *Let $\mathcal{M}$ be a system endowed with a regular filtration, and $x \in X$. Then, it is possible to find a sequence of submanifolds of X*

$$x \in Y_l \subsetneqq Y_{l-1} \subsetneqq \cdots \subsetneqq Y_0 = X$$

and a filtration

$$F^0 = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_x \supset F^1 \supset \cdots \supset F^l \supset F^{l+1} = 0$$

of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_x$, such that

$$\operatorname{gr}_k(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_x) = \mathcal{O}^{m_k}_{Y_k,x}.$$

This was E. Cartan's method of solving partial differential equations. (E. Cartan [2])

Proof. As pointed out in the proof of Theorem 2.4.3, there is a sequence of submanifolds

$$x \in Y_l \subsetneqq Y_{l-1} \subsetneqq \cdots \subsetneqq Y_0 = X$$

and exact sequences

$$0 \to \mathcal{D}^{m_\nu}_{Y_\nu} \to \mathcal{M}_\nu \to \mathcal{N}_\nu \to 0,$$

such that

$$\mathcal{M}_{\nu+1} = (\mathcal{N}_\nu)_{Y_{\nu+1}}, \qquad \mathcal{M}_0 = \mathcal{M}.$$

We may assume that $\mathcal{M}_\nu$ and $\mathcal{N}_\nu$ are coherent $\mathcal{D}_{Y_\nu}$-modules endowed with regular filtrations, and that $\operatorname{char}(\mathcal{N}_\nu) \cap P^*_{Y_{\nu+1}} Y_\nu = \varnothing$. Since $\mathcal{E}xt^1_{\mathcal{D}_{Y_\nu}}(\mathcal{N}_\nu, \mathcal{O}_{Y_\nu})_x = 0$, the sequence

$$0 \to \mathcal{H}om_{\mathcal{D}_{Y_\nu}}(\mathcal{N}_\nu, \mathcal{O}_{Y_\nu})_x \to \mathcal{H}om_{\mathcal{D}_{Y_\nu}}(\mathcal{M}_\nu, \mathcal{O}_{Y_\nu})_x \to \mathcal{H}om_{\mathcal{D}_{Y_\nu}}(\mathcal{D}^{m_\nu}_{Y_\nu}, \mathcal{O}_{Y_\nu})_x \to 0$$

is exact. Moreover, $\mathcal{H}om_{\mathcal{D}_{Y_\nu}}(\mathcal{D}^{m_\nu}_{Y_\nu}, \mathcal{O}_{Y_\nu})_x \simeq \mathcal{O}^{m_\nu}_{Y_\nu,x}$, and $\mathcal{H}om_{\mathcal{D}_{Y_\nu}}(\mathcal{N}_\nu, \mathcal{O}_{Y_\nu})_x \simeq \mathcal{H}om_{\mathcal{D}_{Y_{\nu+1}}}(\mathcal{M}_{\nu+1}, \mathcal{O}_{Y_{\nu+1}})_x$. Therefore,

$$F^\nu = \mathcal{H}om_{\mathcal{D}_{Y_\nu}}(\mathcal{M}_\nu, \mathcal{O}_{Y_\nu})_x$$

defines a filtration of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_x$ such that $\operatorname{gr}_\nu \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_x = \mathcal{O}^{m_\nu}_{Y_\nu,x}$. □

2.5 Application II – Characteristic variety

If $\mathcal{M}$ is a system, then $\operatorname{char}(\mathcal{M})$ is not an arbitrary analytic subset of P^*X, but its properties are not easily clarified. However, for example, we know that the dimension of $\operatorname{char}(\mathcal{M})$ is not in the interval $0, \dots, n-2$.

Theorem 2.5.1. *Let $\mathcal{M}$ be a system. Assume* $\operatorname{char}(\mathcal{M}) = \varnothing$. *Then, as a $\mathcal{D}_X$-module, $\mathcal{M}$ is locally isomorphic to $\mathcal{O}^l_X$.*

Proof. Let $\mathcal{M} = \bigcup_k \mathcal{M}_k$ be a good filtration of $\mathcal{M}$. Since $\operatorname{char}(\mathcal{M}) = \operatorname{supp}(\widetilde{\operatorname{gr}\mathcal{M}}) = \varnothing$, we get $\widetilde{\operatorname{gr}\mathcal{M}} = 0$. Hence, $\operatorname{gr}_k \mathcal{M} = 0$ for $k \gg 0$ (see EGA III [4]). Therefore, since $\mathcal{M} = \mathcal{M}_k$ for $k \gg 0$, $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module. For $x \in X$, using an induction on $l = \dim_{\mathbb{C}} \mathcal{M}(x)$ (where $\mathcal{M}(x) = \mathbb{C} \otimes_{\mathcal{O}_{X,x}} \mathcal{M}_x$), we will show that $\mathcal{M} \simeq \mathcal{O}^l_X$ in a neighborhood of x. If $l = 0$, this follows clearly from Nakayama's lemma. Let $l > 0$. The set $Y = \{x\}$ is non-characteristic for $\mathcal{M}$. Therefore, since the right side in

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_x \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y) \simeq \operatorname{Hom}_{\mathbb{C}}(\mathcal{M}(x), \mathbb{C})$$

is not 0, there is $f \neq 0$ in $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_x$. Before concluding the proof, we need the following lemma.

Lemma 2.5.2. *$\mathcal{O}_{X,x}$ is a simple $\mathcal{D}_{X,x}$-module. (That is, 0 is its only submodule.)*

Proof. Let $N \subset \mathcal{O}_{X,x}$ be a non zero sub $\mathcal{D}_{X,x}$-module. Let us choose $N \ni \varphi \neq 0$. Let us consider the Taylor expansion of φ at x. Since there is at least one non 0 coefficient, taking a system of coordinates $(x_1, \dots, x_n)$ it is possible to find $\alpha_1, \dots, \alpha_n$ such that

$$\partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n} \varphi(x) \neq 0.$$

Set

$$\psi = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n} \varphi \in N.$$

Therefore, $N \supset \mathcal{O}_{X,x}\psi = \mathcal{O}_{X,x}$. □

Since the image of $f : \mathcal{M}_x \to \mathcal{O}_{X,x}$ is not 0, the preceding lemma shows that its image is $\mathcal{O}_{X,x}$. Therefore, we may extend f, in a neighborhood of x, to a surjective homomorphism

$$\mathcal{M} \xrightarrow{f} \mathcal{O}_X \to 0.$$

Let $\mathcal{N}$ be its kernel. Since the sequence

$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{O}_X \to 0$$

is split as a sequence of $\mathcal{O}_X$-modules,

$$\dim \mathcal{N}(x) = \dim \mathcal{M}(x) - 1 = l - 1.$$

Following the induction hypothesis, $\mathcal{N} \simeq \mathcal{O}_X^{l-1}$. On the other hand, since

$$\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{N}) = \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X)^{l-1} = 0,$$

the sequence

$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{O}_X \to 0$$

in a neighborhood of x, is a split exact sequence of $\mathcal{D}_X$-modules. Therefore, $\mathcal{M} \simeq \mathcal{N} \oplus \mathcal{O}_X \simeq \mathcal{O}_X^l$. □

Remark 4. There is a one-to-one correspondence between systems $\mathcal{M}$ with empty characteristic variety, and locally constant sheaves $\mathcal{E}$ (i.e., locally free $\mathbb{C}_X$-module of finite rank), given by

$$\begin{aligned} &\mathcal{E} = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \\ &\mathcal{M} = \mathcal{H}om_{\mathbb{C}_X}(\mathcal{E}, \mathcal{O}_X) = \mathcal{E}^* \otimes_{\mathbb{C}_X} \mathcal{O}_X. \end{aligned}$$

Proof. Since

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X) = \mathbb{C}_X,$$

the result is clear. □

Theorem 2.5.3. [4] *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. If* $\dim \operatorname{char}(\mathcal{M}) \leq \dim X - 2$, *then* $\operatorname{char}(\mathcal{M}) = \varnothing$.

Proof. Denote by $\pi : P^*X \to X$ the projection, and set $Z = \pi(\operatorname{char}(\mathcal{M}))$. Then, $\operatorname{codim} Z \geq 2$. Since we work locally, we may assume that X is simply connected. Hence, $X \setminus Z$ is also simply connected. Since over $X \setminus Z$, $\operatorname{char}(\mathcal{M}) = \varnothing$, there is a locally constant sheaf $\mathcal{E}$ on $X \setminus Z$ such that

$$\mathcal{M}|_{X \setminus Z} = \mathcal{E} \underset{\mathbb{C}}{\otimes} \mathcal{O}_X|_{X \setminus Z}.$$

[4] *Translators' note:* This result can now be considered as a consequence of the involutivity of the characteristic variety, which was first proved in [M. Sato, T. Kawai, and M. Kashiwara, Hyperfunctions and pseudo-differential equations, In H. Komatsu, editor, *Hyperfunctions and Pseudo-Differential Equations*, Lecture Notes in Mathematics **287**, pages 265–529, Springer, 1973]. A purely algebraic proof appears in [O. Gabber, *The integrability of the characteristic variety*, Amer. J. Math. **103** (1981), 445–468].

Since $X \setminus Z$ is simply connected,

$$\mathcal{E} \simeq \mathbb{C}^m_{X\setminus Z}.$$

Therefore, we have the isomorphism

$$F : \mathcal{M}|_{X\setminus Z} \xrightarrow{\sim} \mathcal{O}^m_X|_{X\setminus Z}.$$

Consider a resolution $0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}^{r_0}_X \xleftarrow{P} \mathcal{D}^{r_1}_X$. Then, F defines $f \in \Gamma(X \setminus Z; \mathcal{O}^{r_0 m}_X)$ such that $Pf = 0$. Since $\operatorname{codim} Z \geq 2$, f is extendable to $\overline{f} \in \Gamma(X; \mathcal{O}^{r_0 m}_X)$. Then, because of the analytic continuation property, we have also $P\overline{f} = 0$. Therefore,

$$F : \mathcal{M}|_{X\setminus Z} \to \mathcal{O}^m_X|_{X\setminus Z}$$

may be extended to

$$\overline{F} : \mathcal{M} \to \mathcal{O}^m_X.$$

For the kernel $\mathcal{N}$ of $\overline{F}$, we have $\operatorname{supp}\mathcal{N} \subset Z$. If $Y = \operatorname{supp}\mathcal{N} = \varnothing$, then $Z = \varnothing$, so we may assume $Y \neq \varnothing$. Since we work locally, we may assume Y is smooth. If $\operatorname{char}(\mathcal{N}) \supset P^*_Y X$, then, since $\operatorname{char}(\mathcal{M}) \supset \operatorname{char}(\mathcal{N})$, and $\dim P^*_Y X = n-1$, the proof of the theorem is complete. Therefore, we are reduced to the following Proposition. □

Proposition 2.5.4. *Let $\mathcal{M}$ be a system, and Y a smooth subvariety of X. Assume $Y' = \operatorname{supp}\mathcal{M} \subset Y$. Then, $\operatorname{char}(\mathcal{M}) \supset P^*_Y X \times_X Y'$.*

Proof. It is sufficient to prove the theorem for Y of codimension 1. (As a matter of fact, in the general case $P^*_Y X \times_X Y' = \bigcup_{Y''} P^*_{Y''} X \times_X Y'$, where $Y'' \supset Y$ is any hypersurface). In a suitable system of coordinates $(x_1, \dots, x_n)$, we have $Y = \{x_1 = 0\}$, $Y' \ni 0$. Then, we have to show that $\operatorname{char}(\mathcal{M}) \ni (0, dx_1)$. Since $\operatorname{supp}(\mathcal{M}) \ni 0$, there is a section $u \in \mathcal{M}_0$ such that $\operatorname{supp}(u) \ni 0$. We may assume from the start that $\mathcal{M} = \mathcal{D}_X u$. Since $\operatorname{supp}\mathcal{O}_X u \subset \{x_1 = 0\}$, there is a k such that $x_1^k u = 0$. For the minimum such k, we have $(x_1^{k-1}u)_0 \neq 0$. If we exchange $\mathcal{M}$ and $\mathcal{D}_X x_1^{k-1} u$ from the beginning, we may assume $x_1 u = 0$. If we set $\ker(\cdot u) = \mathcal{I}$, then

$$\mathcal{M} \simeq \mathcal{D}_X/\mathcal{I}.$$

Since $x_1 u = 0$, $\mathcal{I} \ni x_1$. Let us proceed by reductio ad absurdum, and assume $\operatorname{char}(\mathcal{M}) \not\ni (0, dx_1)$. Following the definition, there is a $P \in \mathcal{D}_{X,0}$ such that

$$\begin{cases} Pu = 0 \quad (\text{i.e., } \mathcal{I} \ni P), \\ \text{the principal symbol of } P \text{ is not 0 at } (0, dx_1). \end{cases}$$

Thus,

$$P(x, D) = \partial^m_{x_1} + P_{m-1}(x, \partial_{x'})\partial^{m-1}_{x_1} + \cdots + P_0(x, \partial_{x'}).$$

However, note that $P_j(x, \partial_{x'})$ does not depend on ∂_{x_1}. Since

$$\begin{array}{lcl} \mathcal{I} & \ni & x_1, P, \\ \mathcal{I} & \ni & [P, x_1] = m\partial_{x_1}^{m-1} + (\text{terms of order less than } m-2 \text{ in } \partial_{x_1}), \\ \mathcal{I} & \ni & [[P, x_1], x_1] = m(m-1)\partial_{x_1}^{m-2} + (\text{terms of order less than } m-3 \text{ in } \partial_{x_1}), \\ & \vdots & \\ \mathcal{I} & \ni & \underbrace{[[P, x_1], x_1] \cdots x_1]}_{m \text{ times}} = m!, \end{array}$$

we get $\mathcal{I} = \mathcal{D}_X$, and $\mathcal{M} = 0$. □

2.6 Cauchy problem for hyperfunctions

Let M be a real analytic manifold of dimension n, X a complexification, N a submanifold of M of codimension d, and $Y \subset X$ a complexification of N:

$$\begin{array}{ccc} M^n & \longleftarrow & N^{n-d} \\ \downarrow & & \downarrow \\ X^n & \longleftarrow & Y^{n-d}. \end{array}$$

Let $\mathcal{M}$ be a system. The purpose of this section is to prove that, under suitable conditions, there is an isomorphism

$$\mathcal{E}xt^i_{\mathcal{D}_X, N}(\mathcal{M}, \mathcal{B}_M) \xleftarrow{\sim} \mathcal{E}xt^{i-d}_{\mathcal{D}_Y, N}(\mathcal{M}_Y, \mathcal{B}_N).$$

To this end, we need to develop a suitable formalism.

Let $(X, \mathcal{O}_X)$ be a ringed space, let $Z \subset X$ be a closed set, and let $\mathcal{M}$, $\mathcal{N}$ be $\mathcal{O}_X$-modules. Set

$$\mathcal{E}xt^i_{\mathcal{O}_X, Z}(\mathcal{M}, \mathcal{N}) = H^i(\mathrm{R}\Gamma_Z R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})).$$

Then, we have the spectral sequences

$$\begin{array}{rcl} \mathcal{E}xt^p_{\mathcal{O}_X, Z}(\mathcal{M}, H^q_{Z'}(\mathcal{N})) & \Rightarrow & \mathcal{E}xt^{p+q}_{\mathcal{O}_X, Z\cap Z'}(\mathcal{M}, \mathcal{N}), \\ H^p_Z(\mathcal{E}xt^q_{\mathcal{O}_X, Z'}(\mathcal{M}, \mathcal{N})) & \Rightarrow & \mathcal{E}xt^{p+q}_{\mathcal{O}_X, Z\cap Z'}(\mathcal{M}, \mathcal{N}). \end{array}$$

Let M be a real manifold of dimension n, N a submanifold of dimension $n-d$. Then,

$$or_{N|M} = H^d_N(\mathbb{Z}_M)$$

is locally of isomorphic to $\mathbb{Z}_N$. Then, for any complex of sheaves $\mathcal{F}^\bullet$ on M, we get the morphism

$$\mathcal{F}^\bullet|_N \otimes or_{N|M} \to \mathrm{R}\Gamma_N(\mathcal{F}^\bullet)[d].$$

In fact, since $\mathrm{R}\Gamma_N(\mathbb{Z}_M) = or_{N|M}[-d]$, this is given by the natural morphism

$$\mathcal{F}^\bullet|_N \otimes \mathrm{R}\Gamma_N(\mathbb{Z}_M) \to \mathrm{R}\Gamma_N(\mathcal{F}^\bullet).$$

Consider the diagram

$$\begin{tikzcd} X & Y \arrow[l] \\ M \arrow[u] & N. \arrow[l] \arrow[u] \end{tikzcd}$$

Assume $\mathcal{M}^\bullet$ is a complex of $\mathcal{D}_X$-modules, and $H^i(\mathcal{M}^\bullet)$ is a system, with $H^i(\mathcal{M}^\bullet) = 0$ except for a finite number of i. It follows from the preceding general considerations that we have a map

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{O}_X)|_Y \to \mathrm{R}\Gamma_Y R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{O}_X)[2d].$$

Applying the functor Γ_N, we get

$$\mathrm{R}\Gamma_N(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{O}_X)|_Y) \to \mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{O}_X)[2d].$$

Therefore, if every $H^i(\mathcal{M}^\bullet)$ is non-characteristic for Y, because of

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{O}_X)|_Y \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{O}_Y),$$

we get

$$\mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{O}_Y) \to \mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{O}_X)[2d].$$

Theorem 2.6.1. *Assume $H^i(\mathcal{M}^\bullet)$ is non-characteristic for Y. Then,*

$$\mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{O}_Y) \to \mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{O}_X)[2d] \tag{2.6.1}$$

is an isomorphism.

We will make use of the following proposition.

Proposition 2.6.2. *(Kawai-Komatsu [17][5]) Let P be an operator of degree m, N a submanifold of M of codimension 1 for which P is non-characteristic. Then, locally,*

$$\mathcal{E}xt^i_{\mathcal{D}_X,N}(\mathcal{D}/\mathcal{D}P, \mathcal{B}_M) \simeq \begin{cases} \mathcal{B}_N^m \otimes or_{N|M} & \text{for } i = 1, \\ 0 & \text{for } i \neq 1. \end{cases}$$

[5] *Translators' note:* This result already appeared in [P. Schapira, *Hyperfonctions et valeurs au bord des solutions des équations elliptiques*, Sém. sur le équations aux dérivées partielles, Collège de France (1969/1970); and *Hyperfonctions et problèmes aux limites elliptiques*, Bull. Soc. Math. France **99** (1971), 113–141].

Proof of Theorem 2.6.1. Step one. Assume $d = 1$, $\mathcal{M} = \mathcal{D}/\mathcal{D}P$ for an operator P of degree m. Then, locally, $\mathcal{O}_Y \otimes^L_{\mathcal{O}_X} \mathcal{M} \simeq \mathcal{D}_Y^m$. Therefore,

$$\mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}, \mathcal{O}_Y) \simeq \mathrm{R}\Gamma_N(\mathcal{O}_Y)^m \simeq \mathcal{B}_N^m[1-n] \otimes or_N,$$

where or_N is the orientation sheaf over N. On the other hand,

$$\begin{aligned}\mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) &\simeq \mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathrm{R}\Gamma_M(\mathcal{O}_X)) \\ &\simeq \mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M[-n] \otimes or_M).\end{aligned}$$

Therefore, we have

$$H^i(\mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}, \mathcal{O}_Y)) \simeq \begin{cases} \mathcal{B}_N^m \otimes or_N & \text{for } i = n-1, \\ 0 & \text{for } i \neq n-1, \end{cases}$$

$$\begin{aligned}H^i(\mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)[2d]) &\simeq \mathcal{E}xt^{i+2-n}_{\mathcal{D}_X, N}(\mathcal{M}, \mathcal{B}_M) \otimes or_M \\ &\simeq \begin{cases} \mathcal{B}_N^m \otimes or_N & \text{for } i = n-1, \\ 0 & \text{for } i \neq n-1. \end{cases}\end{aligned}$$

So, (2.6.1) becomes an isomorphism at the level of cohomology (we omit the detailed proof).

Step two. In the case $d = 1$ and $H^i(\mathcal{M}^\bullet)$ is non-characteristic for Y, using techniques of Hartshorne [6], it is possible to reduce to case (A).

Step three. Let $d > 1$. Let us choose a codimension 1 submanifold $N' \supset N$. Let Y' be a complexification of N'. Shrinking X if necessary, we may assume that Y' is non-characteristic for $H^i(\mathcal{M}^\bullet)$. Therefore, because of the induction hypothesis, we get

$$\begin{aligned}\mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{O}_Y) &\xrightarrow{\sim} \mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_{Y'}}(\mathcal{O}_{Y'} \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{O}_{Y'})[2] \\ &\xrightarrow{\sim} \mathrm{R}\Gamma_N \mathrm{R}\Gamma_{N'} R\mathcal{H}om_{\mathcal{D}_{Y'}}(\mathcal{O}_{Y'} \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{O}_{Y'})[2] \\ &\xrightarrow{\sim} \mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{O}_X)[2d].\end{aligned}$$

□

In the same way, we get the following remark.

Remark 5. Assume Y is non-characteristic for $H^i(\mathcal{M}^\bullet)$. Then,

$$\mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{B}_M) \xleftarrow{\sim} R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{B}_N)[-d] \otimes or_{N|M},$$

where $or_{N|M} = or_N \otimes or_M$. Here, or_N, or_M are respectively the orientation sheaves of N and M.

Proof.

$$
\begin{aligned}
&\mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{B}_M)[-n] \otimes or_M \\
&\qquad \simeq \mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathrm{R}\Gamma_M \mathcal{O}_X) \\
&\qquad \simeq \mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{O}_X) \\
&\qquad \simeq \mathrm{R}\Gamma_N R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{O}_Y)[-2d] \\
&\qquad \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathrm{R}\Gamma_N \mathcal{O}_Y)[-2d] \\
&\qquad \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{B}_N \otimes or_N[d-n])[-2d] \\
&\qquad \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{O}_Y \overset{L}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{M}^\bullet, \mathcal{B}_N)[-d-n] \otimes or_N
\end{aligned}
$$

□

Chapter 3

Algebraic aspects of $\mathcal{D}_X$ homology

One of the aims of this chapter is to prove that the global dimension of $\mathcal{D}_{X,x}$ is equal to $\dim X$. This result is related to the fact that the codimension of the characteristic variety of $\mathcal{M}$ can only take the values $0, 1, \ldots, n, \infty$. Here and below, we set $n = \dim X$.

3.1 Global dimension of $\mathcal{D}_X$

Using the first Spencer sequence, the following lemma is clear.

Lemma 3.1.1. *Locally, any system $\mathcal{M}$ has a free resolution*

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}_X^{r_0} \leftarrow \cdots \leftarrow \mathcal{D}_X^{r_{2n}} \leftarrow 0$$

of length $2n$.

Proof. Let $\mathcal{M} = \bigcup_k \mathcal{M}_k$ be a good filtration. Then, the first Spencer sequence

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{M}_k \leftarrow \mathcal{D} \underset{\mathcal{O}}{\otimes} \Theta \underset{\mathcal{O}}{\otimes} \mathcal{M}_{k-1} \leftarrow \cdots \mathcal{D} \underset{\mathcal{O}}{\otimes} \bigwedge^n \Theta \underset{\mathcal{O}}{\otimes} \mathcal{M}_{k-n} \leftarrow 0$$

is exact for $k \gg 0$ (see § 1.6). Moreover, let

$$0 \leftarrow \bigwedge^i \Theta \underset{\mathcal{O}}{\otimes} \mathcal{M}_{k-i} \leftarrow \mathcal{L}_{i,0} \leftarrow \mathcal{L}_{i,1} \leftarrow \cdots \mathcal{L}_{i,n} \leftarrow 0$$

be a free resolution of the coherent $\mathcal{O}_X$-module $\bigwedge^i \Theta \otimes_{\mathcal{O}} \mathcal{M}_{k-i}$. Then, we get the following commutative diagram with exact columns

$$\begin{array}{cccccccc}
0 & & 0 & & & 0 & & \\
\uparrow & & \uparrow & & & \uparrow & & \\
\mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{M}_k & \longleftarrow & \mathcal{D} \underset{\mathcal{O}}{\otimes} \Theta \underset{\mathcal{O}}{\otimes} \mathcal{M}_{k-1} & \longleftarrow & \cdots & \mathcal{D} \underset{\mathcal{O}}{\otimes} \bigwedge^n \Theta \underset{\mathcal{O}}{\otimes} \mathcal{M}_{k-n} & \longleftarrow & 0 \\
\uparrow & & \uparrow & & & \uparrow & & \\
\mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{L}_{0,0} & \longleftarrow & \mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{L}_{1,0} & \longleftarrow & \cdots & \mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{L}_{n,0} & \longleftarrow & 0 \\
\uparrow & & \uparrow & & & \uparrow & & \\
\mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{L}_{0,1} & \longleftarrow & \mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{L}_{1,1} & \longleftarrow & \cdots & \mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{L}_{n,1} & \longleftarrow & 0 \\
\uparrow & & \uparrow & & & \uparrow & & \\
\vdots & & \vdots & & & \vdots & & \\
\mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{L}_{0,n} & \longleftarrow & \mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{L}_{1,n} & \longleftarrow & \cdots & \mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{L}_{n,n} & \longleftarrow & 0 \\
\uparrow & & \uparrow & & & \uparrow & & \\
0 & & 0 & & & 0 & &
\end{array}$$

Therefore,

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{L}_{0,0} \leftarrow (\mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{L}_{1,0}) \oplus (\mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{L}_{0,1}) \leftarrow \cdots \mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{L}_{n,n} \leftarrow 0$$

is an exact sequence. □

It follows from the preceding lemma that for $x \in X$, global dim $\mathcal{D}_{X,x} \leq 2n$.

Theorem 3.1.2. *For any system $\mathcal{M}$,*

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0 \qquad \textit{for } i < \min(n, \operatorname{codim} \operatorname{char}(\mathcal{M})).$$

Proof. Let $\mathcal{M} = \bigcup_k \mathcal{M}_k$ be a good filtration. Let

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{L}_0 \overset{P_0}{\leftarrow} \mathcal{L}_1 \overset{P_1}{\leftarrow} \leftarrow \cdots$$

be a resolution of $\mathcal{M}$ by quasi-free filtered $\mathcal{D}$-modules $\mathcal{L}_i$ and strict homomorphisms (obtained as in Lemma 1.2.4). Hence, we have

$$\mathcal{L}_i = \bigoplus_{k=1}^{r_i} \mathcal{D}(l_{i,k}).$$

Then, $\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$ is the i-th cohomology of

$$\mathcal{L}^{*0} \overset{P_0}{\to} \mathcal{L}^{*1} \overset{P_1}{\to} \cdots \mathcal{L}^{*k} \to \cdots \tag{3.1.1}$$

where $\mathcal{L}^{*i} = \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_i, \mathcal{D})$, endowed with the filtration

$$\mathcal{L}^{*i}_k = \{s \in \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_i, \mathcal{D}) : s(\mathcal{L}_{i,l}) \subset \mathcal{D}_{l+k} \text{ for } \forall l\}.$$

Hence,

$$\mathcal{L}^{*i} = \bigoplus_{k=1}^{r} \mathcal{D}(-l_{i,k}).$$

We have a canonical isomorphism

$$\operatorname{gr} \mathcal{L}^{*i} = \mathcal{H}om_{\overline{\mathcal{D}}}(\operatorname{gr} \mathcal{L}_i, \overline{\mathcal{D}}),$$

and

$$\operatorname{gr} \mathcal{L}^{*0} \to \operatorname{gr} \mathcal{L}^{*1} \to \cdots \tag{3.1.2}$$

is obtained by applying $\mathcal{H}om_{\overline{\mathcal{D}}}(\cdot, \overline{\mathcal{D}})$ to

$$\operatorname{gr} \mathcal{L}_0 \leftarrow \operatorname{gr} \mathcal{L}_1 \leftarrow \cdots .$$

Since

$$0 \leftarrow \operatorname{gr} \mathcal{M} \leftarrow \operatorname{gr} \mathcal{L}_0 \leftarrow \operatorname{gr} \mathcal{L}_1 \leftarrow \cdots$$

is a free resolution of $\operatorname{gr} \mathcal{M}$, the i-th cohomology group of (3.1.2) is equal to

$$\mathcal{E}xt^i_{\overline{\mathcal{D}}}(\operatorname{gr} \mathcal{M}, \overline{\mathcal{D}}).$$

Let us set $Y = \operatorname{Specan}(\overline{\mathcal{D}}) = T^*X$, and consider the faithful exact functor

$$\{\text{coherent } \overline{\mathcal{D}}\text{-module}\} \to \{\text{coherent } \mathcal{O}_Y \text{ Module}\}$$

defined in Houzel [9]. We will denote it by $\sim\!\sim$. Then,

$$(\mathcal{E}xt^i_{\overline{\mathcal{D}}}(\operatorname{gr} \mathcal{M}, \overline{\mathcal{D}}))^{\sim\sim} = \mathcal{E}xt^i_{\mathcal{O}_Y}((\operatorname{gr} \mathcal{M})^{\sim\sim}, \mathcal{O}_Y).$$

Following the theory of regular local rings (EGA IV [4]),

$$\mathcal{E}xt^i_{\mathcal{O}_Y}((\operatorname{gr} \mathcal{M})^{\sim\sim}, \mathcal{O}_Y) = 0 \qquad \text{for } i < \operatorname{codim} \operatorname{supp}((\operatorname{gr} \mathcal{M})^{\sim\sim}).$$

Clearly,

$$\begin{aligned} \operatorname{codim} \operatorname{supp}((\operatorname{gr} \mathcal{M})^{\sim\sim}) &\geq \min(n, \operatorname{codim} \operatorname{supp} \widetilde{\operatorname{gr} \mathcal{M}}) \\ &\geq \min(n, \operatorname{codim} \operatorname{char}(\mathcal{M})) \end{aligned}$$

and

$$\mathcal{E}xt^i_{\mathcal{O}_Y}((\operatorname{gr} \mathcal{M})^{\sim\sim}, \mathcal{O}_Y) = 0 \qquad \text{for } i < \min(n, \operatorname{codim} \operatorname{char}(\mathcal{M})).$$

Therefore, when $i < n$ and $i < \operatorname{codim} \operatorname{char}(\mathcal{M})$, it follows from the fact that $\mathcal{E}xt^i_{\overline{\mathcal{D}}}(\operatorname{gr} \mathcal{M}, \overline{\mathcal{D}}) = 0$, that the i-th cohomology of (3.1.2) vanishes. So, because of Lemma 1.2.2, the i-th cohomology of (3.1.1) also vanishes, and $\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) = 0$. $\square$

Theorem 3.1.3. *For any system $\mathcal{M}$, and for any integer i,*

$$\operatorname{codim}\operatorname{char}(\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M},\mathcal{D}))\geq i.$$

Proof. For a suitable good filtration of $\mathcal{M}$, we get a resolution

$$0\leftarrow\mathcal{M}\leftarrow\mathcal{L}_0\leftarrow\mathcal{L}_1\leftarrow\cdots$$

of $\mathcal{M}$ by quasi-free filtered $\mathcal{D}$-modules $\mathcal{L}_i$ and strict homomorphisms. As in the proof of Theorem 3.1.2, we endow

$$\mathcal{L}^{*i}=\mathcal{H}om_{\mathcal{D}}(\mathcal{L}_i,\mathcal{D})$$

with its natural filtration. Then, we get from the exact sequence

$$\mathcal{L}^{*0}\to\mathcal{L}^{*1}\to\cdots, \tag{3.1.3}$$

the exact sequence of $\overline{\mathcal{D}}$-modules

$$\operatorname{gr}\mathcal{L}_0\leftarrow\operatorname{gr}\mathcal{L}_1\leftarrow\cdots \tag{3.1.4}$$

by applying the functor $\mathcal{H}om_{\overline{\mathcal{D}}}(\cdot,\overline{\mathcal{D}})$ to

$$\operatorname{gr}\mathcal{L}^{*0}\to\operatorname{gr}\mathcal{L}^{*1}\to\cdots.$$

Classical results in commutative algebra show that

$$\operatorname{codim}\operatorname{supp}(\widetilde{\mathcal{E}xt^i_{\overline{\mathcal{D}}}}(\operatorname{gr}\mathcal{M},\overline{\mathcal{D}}))\geq i.$$

Therefore, setting

$$Z=\operatorname{supp}(\widetilde{\mathcal{E}xt^i_{\overline{\mathcal{D}}}}(\operatorname{gr}\mathcal{M},\overline{\mathcal{D}})),$$

we see that the i-th cohomology of

$$\widetilde{\operatorname{gr}\mathcal{L}^{*0}}\to\cdots\to\widetilde{\operatorname{gr}\mathcal{L}^{*i}}\to\cdots \tag{3.1.5}$$

vanishes outside of Z. Therefore, Theorem 3.1.3 follows from the lemma below. $\square$

Lemma 3.1.4. *Let $\mathcal{S}$ be a sheaf of rings satisfying the conditions of §1.1. Set $X^*=\operatorname{Proj}(\operatorname{gr}\mathcal{S})$. Let*

$$\mathcal{M}'\xrightarrow{f}\mathcal{M}\xrightarrow{f'}\mathcal{M}''$$

be a sequence of filtered $\mathcal{S}$-modules endowed with a good filtration such that $f'\circ f=0$. Let $\mathcal{H}$ be its cohomology group, and let $\mathcal{H}'$ be the cohomology group of

$$\operatorname{gr}\mathcal{M}'\to\operatorname{gr}\mathcal{M}\to\operatorname{gr}\mathcal{M}''.$$

Then,

$$\operatorname{char}(\mathcal{H})\subset\operatorname{supp}(\widetilde{\mathcal{H}'}).$$

Proof. Set $\mathcal{Z} = \ker(\mathcal{M} \xrightarrow{f'} \mathcal{M}'')$, and consider the filtration induced by that of $\mathcal{M}$. Set $Z = \operatorname{supp}(\widetilde{\mathcal{H}'})$. Let us show that, outside of Z, the morphism

$$\widetilde{\operatorname{gr}\mathcal{M}'} \to \widetilde{\operatorname{gr}\mathcal{Z}}$$

is surjective. Outside of Z, the sequence

$$\widetilde{\operatorname{gr}\mathcal{M}'} \to \widetilde{\operatorname{gr}\mathcal{M}} \to \widetilde{\operatorname{gr}\mathcal{M}''}$$

is exact. Clearly,

$$\widetilde{\operatorname{gr}\mathcal{Z}} \subset \widetilde{\operatorname{gr}\mathcal{M}},$$

and the morphism $\widetilde{\operatorname{gr}\mathcal{Z}} \to \widetilde{\operatorname{gr}\mathcal{M}''}$ is the zero homomorphism. So,

$$\widetilde{\operatorname{gr}\mathcal{M}'} \to \widetilde{\operatorname{gr}\mathcal{Z}}$$

is surjective outside of Z. Since

$$\mathcal{Z} \to \mathcal{H}$$

is surjective, we may consider the filtration on $\mathcal{H}$ induced by that of $\mathcal{Z}$. Note that the composition of $\widetilde{\operatorname{gr}\mathcal{M}'} \to \widetilde{\operatorname{gr}\mathcal{Z}}$ with $\widetilde{\operatorname{gr}\mathcal{Z}} \to \widetilde{\operatorname{gr}\mathcal{H}}$ is the zero homomorphism. Since $\widetilde{\operatorname{gr}\mathcal{Z}} \to \widetilde{\operatorname{gr}\mathcal{H}}$ is surjective, $\widetilde{\operatorname{gr}\mathcal{H}}$ is 0 outside of Z. Hence, $\operatorname{char}(\mathcal{H}) = \operatorname{supp}\widetilde{\operatorname{gr}\mathcal{H}} \subset Z$. $\square$

We can now prove the following theorem.

Theorem 3.1.5. *For any system $\mathcal{M}$,*

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0 \qquad \textit{for } i > \dim X.$$

Proof. It follows from Theorem 3.1.3 that

$$\operatorname{codim}\operatorname{char}(\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}, \mathcal{D})) \geq i.$$

Therefore, for $i > \dim X$, Theorem 2.5.3 shows that

$$\operatorname{char}(\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}, \mathcal{D})) = \varnothing.$$

Following Theorem 2.5.1, locally

$$\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) \simeq \mathcal{O}_X^r.$$

Let $\mathcal{M} = \bigcup_k \mathcal{M}_k$ be a good filtration on $\mathcal{M}$. Working as in Palamodov [12], this filtration is regular except on a nowhere dense subset. Therefore, the first Spencer sequence of $\mathcal{M}$

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D} \underset{\mathcal{O}}{\otimes} \mathcal{M}_k \leftarrow \cdots \mathcal{D} \underset{\mathcal{O}}{\otimes} \bigwedge^n \Theta \underset{\mathcal{O}}{\otimes} \mathcal{M}_{k-n} \leftarrow 0$$

gives a free resolution of length n of $\mathcal{M}$ at the points x where $\mathcal{M} = \bigcup_k \mathcal{M}_k$ is regular. Therefore,

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)_x = 0,$$

so $r = 0$, and

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$$

on the whole of X. □

Remark 6. For any $x \in X$, global $\dim \mathcal{D}_{X,x} = \dim X$.

It is convenient to set the following definition.

Definition 3.1.6. For any system $\mathcal{M}$,

$$\mathrm{cd}(\mathcal{M}) = \begin{cases} \operatorname{codim} \mathrm{char}(\mathcal{M}) & \text{if } n \geq \operatorname{codim} \mathrm{char}(\mathcal{M}), \\ n & \text{if } \mathcal{M} \neq 0,\ \mathrm{char}(\mathcal{M}) = \varnothing, \\ \infty & \text{if } \mathcal{M} = 0. \end{cases}$$

For any section s of $\mathcal{M}$, we set $\mathrm{cd}(s) = \mathrm{cd}(\mathcal{D}_X s)$.

Then, Theorems 3.1.2, 3.1.3, 3.1.5 show that

$$\begin{aligned} &\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0 \qquad \text{for } i < \mathrm{cd}(\mathcal{M}), \\ &\mathrm{cd}(\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)) \geq i. \end{aligned}$$

Proposition 3.1.7. *For any system* $\mathcal{M}$,

$$\mathrm{char}(\mathcal{M}) \supset \mathrm{char}(\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}, \mathcal{D})).$$

Proof. Since $\widetilde{\mathcal{D}}$ is flat over $\mathcal{D}$,

$$\widetilde{\mathcal{E}xt^i_{\mathcal{D}}}(\mathcal{M}, \mathcal{D}) = \mathcal{E}xt^i_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}).$$

□

3.2 Associated cohomology

(In this section, we follow an idea of Sato [15].) Instead of using spectral sequences, Sato devised in [15, p. 406] a method using associated cohomology. Already at the time when Sato was at Komaba, the following conjecture was made[1].

Conjecture . *Let A be an n-dimensional regular ring, and M an A-module of finite type. Then, $T^0_{p,n}(M) = \{x \in M : \operatorname{codim} \operatorname{supp} x > p\}$ (as for $T^0_{p,n}$, see the explanation below).*

[1] *Translators' note:* For related developments see [M. Kashiwara, *B-functions and holonomic systems*, Invent. Math. **38** (1976), 33–53].

It is possible to give a simple explanation to this conjecture, but we will not give it here. Using this method, it is possible to build a filtration:

$$0 \subset \mathcal{M}_n \subset \mathcal{M}_{n-1} \subset \cdots \subset \mathcal{M}_0 = \mathcal{M}$$

of a coherent $\mathcal{D}_X$-module $\mathcal{M}$. Here,

$$\mathcal{M}_k = \{s \in \mathcal{M} : \mathrm{cd}(s) \geq k\}.$$

So, every $\mathcal{M}_k$ becomes a coherent $\mathcal{D}_X$-module. Hence, it is possible to analyze the characteristic variety of $\mathcal{M}$.

Given a complex $X^\bullet = \{\cdots \to X^n \overset{d_X^n}{\to} X^{n+1} \to \cdots\}$, let us define

$$\sigma_{>n}(X^\bullet), \quad \sigma_{<n}(X^\bullet), \quad \sigma_{\geq n}(X^\bullet), \quad \sigma_{\leq n}(X^\bullet)$$

in the following way (Hartshorne [6]):

$$\begin{aligned}
\sigma_{>n}(X^\bullet) &= (\cdots \to 0 \to \mathrm{im}\, d_X^n \to X^{n+1} \to X^{n+2} \cdots) \\
\sigma_{<n}(X^\bullet) &= (\cdots \to X^{n-2} \to X^{n-1} \to \mathrm{im}\, d_X^{n-1} \to 0 \to \cdots) \\
\sigma_{\geq n}(X^\bullet) &= (\cdots \to 0 \to \mathrm{coker}\, d_X^{n-1} \to X^{n+1} \to X^{n+2} \cdots) \\
\sigma_{\leq n}(X^\bullet) &= (\cdots \to X^{n-2} \to X^{n-1} \to \ker d_X^n \to 0 \to \cdots)
\end{aligned}$$

Then, we have

$$\begin{aligned}
H^i(\sigma_{\gtrless n}(X^\bullet)) &= \begin{cases} H^i(X^\bullet) & \text{for } i \gtrless n, \\ 0 & \text{for } i \lesseqqgtr n, \end{cases} \\
H^i(\sigma_{\substack{\geq \\ \leq} n}(X^\bullet)) &= \begin{cases} H^i(X^\bullet) & \text{for } i \gtreqqless n, \\ 0 & \text{for } i \lessgtr n, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{>n-1}(X^\bullet) &\to \sigma_{\geq n}(X^\bullet), \\
\sigma_{\leq n}(X^\bullet) &\to \sigma_{<n+1}(X^\bullet)
\end{aligned}$$

become quasi-isomorphisms. Furthermore, if we consider

$$X^\bullet \quad \mapsto \quad \sigma_{>n}(X^\bullet),\ \sigma_{<n}(X^\bullet),\ \sigma_{\geq n}(X^\bullet),\ \sigma_{\leq n}(X^\bullet),$$

as functors of the derived category, then we get

$$\sigma_{>n} = \sigma_{\geq n+1}, \qquad \sigma_{<n} = \sigma_{\leq n-1}.$$

For any $p \leq q$, let us set

$${}_{p<}\sigma_{\leq q} = \sigma_{>p} \circ \sigma_{\leq q} = \sigma_{\leq q} \circ \sigma_{>p}.$$

Let X be a topological space, endowed with a sheaf of rings $\mathcal{S}$. For any left (resp. right) $\mathcal{S}$-module $\mathcal{M}$, let us set

$$T^i_{p,q}(\mathcal{M}) = \mathcal{E}xt^i_{\mathcal{S}}({}_{p<}\sigma_{\leq q} R\mathcal{H}om_{\mathcal{S}}(\mathcal{M},\mathcal{S}),\mathcal{S}).$$

Since, for $p \leq q \leq r$, we have the distinguished triangle:

$${}_{p<}\sigma_{\leq q} R\mathcal{H}om_{\mathcal{S}}(\mathcal{M},\mathcal{S}) \to {}_{p<}\sigma_{\leq r} R\mathcal{H}om_{\mathcal{S}}(\mathcal{M},\mathcal{S}) \to {}_{q<}\sigma_{\leq r} R\mathcal{H}om_{\mathcal{S}}(\mathcal{M},\mathcal{S}) \xrightarrow{+1},$$

we get the long exact sequence:

$$\cdots \to T^i_{q,r}(\mathcal{M}) \to T^i_{p,r}(\mathcal{M}) \to T^i_{p,q}(\mathcal{M}) \to T^{i+1}_{q,r}(\mathcal{M}) \to \cdots .$$

Also, we have

$${}_{p-1<}\sigma_{\leq p} R\mathcal{H}om_{\mathcal{S}}(\mathcal{M},\mathcal{S}) = \mathcal{E}xt^p_{\mathcal{S}}(\mathcal{M},\mathcal{S})[-p].$$

Hence,

$$\begin{aligned} T^i_{p-1,p}(\mathcal{M}) &= \mathcal{E}xt^i_{\mathcal{S}}(\mathcal{E}xt^p_{\mathcal{S}}(\mathcal{M},\mathcal{S})[-p],\mathcal{S}) \\ &= \mathcal{E}xt^{i+p}_{\mathcal{S}}(\mathcal{E}xt^p_{\mathcal{S}}(\mathcal{M},\mathcal{S}),\mathcal{S}). \end{aligned}$$

Since ${}_{p<}\sigma_{\leq q} R\mathcal{H}om_{\mathcal{S}}(\mathcal{M},\mathcal{S}) = 0$, for $q < 0$, we get $T^i_{p,q}(\mathcal{M}) = 0$ for $q < 0$. Furthermore, for $i + q < 0$, we have $T^i_{p,q}(\mathcal{M}) = 0$. In fact, this is proved by induction on $q - p$. If $p = q$, or $p = q - 1$, this is obvious. Let us assume $p < q - 1$. Then, we get the exact sequence

$$T^i_{q-1,q} \to T^i_{p,q} \to T^i_{p,q-1},$$

and, by our induction hypothesis, both ends are 0. Therefore, we have $T^i_{p,q} = 0$.

To summarize:

Proposition 3.2.1. *Let X be a topological space, $\mathcal{S}$ a sheaf of rings on X, and $\mathcal{M}$ an $\mathcal{S}$-module. Set*

$$T^i_{p,q}(\mathcal{M}) = \mathcal{E}xt^i_{\mathcal{S}}({}_{p<}\sigma_{\leq q} R\mathcal{H}om_{\mathcal{S}}(\mathcal{M},\mathcal{S}),\mathcal{S})$$

for $p \leq q$. Then,

(i) for $p \leq q \leq r$, we get the long exact sequence:

$$\cdots \to T^i_{q,r} \to T^i_{p,r} \to T^i_{p,q} \to T^{i+1}_{q,r} \to \cdots ,$$

(ii) $T^i_{p,q} = 0$ for $p = q$, $T^i_{p,q} = T^i_{-1,q}$ for $p < 0$,

(iii) $T^i_{q-1,q} = \mathcal{E}xt^{i+q}_{\mathcal{S}}(\mathcal{E}xt^q_{\mathcal{S}}(\mathcal{M},\mathcal{S}),\mathcal{S})$,

(iv) $T^i_{p,q} = 0$ for $q < 0$,

(v) $T^i_{p,q} = 0$ for $i + q < 0$.

Now, let X be an n-dimensional complex manifold, and let $\mathcal{S} = \mathcal{D}_X$. Clearly, if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then $T^i_{p,q}$ is also a coherent $\mathcal{D}_X$-module.

Proposition 3.2.2. *Let $\mathcal{M}$ be coherent $\mathcal{D}_X$-module. Then,*

(vi) $T^i_{p,q}(\mathcal{M}) = 0$ *for* $p \geq n$, $T^i_{p,q} = T^i_{p,n}$ *for* $q \geq n$,

(vii) $T^i_{p,q}(\mathcal{M}) = 0$ *for* $i + p \geq n$,

(viii) $T^i_{p,q}(\mathcal{M}) = 0$ *for* $i < 0$,

(ix) $T^i_{p,q}(\mathcal{M}) = 0$ *for* $i \neq 0$, $p < 0$, $q \geq n$,

(x) $T^i_{p,q}(\mathcal{M}) = \mathcal{M}$ *for* $i = 0$, $p < 0$, $q \geq n$,

(xi) $T^n_{p,0}(\mathcal{M}) = 0$ *for* $p < 0$, $n \geq 1$,

(xii) $T^{n-1}_{p,0}(\mathcal{M}) = 0$ *for* $p < 0$, $n \geq 2$.

Proof. (vi). Since $H^i R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) = 0$ for $i > n$, we get

$$\begin{array}{rcll} \sigma_{>p} R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) & = & 0 & \text{for } p \geq n, \\ \sigma_{\leq q} R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) & = & R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) & \text{for } q \geq n. \end{array}$$

(vii). Using an induction on $q - p$, and (i), we are reduced to the case $p = q - 1$. Then,

$$T^i_{p,q}(\mathcal{M}) = \mathcal{E}xt^{i+q}_{\mathcal{D}}(\mathcal{E}xt^q_{\mathcal{D}}(\mathcal{M}, \mathcal{D}), \mathcal{D}) = 0$$

for $i + q > n$.

(viii). In the same way, working by induction on $q - p$, we may assume $p = q - 1$. It is sufficient to show

$$\mathcal{E}xt^p_{\mathcal{D}}(\mathcal{E}xt^q_{\mathcal{D}}(\mathcal{M}, \mathcal{D}), \mathcal{D}) = 0 \qquad \text{for } p < q.$$

Using Theorem 3.1.3, we get

$$\operatorname{cd} \operatorname{char}(\mathcal{E}xt^q_{\mathcal{D}}(\mathcal{M}, \mathcal{D})) \geq q,$$

and Theorem 3.1.2 shows that

$$\mathcal{E}xt^p_{\mathcal{D}}(\mathcal{E}xt^q_{\mathcal{D}}(\mathcal{M}, \mathcal{D}), \mathcal{D}) = 0 \qquad \text{for } p < q.$$

(ix), (x). If $p < 0$, and $q \geq n$, the conclusion follows from

$$_{p<}\sigma_{\leq q} R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) = R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}),$$

$$\mathcal{M} = R\mathcal{H}om_{\mathcal{D}}(R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}), \mathcal{D}).$$

(xi), (xii). If $p < 0$, we have

$$_{p<}\sigma_{\leq 0} R\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) = \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{D}).$$

Furthermore,

$$T^i_{p,0}(\mathcal{M}) = \mathcal{E}xt^i_{\mathcal{D}}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D}),\mathcal{D}).$$

If we choose a free resolution $0 \leftarrow \mathcal{M} \leftarrow \mathcal{L}_0 \leftarrow \mathcal{L}_1$, we get the exact sequence

$$0 \to \mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D}) \to \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_0,\mathcal{D}) \to \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_1,\mathcal{D}).$$

Let us denote by $\mathcal{N}$ the cokernel of

$$\mathcal{H}om_{\mathcal{D}}(\mathcal{L}_0,\mathcal{D}) \to \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_1,\mathcal{D}).$$

We know that

$$\begin{aligned} \text{proj-dim}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D}) &\leq \max(\text{proj-dim}\,\mathcal{N} - 2, 0) \\ &\leq \max(n-2, 0). \end{aligned}$$

From this, (xi) and (xii) follow. □

Proposition 3.2.3. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then,*

$$\mathrm{cd}(T^i_{p,q}(\mathcal{M})) > i + p$$

Proof. Working by induction on $q - p$, we may assume $q = p + 1$. Then,

$$T^i_{p,q}(\mathcal{M}) = \mathcal{E}xt^{i+p+1}_{\mathcal{D}}(\mathcal{E}xt^{p+1}_{\mathcal{D}}(\mathcal{M},\mathcal{D}),\mathcal{D}).$$

Following Theorem 3.1.3

$$\mathrm{cd}(T^i_{p,q}) \geq i + p + 1.$$

□

Proposition 3.2.4. *Assume $\mathcal{M}$ is a system such that* $\mathrm{cd}(\mathcal{M}) > q$. *Then,*

$$T^i_{p,q}(\mathcal{M}) = 0.$$

Proof. Working by induction on $q - p$, it is enough to consider the case $p = q - 1$. Then,

$$T^i_{p,q}(\mathcal{M}) = \mathcal{E}xt^{i+q}_{\mathcal{D}}(\mathcal{E}xt^{q}_{\mathcal{D}}(\mathcal{M},\mathcal{D}),\mathcal{D}).$$

Following Theorem 3.1.2, $\mathcal{E}xt^q_{\mathcal{D}}(\mathcal{M},\mathcal{D}) = 0$. □

Theorem 3.2.5. *Let $\mathcal{M}$ be a system on X. Then,*

$$0 = T^0_{n,n}(\mathcal{M}) \subset T^0_{n-1,n}(\mathcal{M}) \subset T^0_{n-2,n}(\mathcal{M}) \subset \cdots \subset T^0_{0,n}(\mathcal{M}) \subset T^0_{-1,n}(\mathcal{M}) = \mathcal{M},$$

and

$$T^0_{q,n}(\mathcal{M}) = \{s \in \mathcal{M} : \mathrm{cd}(s) > q\}.$$

Proof. We have already proven in Proposition 3.2.2 that $T^0_{-1,n}(\mathcal{M}) = \mathcal{M}$, $T^0_{n,n}(\mathcal{M}) = 0$. We have the exact sequence

$$T^{-1}_{q-1,q}(\mathcal{M}) \to T^0_{q,n}(\mathcal{M}) \to T^0_{q-1,n}(\mathcal{M}).$$

Following Proposition 3.2.2, we get

$$T^{-1}_{q-1,q}(\mathcal{M}) = 0.$$

Hence, $T^0_{q,n}(\mathcal{M}) \to T^0_{q-1,n}(\mathcal{M})$ is injective. So, we have proven:

$$0 = T^0_{n,n}(\mathcal{M}) \subset T^0_{n-1,n}(\mathcal{M}) \subset \cdots \subset T^0_{-1,n}(\mathcal{M}) = \mathcal{M}.$$

Following Proposition 3.2.3,

$$\mathrm{cd}(T^0_{q,n}(\mathcal{M})) > q.$$

Assume for a while that

$$\mathrm{cd}(\mathcal{M}) > q.$$

Then, the last term of the exact sequence

$$0 \to T^0_{q,n}(\mathcal{M}) \to T^0_{-1,n}(\mathcal{M}) \to T^1_{-1,q}(\mathcal{M})$$

is 0, following Proposition 3.2.4. Therefore, $\mathcal{M} = T^0_{q,n}(\mathcal{M})$.

For any fixed $x \in X$, set

$$M_q = \{s \in \mathcal{M}_x : \mathrm{cd}(s) > q\}.$$

Let $\mathcal{M}_q$ be a coherent sub $\mathcal{D}_X$-module of $\mathcal{M}$, extending M_q in a neighborhood of x. Shrinking this neighborhood if necessary, we may assume

$$\mathrm{cd}(\mathcal{M}_q) > q.$$

Naturally, $T^0_{q,n}(\mathcal{M}) \subset \mathcal{M}_q$. If we consider the diagram

$$\begin{tikzcd} \mathcal{M}_q \arrow[r] & \mathcal{M} \\ T^0_{q,n}(\mathcal{M}_q) \arrow[u] \arrow[r] & T^0_{q,n}(\mathcal{M}), \arrow[u] \end{tikzcd}$$

we get

$$\mathcal{M}_q \subset T^0_{q,n}(\mathcal{M}).$$

Hence, $T^0_{q,n}(\mathcal{M}) = \mathcal{M}_q$. Moreover, for any x, we get $T^0_{q,n}(\mathcal{M})_x = M_q$. $\square$

Theorem 3.2.6. *Let $\mathcal{M}$ be a system, and d an integer. Then, the following two conditions are equivalent.*

(a) $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$ *for* $i \leq d$,

(b) $\mathrm{cd}(\mathcal{M}) > d$.

Proof. We have already proven that (b) $\Rightarrow$ (a). Let us prove that (a) $\Rightarrow$ (b). To this end, let us consider the exact sequence

$$T^0_{d,n}(\mathcal{M}) \to T^0_{-1,n}(\mathcal{M}) \to T^0_{-1,d}(\mathcal{M}).$$

Following Proposition 3.2.3, we have

$$\mathrm{cd}(T^0_{d,n}(\mathcal{M})) > d.$$

From (a), we get

$$T^0_{-1,d}(\mathcal{M}) = 0.$$

Hence, recalling that $\mathcal{M} = T^0_{-1,n}(\mathcal{M})$, we get $\mathrm{cd}(\mathcal{M}) > d$. □

Proposition 3.2.7. *For any system* $\mathcal{M}$,

$$\mathrm{char}(\mathcal{M}) = \bigcup_{i=0}^{n} \mathrm{char}(\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)).$$

Proof. We have already proven the inclusion $\supset$. Let us now prove the opposite inclusion. Set $Z = \bigcup_i \mathrm{char}(\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}, \mathcal{D}))$. For $p = q - 1$, $\mathrm{char}(T^i_{p,q}) \subset Z$. If we use the exact sequence

$$T^i_{p+1,q} \to T^i_{p,q} \to T^i_{p,p+1},$$

and an induction on $q - p$, we get

$$\mathrm{char}(T^i_{p,q}) \subset Z$$

for any i, p, and q. Hence,

$$\mathrm{char}(\mathcal{M}) = \mathrm{char}(T^0_{-1,n}(\mathcal{M})) \subset Z.$$

□

Theorem 3.2.8. *Let* $\mathcal{M}$ *be a coherent* $\mathcal{D}_X$*-module. In order for* $\mathcal{M}$ *to locally admit a resolution of length* k*:*

$$0 \to \mathcal{M} \to \mathcal{D}_X^{r_0} \to \mathcal{D}_X^{r_1} \to \cdots \to \mathcal{D}_X^{r_k},$$

it is necessary and sufficient that

$$T^i_{0,n}(\mathcal{M}) = 0 \qquad \textit{for } i \leq k.$$

Proof. Let $\mathcal{N}$ be the cokernel of $\mathcal{D}^{r_{k-1}} \to \mathcal{D}^{r_k}$. Let

$$\cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{M} \to 0$$

be a free resolution of $\mathcal{M}$. Then,

$$\cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{D}^{r_0} \to \mathcal{D}^{r_1} \cdots \to \mathcal{D}^{r_k} \to \mathcal{N} \to 0$$

becomes a free resolution of $\mathcal{N}$. Moreover,

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}}(\mathcal{N},\mathcal{D}) &= \{\mathcal{D}^{r_k} \to \mathcal{D}^{r_{k-1}} \cdots \to \mathcal{D}^{r_0} \to \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_0,\mathcal{D}) \to \cdots\} \\ R\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D}) &= \{\mathcal{H}om_{\mathcal{D}}(\mathcal{L}_0,\mathcal{D}) \to \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_1,\mathcal{D}) \to \cdots\}. \end{aligned}$$

Hence, denoting by $\mathcal{I}$ the image of $\mathcal{H}om_{\mathcal{D}}(\mathcal{L}_0,\mathcal{D}) \to \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_1,\mathcal{D})$, we get:

$$\begin{aligned} \sigma_{>0} R\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D}) &= \{\mathcal{I} \to \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_1,\mathcal{D}) \to \cdots\} \\ &= (\sigma_{>k+1} R\mathcal{H}om_{\mathcal{D}}(\mathcal{N},\mathcal{D}))[k+1]. \end{aligned}$$

So,

$$R\mathcal{H}om_{\mathcal{D}}(\sigma_{>0} R\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D}),\mathcal{D}) = R\mathcal{H}om_{\mathcal{D}}(\sigma_{>k+1} R\mathcal{H}om_{\mathcal{D}}(\mathcal{N},\mathcal{D}),\mathcal{D})[-k-1]$$

and

$$T^i_{0,n}(\mathcal{M}) = T^{i-k-1}_{k+1,n}(\mathcal{N}).$$

It follows that the left side is zero for $i-k-1<0$, or, in other words, that for $i \leq k$, $T^i_{0,n}(\mathcal{M}) = 0$. Conversely, assuming $T^i_{0,n}(\mathcal{M}) = 0$ for $i \leq k$, let us prove that $\mathcal{M}$ has a right free resolution of length k. Let

$$0 \leftarrow \mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D}) \leftarrow \mathcal{L}_0 \leftarrow \mathcal{L}_1 \leftarrow \cdots \leftarrow \mathcal{L}_k$$

be a free resolution of length k of $\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D})$. Using

$$\mathcal{M} \to \mathcal{H}om_{\mathcal{D}}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D}),\mathcal{D}),$$

it follows that

$$0 \to \mathcal{M} \to \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_0,\mathcal{D}) \to \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_1,\mathcal{D}) \cdots \to \mathcal{H}om_{\mathcal{D}}(\mathcal{L}_k,\mathcal{D}).$$

To prove that this is a free resolution of $\mathcal{M}$, we have to show that

$$\begin{cases} \mathcal{E}xt^i_{\mathcal{D}}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D}),\mathcal{D}) = 0, & \text{for } 0<i<k \text{ (if } k \geq 1\text{)}, \\ \mathcal{M} \to \mathcal{H}om_{\mathcal{D}}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D}),\mathcal{D}), & \text{is injective for } k=0, \\ \mathcal{M} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D}),\mathcal{D}), & \text{for } k \geq 1. \end{cases}$$

However,

$$0 = T^{-1}_{-1,0}(\mathcal{M}) \to T^0_{0,n}(\mathcal{M}) \to T^0_{-1,n}(\mathcal{M}) \to T^0_{-1,0}(\mathcal{M}) \to T^1_{0,n}(\mathcal{M}) \to T^1_{-1,n}(\mathcal{M}) = 0,$$

where $T^0_{-1,n}(\mathcal{M}) = \mathcal{M}$,

$$0 = T^{i-1}_{-1,n}(\mathcal{M}) \to T^{i-1}_{-1,0}(\mathcal{M}) \to T^{i}_{0,n}(\mathcal{M}) \to T^{i}_{-1,n}(\mathcal{M}) = 0$$

for $i > 1$, and

$$\begin{aligned} T^0_{0,n}(\mathcal{M}) &= \ker(\mathcal{M} \to \mathcal{H}om_{\mathcal{D}}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D}),\mathcal{D})), \\ T^1_{0,n}(\mathcal{M}) &= \operatorname{coker}(\mathcal{M} \to \mathcal{H}om_{\mathcal{D}}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D}),\mathcal{D})), \\ T^i_{0,n}(\mathcal{M}) &= \mathcal{E}xt^{i-1}_{\mathcal{D}}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{D}),\mathcal{D}) \text{ for } i > 1. \end{aligned}$$

Moreover, by hypothesis $T^i_{0,n}(\mathcal{M}) = 0$ for $i \leq k$, and the conclusion follows immediately. □

In the constant coefficient case (where the ring is commutative), the preceding facts are found in Palamodov [13].

Chapter 4

Index theorem in dimension 1

In this chapter, X is a complex manifold of dimension 1. Therefore, $P^*X \xrightarrow{\sim} X$.

4.1 Special properties of systems in dimension 1

Proposition 4.1.1. *Let $\mathcal{M}$ be a system on X.*

(a) In order for $\mathcal{M}$ to be projective, it is necessary and sufficient that $\mathcal{M}$ be torsion-free.

(b) In order for $\mathcal{M}$ to be maximally overdetermined[1], it is necessary and sufficient that $\mathcal{M}$ be a torsion module.

Proof. (a). If $\mathcal{M}$ is projective, then $\mathcal{M} \subset \mathcal{D}_X^\ell$ and $\mathcal{M}$ is torsion-free. Conversely, assume $\mathcal{M}$ is torsion-free. We get an inclusion $\mathcal{M} \subset \mathcal{D}_X^\ell$. Since the global dimension of $\mathcal{D}_X^\ell$ is 1, the projective dimension of $\mathcal{D}_X^\ell/\mathcal{M}$ is ≤ 1. Therefore, it follows from

$$0 \to \mathcal{M} \to \mathcal{D}_X^\ell \to \mathcal{D}_X^\ell/\mathcal{M} \to 0$$

that $\mathcal{M}$ is projective.

As for (b), in the one dimensional case, $\mathcal{M}$ is maximally overdetermined if and only if $\operatorname{char}(\mathcal{M}) \neq P^*X$. □

Proposition 4.1.2. *Assume that $\mathcal{M}$ is a maximally overdetermined system. Then, for $x \in X$ the $\mathcal{D}_{X,x}$-module $\mathcal{M}_x$ has finite length.*

Proof. It is sufficient to show that a decreasing sequence $M_1 \supset M_2 \supset \cdots$ of submodules of $\mathcal{M}_x$ is stationary. Let us extend M_i to a maximally overdetermined system $\mathcal{M}_i$ in a neighborhood of x. Let us denote by m_i the order of $\mathcal{M}_i$, and n_i its multiplicity at x (see Definition 4.2.2). Obviously, $m_i \geq m_{i+1}$, $n_i \geq n_{i+1}$.

[1] *Translators' note:* In general, a system $\mathcal{M}$ on X is called maximally overdetermined if $\operatorname{cd}(\mathcal{M}) \geq \dim X$. In today's literature, such systems are called holonomic.

Therefore, $m_i = m_{i+1}$, $n_i = n_{i+1}$ for $i \gg 0$. Hence, the order and the multiplicity of $\mathcal{M}_i/\mathcal{M}_{i+1}$ are 0. Therefore,

$$\mathcal{M}_i/\mathcal{M}_{i+1} = 0, \qquad \text{and } M_i = M_{i+1}.$$

□

4.2 Local index theorem

The aim of this section[2] is to show that for a maximally overdetermined system $\mathcal{M}$, and for $x \in X$, the local index of $\mathcal{M}$ at x, defined by

$$\chi_x(\mathcal{M}) = \sum_{\nu=0}^{1} (-1)^\nu \dim_{\mathbb{C}} \mathcal{E}xt^\nu_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_x$$

may be obtained using "commutative" invariants of $\mathcal{M}$The first step in this direction will be to compute this local index for a maximally overdetermined system $\mathcal{M}$ of a special type. Using the second invariance theorem for the index in Banach spaces, we can obtain a similar result. But, here, we will use a direct method based on Taylor series expansions. (In the following proof, we follow the ideas of Prof. Aomoto.)

Proposition 4.2.1. *Let V be a finite dimensional vector space, $A(t)$ be an* $\mathrm{End}(V)$*-valued holomorphic function defined in a neighborhood of the origin* 0. *Then, the kernel and cokernel of*

$$P = t^n \partial_t - A(t) : V \otimes \mathcal{O}_0 \to V \otimes \mathcal{O}_0$$

are finite dimensional, and

$$\dim \ker P - \dim \operatorname{coker} P = (1-n) \dim V.$$

Proof. For $n = 0$, the theorem is clear (in fact $\dim \ker P = \dim V$, $\operatorname{coker} P = 0$). Below, we assume $n \geq 1$. Let us endow V with a suitable hermitian metric. $A(t)$ is holomorphic for $t < R$ $(R > 0)$, so we may develop it as

$$A(t) = \sum_{\nu=0}^{\infty} A_\nu t^\nu$$

with

$$|A_\nu| \leq cR^{-\nu} \qquad (c > 0).$$

[2] *Translators' note:* The index theorem for holonomic systems in higher dimension was later treated by Kashiwara in [M. Kashiwara, *Index theorem for a maximally overdetermined system of linear differential equations*, Proc. Japan Acad. Ser. A Math. Sci. **49** (1973), 803–804]. For recent developments on the index theorem for $\mathcal{D}$-modules, see [P. Schapira and J.-P. Schneiders, Index theorem for elliptic pairs, Astérisque 224 (1995)]

Set
$$\mathcal{B}_r^N = \{a = (a_\nu)_{\nu \geq N} : a_\nu \in V, |a_\nu| \leq cr^{-\nu} \text{ for some } c \geq 0\}.$$
Using the norm defined by
$$\|a\|_r = \sup_{\nu \geq N} |a_\nu| r^\nu,$$
$\mathcal{B}_r^N$ becomes a Banach space. Below, we assume $0 < r < R$. Let us define the linear operator
$$\begin{array}{rccc} A^* : & \mathcal{B}_r^N & \to & \mathcal{B}_r^N \\ & a & \mapsto & b, \end{array}$$
where
$$b = (b_\nu), \qquad b_\nu = \frac{1}{\nu} \sum_{\mu=N}^{\nu+n-1} A_{\nu+n-1-\mu} a_\mu \qquad \text{for } \nu \geq N.$$
We have
$$\begin{aligned} |b_\nu| &\leq \frac{1}{N} \sum_{\mu=N}^{\nu+n-1} cR^{-(\nu+n-1-\mu)} r^{-\mu} \|a\|_r, \\ &\leq \frac{cr^{1-n-\nu}}{N} (1 - \frac{r}{R})^{-1} \|a\|_r. \end{aligned}$$
Therefore,
$$\|b\|_r \leq \frac{cr^{1-n}}{N} (1 - \frac{r}{R})^{-1} \|a\|_r,$$
and
$$\|A^*\| \leq \frac{cr^{1-n}}{N} (1 - \frac{r}{R})^{-1}.$$
Hence, for $N > cr^{1-n}(1 - r/R)^{-1}$,
$$\|A^*\| < 1.$$
Hence, using the Neumann series,
$$1 - A^* : \mathcal{B}_r^N \to \mathcal{B}_r^N$$
is bijective for $N > cr^{1-n}(1 - r/R)^{-1}$. Let us set
$$\begin{aligned} \mathcal{A}_r^N &= \{u \in V \otimes t^N \mathcal{O}_0 : u = \sum_{\nu=N}^{\infty} a_\nu t^\nu, a_\nu \in V, |a_\nu| \leq \alpha r^{-\nu} \text{ for } {}^\exists \alpha \geq 0\}, \\ \widetilde{\mathcal{A}}_r^N &= \{u \in V \otimes t^N \mathcal{O}_0 : u = \sum_{\nu=N}^{\infty} a_\nu t^\nu, a_\nu \in V, |a_\nu| \leq \alpha(\nu+1) r^{-\nu} \text{ for } {}^\exists \alpha \geq 0\}. \end{aligned}$$
Set also $\mathcal{A}_r = \mathcal{A}_r^0$, $\widetilde{\mathcal{A}}_r = \widetilde{\mathcal{A}}_r^0$. We get
$$\varinjlim_{r \downarrow 0} \mathcal{A}_r^N = \varinjlim_{r \downarrow 0} \widetilde{\mathcal{A}}_r^N = V \otimes t^N \mathcal{O}_0.$$

Let us define $\varphi : \mathcal{A}_r^N \to \mathcal{B}_r^N$ by the formula

$$\mathcal{A}_r^N \ni u = \sum_{\nu \geq N} a_\nu t^\nu \mapsto a = (a_\nu) \in \mathcal{B}_r^N,$$

and $\psi : \widetilde{\mathcal{A}}_r^N \to \mathcal{B}_r^N$ by the formula

$$\widetilde{\mathcal{A}}_r^N \ni u = \sum_{\nu \geq N} a_\nu t^\nu \mapsto b = (b_\nu) \in \mathcal{B}_r^N, \quad b_\nu = \frac{1}{\nu} a_{\nu+n-1}.$$

Then, the diagram

$$\begin{tikzcd} \mathcal{A}_r^N \arrow[r, "P_r"] \arrow[d, "\varphi"] & \widetilde{\mathcal{A}}_r^N \arrow[d, "\psi"'] \\ \mathcal{B}_r^N \arrow[r, "1-A^*"'] & \mathcal{B}_r^N \end{tikzcd}$$

is commutative. In fact, for $u = \sum_{\nu \geq N} a_\nu t^\nu$, we get

$$\begin{aligned} P_r u &= \sum_{\nu \geq N} \nu a_\nu t^{\nu+n-1} - \sum_{N \leq \mu \leq \nu} A_{\nu-\mu} a_\mu t^\nu \\ &= \sum_{\nu \geq N} \left[(\nu - n + 1) a_{\nu-n+1} - \sum_{N \leq \mu \leq \nu} A_{\nu-\mu} a_\mu \right] t^\nu. \end{aligned}$$

Therefore, if we set $\psi(P_r u) = b$, then

$$b_\nu = a_\nu - \frac{1}{\nu} \sum_{N \leq \mu \leq \nu+n-1} A_{\nu+n-1-\mu} a_\mu,$$

and $b = (1 - A^*)a$. So, the diagram is commutative. Since φ is bijective, and ψ is surjective with kernel

$$\sum_{\nu=N}^{N+n-2} V \otimes \mathbb{C} t^\nu = \mathbb{C}^{(n-1)\dim V},$$

it follows that

$$P_r : \mathcal{A}_r^N \to \widetilde{\mathcal{A}}_r^N$$

is injective with cokernel $\mathbb{C}^{(n-1)\dim V}$. Note that $P_r : \mathcal{A}_r \to \widetilde{\mathcal{A}}_r$ induces

$$P : V \otimes \mathcal{O}_0 \to V \otimes \mathcal{O}_0$$

by restriction. The diagram

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & 0 & \longrightarrow & \mathcal{A}_r^N & \longrightarrow & \widetilde{\mathcal{A}}_r^N & \longrightarrow & \mathbb{C}^{(n-1)\dim V} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker P_r & \longrightarrow & \mathcal{A}_r & \longrightarrow & \widetilde{\mathcal{A}}_r & \longrightarrow & \operatorname{coker} P_r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker f & \longrightarrow & \mathcal{A}_r/\mathcal{A}_r^N & \xrightarrow{f} & \widetilde{\mathcal{A}}_r/\widetilde{\mathcal{A}}_r^N & \longrightarrow & \operatorname{coker} f & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

is commutative, and its lines are exact. Since the second and third columns are also exact, we get the exact sequence

$$0 \to \ker P_r \to \ker f \to \mathbb{C}^{(n-1)\dim V} \to \operatorname{coker} P_r \to \operatorname{coker} f \to 0.$$

Since

$$\dim(\mathcal{A}_r/\mathcal{A}_r^N) = \dim(\widetilde{\mathcal{A}}_r^N/\widetilde{\mathcal{A}}_r^N) = N \dim V < \infty,$$

$\dim\ker f = \dim\operatorname{coker} f < \infty$. Therefore, $\dim\ker P_r$, $\dim\operatorname{coker} P_r$ are finite and

$$\dim\ker P_r - \dim\operatorname{coker} P_r = (1-n)\dim V.$$

Note that

$$\ker P = \varinjlim_{r\downarrow 0} \ker P_r, \qquad \operatorname{coker} P = \varinjlim_{r\downarrow 0} \operatorname{coker} P_r.$$

Moreover, $\ker P_r \xrightarrow{\sim} \ker P$ $(0 < r < R)$. This follows from the fact that, since P is non-degenerate for $t \neq 0$, the solutions of P extend naturally. Therefore, $\dim\ker P = \dim\ker P_r < \infty$, so $\dim\operatorname{coker} P_r$ does not depend on r. Below, we prove that $\operatorname{coker} P_r \to \operatorname{coker} P$ is injective. To this end, consider $u \in V \otimes \mathcal{O}_0$. Assume that $v = Pu \in \widetilde{\mathcal{A}}_r$. It is sufficient to show that $u \in \mathcal{A}_r$. Following the definition of $\mathcal{A}_r$, we may assume

$$u = \sum_{\nu=N}^{\infty} a_\nu t^\nu.$$

Working as above, it follows from

$$\widetilde{\mathcal{A}}_r^N = P(\mathcal{A}_r^N) + \sum_{\nu=N}^{N+n-2} V \otimes t^\nu \mathbb{C}$$

that it is possible to write

$$v = P\widetilde{u} + \sum_{\nu=N}^{N+n-2} c_\nu t^\nu, \qquad \widetilde{u} \in \mathcal{A}_r^N, \quad c_\nu \in V$$

Therefore,

$$P(u - \widetilde{u}) = \sum_{\nu=N}^{N+n-2} c_\nu t^\nu,$$

and $u - \widetilde{u}$ is analytic on $D = \{t : |t| < R\}$. Hence, $u - \widetilde{u} \in \mathcal{A}_r$, and so $u \in \mathcal{A}_r$. Therefore, $\operatorname{coker} P_r \to \operatorname{coker} P$ is injective. Hence, for $0 < r' < r$,

$$\operatorname{coker} P_r \to \operatorname{coker} P_{r'}$$

is injective. Since $\dim \operatorname{coker} P_r = \dim \operatorname{coker} P_{r'}$,

$$\operatorname{coker} P_r \xrightarrow{\sim} \operatorname{coker} P_{r'} \qquad \text{for } 0 < r' < r < R$$

and

$$\operatorname{coker} P_r \xrightarrow{\sim} \varinjlim_{r' \downarrow 0} \operatorname{coker} P_{r'} \xrightarrow{\sim} \operatorname{coker} P.$$

Therefore,

$$\begin{aligned} \dim \ker P - \dim \operatorname{coker} P &= \dim \ker P_r - \dim \operatorname{coker} P_r \\ &= (1-n) \dim V. \end{aligned}$$

□

In order to extend the preceding results to the case of a maximally overdetermined system, we introduce the following definition.

Definition 4.2.2. Let $\mathcal{M}$ be a maximally overdetermined system. Outside its characteristic variety, $\mathcal{M}$ is locally of the form $\mathcal{O}_X^m$. For X connected, this m depends only on $\mathcal{M}$. We call it the order of $\mathcal{M}$.

We will extend the preceding discussion to the general case of Proposition 4.1.1. Let $\mathcal{M}$ be a maximally overdetermined system, and $x \in X$. The local index $\chi_x(\mathcal{M})$ of $\mathcal{M}$ at x is defined by

$$\chi_x(\mathcal{M}) = \sum_{i=0}^{1} (-1)^i \dim \mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}, \mathcal{O})_x.$$

For $x \notin \operatorname{char}(\mathcal{M})$, it is clear that the local index of $\mathcal{M}$ is its order.

Theorem 4.2.3. *Let $\mathcal{M}$ be a maximally overdetermined system of order m. Let $x \in X$, and n be the multiplicity of $\mathcal{M}$ at x (see Definition 1.4.7). Then, the local index of $\mathcal{M}$ is $m - n$.*

Proof. Step one. Let V be a finite dimensional vector space, let $x = 0$, let $A(t) \in \Gamma(X; \mathcal{O} \otimes \mathrm{End}(V))$, let $P = t^n \partial_t - A(t)$, and assume $\mathcal{M}$ is the cokernel of

$$\mathcal{D} \underset{\mathbb{C}}{\otimes} V \xleftarrow{P} \mathcal{D} \underset{\mathbb{C}}{\otimes} V.$$

From the exact sequence

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D} \otimes V \leftarrow \mathcal{D} \otimes V \leftarrow 0,$$

it is clear that $\mathrm{char}(\mathcal{M}) = \{0\}$. The group $\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M}, \mathcal{O})_x$ is the cohomology of

$$\mathcal{O}_x \otimes V^* \xrightarrow{P} \mathcal{O}_x \otimes V^*.$$

Following Proposition 4.2.1,

$$\chi_x(\mathcal{M}) = (1 - n) \dim V.$$

If we endow $\mathcal{M}$ with the filtration induced by $\mathcal{D} \otimes V$, we get

$$\mathrm{gr}\, \mathcal{M} = (\overline{\mathcal{D}}/t^n\, \tau\, \overline{\mathcal{D}}) \otimes V,$$

where τ is the section of $\overline{\mathcal{D}}$ associated to ∂_t. Therefore, the multiplicity of $\mathcal{M}$ is $n \dim V$. In a neighborhood of $x \neq 0$, since $\mathcal{M} \xrightarrow{\sim} \mathcal{O}_X^{\dim V}$, the order of $\mathcal{M}$ is $\dim V$. Hence, the theorem holds for the special case treated above.

Step two. The case $\mathrm{supp}\, \mathcal{M} \subset \{x\}$. In this case the order is 0. Since both the multiplicity and the local index are additive, we may assume that $\mathcal{M}$ is generated by a section u. Let us choose a coordinate such that $x = 0$. Since $\mathrm{supp}(\mathcal{O}u) \in \{0\}$, there is n such that

$$t^n u = 0.$$

Hence, we get the filtration

$$\mathcal{M} = \mathcal{D}\, u \supset \mathcal{D}\, t\, u \supset \cdots \supset \mathcal{D}\, t^{n-1}\, u \supset \mathcal{D}\, t^n\, u = 0.$$

Passing to the corresponding graduation, we are reduced to the case where $tu = 0$ and $u \neq 0$. Next we will use the following lemma.

Lemma 4.2.4. *Let* $X = \mathbb{C}^n$, *let* $x \in X$, *and let* $\mathfrak{m}$ *be the maximal ideal of* $\mathcal{O}_x$. *Then,* $\mathcal{D}_x/\mathcal{D}_x\, \mathfrak{m}$ *is a simple* $\mathcal{D}_x$*-module.*

Proof. An element P of $M = \mathcal{D}_x/\mathcal{D}_x\, \mathfrak{m}$ may be represented by

$$\sum_{|\alpha| \leq m} a_\alpha \partial_x^\alpha \qquad a_\alpha \in \mathbb{C}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}.$$

Let $N \subset M$ be a non zero sub $\mathcal{D}_x$-module. We must prove that $N = M$. Let $P = \sum_{|\alpha| \leq m} a_\alpha \partial_x^\alpha \in N$. Moreover, assume that $a_\alpha \neq 0$ for some α such that

$|\alpha| = m$, that is, assume that P has order m. For $m > 0$, there is i such that $a_\alpha \neq 0$, $|\alpha| = m$, $\alpha_i > 0$. For such an i, we have

$$[P, x_i] = \sum_{|\alpha| \le m-1} (\alpha_i + 1) a_{\alpha+\delta_i} \partial_x^\alpha \in N,$$

where $\delta_i = (0, \dots, 1, 0, \dots, 0)$, 1 being at the i-th position. This shows that we may assume $m = 0$. Hence, $N = M$. $\square$

Using the preceding lemma, it follows from the fact that $\mathcal{M}$ is a quotient of $\mathcal{D}/\mathcal{D}t$, that $\mathcal{M} = \mathcal{D}/\mathcal{D}t$. In this case, $\mathcal{M}$ has multiplicity 1,

$$\mathcal{E}xt^0_{\mathcal{D}}(\mathcal{M}, \mathcal{O})_0 = \ker(\mathcal{O}_0 \xrightarrow{t} \mathcal{O}_0) = 0,$$
$$\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M}, \mathcal{O})_0 = \operatorname{coker}(\mathcal{O}_0 \xrightarrow{t} \mathcal{O}_0) = \mathbb{C},$$

and the theorem holds.

Step three. The general case. Let $x = 0$. If m denotes the order of $\mathcal{M}$, it is possible to find in a neighborhood U of x, m sections $u_1, \dots, u_m$ of $\mathcal{M}$, such that $u_1, \dots, u_m$ generate $\mathcal{M}$ as an $\mathcal{O}_X$-module on $U \setminus \{0\}$ except on a discrete subset. Let $\mathcal{F}$ be the coherent $\mathcal{O}_X$-module generated by $u_1, \dots, u_m, \partial_t u_1, \dots, \partial_t u_m$. If we consider on $U \setminus \{0\}$ the locally free sheaf $\mathcal{G} = \sum_i \mathcal{O} u_i$ of rank m, then $\operatorname{supp}(\mathcal{F}/\mathcal{G})$ is a discrete set. Hence, shrinking U if necessary, we may assume that $\operatorname{supp}(\mathcal{F}/\mathcal{G}) \subset \{0\}$. Moreover, for a sufficiently large n,

$$t^n \partial_t u_i \in \mathcal{G}.$$

So, $t^n \partial_t u_i = \sum_{j=1}^m a_{ij} u_j$ for some $a_{ij} \in \Gamma(U; \mathcal{O})$. Set

$$Q = t^n \partial_t - (a_{ij}),$$

and denote by $\mathcal{M}'$ the cokernel of $\mathcal{D}^m \xleftarrow{Q} \mathcal{D}^m$. Consider the homomorphism

$$\mathcal{M} \leftarrow \mathcal{M}'$$

induced by u_i. The support of its kernel is included in $\{0\}$. The support of its cokernel is a discrete subset of $U \setminus \{0\}$. Therefore, shrinking U if necessary, we may assume that $\mathcal{M} \leftarrow \mathcal{M}'$ is an isomorphism outside $\{0\}$. Let $\mathcal{N}'$ and $\mathcal{N}$ be its kernel and cokernel. In the exact sequence $0 \leftarrow \mathcal{N} \leftarrow \mathcal{M} \leftarrow \mathcal{M}' \leftarrow \mathcal{N}' \leftarrow 0$, we have already proven the theorem for $\mathcal{M}'$, $\mathcal{N}'$ and $\mathcal{N}$. Thanks to the additivity property of order, multiplicity, and local index, the theorem also holds for $\mathcal{M}$. $\square$

Theorem 4.2.5. *Let X be a 1-dimensional complex manifold admitting a finite triangulation. Let $\mathcal{M}$ be a maximally overdetermined system of order m. Then,* $\operatorname{char}(\mathcal{M})$ *is a finite subset of X. Let $\chi(X)$ be the Euler characteristic of X. Then,* $\operatorname{Ext}^i_{\mathcal{D}_X}(X; \mathcal{M}, \mathcal{O}_X)$ *is finite dimensional, and*

$$\sum_{i=0}^{2} (-1)^i \dim \operatorname{Ext}^i_{\mathcal{D}_X}(X; \mathcal{M}, \mathcal{O}_X) = m\chi(X) - \sum_{x \in \operatorname{char}(\mathcal{M})} n_x,$$

where n_x is the multiplicity of $\mathcal{M}$ at x.

Proof. Set $Z = \mathrm{char}(\mathcal{M})$. For any $x \in Z$, let D_x denote an open disc centered at x, not containing other elements of Z. Set $U = X \setminus (\bigcup_{x\in Z} \overline{D}_x)$. Because of the exact sequence

$$\cdots \to \mathrm{Ext}^i_{\mathcal{D},\bigcup_{x\in Z}\overline{D}_x}(X;\mathcal{M},\mathcal{O}) \to \mathrm{Ext}^i_{\mathcal{D}}(X;\mathcal{M},\mathcal{O}) \to \mathrm{Ext}^i_{\mathcal{D}}(U;\mathcal{M},\mathcal{O}) \to \cdots,$$

we have

$$\begin{aligned}\sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathcal{D}}(X;\mathcal{M},\mathcal{O}) &= \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathcal{D}}(U;\mathcal{M},\mathcal{O}) + \\ &\quad + \sum_i (-1)^i \sum_{x\in Z} \dim \mathrm{Ext}^i_{\mathcal{D},\overline{D}_x}(X;\mathcal{M},\mathcal{O}).\end{aligned}$$

From

$$\mathrm{Ext}^i_{\mathcal{D}}(U;\mathcal{M},\mathcal{O}) = H^i(U;\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{O})),$$

it follows that

$$\begin{aligned}\sum_i (-1)^i \dim H^i(U;\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{O})) &= m\,\chi(U) \\ &= m\,(\chi(X) - \#Z).\end{aligned}$$

Following the exact sequence

$$\cdots \to \mathrm{Ext}^i_{\mathcal{D},\partial D_x}(X;\mathcal{M},\mathcal{O}) \to \mathrm{Ext}^i_{\mathcal{D},\overline{D}_x}(X;\mathcal{M},\mathcal{O}) \to \mathrm{Ext}^i_{\mathcal{D}}(D_x;\mathcal{M},\mathcal{O}) \to \cdots,$$

we get

$$\begin{aligned}\sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathcal{D},\overline{D}_x}(X;\mathcal{M},\mathcal{O}) &= \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathcal{D},\partial D_x}(X;\mathcal{M},\mathcal{O}) + \\ &\quad + \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathcal{D}}(D_x;\mathcal{M},\mathcal{O}).\end{aligned}$$

Moreover,

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{D},\partial D_x}(X;\mathcal{M},\mathcal{O}) &= H^i_{\partial D_x}(X;\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{O})) \\ &= H^{i-1}(\partial D_x; H^1_{\partial D_x}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{O}))).\end{aligned}$$

Therefore,

$$\sum_i (-1)^i \mathrm{Ext}^i_{\mathcal{D},\partial D_x}(X;\mathcal{M},\mathcal{O}) = -m\,\chi(\partial D_x) = 0.$$

Moreover,

$$\mathrm{Ext}^i_{\mathcal{D}}(D_x;\mathcal{M},\mathcal{O}) \xrightarrow{\sim} \mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M},\mathcal{O})_x.$$

As a matter of fact, denoting by D_ε the disc centered at x of radius ε, we get

$$\mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M},\mathcal{O})_x = \varinjlim_{\varepsilon} \mathrm{Ext}^i_{\mathcal{D}}(D_\varepsilon;\mathcal{M},\mathcal{O}).$$

We have the exact sequence

$$\cdots \to \mathrm{Ext}^i_{\mathcal{D},D_x\setminus D_\varepsilon}(D_x;\mathcal{M},\mathcal{O}) \to \mathrm{Ext}^i_{\mathcal{D}}(D_x;\mathcal{M},\mathcal{O}) \to \mathrm{Ext}^i_{\mathcal{D}}(D_\varepsilon;\mathcal{M},\mathcal{O}) \to \cdots,$$

and the isomorphism

$$\mathrm{Ext}^i_{\mathcal{D},D_x\setminus D_\varepsilon}(D_x;\mathcal{M},\mathcal{O}) = H^i_{D_x\setminus D_\varepsilon}(D_x;\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{O})).$$

Since $H^i_{D_x\setminus D_\varepsilon}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{O})) = 0$, the second term vanishes. Therefore,

$$\mathrm{Ext}^i_{\mathcal{D}}(D_x;\mathcal{M},\mathcal{O}) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{D}}(D_\varepsilon;\mathcal{M},\mathcal{O}).$$

So,

$$\mathrm{Ext}^i_{\mathcal{D}}(D_x;\mathcal{M},\mathcal{O}) \xrightarrow{\sim} \varinjlim_{\varepsilon} \mathrm{Ext}^i_{\mathcal{D}}(D_\varepsilon;\mathcal{M},\mathcal{O}) = \mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M},\mathcal{O})_x,$$

and hence

$$\begin{aligned}\sum_i(-1)^i \dim \mathrm{Ext}^i_{\mathcal{D},\overline{D}_x}(X;\mathcal{M},\mathcal{O}) &= \sum_i(-1)^i \dim \mathcal{E}xt^i_{\mathcal{D}}(\mathcal{M},\mathcal{O})_x \\ &= m - n_x.\end{aligned}$$

Finally,

$$\begin{aligned}\sum_i(-1)^i \dim \mathrm{Ext}^i_{\mathcal{D}_X}(X;\mathcal{M},\mathcal{O}_X) &= m\,(\chi(X) - \#Z) + \sum_{x\in Z} m - n_x \\ &= m\,\chi(X) - \sum_{x\in Z} n_x.\end{aligned}$$

□

Proposition 4.2.6. *Let M be a real analytic manifold of dimension 1, let X be a complex neighborhood of M, and let $\mathcal{M}$ be any system over X. Then,*

$$\begin{aligned}\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M},\widetilde{\mathcal{A}}_M) &= 0,\\ \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M) &= 0,\\ \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M},\mathcal{C}_M) &= 0,\end{aligned}$$

for $i > 0$.

Proof. We have an exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}''$, where $\mathcal{M}'$ is maximally overdetermined and $\mathcal{M}''$ is projective. Hence, we may assume, from the beginning, that $\mathcal{M}$ is a maximally overdetermined system. Furthermore, we may assume $\mathrm{char}(\mathcal{M}) \subset M$. Let us show that $\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M},\widetilde{\mathcal{A}}) = 0$ on the upper half part $M_+ \subset S^*M$. Consider a resolution

$$0 \leftarrow \mathcal{M} \leftarrow \mathcal{D}^{r_0} \xleftarrow{P_0} \mathcal{D}^{r_1} \xleftarrow{P_1} \mathcal{D}^{r_2}.$$

Then, $\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M},\widetilde{\mathcal{A}})$ is the cohomology of $\widetilde{\mathcal{A}}^{r_0} \xrightarrow{P_0} \widetilde{\mathcal{A}}^{r_1} \xrightarrow{P_1} \widetilde{\mathcal{A}}^{r_2}$. Let $f_1 \in (\widetilde{\mathcal{A}}_{x+i0})^{r_1}$, and assume $P_1 f_1 = 0$. In the upper half part of $X \setminus M$, we may assume $\mathcal{M} \xrightarrow{\sim} \mathcal{O}_X^{\ell}$. If we consider f_1 as defined on a simply connected region U, then $f_1 = P_0 f_0$ with $f_0 \in \mathcal{O}(U)^{r_0}$. So, $\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M},\widetilde{\mathcal{A}}) = 0$. From the exact sequence

$$0 \to \pi^{-1}\mathcal{A} \to \widetilde{\mathcal{A}} \to \mathcal{C} \to 0,$$

we deduce the exact sequence

$$0 = \mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M},\widetilde{\mathcal{A}}) \to \mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M},\mathcal{C}) \to \mathcal{E}xt^2_{\mathcal{D}}(\mathcal{M},\pi^{-1}\mathcal{A}) = 0.$$

Hence, $\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M},\mathcal{C}) = 0$. Since $\mathcal{B} = \pi_*\widetilde{\mathcal{A}}/\mathcal{A}$, we have the exact sequence

$$0 = \mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M},\pi_*\widetilde{\mathcal{A}}) \to \mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M},\mathcal{B}) \to \mathcal{E}xt^2_{\mathcal{D}}(\mathcal{M},\mathcal{A}) = 0,$$

and $\mathcal{E}xt^1_{\mathcal{D}}(\mathcal{M},\mathcal{B}) = 0$. □

Theorem 4.2.7. *Let M be a real analytic curve, X a complexification of M, let $\mathcal{M}$ be a maximally overdetermined system on X, and consider $x \in M$. Let m be the order of $\mathcal{M}$, and n its multiplicity at x. Then,*

$$\dim \mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{B})_x = n + m,$$

$$\dim \mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{C})_{x\pm i0} = n.$$

Proof. We may assume that $\operatorname{char}(\mathcal{M}) = \{0\}$, $X = \{t : |t| < \varepsilon\}$, $x = 0$. We have $\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{B})_x = \operatorname{Hom}_{\mathcal{D}}(X;\mathcal{M},\mathcal{B}) = \operatorname{Ext}^1_{\mathcal{D},M}(X;\mathcal{M},\mathcal{O})$. From the exact sequence

$$\cdots \to \operatorname{Ext}^i_{\mathcal{D},M}(X;\mathcal{M},\mathcal{O}) \to \operatorname{Ext}^i_{\mathcal{D}}(X;\mathcal{M},\mathcal{O}) \to \operatorname{Ext}^i_{\mathcal{D}}(X \setminus M;\mathcal{M},\mathcal{O}) \to \cdots,$$

it follows that

$$\begin{aligned}
\dim \mathcal{H}om_{\mathcal{D}}(\mathcal{M};\mathcal{B})_x &= \dim \operatorname{Ext}^1_{\mathcal{D},M}(X;\mathcal{M},\mathcal{O}) \\
&= \sum_i (-1)^i \dim \operatorname{Ext}^i_{\mathcal{D}}(X \setminus M;\mathcal{M},\mathcal{O}) \\
&\quad - \sum_i (-1)^i \dim \operatorname{Ext}^i_{\mathcal{D}}(X;\mathcal{M},\mathcal{O}) \\
&= 2m - (m-n) \\
&= m + n.
\end{aligned}$$

Moreover, thanks to the exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \pi_*\mathcal{C} \to 0,$$

we get

$$\begin{aligned}\dim \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \pi_*\mathcal{C})_x &= \dim \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{B}) - \chi_x(\mathcal{M}) \\ &= (m+n) - (m-n) \\ &= 2n.\end{aligned}$$

Therefore, to conclude, it is sufficient to show that

$$\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{C})_{x+i0} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{C})_{x-i0}.$$

To define an isomorphism

$$T : \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{C})_{x+i0} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{C})_{x-i0},$$

consider $f_+(x+i0) \in \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{C})_{x+i0}$. Since f_+ is a multivalued analytic function on $x \neq 0$, it may be extended as an analytic function on $X \setminus \{x \leq 0\}$. We denote its restriction to the lower half plane by $f_-(x - i0)$. If we set

$$T(f_+(x+i0)) = f_-(x - i0),$$

T is clearly an isomorphism. □

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Translators' note: The starred references where yet "to appear" at the time of publication of Kashiwara's thesis.

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