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Micro-Local Analysis of Prehomogeneous Vector Spaces

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Introduction

The purpose of this paper is to give an explicit method to calculate the b-functions of the relative invariants of regular prehomogeneous vector spaces by using the theory of simple holonomic systems of micro-differential equations.

It is proved in [7], [14] and [16] that for a holomorphic function $f(x)$ in x there exist a non-zero polynomial $b(s)$ in s and a polynomial $P(s, x, D)$ in s with differential operators as coefficients satisfying

$$(0.1) \qquad P(s,x,D)\,f(x)^{s+1} = b(s)\,f(x)^s.$$

Such $b(s)$ with the smallest degree is called the *b-function* of $f(x)$. The above equation (0.1) is equivalent to $(P(s,x,D)f(x)-b(s))f(x)^s=0$. Therefore, in principle, we can calculate $b(s)$ if we know the system of differential equations to which $f(x)^s$ is a solution. When $f(x)$ is a relative invariant of a regular prehomogeneous vector space, $f(x)^s$ satisfies the system of the first-order differential equations derived from the relative invariance of $f(x)$. This is the case that we treat in this paper.

Now we shall explain how the micro-local analysis is applied to obtain $b(s)$. Let V be a vector space over $\mathbb{C}$ and G a closed subgroup of $GL(V)$. If V has an open dense orbit, (G, V) is called a prehomogeneous vector space. We assume that there is a unique relatively invariant irreducible polynomial $f(x)$ on V. We assume further that the Hessian of $f(x)$ is not identically zero. Let χ be the character of $f(x)$ and $\delta\chi$ its infinitesimal character. Then $f(x)^s$ satisfies the following system of differential equations

$$(0.2) \qquad (\langle Ax, D_x\rangle - s\,\delta\chi(A))\,u(x) = 0 \qquad \text{for } A \in \mathfrak{g}$$

where $\mathfrak{g}$ denotes the Lie algebra of G.

A subvariety Λ of $V \times V^*$ is called a good Lagrangian subvariety if $\dim V = \dim \Lambda$ and Λ is a G-orbit contained in the characteristic variety of the system (0.2). For such Λ, there exists a polynomial $b_\Lambda(s)$ in s and an invertible micro-

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differential operator P_Λ such that $P_\Lambda f(x)^{s+1} = b_\Lambda(s) f(x)^s$ holds in a neighborhood of Λ. This $b_\Lambda(s)$ is uniquely determined and called the local b-function of Λ. We have $b_{V\times\{0\}}(s)=1$ and $b_{\{0\}\times V^*}(s)=b(s)$. Hence if we know the relation among the local b-functions, we can obtain the b-function $b(s)$ of $f(x)$. In fact we can get the ratio of $b_{\Lambda_0}(s)$ and $b_{\Lambda_1}(s)$ for two good Lagrangian varieties Λ_0 and Λ_1 with $\dim(\Lambda_0 \cap \Lambda_1) = \dim V - 1$. The idea is as follows. For simplicity, we assume that $b_{\Lambda_0}(s)$ divides $b_{\Lambda_1}(s)$. Then there exists a micro-differential operator P satisfying $Pf(x)^{s+1} = b_{\Lambda_1}(s) f(x)^s$. Let α be a complex number and let v be a solution to the system of micro-differential equations satisfied by $f(x)^\alpha$. Then we have $Pfv = b_{\Lambda_1}(\alpha) v$. Hence if $v \neq 0$ and $fv=0$, we obtain $b_{\Lambda_1}(\alpha)=0$. Under some additional conditions, it is proved that α is a root of $b_{\Lambda_1}(s)/b_{\Lambda_0}(s)$. In order to construct such a solution v, we shall use the theory of holonomic systems of micro-differential equations.

Contents

In §1, we shall give elementary properties of prehomogeneous vector spaces. In §2 and §3, we review the theory of systems of micro-differential equations. In §4 and §5, we shall study properties of a good Lagrangian variety and define the local b-function of a good Lagrangian variety Λ. In §6 and §7, we study the relation of two local b-functions $b_{\Lambda_0}(s)$ and $b_{\Lambda_1}(s)$ when $\dim(\Lambda_0 \cap \Lambda_1) = \dim V - 1$. In §8, we give the proof of the theorems which are used in §6 and §7 in order to construct the solution v to the system of micro-differential equations satisfied by $f(x)^s$. In §9, we show two examples of the calculation of b-functions.

The appendix is to give the proof of a theorem used in §4.

§1. Λ-functions, b-functions and c-functions of Regular Prehomogeneous Vector Spaces

Let G be a connected linear algebraic group and $\rho: G \to GL(V)$ its finite-dimensional linear rational representation all defined over the complex number field $\mathbb{C}$. If V has a Zariski-dense G-orbit, the triplet (G, ρ, V) is called *a prehomogeneous vector space* (abbrev. P.V.). In this case, the complement S of this orbit is Zariski-closed, which we call *the singular set* of this P.V. We denote by $X(G)$ the group of all rational characters of G. A rational function $f(x)$ on V is called *a relative invariant* if there exists some $\chi \in X(G)$ satisfying $f(\rho(g)x) = \chi(g) f(x)$ for all $g \in G$ and $x \in V$. Let $S_1, \ldots, S_l$ be the irreducible components of

S with codimension one. Then each S_i is the zeros of some irreducible polynomial $f_i(x)$: $S_i=\{x\in V; f_i(x)=0\}$ $(1\leqq i\leqq l)$. These $f_1(x),\ldots,f_l(x)$ are algebraically independent relative invariants and any relative invariant $f(x)$ is of the form $f(x)=cf_1(x)^{m_1}\ldots f_l(x)^{m_l}$ $(c\in\mathbb{C},\ (m_1,\ldots,m_l)\in\mathbb{Z}^l)$ (See [1]). Let G_1 be the subgroup of G generated by the commutator subgroup and a generic isotropy subgroup $G_x=\{g\in G;\ \rho(g)x=x\}$ for $x\in V-S$. It is easy to see that G_1 does not depend on the choice of a generic point $x\in V-S$. Then $\chi\in X(G)$ corresponds to some relative invariant if and only if $\chi|_{G_1}=1$ (See [1]). Therefore $X_1(G)$ is a free abelian group of rank l generated by $\chi_1,\ldots,\chi_l$ where $X_1(G)=\{\chi\in X(G);\ \chi|_{G_1}=1\}$. A triplet (G,ρ^*,V^*) is called *the dual* of (G,ρ,V) if ρ^* is the contragredient representation of ρ on the dual V^* of V. We shall consider the relation of a P.V. and its dual. Let $\mathfrak{g}$ (resp. $\mathfrak{g}_1$) be the Lie algebra of G(resp. G_1) and let $\delta\chi_1,\ldots,\delta\chi_l$ be the infinitesimal characters of $\chi_1,\ldots,\chi_l$. Then each $\delta\chi_i$ is an element of the dual vector space $\mathfrak{g}^*$ of $\mathfrak{g}$ satisfying $\delta\chi_i|\mathfrak{g}_1=0$ $(1\leqq i\leqq l)$. Put $\bar{X}_1=\{\omega\in\mathfrak{g}^*;\ \omega|\mathfrak{g}_1=0\}$ and let $\bar{X}_0$ be its subspace generated by $\delta\chi_1,\ldots,\delta\chi_l$.

Lemma 1.1. *For $\omega\in\mathfrak{g}^*$, the following conditions are equivalent.*

(1) $\omega\in\bar{X}_1$

(2) *There exists a unique rational map φ_ω: $V-S\to V^*$ satisfying $\varphi_\omega(\rho(g)x)=\rho^*(g)\varphi_\omega(x)$ and $\langle\varphi_\omega(x), d\rho(A)x\rangle=\omega(A)$ for all $x\in V-S$, $g\in G$, $A\in\mathfrak{g}$.*

Proof. (1) ⇒ (2): Take $x\in V-S$, and let $\mathfrak{g}_x$ be the isotropy subalgebra of $\mathfrak{g}$ at x. Then $A\to d\rho(A)x$ $(A\in\mathfrak{g})$ induces a bijective linear map from $\mathfrak{g}/\mathfrak{g}_x$ onto V. Since $\omega(A)=0$ whenever $d\rho(A)x=0$, there exists uniquely an element $\varphi_\omega(x)$ of V^* satisfying $\langle\varphi_\omega(x), d\rho(A)x\rangle=\omega(A)$ for all $A\in\mathfrak{g}$. Since $\omega(B)=0$ for $B\in[\mathfrak{g},\mathfrak{g}]\subset\mathfrak{g}_1$, we have $\omega(Ad(g)\cdot A)=\omega(A)$ for $g\in G$, $A\in\mathfrak{g}$ where Ad denotes the adjoint representation, and hence

$$\begin{aligned}\langle\varphi_\omega(\rho(g)x), d\rho(A)\rho(g)x\rangle &=\omega(A)=\omega(Ad(g^{-1})\cdot A)\\ &=\langle\rho^*(g)\varphi_\omega(x), d\rho(A)\cdot\rho(g)x\rangle\end{aligned}$$

for all $g\in G$, $A\in\mathfrak{g}$, i.e., $\varphi_\omega(\rho(g)x)=\rho^*(g)\varphi_\omega(x)$. Since $\varphi_\omega(x)$ is clearly a regular rational map on $V-S$, we have (2).

(2) ⇒ (1): For $A\in\mathfrak{g}_x$, we have $\omega(A)=\langle\varphi_\omega(x),0\rangle=0$. On the other hand, $\varphi_\omega(\rho(g)x)=\rho^*(g)\varphi_\omega(x)$ implies $\omega(A)=\omega(Ad(g)A)$ and hence $\omega(A)=0$ for $A\in[\mathfrak{g},\mathfrak{g}]$. Since $\mathfrak{g}_1=\mathfrak{g}_x+[\mathfrak{g},\mathfrak{g}]$, we get our assertion. Q.E.D.

Definition 1.2. A P.V. is called *regular* (resp. *quasi-regular*) if there exists $\omega\in\bar{X}_0$ (resp. $\omega\in\bar{X}_1$) such that φ_ω is generically surjective, i.e., the image of φ_ω is Zariski-dense in V^*. In this case, ω is called *non-degenerate*.

Proposition 1.3. 1) *For $\omega=\sum_{i=1}^{l} s_i\delta\chi_i\in\bar{X}_0$, we have*

$$\varphi_\omega(x)=\operatorname{grad}\log f_1(x)^{s_1}\ldots f_l(x)^{s_l}\qquad \textit{for all } x\in V-S.$$

2) *If G is reductive, the following conditions are equivalent.*

i) *regular,*

ii) *quasi-regular,*

iii) *The singular set S is a hypersurface.*

Proof. 1) Since grad $\log f_1(x)^{s_1}\dots f_l(x)^{s_l}$ satisfies the conditions of Lemma 1.1 (See [1]), it follows from the uniqueness.

2) The equivalence of i) and iii) is proved in [1] or [6]. The connected commutative algebraic group G/G_1 is the direct product of its torus $(G/G_1)_t$ and its unipotent subgroup $(G/G_1)_a$. Let $\mathfrak{g} \bmod \mathfrak{g}_1 = (\mathfrak{g} \bmod \mathfrak{g}_1)_t + (\mathfrak{g} \bmod \mathfrak{g}_1)_a$ be the corresponding decomposition of the Lie algebra $\mathfrak{g} \bmod \mathfrak{g}_1$ of G/G_1. Then $\bar{X}_0 = \{\omega \in \bar{X}_1;\ \omega(\mathfrak{g} \bmod \mathfrak{g}_1)_a = 0\}$. Since G is reductive, we have $(\mathfrak{g} \bmod \mathfrak{g}_1)_a = 0$ and hence $\bar{X}_1 = \bar{X}_0$. This implies i) $\Leftrightarrow$ ii). Q.E.D.

Proposition 1.4. *Let (G, ρ, V) be a regular (resp. quasi-regular) P.V. Then its dual triplet (G, ρ^*, V^*) is also a regular (resp. quasi-regular) P.V.*

Proof. Let $\omega \in \bar{X}_0$ (resp. $\omega \in \bar{X}_1$) be a non-degenerate element. Then $\varphi_\omega(V-S)$ is a Zariski-dense G-orbit in V^* and hence the dual triplet (G, ρ^*, V^*) is a P.V., and $\varphi_\omega(V-S) = V^* - S^*$ where S^* is its singular set. For $x \in V-S$, put $y = \varphi_\omega(x)$ and $G_y^* = \{g \in G;\ \rho^*(g)\,y = y\}$. Then we have $G_y^* \supset G_x = \{g \in G;\ \rho(g)\,x = x\}$ and $\dim G_y^* = \dim G - \dim V^* = \dim G_x$. Hence the Lie algebra of G_1^* $(=[G,G]\cdot G_y^*)$ is $\mathfrak{g}_1$. By Lemma 1.1, there exists a map $\psi_\omega: V^* - S^* \to V$ satisfying $\psi_\omega(\rho^*(g)y) = \rho(g)\psi_\omega(y)$ and $\langle \psi_\omega(y), d\rho^*(A)\,y\rangle = -\omega(A)$ for $y \in V^* - S^*$, $g \in G$, $A \in \mathfrak{g}$. Therefore, we have

$$\langle x, d\rho^*(A)\,\varphi_\omega(x)\rangle = -\langle d\rho(A)\,x, \varphi_\omega(x)\rangle = -\omega(A) = \langle \psi_\omega(y), d\rho^*(A)\,y\rangle.$$

This implies $y = \varphi_\omega(x)$ if and only if $x = \psi_\omega(y)$, and hence φ_ω is a biregular rational map. Moreover, we have $G_y^* = G_x$ and $G_1^* = G_1$. Q.E.D.

Corollary 1.5. *Let (G, ρ, V) be a quasi-regular P.V. and ω a non-degenerate element of $\bar{X}_1$. Then φ_ω is a biregular rational map from $V-S$ to V^*-S^*.*

Corollary 1.6. *Let (G, ρ, V) be a quasi-regular P.V. Then the number l^* of one-codimensional irreducible components of S^* coincides with that of S, i.e., $l^* = l$.*

Proof. Since l^* is the rank of the character group of G/G_1^*, and $G_1^* = G_1$, we have $l^* = l$. Q.E.D.

Definition 1.7. Let (G, ρ, V) be a quasi-regular P.V. Let $f_1^*(y), \dots, f_l^*(y)$ be the irreducible relative invariant polynomial of the dual P.V., and $\chi_1^*, \dots, \chi_l^*$ their characters.

Then $X_1(G)$ is a free abelian group generated by $\chi_1^*, \dots, \chi_l^*$. For any $\chi \in X_1(G)$, we have

$$\chi = \prod_{i=1}^{l} \chi_i^{n_i} = \prod_{i=1}^{l} \chi_i^{*\,m_i} \qquad (n_1, \dots, n_l, m_1, \dots, m_l \in \mathbb{Z}).$$

We shall fix these polynomials $f_i(x)$, $f_i^*(y)$ $(1 \leqq i \leqq l)$, and put

$$f^\chi(x) = \prod_{i=1}^{l} f_i(x)^{n_i}, \qquad f^{*\chi}(y) = \prod_{i=1}^{l} f_i^*(y)^{m_i}.$$

From now on, we shall consider regular P.V.'s. We identify $\bar{X}_0$ with $\mathbb{C}^l$ by $\omega = \sum_{i=1}^{l} s_i\,\delta\chi_i \mapsto s = (s_1, \dots, s_l) \in \mathbb{C}^l$. We denote $f_1(x)^{s_1}\dots f_l(x)^{s_l}$ by $f(x)^s$ for $s \in \bar{X}_0$. We have $\varphi_s(x)=\mathrm{grad}\log f(x)^s$ for $s\in\bar{X}_0$. For $\chi=\prod_{i=1}^{l}\chi_i^{n_i}\in X_1(G)$, we also denote $\delta\chi=\Sigma n_i\,\delta\chi_i\in\bar{X}_0$ by χ.

Proposition 1.8. *Let (G,ρ,V) be a regular P.V. Then, for each $\chi\in X_1(G)$, there exists a homogeneous rational function $a_\chi(s)$ on $\bar{X}_0$ satisfying $f^\chi(x)f^{*\chi^{-1}}(\varphi_s(x))=a_\chi(s)$ for $x\in V-S$. If $f^{*\chi^{-1}}(y)$ is a polynomial, then $a_\chi(s)$ is a polynomial of the same degree as $f^{*\chi^{-1}}(y)$.*

Proof. Since $f^\chi(\rho(g)x)f^{*\chi^{-1}}(\varphi_s(\rho(g)x))=f^\chi(x)f^{*\chi^{-1}}(\varphi_s(x))$ for $x\in V-S$, this depends only on χ and s. Since $f^{*\chi^{-1}}(y)$ is a homogeneous rational function, and φ_s is linear in $s\in\bar{X}_0$, $a_\chi(s)$ is a homogeneous rational function. The remaining part is obvious. Q.E.D.

Definition 1.9. We call $a_\chi(s)$ an *a-function.*

Proposition 1.10. *Let χ be a character such that $f^{*\chi^{-1}}(y)$ is a polynomial and $f^{*\chi^{-1}}(\mathrm{grad}_x)$ a differential operator on V with constant coefficients satisfying $f^{*\chi^{-1}}(\mathrm{grad}_x)e^{\langle x,y\rangle}=f^{*\chi^{-1}}(y)e^{\langle x,y\rangle}$ $(x\in V,\ y\in V^*)$. Then there exists a polynomial $b_\chi(s)$ on $\bar{X}_0$ satisfying $f^{*\chi^{-1}}(\mathrm{grad}_x)f(x)^{s+\chi}=b_\chi(s)f(x)^s$.*

Proof. If $\chi=\prod_{i=1}^{l}\chi_i^{n_i}$, we have

$$\frac{\partial}{\partial x_i}f^{s+\chi}=\sum_k(s_k+n_k)f_1^{s_1+n_1}\dots f_k^{s_k+n_k-1}\dots f_l^{s_l+n_l}\cdot\left(\frac{\partial f_k}{\partial x_i}\right).$$

Since $f^{*\chi^{-1}}(y)$ is a polynomial, by repeating this procedure, we obtain that

$$f^{*\chi^{-1}}(\mathrm{grad}_x)f(x)^{s+\chi}=g(s,x)f(x)^{s-\lambda}$$

for some λ and a polynomial $g(s,x)$. Then, since

$$f^{*\chi^{-1}}(\mathrm{grad}_{\rho(g)x})=\chi^{-1}(g)f^{*\chi^{-1}}(\mathrm{grad}_x),$$

the function $f(x)^{-s}\cdot f^{*\chi^{-1}}(\mathrm{grad}_x)f(x)^{s+\chi}=g(s,x)f(x)^{-\lambda}$ is an absolute invariant rational function on x. This implies that $g(s,x)f(x)^{-\lambda}$ does not depend on x but only on s. Therefore, putting $b_\chi(s)=g(s,x)f(x)^{-\lambda}$, we have $f^{*\chi^{-1}}(\mathrm{grad}_x)f(x)^{s+\chi}=b_\chi(s)f(x)^s$. Q.E.D.

Definition 1.11. We call $b_\chi(s)$ a *b-function.*

Proposition 1.12. *An a-function $a_\chi(s)$ is the leading homogeneous part of a b-function $b_\chi(s)$.*

Proof. Assume that $\chi=\prod_{i=1}^{l}\chi_i^{n_i}$. Since

$$\frac{\partial}{\partial x_i}f^{s+\chi}(x)=\sum_{k=1}^{l}(s_k+n_k)f_1(x)^{s_1+n_1}\dots f_k(x)^{s_k+n_k-1}\dots f_l(x)^{s_l+n_l}\left(\frac{\partial f_k}{\partial x_i}(x)\right)$$
$$=f^{s+\chi}(x)\cdot\frac{\partial}{\partial x_i}\log f^s(x)+\text{lower term in } s,$$

we have $\left(\frac{\partial}{\partial x_i}\right)^k f^{s+\chi}(x) = f^{s+\chi}(x)\cdot\left(\frac{\partial}{\partial x_i}\log f^s(x)\right)^k$ + lower term in s, and hence $f^{*\chi^{-1}}(\mathrm{grad}_x) f(x)^{s+\chi} = f^{*\chi^{-1}}(\mathrm{grad}\log f^s(x))\cdot f(x)^{s+\chi}$ + lower term in $s = a_\chi(s)\cdot f(x)^s$ + lower term in s. Since $f^{*\chi^{-1}}(\mathrm{grad}_x) f(x)^{s+\chi} = b_\chi(s) f(x)^s$, we obtain our assertion. Q.E.D.

Proposition 1.13. *Let (G, ρ, V) be a regular P.V. Then we have $\chi_0^2 \in X_1(G)$ where $\chi_0(g) = \det_V \rho(g)$ for $g \in G$, and there exists a homogeneous polynomial $C(s)$ of degree n ($n = \dim V$) on $\bar{X}_0$ satisfying $C(s) = f^{\chi_0^2}(x)\cdot\det d\varphi_s(x)$ for $x \in V - S$.*

Proof. By taking a basis of V and its dual basis of V^*, we identify V and V^* with $\mathbb{C}^n$. Moreover, we assume that $G \subset GL(n, \mathbb{C})$. Then we may regard the differential map $d\varphi_s(x)$ $(x \in V - S)$ of φ_s as a linear transformation of $\mathbb{C}^n$. Since $d\varphi_s(gx) = {}^t g^{-1} d\varphi_s(x) g^{-1}$ for $g \in G$, $x \in V - S$, the determinant $J_s(x) = \det d\varphi_s(x)$ is a rational function on V satisfying $J_s(gx) = (\det g)^{-2}\cdot J_s(x)$. Since this P.V. is regular, $J_s(x)$ is not identically zero and we have $\chi_0^2 \in X_1(G)$. The remaining part is obvious. Q.E.D.

Definition 1.14. We call $C(s)$ a *c-function.*

Remark 1.15. Since this polynomial $C(s)$ is not identically zero, there exists $s = (n_1, \ldots, n_l) \in \mathbb{Z}^l$ satisfying $C(s) \neq 0$. Then $f(x) = f_1(x)^{n_1} \ldots f_l(x)^{n_l}$ is a relative invariant such that $\mathrm{grad}\log f\colon V - S \to V^*$ is dominant, i.e., generically surjective. Therefore the definition of regularity here coincides with that of [1].

Remark 1.16. Although we have defined a-functions, b-functions and c-functions over $\bar{X}_0$, it is possible to define them over $\bar{X}_1$ for a quasi-regular P.V. (See [4].)

Now we shall consider an irreducible regular P.V. (G, σ, V). Then there exists an irreducible relative invariant polynomial $f(x)$ which is unique up to a constant multiple. If $\chi \in X_1(G)$ is its corresponding character, then we have $d|2n$ and $\chi(g)^{\frac{2n}{d}} = \det_V \rho(g)^2$ for $g \in G$ where $d = \deg f$ and $n = \dim V$ (See [1]). There exists an irreducible relative invariant polynomial $f^*(y)$ of the dual P.V. corresponding to χ^{-1}: $f^*(\rho^*(g) y) = \chi^{-1}(g) f^*(y)$ for $y \in V^*$ and $g \in G$. Then we have $f^*(\mathrm{grad}_x) f(x)^{s+1} = b_\chi(s) f(x)^s$.

Definition 1.17. We fix $f(x)$ and $f^*(y)$ so that $b_\chi(s)$ is a monic polynomial in s. In this case, we denote $b_\chi(s)$ by $b(s)$ and call $b(s)$ *the b-function* of an irreducible regular P.V. (G, ρ, V).

Our main purpose is to calculate the b-function of each irreducible regular P.V.

§ 2. Micro-differential Operators

Let X be a complex manifold of dimension n, T^*X the cotangent vector bundle of X. The projection from T^*X onto X will be denoted by π_X. Let $(z_1, \ldots, z_n)$ be a local coordinate system of X and $(z_1, \ldots, z_n, \xi_1, \ldots, \xi_n)$ the corresponding coordinate system of T^*X so that $\omega_X = \xi_1 dz_1 + \ldots + \xi_n dz_n$ is the canonical 1-form on T^*X.

For $\lambda \in \mathbb{C}$, we define the sheaf $\mathscr{E}_X^\infty(\lambda)$ on T^*X as follows: for any open set Ω of T^*X, a section of $\mathscr{E}_X^\infty(\lambda)$ on Ω is a set $\{P_{\lambda+j}(z, \xi)\}_{j \in Z}$ of holomorphic functions $P_{\lambda+j}(z, \xi)$ defined on Ω satisfying the following conditions:

(2.1) $P_{\lambda+j}(z, \xi)$ is homogeneous of degree $\lambda+j$ with respect to ξ, i.e.,

$$\left(\sum \xi_k \frac{\partial}{\partial \xi_k}\right) P_{\lambda+j}(z, \xi) = (\lambda+j) P_{\lambda+j}(z, \xi)$$

(2.2) for any positive constant ε and any compact subset K of Ω, there is a constant $C_{K,\varepsilon}$ such that

$$\sup_K |P_{\lambda+j}(z, \xi)| \leqq \frac{1}{j!} C_{K,\varepsilon} \cdot \varepsilon^j \qquad \text{for } j \geqq 0$$

(2.3) for any compact subset K of Ω, there is a constant R_K such that

$$\sup_K |P_{\lambda+j}(z, \xi)| \leqq (-j)!\, R_K^{-j} \qquad \text{for } j < 0.$$

It is obvious that $\mathscr{E}_X^\infty(\lambda)$ is a $\mathbb{C}$-*Module* by $a\{P_{\lambda+j}\} + b\{Q_{\lambda+j}\} = \{aP_{\lambda+j} + bQ_{\lambda+j}\}$.

We have $\mathscr{E}_X^\infty(\lambda) = \mathscr{E}_X^\infty(\lambda+m)$ for any integer m. A section $\{P_{\lambda+j}(z, \xi)\}$ is usually written by $\Sigma P_{\lambda+j}(z, D)$, and called *a micro-differential operator*.

We define the product $R(z, D) = \Sigma R_{\lambda+\mu+j}(z, D) \in \mathscr{E}_X^\infty(\lambda+\mu)$ of two micro-differential operators $P(z, D) = \Sigma P_{\lambda+j}(z, D) \in \mathscr{E}_X^\infty(\lambda)$ and $Q(z, D) = \Sigma Q_{\mu+j}(z, D) \in \mathscr{E}_X^\infty(\mu)$ as follows:

$$R_{\lambda+\mu+l}(z, \xi) = \sum_{\substack{\alpha \in Z_+^n \\ l = j+k-|\alpha|}} \frac{1}{\alpha!} (D_\xi^\alpha P_{\lambda+j}(z, \xi))(D_z^\alpha Q_{\mu+k}(z, \xi)). \tag{2.4}$$

This power series converges uniformly on any compact set. This product satisfies the associative law and distributive law: $(PQ)R = P(QR)$ for $P \in \mathscr{E}_X^\infty(\lambda)$, $Q \in \mathscr{E}_X^\infty(\mu)$ and $R \in \mathscr{E}_X^\infty(\nu)$. $1 \in \mathscr{E}_X^\infty(0)$, i.e., $1 = \{P_j\}$ where $P_j = 0$ for $j \neq 0$ and $P_0 = 1$, is the identity, namely $1 \cdot P = P \cdot 1 = P$ for any $P \in \mathscr{E}_X^\infty(\lambda)$ (§ 1-4 and § 1-5 Chap. II in [2]).

Put $\mathscr{E}_X^\infty = \mathscr{E}_X^\infty(0)$. Then $\mathscr{E}_X^\infty$ has a structure of $\mathbb{C}$-Algebra and $\mathscr{E}_X^\infty(\lambda)$ is an $\mathscr{E}_X^\infty$-*bi-Module*.

The subsheaf of $\mathscr{E}_X^\infty(\lambda)$ consisting of $P(z, D) = \Sigma P_{\lambda+j}(z, D)$ satisfying $P_{\lambda+j}(z, \xi) = 0$ for $j > 0$ will be denoted by $\mathscr{E}_X(\lambda)$. For $P(z, D) = \Sigma P_{\lambda+j}(z, D)$, $\lambda + j_0$ is called the *order* of P and denoted by ord (P) if j_0 is the largest j satisfying $P_{\lambda+j}(z, \xi) \not\equiv 0$.

By the homomorphism $P = P_\lambda(z, D) + P_{\lambda-1}(z, D) + \ldots \mapsto P_\lambda(z, \xi)$, the quotient $\mathscr{E}_X(\lambda)/\mathscr{E}_X(\lambda-1)$ is isomorphic to the sheaf $\mathscr{O}(\lambda)$ of homogeneous holomorphic functions of degree λ in ξ.

We denote this homomorphism by σ_λ or simply by σ if there is no confusion. Then we have the following relations:

$$\sigma_{\lambda+\mu}(PQ) = \sigma_\lambda(P)\sigma_\mu(Q) \tag{2.5}$$

$$\sigma_{\lambda+\mu-1}([P, Q]) = \{\sigma_\lambda(P), \sigma_\lambda(Q)\} \qquad \text{for } P \in \mathscr{E}_X(\lambda) \text{ and } Q \in \mathscr{E}_X(\mu) \tag{2.6}$$

where $[P, Q]=PQ-QP$ and $\{f, g\}$ is the Poisson bracket of f and g, i.e.,

$$\{f, g\}=\sum_j \left(\frac{\partial f}{\partial \xi_j}\cdot\frac{\partial g}{\partial z_j}-\frac{\partial f}{\partial z_j}\cdot\frac{\partial g}{\partial \xi_j}\right).$$

Put $\mathscr{E}_X=\bigcup_{j\in\mathbb{Z}}\mathscr{E}_X(j)$.

The direct image $\pi_{X_*}(\mathscr{E}_X^\infty)$ is isomorphic to the sheaf $\mathscr{D}_X^\infty$ of differential operators of infinite order because all P_j are polynomials with respect to ξ. The direct image of $\mathscr{E}_X$ by π_X is isomorphic to the sheaf $\mathscr{D}_X$ of differential operators of finite order. Note that $\mathscr{E}_X^\infty|_X=\mathscr{D}_X^\infty$ and $\mathscr{E}_X|_X=\mathscr{D}_X$ by the same reason.

Since definitions above depend heavily on the choice of local coordinate systems, it is necessary to give the transformation law under coordinate transformations.

Let $\tilde{z}=(\tilde{z}_1, \ldots, \tilde{z}_n)$ be another local coordinate system and $(\tilde{z}, \tilde{\xi})=(\tilde{z}_1, \ldots, \tilde{z}_n, \tilde{\xi}_1, \ldots, \tilde{\xi}_n)$ the corresponding local coordinate system of T^*X. We have $\tilde{\xi}_j=\sum_k \frac{\partial z_k}{\partial \tilde{z}_j}\xi_k$.

Then we identify $P(z, D_z)=\sum P_{\lambda+j}(z, D_z)$ with $\tilde{P}(\tilde{z}, D_{\tilde{z}})=\sum \tilde{P}_{\lambda+j}(\tilde{z}, \tilde{\xi})$ by the following formula:

$$\tilde{P}_{\lambda+l}(\tilde{z}, \tilde{\xi})=\sum\frac{1}{\nu!\,\alpha_1!\ldots\alpha_\nu!}\langle\tilde{\xi}, D_z^{\alpha_1}\tilde{z}\rangle\ldots\langle\tilde{\xi}, D_z^{\alpha_\nu}\tilde{z}\rangle D_\xi^\alpha P_{\lambda+j}(z, \xi) \tag{2.7}$$

where the indices run over $j\in\mathbb{Z}$, $\nu\in\mathbb{Z}_+$, $(\alpha_1, \ldots, \alpha_\nu)\in(\mathbb{Z}_+^n)^\nu$ and $\alpha=\alpha_1+\ldots+\alpha_\nu$ such that $|\alpha_1|, \ldots, |\alpha_\nu|\geqq 2$ and $l=j+\nu-|\alpha_1|-\ldots-|\alpha_\nu|$. For $\beta\in\mathbb{Z}_+^n$, $\langle\tilde{\xi}, D_z^\beta\tilde{z}\rangle$ means $\sum_j \tilde{\xi}_j D_z^\beta \tilde{z}_j$. For example, for $P\in\mathscr{E}_X(\lambda)$, the first two terms are

$$\begin{aligned}
&\tilde{P}_\lambda(\tilde{z}, \tilde{\xi})=P_\lambda(z, \xi)\\
&\tilde{P}_{\lambda-1}(\tilde{z}, \tilde{\xi})=P_{\lambda-1}(z, \xi)+\tfrac{1}{2}\sum_{i,j,k}\tilde{\xi}_k\frac{\partial^2\tilde{z}_k}{\partial z_i\partial z_j}\frac{\partial^2}{\partial\xi_i\partial\xi_j}P_\lambda(z, \xi).
\end{aligned} \tag{2.8}$$

By this formula, $\mathscr{E}_X^\infty(\lambda)$ and $\mathscr{E}_X(\lambda)$ are defined all over T^*X. The homomorphism $\sigma_\lambda : \mathscr{E}_X(\lambda)\to\mathcal{O}(\lambda)$ does not depend on the choice of local coordinate systems (Theorem 1.5.5 Chap. II in [2]).

Now we shall give several properties on micro-differential operators. We can find their proofs in Chap. II in [2].

Proposition 2.1 (Boutet de Monvel-Kree [17], Theorem 2.1 Chap. II in [2]). *Let P be a section of $\mathscr{E}_X(\lambda)$ defined near $p\in T^*X$. If $\sigma_\lambda(P)(p)\neq 0$, then P is invertible at p, i.e., there exists a unique micro-differential operator $Q\in\mathscr{E}_X(-\lambda)$ defined near p satisfying $QP=PQ=1$.*

Proposition 2.2 (Theorem 2.1.2 Chap. II in [2]). *Let P and Q be micro-differential operators defined near p. Assume that* (1) $\sigma_\lambda(P)=\sigma_\lambda(Q)$ *and* $\sigma_\lambda(P)(p)=0$ (2) $d\sigma_\lambda(P)$ *and ω_X are not parallel at p. Then there is an invertible micro-differential operator R of order* 0 *defined near p such that $Q=RPR^{-1}$.*

The $\mathbb{C}$-*Algebras* $\mathscr{E}_X^\infty$ and $\mathscr{E}_X$ have the following algebraic properties.

Proposition 2.3 (§ 3 Chap. II in [2]). i) *$\mathscr{E}_X$ and $\mathscr{E}_X(0)$ are coherent Rings and their stalks are noetherian ring from the both sides.*

ii) *$\mathscr{E}_X$ contains $\pi_X^{-1}\mathscr{D}_X$ as a subRing and $\mathscr{E}_X$ is flat over $\pi_X^{-1}\mathscr{D}_X$.*

iii) *$\mathscr{E}_X^\infty$ is faithfully flat over $\mathscr{E}_X$.*

iv) *Let $\mathscr{M}' \xrightarrow{\varphi} \mathscr{M} \xrightarrow{\psi} \mathscr{M}''$ be a complex of coherent $\mathscr{E}_X$-Modules and let $\mathscr{M}'_0$, $\mathscr{M}_0$ and $\mathscr{M}''_0$ be coherent sub-$\mathscr{E}_X(0)$-Modules of $\mathscr{M}'$, $\mathscr{M}$ and $\mathscr{M}''$ respectively, such that $\mathscr{E}_X\mathscr{M}'_0=\mathscr{M}'$, $\mathscr{E}_X\mathscr{M}_0=\mathscr{M}$, $\mathscr{E}_X\mathscr{M}''_0=\mathscr{M}''$, $\varphi(\mathscr{M}'_0)\subset\mathscr{M}_0$ and $\psi(\mathscr{M}_0)\subset\mathscr{M}''_0$. Put $\overline{\mathscr{M}}'=\mathscr{M}'_0/\mathscr{E}_X(-1)\mathscr{M}'_0$, $\overline{\mathscr{M}}=\mathscr{M}_0/\mathscr{E}_X(-1)\mathscr{M}_0$ and $\overline{\mathscr{M}}''=\mathscr{M}''_0/\mathscr{E}_X(-1)\mathscr{M}''_0$. If $\overline{\mathscr{M}}'\to\overline{\mathscr{M}}\to\overline{\mathscr{M}}''$ is an exact sequence, then $\mathscr{M}'_0\to\mathscr{M}_0\to\mathscr{M}''_0$ and $\mathscr{M}'\to\mathscr{M}\to\mathscr{M}''$ are exact.*

Definition 2.4. A coherent $\mathscr{E}_X$-Module $\mathscr{M}$ is called *a system of micro-differential equations* and its support is called its *characteristic variety.* A characteristic variety is, therefore, an analytic subset invariant under the action of $\mathbb{C}^\times$.

Definition 2.5. A variety V of T^*X is called involutory if for any two functions f, g vanishing on V, their Poisson bracket $\{f, g\}$ vanishes on V. If V is involutory, we have $\operatorname{codim} V \leqq \dim X$. The following theorem is one of the most fundamental theorem.

Theorem 2.6 (Theorem 5.3.2 Chap. II in [2]). *A characteristic variety of any system of micro-differential equations is involutory.*

Definition 2.7. Let $\mathscr{M}$ be a system of micro-differential equations with one unknown function u, i.e., a coherent $\mathscr{E}_X$-Module generated by u. Let $\mathscr{J}$ be an annihilator of u. Then $\mathscr{M}$ is isomorphic to $\mathscr{E}_X/\mathscr{J}$. *The symbol Ideal* $\bar{\mathscr{J}}$ is an Ideal of $\mathscr{O}_{T^*X}$ generated by principal symbols $\sigma(P)$ of micro-differential operators P in $\mathscr{J}$. The characteristic variety of $\mathscr{M}$ is the zeros of $\bar{\mathscr{J}}$ (See Proposition 2.1).

Definition 2.8. A system $\mathscr{M}=\mathscr{E}_X u=\mathscr{E}_X/\mathscr{J}$ of micro-differential operators with one unknown function u is called *simple* if the symbol Ideal $\bar{\mathscr{J}}$ is reduced, i.e., $\bar{\mathscr{J}}$ coincides with the Ideal of all functions vanishing on the characteristic variety of $\mathscr{M}$.

Definition 2.9. A system $\{P_1, \ldots, P_N\}$ of sections of $\mathscr{J}$ is called an *involutive* base if $\sigma(P_1), \ldots, \sigma(P_N)$ generate the symbol Ideal $\bar{\mathscr{J}}$.

Proposition 2.10. *Let P_j be a section of $\mathscr{J}$ of order m_j for $j=1, \ldots, N$. Then the following conditions are equivalent.*

(1) *$\{\sigma_{m_j}(P_j)\}_j$ generates the symbol Ideal $\bar{\mathscr{J}}$, i.e., $\{P_j\}_j$ is an involutive base.*

(2) *$\{P_j\}_j$ generates $\mathscr{J}$ as $\mathscr{E}_X$-Module and for any set of homogeneous functions g_j of degree $m-m_j$ such that $\sum g_j\sigma_{m_j}(P_j)=0$, there exists a set $\{G_j\}$ of micro-differential operators of order $\leqq m-m_j$ such that $g_j=\sigma_{m-m_j}(G_j)$ and $\sum G_jP_j=0$.*

Corollary 2.11. *Let $P_1, \ldots, P_N$ be micro-differential operators of order $m_1, \ldots, m_N$ respectively, which generate $\mathscr{J}$. Assume that $d\sigma(P_1), \ldots, d\sigma(P_N)$ are linearly independent, then $\{P_j\}_j$ is an involutive base if and only if there are micro-differential operators $G_{i,j,k}$ of order $\leqq m_i+m_j-m_k-1$ such that*

$$[P_i, P_j]=\sum G_{i,j,k}P_k.$$

Definition 2.12. A system of micro-differential equations are called *holonomic* if its characteristic variety has codimension n ($=\dim X$).

A subvariety (resp. submanifold) V of T^*X is called *holonomic subvariety* (resp. holonomic submanifold) or *Lagrangian subvariety* (resp. Lagrangian submanifold) if it is involutory and $\dim V = \dim X$. Therefore the characteristic variety of a holonomic system is a holonomic variety.

§ 3. Principal Symbols of Simple Holonomic Systems

Let X be an n-dimensional complex manifold and T^*X its cotangent bundle. Let $\mathscr{M} = \mathscr{E}_X/\mathscr{J}$ be a simple holonomic system and Λ its characteristic variety. Since we consider everything micro-locally around a non-singular point of Λ in T^*X, we shall assume that Λ is non-singular. Let $(z_1, \dots, z_n, \xi_1, \dots, \xi_n)$ be a local coordinate system of T^*X such that $\omega_X = \sum_{i=1}^{n} \xi_i dz_i$ is the canonical 1-form.

Definition 3.1. For a function $f(z, \xi)$ on T^*X, we define the *Hamilton vector field* H_f by $H_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial \xi_i} \cdot \frac{\partial}{\partial z_i} - \frac{\partial f}{\partial z_i} \cdot \frac{\partial}{\partial \xi_i}\right)$. Therefore we have $\{f, g\} = H_f(g)$ where $\{\ ,\ \}$ denotes the Poisson bracket.

For a micro-differential operator $P(z, D) = \sum_{j \leq m} P_j(z, D)$ in $\mathscr{E}_X$ where $P_j(z, \xi)$ is homogeneous of degree j in ξ, we define $L_p^{(m)}(z, \xi)$ by

$$L_P^{(m)}(z, \xi) = H_{P_m(z,\xi)} + \left(P_{m-1}(z, \xi) - \tfrac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 P_m(z, \xi)}{\partial z_i \partial \xi_i}\right).$$

We denote $L_P^{(m)}(z, \xi)$ by $L_P(z, \xi)$ when there is no confusion.

Lemma 3.2. *For* $P, Q \in \mathscr{E}_X$ *with* $\operatorname{ord}(P) = m$ *and* $\operatorname{ord}(Q) = l$,

(1) $L_{PQ}^{(m+l)} = \sigma_m(P) L_Q^{(l)} + \sigma_l(Q) L_P^{(m)} + \frac{1}{2}\{\sigma_m(P), \sigma_l(Q)\}$

(2) $L_{[P,Q]}^{(m+l-1)} = [L_P^{(m)}, L_Q^{(l)}]$.

Proof. (1) Put $R = PQ = \sum_{j \leq m+l} R_j(z, D_z)$. Then we have $R_{m+l} = P_m Q_l$ and

$$R_{m+l-1} = P_m Q_{l-1} + P_{m-1} Q_l + \langle d_\xi P_m, d_z Q_l \rangle$$

where

$$\langle d_\xi P_m, d_z Q_l \rangle = \sum_{i=1}^{n} \frac{\partial P_m}{\partial \xi_i} \frac{\partial Q_l}{\partial z_i}.$$

Since

$$\sum_{i=1}^{n} \frac{\partial^2 R_{m+l}}{\partial z_i \partial \xi_i} = Q_l \sum_{i=1}^{n} \frac{\partial^2 P_m}{\partial z_i \partial \xi_i} + P_m \sum_{i=1}^{n} \frac{\partial^2 Q_l}{\partial z_i \partial \xi_i} + \langle d_z P_m, d_\xi Q_l \rangle + \langle d_\xi P_m, d_z Q_l \rangle$$

and $H_{fg} = f H_g + g H_f$, we have

$$L_R = H_{R_{m+l}} + R_{m+l-1} - \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 R_{m+l}}{\partial z_i \partial \xi_i}$$
$$= P_m\left(H_{Q_l} + Q_{l-1} - \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 Q_l}{\partial z_i \partial \xi_i}\right) + Q_l\left(H_{P_m} + P_{m-1} - \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 P_m}{\partial z_i \partial \xi_i}\right)$$
$$+\frac{1}{2}\{P_m, Q_l\} = P_m L_Q^{(l)} + Q_l L_P^{(m)} + \frac{1}{2}\{P_m, Q_l\}.$$

(2) Put $R=[P,Q]=PQ-QP$. Since $(PQ)_{m+l}=(QP)_{m+l}$, we have $R_{m+l}=0$. Since

$$(PQ)_{m+l-1} = P_m Q_{l-1} + P_{m-1} Q_l + \langle d_\xi P_m, d_z Q_l\rangle,$$

we have $R_{m+l-1}=\{P_m, Q_l\}$. Since

$$(PQ)_{m+l-2} = P_m Q_{l-2} + P_{m-1} Q_{l-1} + P_{m-2} Q_l + \langle d_\xi P_m, d_z Q_{l-1}\rangle$$
$$+\langle d_\xi P_{m-1}, d_z Q_l\rangle + \frac{1}{2}\sum \frac{\partial^2 P_m}{\partial \xi_i \partial \xi_j}\cdot\frac{\partial^2 Q_l}{\partial z_i \partial z_j},$$

we have

$$R_{m+l-2} = \{P_m, Q_{l-1}\} + \{P_{m-1}, Q_l\} + \frac{1}{2}\sum_{i,j}\left(\frac{\partial^2 P_m}{\partial \xi_i \partial \xi_j}\cdot\frac{\partial^2 Q_l}{\partial z_i \partial z_j} - \frac{\partial^2 Q_l}{\partial \xi_i \partial \xi_j}\cdot\frac{\partial^2 P_m}{\partial z_i \partial z_j}\right).$$

Since $H_{\{f,g\}}=[H_f, H_g](=H_f H_g - H_g H_f)$, we have

$$L_R^{(m+l-1)} = [H_{P_m}, H_{Q_l}] + \left\{P_m, Q_{l-1} - \frac{1}{2}\sum_k \frac{\partial^2 Q_l}{\partial z_k \partial \xi_k}\right\} + \left\{P_{m-1} - \frac{1}{2}\sum_k \frac{\partial^2 P_m}{\partial z_k \partial \xi_k}, Q_l\right\}$$
$$= \left[H_{P_m} + P_{m-1} - \frac{1}{2}\sum_k \frac{\partial^2 P_m}{\partial z_k \partial \xi_k},\ H_{Q_l} + Q_{l-1} - \frac{1}{2}\sum_k \frac{\partial^2 Q_l}{\partial z_k \partial \xi_k}\right]$$
$$= [L_P^{(m)}, L_Q^{(l)}].\quad \text{Q.E.D.}$$

Lemma 3.3. *$\sqrt{dz}^{-1} L_P \sqrt{dz}$ does not depend on a local coordinate system, i.e., if $\tilde{L}_P$ is a corresponding one obtained from another coordinate system $(\tilde{z}_1, \dots, \tilde{z}_n)$, we have $\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-\frac{1}{2}} \tilde{L}_P \left(\frac{\partial \tilde{z}}{\partial z}\right)^{\frac{1}{2}} = L_P$.*

Proof. Let P^* (resp. $\tilde{P}^*$) be the adjoint operator of P with respect to $z=(z_1, \dots, z_n)$ (resp. $\tilde{z}=(\tilde{z}_1, \dots, \tilde{z}_n)$). Then we have $P^*(z, D)=\sum P_k^*(z, D)$ where

$$P_k^*(z, \xi) = \sum_{k=j-|\alpha|} \frac{(-1)^j}{\alpha!} D_\xi^\alpha D_z^\alpha P_j(z, \xi)$$

(See Th. 1.5.1 Chap. II in [2]), and hence

$$P_{m-1} - \frac{1}{2}\sum_j \frac{\partial^2 P_m}{\partial z_j \partial \xi_j} = \frac{1}{2}\sigma_{m-1}(P - (-1)^m P^*).$$

Put $P=\sum_{j\leq m} P_j(z, D_z) = \sum_{j\leq m} \tilde{P}_j(\tilde{z}, D_{\tilde{z}})$. Since $\tilde{P}^* = \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1}\cdot P^* \cdot \left(\frac{\partial \tilde{z}}{\partial z}\right)$, we have

$$\tilde{P}_{m-1}-\tfrac{1}{2}\sum\frac{\partial^2\tilde{P}_m}{\partial\tilde{z}_j\partial\tilde{\xi}_j}=\tfrac{1}{2}\sigma_{m-1}(P-(-1)^m\tilde{P}^*)=\tfrac{1}{2}\sigma_{m-1}\left(P-(-1)^m\cdot\left(\frac{\partial\tilde{z}}{\partial z}\right)^{-1}\cdot P^*\cdot\left(\frac{\partial\tilde{z}}{\partial z}\right)\right)$$

$$=\tfrac{1}{2}\sigma_{m-1}(P-(-1)^mP^*)+(-1)^m\tfrac{1}{2}\left(\frac{\partial\tilde{z}}{\partial z}\right)^{-1}\cdot\sigma_{m-1}\left(\frac{\partial\tilde{z}}{\partial z}P^*-P\frac{\partial\tilde{z}}{\partial z}\right)$$

$$=\tfrac{1}{2}\sigma_{m-1}(P-(-1)^mP^*)-\tfrac{1}{2}\left(\frac{\partial\tilde{z}}{\partial z}\right)^{-1}\cdot\left\{P_m,\frac{\partial\tilde{z}}{\partial z}\right\}$$

and hence we have

$$\left(\frac{\partial\tilde{z}}{\partial z}\right)^{\frac{1}{2}}L_P\left(\frac{\partial\tilde{z}}{\partial z}\right)^{-\frac{1}{2}}=H_P-\tfrac{1}{2}\left(\frac{\partial\tilde{z}}{\partial z}\right)^{-1}\cdot\left\{P_m,\left(\frac{\partial\tilde{z}}{\partial z}\right)\right\}+\tfrac{1}{2}\sigma_{m-1}(P-(-1)^mP^*)$$

$$=H_P+\tfrac{1}{2}\sigma_{m-1}(P-(-1)^m\tilde{P}^*)=\tilde{L}_P.\quad \text{Q.E.D.}$$

Lemma 3.4. *Let f be a function on T^*X such that $f|_\Lambda=0$, i.e., f vanishes on Λ. Then the Hamilton vector field H_f can be regarded as a vector field on Λ.*

Proof. Since $f|_\Lambda=0$, we have $df=0$ as an element of $T^*\Lambda$ and hence $df\in T^*_\Lambda(T^*X)$ where $T^*_\Lambda(T^*X)$ is the conormal bundle of T^*X with respect to Λ. On the other hand, by the correspondence $T(T^*X)\ni v\mapsto v^*\in T^*(T^*X)$ where $d\omega(v'\wedge v)=v^*(v')$ for all $v'\in T(T^*X)$, ω being the fundamental 1-form, we shall identify $T(T^*X)$ with $T^*(T^*X)$. By this correspondence, we have $H_f=df\in T^*_\Lambda(T^*X)=(T\Lambda)^\perp$. Since Λ is holonomic, i.e., Lagrangian, we have $(T\Lambda)^\perp=T\Lambda$ and hence $H_f\in T\Lambda$. These relations can be shown by the following diagram.

$$(3.1)\qquad \begin{array}{ccc} 0\to T\Lambda\longrightarrow & T(T^*X) & \to T_\Lambda(T^*X)\\ & \text{\rotatebox{90}{$\in$}} & \\ & H_f & \\ & \updownarrow\; d\omega & \\ & df & \\ & \text{\rotatebox{-90}{$\in$}} & \\ 0\leftarrow T^*\Lambda\leftarrow & T^*(T^*X) & \leftarrow T^*_\Lambda(T^*X)=(T\Lambda)^\perp \end{array}$$

Q.E.D.

Definition 3.5. Let $\Omega_\Lambda^{\otimes\frac{1}{2}}$ be a line bundle F on Λ such that $F^{\otimes 2}$ is isomorphic to the sheaf Ω_Λ of n-forms. In general, F does not exist globally. Therefore, *we treat a section of F only up to a constant multiple*, and then everything goes well.

Definition 3.6. Let v (resp. φ) be a vector (resp. scalar) field on Λ. Then $v+\varphi$ acts on $\Omega_\Lambda^{\otimes\frac{1}{2}}$ as $\Omega_\Lambda^{\otimes\frac{1}{2}}\ni s\mapsto\frac{1}{2s}L_v(s^2)+\varphi s\in\Omega_\Lambda^{\otimes\frac{1}{2}}$ where L_v denotes the Lie derivative along v and $\frac{1}{2s}L_v(s^2)$ denotes an element $t\in\Omega_\Lambda^{\otimes\frac{1}{2}}$ satisfying $t\otimes s=\frac{1}{2}L_v(s^2)\in\Omega_\Lambda$. Fix a section ω of Ω_Λ and take $s\in\Omega_\Lambda^{\otimes\frac{1}{2}}$ satisfying $s^2=\omega$. Although this is not unique, fix one of them and denote it by $\sqrt{\omega}$. Then a section of $\Omega_\Lambda^{\otimes\frac{1}{2}}$ is of the form $f\sqrt{\omega}$ where f is a function. Then by simple calculation we have

$$\widetilde{v+\varphi}\colon f\sqrt{\omega}\mapsto\left(v(f)+\frac{f}{2}\cdot\frac{L_v(\omega)}{\omega}+\varphi f\right)\sqrt{\omega}. \tag{3.2}$$

In general, we denote $v+\varphi$ by $\widetilde{v+\varphi}$ when it is considered as an operator on $\Omega_A^{\otimes\frac{1}{2}}$. Recall that $\mathscr{M}=\mathscr{E}_X/\mathscr{J}$ is a simple holonomic system with the characteristic variety Λ. Then by Proposition 2.1 and Lemma 3.4, H_{P_m} is a vector field on Λ for $P=\sum_{j\leq m}P_j(z,D)\in\mathscr{J}$, and hence L_p acts on $\Omega_A^{\otimes\frac{1}{2}}$ for $P\in\mathscr{J}$. As an operator on $\Omega_A^{\otimes\frac{1}{2}}$, we denote L_p by $\tilde{L}_p$.

Lemma 3.7. *Let a and φ be functions, v a vector field. Then we have* $\widetilde{aP}=a\tilde{P}+\frac{1}{2}v(a)=\frac{1}{2}(a\tilde{P}+\tilde{P}a)$ *where* $P=v+\varphi$.

Proof. Take a local coordinate $(z_1,\dots,z_n)$ so that $\omega=dz_1\wedge\dots\wedge dz_n$ and $v=\sum c_j(z)\frac{\partial}{\partial z_j}$. Then we have

$$L_{av}(\omega)=\sum_{j=1}^{n}\frac{\partial ac_j}{\partial z_j}dz_1\wedge\dots\wedge dz_n=a\sum\frac{\partial c_j}{\partial z_j}\omega+\sum c_j\frac{\partial a}{\partial z_j}\omega=aL_v(\omega)+v(a)\,\omega$$

and hence

$$\begin{aligned}\widetilde{av}(\varphi\sqrt{\omega})&=\left(a(v\varphi)+\frac{1}{2}\frac{\varphi L_{av}(\omega)}{\omega}\right)\sqrt{\omega}=a\left(v(\varphi)+\frac{1}{2}\frac{\varphi L_v(\omega)}{\omega}\right)\sqrt{\omega}+\frac{v(a)}{2}\varphi\sqrt{\omega}\\&=(a\tilde{v}+\tfrac{1}{2}v(a))\,\varphi\sqrt{\omega}.\end{aligned}$$

This implies that $\widetilde{aP}=\widetilde{av}+\widetilde{a\varphi}=a\tilde{P}+\frac{1}{2}v(a)$. Since $\tilde{v}a=a\tilde{v}+v(a)$, we obtain the second equality. Q.E.D.

Lemma 3.8. (1) $[\tilde{L}_P,\tilde{L}_Q]=\tilde{L}_{[P,Q]}$ *for* $P,Q\in\mathscr{J}$.

(2) $\tilde{L}_{AP}=\sigma(A)\tilde{L}_P$ *for* $P\in\mathscr{J}$, $A\in\mathscr{E}_X$.

Proof. (1) For $P_1=v_1+\varphi_1$ and $P_2=v_2+\varphi_2$, we have $[\widetilde{P_1,P_2}]=[\tilde{P}_1,\tilde{P}_2]$. By Lemma 3.2 and this fact, we obtain our result.

(2) By Lemma 3.2, we have $L_{AP}=\sigma(A)L_P+\sigma(P)L_A+\frac{1}{2}\{\sigma(A),\ \sigma(P)\}$. Since $\sigma(P)|_\Lambda=0$, we have

$$\begin{aligned}\tilde{L}_{AP}&=\widetilde{\sigma(A)L_P}+\tfrac{1}{2}\widetilde{\{\sigma(A),\sigma(P)\}}\\&=\sigma(A)\tilde{L}_P+\tfrac{1}{2}H_{\sigma(P)}(\sigma(A))+\tfrac{1}{2}\{\sigma(A),\sigma(P)\}=\sigma(A)\tilde{L}_P\end{aligned}$$

by Lemma 3.7 and $H_{\sigma(P)}(\sigma(A))=\{\sigma(P),\sigma(A)\}$. Q.E.D.

The following lemma of Pfaff is well-known.

Lemma 3.9 (Pfaff). *Let $G_j=v_j+\varphi_j$ $(j=1,\dots,n)$ be a differential operator near x_0 of an n-dimensional manifold X where v_j (resp. φ_j) is a vector (resp. scalar) field. Assume that* (1) *$\{v_1(x_0),\dots,v_n(x_0)\}$ is a basis of $T_{x_0}X$* (2) *there exist functions a_{jkl} defined near x_0 satisfying*

$$[G_j,G_k]=\sum_l a_{jkl}G_l\qquad(j,k=1,\dots,n).$$

Then the solutions of $G_1 u = \dots = G_n u = 0$ *near* x_0 *form a one-dimensional vector space.*

Using above lemmas, we shall prove the following theorem.

Theorem 3.10. *Let* $\mathscr{M} = \mathscr{E}_X/\mathscr{J}$ *be a simple holonomic system with the characteristic variety* Λ. *Then a solution* $s \in \Omega_\Lambda^{\otimes \frac{1}{2}}$ *of* $\tilde{L}_P s = 0$ *for all* $P \in \mathscr{J}$ *exists locally and it is unique up to a constant multiple.*

Proof. By assumption, there exists an involutory basis $\{P_1, \dots, P_n\}$ of $\mathscr{J}$. Since $\sigma(P_j)|_\Lambda = 0$, we have $H_{\sigma(P_i)} \in T\Lambda$ by Lemma 3.4. Since $\{d\sigma(P_1), \dots, d\sigma(P_n)\}$ is a basis of $T_\Lambda^*(T^*X)$, $\{H_{\sigma(P_1)}, \dots, H_{\sigma(P_n)}\}$ is a basis of T_Λ. For $s = f\sqrt{\omega} \in \Omega_\Lambda^{\otimes \frac{1}{2}}$, we have $\tilde{L}_{P_j} s = \left(v_j(f) + \frac{f}{2\omega} L_{v_j}(\omega) + \varphi_j f\right)\sqrt{\omega} = 0$ if and only if $\left(v_j + \frac{1}{2\omega} L_{v_j}(\omega) + \varphi_j\right) f = 0$ where $L_{P_j} = v_j + \varphi_j$, $v_j = H_{\sigma(P_j)}$ $(j = 1, \dots, n)$. Put $G_j = v_j + \frac{1}{2\omega} L_{v_j}(\omega) + \varphi_j$ for $j = 1, \dots, n$. Then $\tilde{L}_p s = 0$ $(P \in \mathscr{J})$ for $s = f\sqrt{\omega} \in \Omega_\Lambda^{\otimes \frac{1}{2}}$ and $G_1 f = \dots = G_n f = 0$ are equivalent. These G_j $(j = 1, \dots, n)$ satisfy the first assumption of Lemma 3.9. About the second assumption, it is enough to show that $[\tilde{L}_{P_j}, \tilde{L}_{P_k}] = \sum a_{jkl} \tilde{L}_{P_l}$ since $[\tilde{L}_{P_j}, \tilde{L}_{P_k}] f\sqrt{\omega} = ([G_j, G_k] f)\sqrt{\omega}$. Since $\{P_1, \dots, P_n\}$ is a basis of $\mathscr{J}$ by Proposition 2.10, there exist $A_{jkl} \in \mathscr{E}_X$ satisfying $[P_j, P_k] = \sum_l A_{jkl} P_l$. Then by Lemma 3.8, we have

$$[\tilde{L}_{P_j}, \tilde{L}_{P_k}] = \tilde{L}_{[P_j, P_k]} = \tilde{L}_{\sum_l A_{jkl} P_l} = \sum_l \tilde{L}_{A_{jkl} P_l} = \sum_l \sigma(A_{jkl}) \tilde{L}_{P_l}$$

and hence we obtain our assertion by Lemma 3.9.

Definition 3.11. For a given simple holonomic system $\mathscr{M} = \mathscr{E}_X u = \mathscr{E}_X/\mathscr{J}$, let s be a solution of $\tilde{L}_P s = 0$ $(P \in \mathscr{J})$ in $\Omega_\Lambda^{\otimes \frac{1}{2}}$. Although this depends on a local coordinate system $z = (z_1, \dots, z_n)$, $s \otimes \sqrt{dz}^{-1} \in \Omega_\Lambda^{\otimes \frac{1}{2}} \otimes \Omega_X^{\otimes -\frac{1}{2}}$ does not depend on a local coordinate system by Lemma 3.3, and it is unique up to a constant multiple. We denote $s \otimes \sqrt{dz}^{-1}$ by $\sigma_\Lambda(u)$ or simply $\sigma(u)$ and call it *the principal symbol* of $\mathscr{M} = \mathscr{E}_X u$. This is unique up to a constant multiple. Recall that we always consider a section of $\Omega_\Lambda^{\otimes \frac{1}{2}}$ up to a constant multiple.

Lemma 3.12. $\tilde{L}_{PA} = \tilde{L}_P \sigma(A)$ *for all* $P \in \mathscr{J}$.

Proof. Put $l = \operatorname{ord}(A)$ and $m = \operatorname{ord}(P)$. Then by (2.6), we have

$$\begin{aligned} L_{PA}^{(m+l)} &= L_{[P,A]}^{(m+l)} + L_{AP}^{(m+l)} = \sigma_{m+l-1}([P, A]) + L_{AP}^{(m+l)} \\ &= \{\sigma(P), \sigma(A)\} + L_{AP}^{(m+l)}. \end{aligned}$$

Therefore by Lemma 3.8, we have $\tilde{L}_{PA} = \sigma(A)\tilde{L}_P + \{\sigma(P), \sigma(A)\}$. On the other hand,

$$\tilde{L}_P \sigma(A) s = H_{\sigma(P)} \sigma(A) s + \sigma(A) \left(P_{m-1} - \tfrac{1}{2} \sum_i \frac{\partial^2 P_m}{\partial z_i \partial \xi_i}\right) s = \sigma(A) \tilde{L}_P s + \{\sigma(P), \sigma(A)\} s$$

for all $s \in \Omega_\Lambda^{\otimes \frac{1}{2}}$ and hence we have $\tilde{L}_{PA} = \tilde{L}_P \sigma(A)$. Q.E.D.

Proposition 3.13. *If $\sigma(Q)|_A \neq 0$, i.e., Q is invertible, then we have $\sigma_A(Qu) = \sigma(Q)\sigma_A(u)$. Note that we always consider $\sigma_A(u)$ up to a constant multiple so this equality has a meaning only up to a constant multiple.*

Proof. Since $R(Qu)=0$ if and only if $R=PQ^{-1}$ with $P \in \mathscr{J}$, we have $\tilde{L}_R(\sigma_A(Qu) \otimes \sqrt{dz})=0$ and

$$\hat{L}_R(\sigma(Q)\,\sigma_A(u)\otimes\sqrt{dz})=\tilde{L}_P\sigma(Q)^{-1}\,\sigma(Q)\,\sigma_A(u)\otimes\sqrt{dz}=\tilde{L}_P\sigma_A(u)\otimes\sqrt{dz}=0$$

by Lemma 3.12. Since a solution s of $\tilde{L}_R s=0$ for all R, is unique by Theorem 3.10, we obtain our result. Q.E.D.

Lemma 3.14. *Put $v=\sum_{i=1}^{n}\xi_i\frac{\partial}{\partial\xi_i}$. Then we have $[v, L_P^{(m)}]=(m-1)L_P^{(m)}$.*

Proof. Put $\sigma_{m-1}(P)=P_{m-1}-\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 P_m}{\partial z_i\partial\xi_i}$. Then

$$L_P^{(m)}=\sum_i\left(\frac{\partial P_m}{\partial\xi_i}\cdot\frac{\partial}{\partial z_i}-\frac{\partial P_m}{\partial z_i}\cdot\frac{\partial}{\partial\xi_i}\right)+\sigma_{m-1}(P).$$

We have

$$\left[v,\frac{\partial P_m}{\partial\xi_i}\frac{\partial}{\partial z_i}\right]=v\left(\frac{\partial P_m}{\partial\xi_i}\right)\frac{\partial}{\partial z_i}+\frac{\partial P_m}{\partial\xi_i}v\frac{\partial}{\partial z_i}-\frac{\partial P_m}{\partial\xi_i}\cdot\frac{\partial}{\partial z_i}\cdot v$$
$$=v\left(\frac{\partial P_m}{\partial\xi_i}\right)\frac{\partial}{\partial z_i}=(m-1)\frac{\partial P_m}{\partial\xi_i}\cdot\frac{\partial}{\partial z_i}$$

and

$$\left[v,\frac{\partial P_m}{\partial z_i}\cdot\frac{\partial}{\partial\xi_i}\right]=(m-1)\frac{\partial P_m}{\partial z_i}\cdot\frac{\partial}{\partial\xi_i}$$

similarly. Since we have also

$$[v,\sigma_{m-1}(P)]=(v(\sigma_{m-1}(P))+\sigma_{m-1}(P)\,v)-\sigma_{m-1}(P)\,v$$
$$=v(\sigma_{m-1}(P))=(m-1)\,\sigma_{m-1}(P),$$

we have $[v, L_P^{(m)}]=(m-1)L_P^{(m)}$. Q.E.D.

Proposition 3.15. *The principal symbol $\sigma_A(u)$ is homogeneous with respect to ξ.*

Proof. It is enough to prove that $\tilde{L}_P(\tilde{v}s)=0$ for all $P\in\mathscr{J}$ if $\tilde{L}_P s=0$ for all $P\in\mathscr{J}$ where $v=\sum_{i=1}^{n}\xi_i\frac{\partial}{\partial\xi_i}$. Since $\tilde{L}_p\tilde{v}=\tilde{v}\tilde{L}_P+[\tilde{L}_P,\tilde{v}]$, we have $\tilde{L}_p\tilde{v}s=[\tilde{L}_p,\tilde{v}]\,s=[\widetilde{L_p, v}]\,s=-(m-1)\tilde{L}_P s=0$ by Lemma 3.14. Q.E.D.

Definition 3.16. The homogeneous degree of $\sigma_A(u)$ with respect to ξ is called the *order* of u at A and is denoted by $\mathrm{ord}_A(u)$.

Proposition 3.17. *Let P be a section of $\mathscr{J}$ satisfying $d\sigma_m(P)\equiv\varphi\omega \bmod J_A$ for some function φ where J_A denotes the all functions vanishing on A, and ω is the canonical 1-form. Then we have*

$$\left(\mathrm{ord}_A(u)+\frac{m-1}{2}\right)\varphi\equiv\left(P_{m-1}-\frac{1}{2}\sum_i\frac{\partial^2 P_m}{\partial z_i\partial\xi_i}\right)\bmod J_A. \tag{3.3}$$

Proof. We identified $T(T^*X)$ with $T^*(T^*X)$ by $v \mapsto v^*$ where $d\omega(v' \wedge v) = v^*(v')$ for all $v' \in T(T^*X)$ (See the proof of Lemma 3.4), and hence we have $d\sigma_m(P) = H_{\sigma_m(P)}$ and $\omega = -v$ where $v = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i}$. Since $d\sigma_m(P) \equiv \varphi\omega \bmod \Lambda$, we have $H_{\sigma_m(P)} = -\varphi v$ on Λ, and hence $\tilde{L}_P = (\widetilde{-\varphi v}) + \sigma_{m-1}(P) = -\varphi \tilde{v} - \frac{1}{2} v(\varphi) + \sigma_{m-1}(P)$ by Lemma 3.7, where $\sigma_{m-1}(P) = P_{m-1} - \frac{1}{2} \sum_i \frac{\partial^2 P_m}{\partial z_i \partial \xi_i}$. Note that $v(\varphi) = (m-1)\varphi$ because $d\sigma_m(P)$ (resp. ω) is homogeneous of degree m (resp. 1) and $d\sigma_m(P) \equiv \varphi\omega \bmod J_\Lambda$. By definition, we have $\tilde{v}\sigma_\Lambda(u) = \mathrm{ord}_\Lambda(u) \cdot \sigma_\Lambda(u)$ and hence

$$\tilde{L}_P \cdot \sigma_\Lambda(u) = \left(-\varphi\, \mathrm{ord}_\Lambda(u) - \frac{m-1}{2}\varphi + \sigma_{m-1}(P)\right) \sigma_\Lambda(u) = 0.$$

Therefore we have $\left(\mathrm{ord}_\Lambda(u) + \frac{m-1}{2}\right) \varphi \equiv \sigma_{m-1}(P) \bmod J_\Lambda$. Q.E.D.

Corollary 3.18. *Let P be a section of $\mathscr{J}$ of order 1 satisfying $d\sigma(P) \equiv \omega \bmod J_\Lambda$. Then we have*

$$\mathrm{ord}_\Lambda(u) = \left(P_0(x, \xi) - \frac{1}{2}\sum_i \frac{\partial^2 \sigma(P)}{\partial z_i \partial \xi_i}\right)\Big|_\Lambda.$$

Remark 3.19. Note that such P in Corollary 3.18 always exists. By a contact transformation, we may assume that $\Lambda = \{z_1 = \ldots = z_n = 0\}$. In this case, take P so that $\sigma_1(P) = \sum_{i=1}^{n} z_i \xi_i$.

Example 3.20. Let $(z_1, \ldots, z_n)$ be a local coordinate system, and consider the following equations $\mathscr{M}$. $\mathscr{M}$: $z_1 u = \ldots = z_r u = 0$, $D_{r+1} u = \ldots = D_n u = 0$ where $D_i = \frac{\partial}{\partial z_i}$. Then the Dirac δ-function $u = \delta(z_1, \ldots, z_r)$ is a generator of $\mathscr{M}$ and $\mathscr{M}$ is a simple holonomic system with the characteristic variety Λ where $\Lambda = \{(z, \xi);\ z_1 = \ldots = z_r = \xi_{r+1} = \ldots = \xi_n = 0\}$. We shall calculate the principal symbol

$$\sigma_\Lambda(u) = \varphi \sqrt{d\xi_1 \ldots d\xi_r\, dz_{r+1} \ldots dz_n} / \sqrt{dz_1 \ldots dz_n} \in \Omega_\Lambda^{\otimes \frac{1}{2}} \otimes \Omega_X^{\otimes -\frac{1}{2}}.$$

Since $L_{z_j} = H_{z_j} = -\frac{\partial}{\partial \xi_j}$ and $L_{D_j} = H_{\xi_j} = \frac{\partial}{\partial z_j}$, we have $\frac{\tilde{\partial}}{\partial \xi_j} \sigma_\Lambda(u) \sqrt{dz} = 0$ $(1 \leqq j \leqq r)$ and $\frac{\tilde{\partial}}{\partial z_k} \sigma_\Lambda(u) \sqrt{dz} = 0$ $(r+1 \leqq k \leqq n)$. Using $L_{\frac{\partial}{\partial \xi_j}}(d\xi_1 \ldots dz_n) = L_{\frac{\partial}{\partial z_k}}(d\xi_1 \ldots dz_n) = 0$, we have $\frac{\partial \varphi}{\partial \xi_j} = 0$ $(1 \leqq j \leqq r)$ and $\frac{\partial \varphi}{\partial z_k} = 0$ $(r+1 \leqq k \leqq n)$. This implies that $\varphi = \text{const.}$, and hence we have

$$(3.4) \quad \begin{aligned} &\sigma_\Lambda(\delta(z_1, \ldots, z_r)) = \sqrt{d\xi_1 \ldots d\xi_r\, dz_{r+1} \ldots dz_n} / \sqrt{dz_1 \ldots dz_n} \\ &\mathrm{ord}_\Lambda(\delta(z_1, \ldots, z_r)) = \frac{r}{2}. \end{aligned}$$

Note that the principal symbol $\sigma_\Lambda(u)$ is always defined modulo constant multiple.

Example 3.21. Consider the following equations $\mathscr{M}$ micro-locally at $\xi_1 \neq 0$.

$$\mathscr{M}: (x_1 D_1 - \alpha)u = (x_2 D_2 - \beta)u = D_3 u = \dots = D_n u = 0.$$

Then $\mathscr{M}$ is a simple holonomic system with the characteristic variety $\Lambda = \Lambda_1 \cup \Lambda_2$ where $\Lambda_1 = \{(z,\xi);\ z_1 = z_2 = \xi_3 = \dots = \xi_n = 0\}$ and $\Lambda_2 = \{(z,\xi);\ z_1 = \xi_2 = \dots = \xi_n = 0\}$. For $P_1(z,D) = z_1 D_1 - \alpha \in \mathscr{J}$, we have $L_{P_1} = H_{z_1\xi_1} - \alpha - \frac{1}{2} = z_1 \frac{\partial}{\partial z_1} - \xi_1 \frac{\partial}{\partial \xi_1} - \alpha - \frac{1}{2}$ and hence $L_{P_1} = -\left(\xi_1 \frac{\partial}{\partial \xi_1} + \alpha + \frac{1}{2}\right)$ on $\Lambda_1 \cup \Lambda_2$. For $P_2(z,D) = z_2 D_2 - \beta \in \mathscr{J}$, we have $L_{P_2} = z_2 \frac{\partial}{\partial z_2} - \xi_2 \frac{\partial}{\partial \xi_2} - \beta - \frac{1}{2}$ and hence $L_{P_2} = -\left(\xi_2 \frac{\partial}{\partial \xi_2} + \beta + \frac{1}{2}\right)$ on Λ_1 and $L_{P_2} = \left(z_2 \frac{\partial}{\partial z_2} - \beta - \frac{1}{2}\right)$ on Λ_2. For $P_j(z,D) = D_j$ for $j = 3,\dots,n$, we have $L_{P_j} = \frac{\partial}{\partial x_j}$. Put $\omega_1 = d\xi_1 d\xi_2 dz_3 \dots dz_n$ and $\omega_2 = d\xi_1 dz_2 \dots dz_n$. We shall calculate $\sigma_{\Lambda_1}(u) = f_1\sqrt{\omega_1}/\sqrt{dz}$ and $\sigma_{\Lambda_2}(u) = f_2\sqrt{\omega_2}/\sqrt{dz}$ where $dz = dz_1 \dots dz_n$. Using (3.2) and $L_{\xi_1 \frac{\partial}{\partial \xi_1}} \omega_i = \omega_i$ $(i = 1, 2)$, we have $\tilde{L}_{P_1} f_1 \sqrt{\omega_1} = -\left(\xi_1 \frac{\partial f_1}{\partial \xi_1} + (\alpha + 1) f_1\right)\sqrt{\omega_1} = 0$ and $\tilde{L}_{P_1} f_2 \sqrt{\omega_2} = -\left(\xi_1 \frac{\partial f_2}{\partial \xi_1} + (\alpha+1) f_2\right)\sqrt{\omega_2} = 0$. Since $L_{P_2} = -\left(\xi_2 \frac{\partial}{\partial \xi_2} + \beta + \frac{1}{2}\right)$ on Λ_1, we have $\tilde{L}_{P_2} f_1 \sqrt{\omega_1} = -\left(\xi_2 \frac{\partial f_1}{\partial \xi_2} + (\beta+1) f_1\right)\sqrt{\omega_1} = 0$. On the other hand, since $L_{z_2 \frac{\partial}{\partial z_2}} \omega_2 = \omega_2$ and $L_{P_2} = z_2 \frac{\partial}{\partial z_2} - \beta - \frac{1}{2}$ on Λ_2, we have $\tilde{L}_{P_2} f_2 \sqrt{\omega_2} = \left(z_2 \frac{\partial f_2}{\partial z_2} - \beta f_2\right)\sqrt{\omega_2} = 0$. Finally we have $\tilde{L}_{P_j} f_i \sqrt{\omega_i} = \frac{\partial f_i}{\partial x_j}\sqrt{\omega_i} = 0$ for $i = 1, 2;\ j = 3,\dots,n$. Therefore $f_1 = f_1(\xi_1, \xi_2, x_3, \dots, x_n)$ satisfies the equations:

$$\xi_1 \frac{\partial f_1}{\partial \xi_1} = -(\alpha+1) f_1, \qquad \xi_2 \frac{\partial f_1}{\partial \xi_2} = -(\beta+1) f_1, \qquad \frac{\partial f_1}{\partial x_j} = 0 \qquad (3 \leqq j \leqq n),$$

and hence we have $f_1 = \xi_1^{-\alpha-1} \xi_2^{-\beta-1}$ up to a constant multiple. Therefore we have

$$\begin{aligned} &\sigma_{\Lambda_1}(u) = \xi_1^{-\alpha-1} \xi_2^{-\beta-1} \sqrt{d\xi_1 d\xi_2 dz_3 \dots dz_n}/\sqrt{dz_1 \dots dz_n} \quad \text{and} \\ &\mathrm{ord}_{\Lambda_1}(u) = -\alpha - \beta - 1. \end{aligned} \tag{3.5}$$

On the other hand, $f_2 = f_2(\xi_1, z_2, \dots, z_n)$ satisfies the equations:

$$\xi_1 \frac{\partial f_2}{\partial \xi_1} = -(\alpha+1) f_2, \quad z_2 \frac{\partial f_2}{\partial z_2} = \beta f_2, \quad \frac{\partial f_2}{\partial z_j} = 0 \quad \text{for } j = 3, \dots, n,$$

and hence $f_2 = \xi_1^{-\alpha-1} z_2^\beta$ up to a constant multiple. Therefore we have

$$\begin{aligned} \sigma_{A_2}(u) &= \xi_1^{-\alpha-1} z_2^\beta \sqrt{d\xi_1 dz_2 \dots dz_n} / \sqrt{dz_1 \dots dz_n} \\ \operatorname{ord}_{A_2}(u) &= -\alpha - \tfrac{1}{2}. \end{aligned} \tag{3.6}$$

We shall see later that $\operatorname{ord}_{A_1}(u) - \operatorname{ord}_{A_2}(u) - \frac{1}{2} = \beta - 1$ is very important.

§4. Simple Holonomic Systems of Irreducible Regular Prehomogeneous Vector Spaces

Let (G, ρ, V) be an irreducible regular P.V. with $n = \dim V$. Then there exists a relative invariant polynomial $f(x)$; $f(\rho(g)x) = \chi(g) f(x)$ for $g \in G$, $x \in V$. It is unique up to a constant multiple. (See Definition 1.17 in §1.)

For simplicity, we identify V and V^* with $\mathbb{C}^n$ by a dual basis. We also assume that $G \subset GL(n, \mathbb{C})$ and $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$ where $\mathfrak{g}$ denotes the Lie algebra of G. By differentiating

$$f(\exp tAx)^s = \chi(\exp tA)^s f(x)^s = \exp ts\delta\chi(A) \cdot f(x)^s \qquad (A \in \mathfrak{g})$$

at $t=0$, we have $\sum_{i,j} a_{ij} x_j \frac{\partial}{\partial x_j} f(x)^s = s\delta\chi(A) f(x)^s$ for $A = (a_{ij}) \in \mathfrak{g}$. Therefore, $f(x)^s$ satisfies the following differential equations.

$$(\langle d\rho(A)x, D_x \rangle - s\delta\chi(A))u = 0 \qquad \text{for } A \in \mathfrak{g}. \tag{4.1}$$

Definition 4.1. We denote $\mathscr{D}_V$ (resp. $\mathscr{E}_V$) by $\mathscr{D}$ (resp. $\mathscr{E}$). Let $\mathscr{D}[s]$ (resp. $\mathscr{E}[s]$) be the sheaf of polynomials on s with coefficients in $\mathscr{D}$ (resp. $\mathscr{E}$). Denote by $\mathscr{J}$ *the Ideal* of $\mathscr{D}[s]$ generated by all $P(s) \in \mathscr{D}[s]$ satisfying $P(s) f(x)^s = 0$ for all s and $x \in V - S$. In particular, we have $\langle d\rho(A)x, D_x \rangle - s\delta\chi(A) \in \mathscr{J}$ for $A \in \mathfrak{g}$. For $\alpha \in \mathbb{C}$, put $\mathscr{J}[\alpha] = \{P(\alpha);\ P(s) \in \mathscr{J}\}$, and $\mathscr{M}_\alpha = \mathscr{E} / \mathscr{E} \cdot \mathscr{J}[\alpha]$. Its generator 1 modulo $\mathscr{E} \cdot \mathscr{J}[\alpha]$ will be denoted by $f(x)^\alpha$.

Put $\mathscr{M}'_\alpha = \mathscr{E} / \sum_{A \in \mathfrak{g}} \mathscr{E}(\langle d\rho(A)x, D_x \rangle - \alpha\delta\chi(A))$.

We identify the cotangent bundle T^*V of V with $V \times V^*$.

Definition 4.2. We define two Zariski-closed subsets W and W' of $V \times V^*$ as follows.

$$W' = \{(x, y) \in V \times V^*;\ \langle d\rho(A)x, y \rangle = 0 \text{ for all } A \in \mathfrak{g}_0\}$$

where $\mathfrak{g}_0 = \{A \in \mathfrak{g};\ \delta\chi(A) = 0\}$.

$W =$ the Zariski-closure of $\{(x, \operatorname{grad} \log f(x)^s) \in V \times V^*;\ s \in \mathbb{C},\ x \in V - S\}$. Then we have $W \subset W'$ since $\langle d\rho(A)x, \operatorname{grad} \log f(x)^s \rangle = s\delta\chi(A)$ for all $A \in \mathfrak{g}$. We define the function $\sigma(x, y)$ on W' by $\langle d\rho(A)x, y \rangle = \sigma(x, y)\,\delta\chi(A)$ for $A \in \mathfrak{g}$, i.e., $\sigma(x, y) = \langle d\rho(A_1)x, y \rangle$ if $\delta\chi(A_1) = 1$. Put $W_0 = \{(x, y) \in W;\ \sigma(x, y) = 0\}$.

Theorem 4.3. (1) $P \cdot f^\alpha = 0$ *on* $V - S$ *if and only if* $P \in \mathscr{J}[\alpha]$ *for a generic* $\alpha \in \mathbb{C}$, *i.e.,* $\alpha \in \mathbb{C} - S'$ *where* S' *is a discrete subset of* $\mathbb{C}$.

(2) $SS(\mathscr{M}_\alpha) = W_0$ *for any* $\alpha \in \mathbb{C}$ *where SS denotes the singular spectrum, i.e., the support in* $V \times V^*$.

Proof. See proposition 6.2. in [7] and Appendix.

Definition 4.4. Let x_0 be a point of V, $\rho(G)x_0$ the G-orbit of x_0. The *conormal vector space* $V_{x_0}^*$ is, by definition,

$$V_{x_0}^* = (d\rho(\mathfrak{g})x_0)^\perp = \{y \in V^*;\ \langle d\rho(A)x_0, y\rangle = 0 \quad \text{for all } A \in \mathfrak{g}\}.$$

Since $V_{\rho(g)x_0}^* = \rho^*(g)V_{x_0}^*$, the isotropy subgroup G_{x_0} at x_0 acts on $V_{x_0}^*$, and hence we obtain a triplet $(G_{x_0}, \rho_{x_0}, V_{x_0}^*)$ where $\rho_{x_0} = \rho^* | G_{x_0}$. This triplet $(G_{x_0}, \rho_{x_0}, V_{x_0}^*)$ is called *the colocalization* of (G, ρ, V) at x_0. Note that $(G_{x_0}, \rho_{x_0}, V_{x_0}^*) \cong (G_{x_1}, \rho_{x_1}, V_{x_1}^*)$ for $x_1 \in \rho(G)x_0$. The *conormal bundle* $T(\rho(G)x_0)^\perp$ (or $T^*_{\rho(G)x_0}V$) of the orbit $\rho(G)x_0$ is, by definition, the Zariski-closure of $\{(x, y) \in V \times V^*;$ $x \in \rho(G)x_0, y \in V_x^*\}$. Then G acts on $T(\rho(G)x_0)^\perp$ by $(x, y) \mapsto (\rho(g)x,\ \rho^*(g)y)$ for $g \in G$. It is clear that G acts on $T(\rho(G)x_0)^\perp$ prehomogeneously, i.e., $T(\rho(G)x_0)^\perp$ has a Zariski-dense orbit if and only if the colocalization $(G_{x_0}, \rho_{x_0}, V_{x_0}^*)$ is a P.V. Note that the union of cotangent bundles of all G-orbits coincides with $\{(x, y) \in W';\ \sigma(x, y) = 0\}$, and by (4.1) and Proposition 2.1 in §2, the supports of $\mathcal{M}_\alpha$ and $\mathcal{M}'_\alpha$ are contained in this set.

Definition 4.5. The conormal bundle $\Lambda = T(\rho(G)x_0)^\perp$ of an orbit $\rho(G)x_0$ is called a *good holonomic variety* (or a *good Lagrangian*) if (1) G acts on Λ prehomogeneously (2) $\Lambda \subset W$, i.e., $\Lambda \subset W_0$.

Proposition 4.6. *Let (G, ρ, V) be a P.V. without assumption of irreducibility. Then it is a regular P.V. if and only if the conormal bundle $\{0\} \times V^*$ of the origin is a good holonomic variety.*

Proof. Assume that (G, ρ, V) is regular and take $\operatorname{grad}\log f^s$: $V - S \to V^*$ to be generically surjective. Since $(\varepsilon x, \operatorname{grad}\log f^s(x)) = (\varepsilon x,\ \operatorname{grad}\log f^s(\varepsilon x)) \in W$ for any $\varepsilon \neq 0$, we have $(0, \operatorname{grad}\log f^s(x)) = \lim_{\varepsilon \to 0}(\varepsilon x,\ \operatorname{grad}\log f^s(x)) \in W$ for all $x \in V - S$, and hence $\{0\} \times V^* \subset W$. Since the dual of (G, ρ, V) is also a P.V. (See Proposition 4 in §1), $\Lambda = \{0\} \times V^*$ is a good holonomic variety. Conversely, if $\{0\} \times V^*$ is a good holonomic variety, then the dual (G, ρ^*, V^*) is clearly a P.V. Let S^* be its singular set. Assume that (G, ρ, V) is not regular. Then we have $\operatorname{grad}\log f^s(V - S) \subset S^*$ for all $s \in \mathbb{C}^l$ and hence $\{0\} \times V^* \subset W \subset V \times S^*$. This is a contradiction. Q.E.D.

Now we shall prove that two systems $\mathcal{M}_s = \mathcal{E} f^s$ and $\mathcal{M}'_s = \mathcal{E} / \sum_{A \in \mathfrak{g}} \mathcal{E}(\langle d\rho(A)x, D_x\rangle - s\delta\chi(A))$ coincide on a good holonomic variety for $s \in \mathbb{C}$, and they are simple holonomic systems.

Proposition 4.7. *Let Λ be a good holonomic variety, and $p = (x_0, y_0)$ a generic point of Λ, i.e., a point of the Zariski-dense orbit. Then the following properties hold in a neighborhood of p.*

(1) *$W = W'$ and W is a non-singular manifold of dimension $(n+1)$. Moreover the ideal J_W of all functions vanishing on W is generated by $\{\langle d\rho(A)x, y\rangle;\ A \in \mathfrak{g}_0\}$.*

(2) *The ideal J_Λ of all functions vanishing on Λ is generated by $\{\langle d\rho(A)x, y\rangle;$ $A \in \mathfrak{g}\}$.*

(3) *$d\sigma$ is a non-zero 1-form on W, and $\Lambda = \{(x, y) \in W;\ \sigma(x, y) = 0\}$.*

Proof. Let $\mathscr{K}$ (resp. $\mathscr{K}_0$) $=\{d\langle d\rho(A)x, y\rangle_{p=(x_0, y_0)};\ A\in\mathfrak{g}$ (resp. $\mathfrak{g}_0$)$\}$. Then $\mathscr{K}$ (resp. $\mathscr{K}_0$) is isomorphic to $\{(d\rho(A)x_0, d\rho^*(A)y_0)\in V\times V^*;\ A\in\mathfrak{g}$ (resp. $\mathfrak{g}_0$)$\}$. Since Λ is G-prehomogeneous of dimension n and $\mathscr{K}\cong T_p\Lambda$, we have $\dim\mathscr{K}=n$. This implies immediately (2). By $\dim(\mathfrak{g}/\mathfrak{g}_0)=1$, we have $\dim\mathscr{K}_0\geqq n-1$. Therefore there exist $A_1,\ldots,A_{n-1}\in\mathfrak{g}_0$ such that $d\langle d\rho(A_1)x, y\rangle,\ldots,d\langle d\rho(A_{n-1})x, y\rangle$ are linearly independent at p. Then $W''=\{(x,y)\in V\times V^*;\ \langle d\rho(A_1)x, y\rangle=\ldots=\langle d\rho(A_{n-1})x, y\rangle=0\}$ is a non-singular submanifold of dimension $(n+1)$ in a neighborhood of p. On the other hand we have $p\in\Lambda\subset W\subset W'\subset W''$ and W is an irreducible algebraic set of dimension $(n+1)$. Therefore $W''=W$ in a neighborhood of p and hence $W=W'=W''$. This implies (1). Since W' is a non-singular submanifold of codimension $(n-1)$, we have $\dim\mathscr{K}_0=n-1$. On the other hand, $\mathscr{K}/\mathscr{K}_0=\mathbb{C}\,d\sigma$ and hence $d\sigma$ is non-zero on W. This implies that $\{(x,y)\in W;\ \sigma(x,y)=0\}$ is an n-dimensional manifold containing Λ, and hence they coincide. Q.E.D.

Now we shall prove that $\mathscr{M}_\alpha=\mathscr{M}'_\alpha$.

Proposition 4.8. *Let Λ be a good holonomic variety, and p a generic point of Λ. Then $\mathscr{M}_\alpha$ is a simple holonomic system and isomorphic to $\mathscr{M}'_\alpha$ for any $\alpha\in\mathbb{C}$ in a neighborhood of p (See Definition 4.1).*

Proof. Put $\mathscr{J}'=\sum_{A\in\mathfrak{g}}\mathscr{E}(\langle d\rho(A)x, D_x\rangle-\alpha\delta\chi(A))$. Then $\mathscr{J}'$ is contained in $\mathscr{E}\mathscr{J}[\alpha]$. The symbol Ideal $\bar{\mathscr{J}}'$ of $\mathscr{J}'$ is therefore contained in the symbol ideal $\overline{\mathscr{E}\mathscr{J}[\alpha]}$ of $\mathscr{E}\mathscr{J}[\alpha]$. Let J_Λ be the Ideal of functions vanishing on Λ. Then by Proposition 4.7, J_Λ is generated by $\{\langle d\rho(A)x, y\rangle;\ A\in\mathfrak{g}\}$ and hence we have $J_\Lambda\subset\bar{\mathscr{J}}'\subset\overline{\mathscr{E}\mathscr{J}[\alpha]}$. Since $\mathscr{M}_\alpha=\mathscr{E}/\mathscr{E}\cdot\mathscr{J}[\alpha]$ has support Λ by Theorem 4.3, we have $\overline{\mathscr{E}\mathscr{J}[\alpha]}\subset J_\Lambda$. It follows that $J_\Lambda=\bar{\mathscr{J}}'=\overline{\mathscr{E}\mathscr{J}[\alpha]}$, and hence we have $\mathscr{J}'=\mathscr{E}\cdot\mathscr{J}[\alpha]$. Therefore $\mathscr{M}_\alpha$ is a simple holonomic system and isomorphic to $\mathscr{M}'_\alpha$. Q.E.D.

Proposition 4.9. *The function $\sigma(x,y)$ on W' is G-invariant* (See Definition 4.2).

Proof. Let $A_1\in\mathfrak{g}$ be an element such that $\sigma(x,y)=\langle d\rho(A_1)x, y\rangle$. Then we have

$$\begin{aligned}\sigma(\rho(g)x, \rho^*(g)y)-\sigma(x,y)&=\langle d\rho(A_1)\rho(g)x, \rho^*(g)y\rangle-\langle d\rho(A_1)x, y\rangle\\&=\langle[\rho(g)^{-1}d\rho(A_1)\rho(g)-d\rho(A_1)]x, y\rangle=0\end{aligned}$$

for $(x,y)\in W'$ since $\rho(g)^{-1}d\rho(A_1)\rho(g)-d\rho(A_1)\in\mathfrak{g}_0$. Q.E.D.

Proposition 4.10. *Let Λ be a good holonomic variety. Then there exists a non-negative integer m_Λ such that $\varphi(x,y)=f(x)/\sigma(x,y)^{m_\Lambda}$ is a non-zero regular G-relative invariant function on Λ.*

Proof. Since W is smooth near a generic point p of Λ, and Λ is defined by $\sigma(x,y)=0$ on W, we may choose a local coordinate system $(\sigma, t_1,\ldots,t_n)$ of W near p such that p is the origin $(\sigma,t)=(0,\ldots,0)$ where $t=(t_1,\ldots,t_n)$. By the projection π: $W\to V$, we regard the relative invariant $f(x)$ on V as a function $f(\sigma,t)$ on W. Since the G-orbit of p is of n-dimension, the dimension of a G-orbit of (σ,t) which is sufficiently near p is at least n. On the other hand, since $\sigma(x,y)$ is G-invariant and W is G-admissible, i.e., $(\rho(g)x, \rho^*(g)y)\in W$ for all $(x,y)\in W$, the dimension of the G-orbit of (σ, t) does not exceed n and hence it is equal to n. This implies that there exists an element g of G satisfying $(\sigma, t)=g\cdot(\sigma, 0)$ for a point (σ, t) sufficiently near p. Hence we have $f(\sigma, t)=f(g(\sigma, 0))=\chi(g)f(\sigma, 0)$. Since $f(0,0)=f(p)\neq\infty$, there exists a non-negative integer m_A such that $f(\sigma, 0)=\sigma^{m_A}\cdot\tilde{f}(\sigma)$ where $\tilde{f}(0)\neq 0, \infty$. Then $\varphi(x,y)=\dfrac{f(x)}{\sigma^{m_A}}=\chi(g)\tilde{f}(\sigma)$ is a G-relative invariant function on Λ. Q.E.D.

Definition 4.11. We denote $\varphi(x,y)$ in Proposition 4.10 by $f_\Lambda(x,y)$. This function $f_\Lambda(x,y)$ on Λ is a relative invariant with the character χ by Proposition 4.9. Note that f_Λ might have a singularity outside of the open G-orbit in Λ.

Now let (G,ρ,V) be an irreducible regular P.V. and let $f(x)$ and $f^*(y)$ be as in the end of § 1.

Proposition 4.12. *There exists a constant c such that* $f_\Lambda(x,y)=\dfrac{c}{f^*(y)}$ *for* $\Lambda=\{0\}\times V^*$.

Proof. Since $f_\Lambda(x,y)$ is a relative invariant on V^* with the character χ, it must coincide with $f^*(y)^{-1}$ up to a constant multiple. Q.E.D.

Let $x=(x_1,\dots,x_n)$ be a coordinate system of V and $dx=dx_1\dots dx_n$ an n-form on V. Let $\pi\colon W\to V$ be the projection. Then $\pi^*(dx)\wedge d\sigma$ is an $(n+1)$-form on W.

Proposition 4.13. *Let Λ be a good holonomic variety. Then there exists a non-negative integer μ_Λ such that $\dfrac{1}{\sigma^{\mu_\Lambda}}\pi^*(dx)\wedge d\sigma$ is non-zero regular near a generic point of Λ. Moreover, $\omega_\Lambda=\left\{\dfrac{1}{\sigma^{\mu_\Lambda}}\pi^*(dx)\wedge d\sigma\right\}\Big/ d\sigma$ is a relative invariant n-form on Λ corresponding to the character $\chi_0(g)=\det_V\rho(g)$. This ω_Λ is unique up to a constant multiple.*

Proof. The proof goes just same way as that of Proposition 4.10. Q.E.D.

Proposition 4.14. *Let Λ be a good holonomic variety. Note that $\mathcal{M}_s=\mathscr{E}f^s$ is a simple holonomic system on Λ in this case. Then we have $\sigma_\Lambda(f^s)=f_\Lambda^s\sqrt{\omega_\Lambda}/\sqrt{dx}$ and $\operatorname{ord}_\Lambda f^s=-m_\Lambda s-\mu_\Lambda/2=s\delta\chi(A_0)-tr_{V^*_{x_0}}d\rho_{x_0}(A_0)+\frac{1}{2}\dim V^*_{x_0}$ where A_0 is any element of* $\mathfrak{g}$ *satisfying $d\rho(A_0)x_0=0$ and $d\rho^*(A_0)y_0=y_0$ for a generic point $p=(x_0,y_0)$ of Λ. In particular, we have $\operatorname{ord}_{V\times\{0\}}f^s=0$.*

Proof. Put $P_A(s)=\langle d\rho(A)x, D_x\rangle-s\delta\chi(A)$ for $A\in\mathfrak{g}$. Since the Hamilton vector field $H_{\langle d\rho(A)x,y\rangle}=\langle d\rho(A)x, D_x\rangle+\langle d\rho^*(A)y, D_y\rangle$ is the infinitesimal transformation of $(x,y)\mapsto(\rho(g)x, \rho^*(g)y)\,(g\in G)$, we have $H_{\langle d_\rho(A)x,y\rangle}f_\Lambda^s=s\delta\chi(A)\cdot f_\Lambda^s$ and $L_{H_{\langle d\rho(A)x,y\rangle}}\omega_\Lambda=tr_V d\rho(A)\cdot\omega_\Lambda$ where L_v denotes the Lie derivative along v, and hence

$$\tilde{L}_{H_{\langle d\rho(A)x,y\rangle}}f_\Lambda^s\sqrt{\omega_\Lambda}=(s\delta\chi(A)+\tfrac{1}{2}tr_V\,d\rho(A))f_V^s\sqrt{\omega_\Lambda}.$$

On the other hand, since $\sum_i\dfrac{\partial^2\langle d\rho(A)x,y\rangle}{\partial x_i\,\partial y_i}=tr_V\,d\rho(A)$, we have

$$\tilde{L}_{P_A(s)}=\tilde{L}_{H_{\langle d\rho(A)x,y\rangle}}-s\delta\chi(A)-\tfrac{1}{2}tr_V\,d\rho(A)$$

and hence $\tilde{L}_{P_A(s)} f_A^s \sqrt{\omega_A} = 0$ for all $A \in \mathfrak{g}$. Since Λ is a good holonomic variety, $P_A(s)$ ($A \in \mathfrak{g}$) is an involutive basis of $\mathscr{E}\mathscr{J}[s]$ and hence we have $\tilde{L}_P f_A^s \sqrt{\omega_A} = 0$ for all $P \in \mathscr{E}\mathscr{J}[s]$. Therefore we have $\sigma_\Lambda(f^s) = f_\Lambda^s \sqrt{\omega_\Lambda}/\sqrt{dx}$. Note that $\sigma_\Lambda(f^s)$ is uniquely determined up to a constant multiple. The order $\mathrm{ord}_\Lambda f^s$ is, by the definition, $\deg_y \sigma_\Lambda(f^s) = \deg_y f_\Lambda^s + \frac{1}{2}\deg_y \omega_\Lambda$. On the other hand, $f(x)$ and $\sigma(x, y)$ are homogeneous of degree 0 and 1 in y respectively, and hence f_Λ and ω_Λ are homogeneous of degree $-m_\Lambda$ and $-\mu_\Lambda$ respectively, i.e., $\mathrm{ord}_\Lambda f^s = -m_\Lambda s - \dfrac{\mu_\Lambda}{2}$ (See Propositions 4.10 and 4.13). Since

$$(\langle d\rho(A)x, D_x\rangle + \langle d\rho^*(A)y, D_y\rangle) f_\Lambda^s = s\delta\chi(A) \cdot f_\Lambda^s$$

and $\langle y, D_y\rangle f_\Lambda^s = (\deg_y f_\Lambda^s) f_\Lambda^s$, we have $\deg_y f_\Lambda^s = s\delta\chi(A_0)$ with $d\rho(A_0)x_0 = 0$ and $d\rho^*(A_0)y_0 = y_0$. Let $\pi\colon V \times V^* \to V$ be the projection, $(t_1, \dots, t_n)$ a local coordinate system of V near x_0 satisfying $\pi(\Lambda) = \{(t_1, \dots, t_n);\ t_1 = \dots = t_r = 0\}$ and $\Lambda = \{(t, \tau);\ t_1 = \dots = t_r = 0,\ \tau_{r+1} = \dots = \tau_n = 0\}$ where $r = \dim V_{x_0}^*$. Then we have $\omega_\Lambda = \varphi(t, \tau)\, d\tau_1 \dots d\tau_r\, dt_{r+1} \dots dt_n$ near (x_0, y_0). Since $d\tau_1 \dots d\tau_r$ (resp. $dt_{r+1} \dots dt_n$) is a volume element of $V_{x_0}^*$ (resp. $\mathfrak{g}_{x_0}$) and hence $d\tau_1 \dots d\tau_r\, dt_{r+1} \dots dt_n$ is a relative invariant n-form corresponding to the character $tr_{V_{x_0}^*}\, d\rho_{x_0}(A) + tr_{\mathfrak{g}_{x_0}}\, d\rho(A) = 2\,tr_{V_{x_0}^*}\, d\rho_{x_0}(A) + tr_V\, d\rho(A)$ for $A \in \mathfrak{g}_{x_0}$. Note that $tr_V\, d\rho(A) - tr_{\mathfrak{g}_{x_0}}\, d\rho(A) = tr_{V/\mathfrak{g}_{x_0}}\, d\rho(A) = -tr_{V_{x_0}^*}\, d\rho_{x_0}(A)$. Since ω_Λ is a relative invariant n-form corresponding to the character $tr_V\, d\rho(A)$, $\varphi(t, \tau)$ is a relative invariant corresponding to the character $-2\,tr_{V_{x_0}^*}\, d\rho_{x_0}(A)$. Therefore, $\deg_\tau \varphi(t, \tau) = -2\,tr_{V_{x_0}^*}\, d\rho_{x_0}(A_0)$ where $d\rho(A_0)x_0 = 0$, $d\rho^*(A_0)y_0 = y_0$, and hence we have $\deg_y \omega_\Lambda = \deg_\tau \varphi(t, \tau) + r = -2\,tr_{V_{x_0}^*}\, d\rho_{x_0}(A_0) + \dim V_{x_0}^*$. Since $\mathrm{ord}_\Lambda f^s = \deg_y f_\Lambda^s + \frac{1}{2}\deg_y \omega_\Lambda$, we obtain our assertion. Q.E.D.

Remark 4.15. For a good holonomic variety Λ, we can define the local a-function $a_\Lambda^\chi(s)$ and the local c-function $c_\Lambda(s)$. In the irreducible case, we have $a_\Lambda^\chi(s) = s^{m_\Lambda}$ and $c_\Lambda(s) = s^{\mu_\Lambda}$ (See Proposition 4.10 and 4.13).

§ 5. Local b-functions

First, we shall define the local b-functions without assumption of irreducibility.

Definition 5.1. Let Λ be a good holonomic variety. Assume that f^χ is a polynomial (See Definition 1.7). A polynomial $b_\Lambda^\chi(s)$ in $s = (s_1, \dots, s_l) \in \bar{X}_0$ is called a *local b-function* of Λ if there exists an invertible micro-differential operator P_χ (i.e., $\sigma(P_\chi)|_\Lambda \neq 0$) defined near a generic point of Λ which is independent of s and satisfies $P_\chi f^{s+\chi} = b_\Lambda^\chi(s) f^s$. Namely $P_\chi f^\chi u_s = b_\Lambda^\chi(s) u_s$ where u_s is a generator of $\mathscr{M}_s = \mathscr{E} / \sum_{A \in \mathfrak{g}} \mathscr{E}\,(\langle d\rho(A), D_x\rangle - \sum s_i \delta\chi_i(A))$.

Remark 5.2. Let (G, ρ, V) be a regular P.V. Then, by Proposition 4.6, $\Lambda = \{0\} \times V^*$ is a good holonomic variety, and a b-function $b_\chi(s)$ of (G, σ, V) is a local b-function of $\Lambda = \{0\} \times V^*$. In fact, $f^{*\chi^{-1}}(\mathrm{grad}_x)$ is an invertible operator near a generic point of Λ since $f^{*\chi^{-1}}(y) \neq 0$.

From now on, we shall assume that (G, ρ, V) is an irreducible regular P.V. Hence this has a unique relative invariant irreducible polynomial $f(x)$. We write $b_\Lambda(s)$ for $b_\Lambda^\chi(s)$ with the character χ corresponding to $f(x)$.

Lemma 5.3. *Let P be an invertible operator satisfying $Pfu_s = b_A(s)u_s$. Then we have* $\operatorname{ord}(P) = m_A$ *and* $\sigma_{m_A}(P)|_A = c \cdot f_A^{-1}$ *for some* $c \in \mathbb{C}^\times$. *Here* m_A *is defined in Proposition* 4.10.

Proof. By Proposition 4.14, we have $\sigma_A(u_s) = f_A^s\sqrt{\omega_A}$. In the same way, we have $\sigma_A(fu_s) = f_A^{s+1}\sqrt{\omega_A}$ if $fu_s \neq 0$ because fu_s satisfies the same system of equations as u_{s+1} does. On the other hand, fu_s does not vanish for s such that $b_A(s) \neq 0$, because this equals to $b_A(s)P_A^{-1}u_s$. Note that $\sigma_A(u_s)$ and $\sigma_A(fu_s)$ are defined modulo constant. Therefore we have

$$\sigma_A(Pfu_s) = \sigma(P)\sigma_A(fu_s) = \sigma(P)f_A^{s+1}\sqrt{\omega_A} = \sigma_A(b_A(s)u_s) = f_A^s\sqrt{\omega_A}$$

modulo constant, and hence $\sigma_{m_A}(P) = c \cdot f_A^{-1}$ for some $c \in \mathbb{C}^\times$. Q.E.D.

Theorem 5.4 (*Uniqueness of a local b-function*) *A local b-function $b_A(s)$ of A is unique up to a constant multiple.*

Proof. Assume that $P_i fu_s = b_i(s)u_s$ $(i=1,2)$ where P_1 and P_2 are invertible. Since $fu_s = b_1(s)P_1^{-1}u_s$, we have $b_2(s)u_s = b_1(s)P_2P_1^{-1}u_s$, i.e., $(b_2(s) - b_1(s)P_2P_1^{-1})u_s = 0$. Since the principal symbol of $P_2P_1^{-1}$ is constant by Lemma 5.3, we have $\sigma_0(b_2(s) - b_1(s)P_2P_1^{-1}) = b_2(s) - cb_1(s)$ for some $c \in \mathbb{C}^\times$. Therefore we have $b_2(s) = cb_1(s)$ since otherwise $b_2(s) - b_1(s)P_2P_1^{-1}$ is invertible and hence $u_s = 0$, which is a contradiction. Q.E.D.

Theorem 5.5 (*Existence of a local b-function*). *There exist an invertible operator P_A of order $(-m_A)$ defined on a neighborhood of a generic point of A satisfying $\sigma_{-m_A}(P_A)|_A = f_A$ and a monic polynomial $b_A(s)$ of degree m_A satisfying $fu_s = b_A(s)P_Au_s$.*

The rest of this section will be devoted to prove this existence theorem.

Definition 5.6. For a micro-differential operator $T(s) = \sum_{\alpha \in \mathbb{Z}} s^\alpha T_\alpha$, which is a polynomial in s, we define $\operatorname{ord} T(s)$ and $\underline{\underline{\operatorname{ord}}}\, T(s)$ by $\operatorname{ord} T(s) = \max_\alpha \operatorname{ord} T_\alpha$ and $\underline{\underline{\operatorname{ord}}}\, T(s) = \max_\alpha (\alpha + \operatorname{ord} T_\alpha)$ respectively.

Lemma 5.7. *Let $G(s) = \sum s^\alpha G_\alpha$ be a micro-differential operator satisfying* $\operatorname{ord} G(s) \leqq m$, $\underline{\underline{\operatorname{ord}}}\, G(s) \leqq k$ *and $\sigma_m(G(s))|_A = 0$. Then there exists a micro-differential operator $T(s)$ such that $T(s)u_s = G(s)u_s$ and* $\operatorname{ord} T(s) < m$, $\underline{\underline{\operatorname{ord}}}\, T(s) \leqq k$.

Proof. First we shall consider the case when $G(s)$ does not contain s, i.e., $G(s) = G$. Then we have $\operatorname{ord} G = \underline{\underline{\operatorname{ord}}}\, G \leqq \min(m, k)$. If $m > k$, then we may take $T(s) = G$. If $m \leqq k$, we may assume that $m = k$ since $\operatorname{ord} G = \underline{\underline{\operatorname{ord}}}\, G$. Since $\sigma_m(G)|_A = 0$ and $\{\langle d\rho(A)x, D_x\rangle - s\delta\chi(A);\, A \in \mathfrak{g}\}$ generates the defining ideal of A, we have $\sigma_m(G) = \sum_{A \in \mathfrak{g}} \varphi_A(x, y)\langle d\rho(A)x, y\rangle$ and $G = \sum_{A \in \mathfrak{g}} \varphi_A(x, D_x)\langle d\rho(A)x, D_x\rangle + K$ where $\operatorname{ord} \varphi_A(x, D_x) = m-1$ and $\operatorname{ord} K \leqq m-1$. Hence $Gu_s = \sum_{A \in \mathfrak{g}} \varphi_A(x, D_x)s\delta\chi(A)u_s + Ku_s$. Now put $T(s) = \sum_{A \in \mathfrak{g}} \varphi_A(x, D_x)s\delta\chi(A) + K$. Then we have $T(s)u_s = Gu_s$, $\operatorname{ord} T(s) \leqq m-1$, and $\underline{\underline{\operatorname{ord}}}\, T(s) \leqq m$. Next we shall consider the general case. Since $\operatorname{ord} G_\alpha \leqq m$, $\underline{\underline{\operatorname{ord}}}\, G_\alpha = \operatorname{ord} G_\alpha \leqq k - \alpha$ and $\sigma_m(G_\alpha)|_A = 0$, there exists $T_\alpha(s)$ such that

$T_\alpha(s)u_s = G_\alpha u_s$, ord $T_\alpha(s) < m$ and $\underline{\underline{\mathrm{ord}}}\, T_\alpha(s) \leqq k - \alpha$. Now put $T(s) = \sum s^\alpha T_\alpha(s)$. Then we have $T(s)u_s = G(s)u_s$, ord $T(s) < m$ and $\underline{\underline{\mathrm{ord}}}\, T(s) = \max(\alpha + \underline{\underline{\mathrm{ord}}}\, T_\alpha(s)) \leqq k$. Q.E.D.

Lemma 5.8. *For $\alpha \in \mathbb{C}$ and an operator G such that $Gu_\alpha \neq 0$, there exists a number r such that* ord $T \geqq r$ *for any operator T satisfying $Tu_\alpha = Gu_\alpha$.*

Proof. Since Λ is a good holonomic variety, $\mathcal{M} = \mathscr{E} u_\alpha$ is a simple holonomic system. Hence it is simple as a module (See [2]) so that $\mathscr{E} G u_\alpha = \mathscr{E} u_\alpha$. This implies that there exists an invertible operator K satisfying $Gu_\alpha = Ku_\alpha$. Put $r = \mathrm{ord}\, K$. If there exists T such that $Gu_\alpha = Tu_\alpha$ and ord $T < r$, then we have $(K - T)u_\alpha = 0$ and $\sigma(K - T) = \sigma(K) \neq 0$. This implies $u_\alpha = 0$, which is a contradiction. Q.E.D.

Lemma 5.9. *If $G(s)u_s \neq 0$, then there exists an operator $T(s)$ which is invertible at a generic point of Λ for a generic s such that $T(s)u_s = G(s)u_s$ and* $\underline{\underline{\mathrm{ord}}}\, T(s) < \underline{\underline{\mathrm{ord}}}\, G(s)$.

Proof. It is obvious from Lemma 5.7 and Lemma 5.8. Q.E.D.

Let P be a micro-differential operator of order $(-m_\Lambda)$ such that $\sigma_{-m_\Lambda}(P)|_W = f(x)/\sigma(x, y)^{m_\Lambda}$, i.e., $f = \sigma_{-m_\Lambda}(P) \cdot \sigma^{m_\Lambda}$ on W. Note that $\sigma(x, y) = \langle d\rho(A_1)x, y \rangle$ with $\delta\chi(A_1) = 1$ (See Definition 4.2). Let $B_1, \ldots, B_r$ be a basis of $\mathfrak{g}_0$. Then since functions vanishing on W are linear combinations of $\langle d\rho(B_i)x, y \rangle$ $(i = 1, \ldots, r)$, we have

$$f(x) - P(x, D_x) \cdot \langle d\rho(A_1)x, D_x \rangle^{m_\Lambda} = \sum_{j=1}^{r} T_j(x, D_x) \langle d\rho(B_j)x, D_x \rangle + K$$

with ord $K \leqq -1$, and hence $fu_s = s^{m_\Lambda} \cdot Pu_s + Ku_s$ (ord $K \leqq -1$).

If $Ku_s = 0$, we obtain our assertion of Theorem 5.5. We shall assume that $Ku_s \neq 0$.

Then, by Lemma 5.9, there exists an operator $G(s)$ of $\underline{\underline{\mathrm{ord}}}\, G(s) \leqq -1$ such that $fu_s = s^{m_\Lambda} Pu_s + G(s)u_s$ and that $G(s)$ is invertible at a generic point of Λ for a generic s.

Lemma 5.10. *We have* ord $G(s) \leqq -m_\Lambda$.

Proof. Assume that ord $G(s) > -m_\Lambda$. Then we have

$$\begin{aligned} \mathrm{ord}\,(s^{m_\Lambda} Pu_s + G(s)u_s) &= \mathrm{ord}\, G(s) + \mathrm{ord}\, u_s \\ = \mathrm{ord}\, fu_s = \deg_y f_\Lambda^{s+1} \sqrt{\omega_\Lambda} &= \deg_y f_\Lambda + \deg_y f_\Lambda^s \sqrt{\omega_\Lambda} = -m_\Lambda + \mathrm{ord}\, u_s, \end{aligned}$$

and hence ord $G(s) = -m_\Lambda$, which is a contradiction. Q.E.D.

Lemma 5.11. $\sigma_{-m_\Lambda}(G(s)) = \sum_{\alpha \lneqq m_\Lambda} s^\alpha \sigma_{-m_\Lambda}(G_\alpha)$ *where* $G(s) = \sum s^\alpha G_\alpha$.

Proof. Since $\underline{\underline{\mathrm{ord}}}\, G(s) \leqq -1$ and ord $G(s) \leqq -m_\Lambda$, we have ord $G_\alpha \leqq -m_\Lambda$ and $\alpha + \mathrm{ord}\, G_\alpha \leqq -1$. Since $\sigma_{-m_\Lambda}(G(s)) = \sum s^\alpha \sigma_{-m_\Lambda}(G_\alpha)$ where the sum is taken over α satisfying ord $G_\alpha = -m_\Lambda$, we have $|\alpha| \leqq -1 - \mathrm{ord}\, G_\alpha = m_\Lambda - 1$. Q.E.D.

Lemma 5.12. *There exists a micro-differential operator $P(s)$ and a monic polynomial $b_A(s)$ of degree m_A in s satisfying* (1) $f u_s = P(s) u_s$ (2) $\operatorname{ord} P(s) = -m_A$ (3) $\underline{\underline{\operatorname{ord}}}\, P(s) \leqq 0$ (4) $\sigma_{-m_A}(P(s))|_A = b_A(s) f_A$.

Proof. Put $P(s) = s^{m_A} \cdot P + G(s)$. By Lemma 5.10 and Lemma 5.11, we have (1), (2) and (3). Note that $f u_s \neq 0$ for s such that $\sigma_{-m_A}(P(s))|_A \neq 0$. By taking the principal symbol, we have, for such s,

$$f_A^{s+1} \sqrt{\omega_A} = \sigma_A(f u_s) = \sigma_A(P(s) u_s) = \sigma_{-m_A}(P(s))\, f_A^s \sqrt{\omega_A}$$

modulo constants, and hence we have $\sigma_{-m_A}(P(s)) = b_A(s) f_A$ for some constant $b_A(s)$. On the other hand,

$$\sigma_{-m_A}(P(s)) = s^{m_A} \sigma_{-m_A}(P) + \sigma_{-m_A}(G(s)) = s^{m_A} f_A + \sum_{\alpha \lneqq m_A} s^\alpha \sigma_{-m_A}(G_\alpha)$$

and hence, we have $b_A(s) = s^{m_A} + \text{lower term}$. Q.E.D.

Lemma 5.13. *Suppose that $Q(s)$ is of order $-m_A$ and $Q(s) u_s$ satisfies the same equation as u_{s+1}.* If $\sigma_{-m_A}(Q(\alpha))|_A = 0$ for $\alpha \in \mathbb{C}$, then $Q(\alpha) u_\alpha = 0$.

Proof. Let $\{A_j\}_j$ be a basis of $\mathfrak{g}$. Since $\sigma_{-m_A}(Q(\alpha))|_A = 0$, we have $\sigma_{-m_A}(Q(\alpha)) = \sum_j \varphi_j \langle d\rho(A_j) x, y\rangle$ and hence $Q(\alpha) = \sum_j \Phi_j(\langle d\rho(A_j) x, D_x\rangle - \alpha \delta\chi(A_j)) + K$ with $\operatorname{ord} K \leqq -m_A - 1$. Therefore we have $Q(\alpha) u_\alpha = K u_\alpha$, and hence $\operatorname{ord} Q(\alpha) u_\alpha \leqq \operatorname{ord} K + \operatorname{ord} u_\alpha \leqq \operatorname{ord} u_\alpha - m_A - 1$. Assume that $Q(\alpha) u_\alpha \neq 0$. Then $\mathscr{E} Q(\alpha) u_\alpha \simeq \mathscr{E} u_{\alpha+1}$ and hence $\operatorname{ord} Q(\alpha) u_\alpha = \operatorname{ord} u_{\alpha+1} = \operatorname{ord} u_\alpha - m_A$, which is a contradiction. Q.E.D.

Lemma 5.14. *Let $T(s)$ be a micro-differential operator satisfying $T(\alpha) u_\alpha = 0$. Then there exists a micro-differential operator $R(s)$ satisfying* (1) $T(s) u_s = (s - \alpha) R(s) u_s$ (2) $\underline{\underline{\operatorname{ord}}}\, R(s) \leqq \underline{\underline{\operatorname{ord}}}\, T(s) - 1$ (3) $\operatorname{ord} R(s) \leqq \operatorname{ord} T(s)$.

Proof. Let $B_1, \dots, B_r$ be a basis of $\mathfrak{g}_0$, and let A_1 be as $\delta\chi(A_1) = 1$. Since Λ is a good holonomic variety, we have

$$T(\alpha) = \sum_{j=1}^{r} \Phi_j(x, D_x) \langle d\rho(B_j) x, D_x\rangle + M(\langle d\rho(A_1) x, D_x\rangle - \alpha)$$

with $\operatorname{ord} M \leqq \operatorname{ord} T(\alpha) - 1$. In general, we have $T(s) = T(\alpha) + (s - \alpha) R_1(s)$ with $\operatorname{ord} R_1(s) \leqq \operatorname{ord} T(s)$ and $\underline{\underline{\operatorname{ord}}}\, R_1(s) \leqq \underline{\underline{\operatorname{ord}}}\, T(s) - 1$. Applying this to u_s, we obtain that $T(s) u_s = (s - \alpha) R_1(s) u_s + M(s - \alpha) u_s = (s - \alpha)(R_1(s) + M) u_s$. Put $R(s) = R_1(s) + M$. This satisfies our assertion. Q.E.D.

Lemma 5.15. *Suppose that $Q(s)$ is of order $-m_A$ and $Q(s) u_s$ satisfies the same equation as u_{s+1}. If a monic polynomial $c(s)$ divides $\sigma_{-m_A}(Q(s))$, then there exists $\tilde{Q}(s)$ satisfying* (1) $Q(s) u_s = c(s) \tilde{Q}(s) u_s$ (2) $\operatorname{ord} \tilde{Q}(s) = -m_A$ (3) $\underline{\underline{\operatorname{ord}}}\, \tilde{Q}(s) \leqq \underline{\underline{\operatorname{ord}}}\, Q(s) - \deg c(s)$.

Proof. Put $c(s) = (s - \alpha_1) \dots (s - \alpha_k)$. We shall prove this by induction on k. If $k = 1$, it is obvious from Lemma 5.13 and Lemma 5.14. Assume that $Q(s) u_s = (s - \alpha_1) \dots (s - \alpha_{k-1}) Q_k(s) u_s$. Then $Q_k(s) u_s$ satisfies the same equation as u_{s+1} for a generic s and hence for any s by Lemma 5.9. Since $\sigma(Q_k(\alpha_k))|_\Lambda=0$, Lemma 5.13 and Lemma 5.14 guarantee the existence of $\tilde{Q}(s)$ such that $Q_k(s)u_s=(s-\alpha_k)\tilde{Q}(s)u_s$. Q.E.D.

We shall apply this Lemma 5.15 to $P(s)$ in Lemma 5.12. Since $\sigma_{-m_\Lambda}(P(s))|_\Lambda=b_\Lambda(s)f_\Lambda$, there exists $P_1(s)$ satisfying $fu_s=P(s)u_s=b_\Lambda(s)P_1(s)u_s$ where $\operatorname{ord}P_1(s)=-m_\Lambda$, $\sigma_{-m_\Lambda}(P_1(s))=f_\Lambda$ and $\underline{\underline{\operatorname{ord}}}\,P_1(s)\leqq\underline{\underline{\operatorname{ord}}}\,P(s)-m_\Lambda\leqq-m_\Lambda$. Therefore we have $P_1(s)=\sum_{j\geqq 0}s^jQ_j$ with $\operatorname{ord}Q_j\leqq-m_\Lambda-j$, and hence

$$P_1(s)u_s=\sum_{j\geqq 0}Q_j(x,D_x)\langle d\rho(A_1)x,D_x\rangle^j u_s$$

where $\delta\chi(A_1)=1$. Put $P_\Lambda(x,D_x)=\sum_{j\geqq 0}Q_j(x,D_x)\langle d\rho(A_1)x,D_x\rangle^j$. Then $\operatorname{ord}P_\Lambda\leqq\underline{\underline{\operatorname{ord}}}\,P_1(s)\leqq-m_\Lambda$ and $\sigma_{-m_\Lambda}(P_\Lambda)|_\Lambda=\sigma_{-m_\Lambda}(Q_0)=\sigma_{-m_\Lambda}(P_1(s))=f_\Lambda$. Note that $\langle d\rho(A_1)x,y\rangle=\sigma(x,y)=0$ on Λ. Hence we have $fu_s=b_\Lambda(s)P_\Lambda u_s$ with $\operatorname{ord}P_\Lambda==m_\Lambda$ and $\sigma_{-m_\Lambda}(P_\Lambda)|_\Lambda=f_\Lambda$. This completes the proof of Theorem 5.5.

Remark 5.16. (1) The conormal bundle $\Lambda=V\times\{0\}$ is a good holonomic variety and $b_\Lambda(s)=1$. In fact, $1/f$ is an invertible operator near a generic point of Λ.

(2) By Remark 5.2 and Theorem 5.4, we have $b(s)=b_\Lambda(s)$ with $\Lambda=\{0\}\times V^*$. In the following, we are concerned to investigate the relation between $b_\Lambda(s)$ and $b_{\Lambda'}(s)$ when Λ and Λ' intersect with codimension one.

§ 6. Holonomy Diagrams

Let (G,ρ,V) be a regular irreducible P.V., and let $\Lambda_1,\dots,\Lambda_l$ be the irreducible components of W_0 (See Definition 4.2).

Definition 6.1. To each Λ_i, associate a circle Λ_i and connect the two circles associated to Λ_i and Λ_j if and only if $\dim\Lambda_i\cap\Lambda_j=n-1$. Thus we obtain a diagram which is called *the holonomy diagram* of (G,ρ,V). If Λ is G-prehomogeneous, i.e., a good holonomic variety, then Λ is the conormal bundle of some G-orbit on V, and $\mathscr{M}=\mathscr{E}f^s$ is a simple holonomic system on Λ. Moreover we have $\mathscr{M}=\mathscr{M}'$ on Λ where $\mathscr{M}'$: $(\langle d\rho(A)x,D_x\rangle-s\delta\chi(A))u=0$ $(A\in\mathfrak{g})$ (See § 4). Note that $V\times\{0\}$ and $\{0\}\times V^*$ are good holonomic varieties. We sometimes write the order $\operatorname{ord}_\Lambda f^s$ beside the circle associated to a good holonomic variety.

Remark 6.2 (a typical method to obtain the holonomy diagram).

Let x_0 be a point of V, $(G_{x_0},\rho_{x_0},V_{x_0}^*)$ the colocalization of (G,ρ,V) at x_0 (See Definition 4.4). Assume that it is a P.V., and let y_0 be its generic point. Then the conormal bundle Λ_0 of the orbit $\rho(G)x_0$ of x_0 is given by $\Lambda_0=T(\rho(G)x_0)^\perp=\overline{G(x_0,y_0)}$ where $G(x_0,y_0)=\{(\rho(g)x_0,\ \rho^*(g)y_0);\ g\in G\}$ and $\overline{}$ denotes the Zariski-closure. Let y_1 be a point of $V_{x_0}^*$ such that $\operatorname{codim}_{V_{x_0}^*}\rho_{x_0}(G_{x_0})y_1=1$. Then the orbit $G(x_0,y_1)$ of (x_0,y_1) is of $(n-1)$-dimension ($n=\dim V$) and contained in Λ_0. Let $(G_{y_1},\rho_{y_1}^*,V_{y_1})$ be the colocalization of the dual P.V. (G,ρ^*,V^*) at y_1. Assume that it is a P.V., and let x_1 be its generic point. Then we have $x_1\in V$, and the conormal bundle Λ_1 of the orbit $\rho(G)x_1$ is given by $\Lambda_1=T(\rho(G)x_1)^\perp$

$=\overline{G(x_1, y_1)}$. Note that Λ_0 (resp. Λ_1) coincides with the conormal bundle of $\rho^*(G)y_0$ (resp. $\rho^*(G)y_1$). We also assume that $\rho(G)x_0 \neq \rho(G)x_1$. Since $G(x_0, y_1) \subset \Lambda_0 \cap \Lambda_1$ and $\dim G(x_0, y_1) = n-1$, we have $\dim \Lambda_0 \cap \Lambda_1 = n-1$. We can obtain the holonomy diagram by this method in many cases (See § 9 and [15]).

We shall prove the following theorem later in § 8.

Theorem 6.3. *Let Λ_0 and Λ_1 be holonomic varieties in the cotangent bundle of a complex manifold of dimension n, and let p be a point of $\Lambda_0 \cap \Lambda_1$. Assume that* (1) *Λ_0 is non-singular near p,* (2) *$\dim \Lambda_0 \cap \Lambda_1 = n-1$,* (3) *$\Lambda_0 \cup \Lambda_1 \subset U$ for some $(n+1)$-dimensional non-singular variety U invariant under the action of $\mathbb{C}^\times$,* (4) *Λ_1 is irreducible near p.*

Then there exist integers μ and ν with $\mu \geqq 1$ and $\nu \geqq 0$ such that any simple holonomic system $\mathscr{M} = \mathscr{E}u$ $(= \mathscr{E}/\mathscr{J})$ defined near p with support $\Lambda_0 \cup \Lambda_1$ can be transformed to the following form by a quantized contact transformation.

$$\mathscr{M}: \left(\frac{1}{\nu+\mu} x_1 D_1 + \frac{1}{\mu} x_2 D_2 - \lambda\right) u = 0$$
$$[x_1(D_1^\mu - x_1^\nu D_2^\mu) + \lambda' D_1^{\mu-1}] u = 0$$
$$D_3 u = \dots = D_n u = 0$$

$$\Lambda_0 = \{(x, \xi);\ x_1 = x_2 = \xi_3 = \xi_4 = \dots = \xi_n = 0\}$$
$$\Lambda_1 = \left\{(x, \xi);\ x_2 + \frac{\mu}{\nu+\mu} x_1 \frac{\xi_1}{\xi_2} = 0,\ \left(\frac{\xi_1}{\xi_2}\right)^\mu = x_1^\nu,\ \xi_3 = \dots = \xi_n = 0\right\}$$

If $\mu = 1$, for any ν, $\mathscr{M}$ is isomorphic to the case of $\nu = 0$, and hence we shall make a convention that $\nu = 0$ whenever $\mu = 1$. If $\mu \geqq 2$ and $\nu \geqq 1$ we may assume that $(\mu, \nu) = 1$.

Definition 6.4. We shall call $(\mu : \nu)$ the intersection exponent of Λ_1 to Λ_0.

Note that these μ and ν depend only on Λ_0 and Λ_1. Let $\Lambda_0 = \overline{G(x_0, y_0)}$, $\Lambda_1 = \overline{G(x_1, y_1)}$ and $p = (x_0, y_1)$ be as in Remark 6.2. We shall consider how to calculate these μ and ν in this case.

Proposition 6.5. *Let $\tilde{A}$ be an element of $\mathfrak{g}$ satisfying $d\rho(\tilde{A})x_0 = 0$ and $d\rho^*(\tilde{A})y_1 = y_1$. Then $\tilde{A}$ acts on the one-dimensional vector space $\tilde{V} = V_{x_0}^*$ modulo $d\rho_{x_0}(\mathfrak{g}_{x_0})y_1$. Let β be its eigen value, i.e., $\beta = \mathrm{tr}_{\tilde{V}} \tilde{A}$. Then μ and ν are given by $\beta = \frac{\mu}{\mu+\nu}$, $(\mu, \nu) = 1$. If β is not determined uniquely, i.e., β depends on $\tilde{A}$, then we have $\mu = 1$, $\nu = 0$, and Λ_0, Λ_1 intersect regularly.*

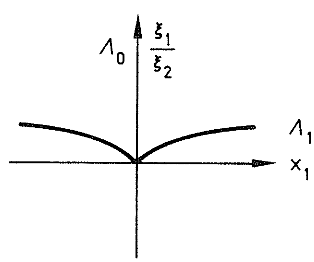


Fig. 6.1

Proof. Let $\bar{\mathscr{J}}$ be the symbol ideal of $\mathscr{J}$. Then by Theorem 6.3, $\bar{\mathscr{J}}$ is spanned by $\frac{1}{\nu+\mu}x_1\xi_1+\frac{1}{\mu}x_2\xi_2$, $x_1(\xi_1^\mu-x_1^\nu\xi_2^\mu)$, $\xi_3, \xi_4, \ldots, \xi_n$. Let ψ be an element of $\bar{\mathscr{J}}$, i.e., $\psi=\varphi_1\left(\frac{1}{\mu+\nu}x_1\xi_1+\frac{1}{\mu}x_2\xi_2\right)+\varphi_2x_1(\xi_1^\mu-x_1^\nu\xi_2^\mu)+\varphi_3\xi_3+\ldots+\varphi_n\xi_n$.

Since $\psi|_{\Lambda_0}=0$ and Λ_0 is holonomic, H_ψ can be regarded as a vector field on Λ_0 (See Lemma 3.4). We have

$$H_\psi=\left(-\frac{\varphi_1}{\nu+\mu}\xi_1+\varphi_2\xi_1^\mu\right)\frac{\partial}{\partial\xi_1}-\frac{\varphi_1}{\mu}\xi_2\frac{\partial}{\partial\xi_2}+\varphi_3\frac{\partial}{\partial x_3}+\ldots+\varphi_n\frac{\partial}{\partial x_n}$$

on Λ_0. Put $S=\Lambda_0\cap\Lambda_1$. Then for any $q\in S$ near p, we have

$$T_q\Lambda_0\supset T_qS=\left\langle\frac{\partial}{\partial\xi_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \ldots, \frac{\partial}{\partial x_n}\right\rangle$$

since $S=\{(x,\xi); x_1=x_2=\xi_1=\xi_3=\ldots=\xi_n=0\}$. Hence we have $(H_\psi)_q\in T_qS$. On the other hand, we have $\left(\sum_i\xi_i\frac{\partial}{\partial\xi_i}\right)_q\in T_qS$. Assume that $\varphi_3(p)=\varphi_4(p)=\ldots=\varphi_n(p)=0$ and put $v_\psi=H_\psi-a\sum_i\xi_i\frac{\partial}{\partial\xi_i}$ where $a=-\frac{\varphi_1(p)}{\mu}$. Then $v_\psi(q)\in T_qS\subset T_q\Lambda_0$ and $v_\psi(p)=0$. In general, let X be a manifold and $v=\sum a_j(x)\frac{\partial}{\partial x_j}$ a vector field defined near $p\in X$ satisfying $v(p)=0$. Then v acts on T_pX, i.e.,

$$A_v\colon T_pX\to T_pX, \qquad A_v(w)=[w,v](p)\left(A_v\left(\frac{\partial}{\partial x_i}\right)=\sum\frac{\partial a_k}{\partial x_i}(p)\frac{\partial}{\partial x_k}\right).$$

In our case, v_ψ acts on the one-dimensional vector space $T_p\Lambda_0/T_pS$:

$$A_{v_\psi}\colon T_p\Lambda_0/T_pS\to T_p\Lambda_0/T_pS,$$

$$A_{v_\psi}\left(\frac{\partial}{\partial\xi_1}\right)=\left(-\varphi_1(p)\left(\frac{1}{\mu+\nu}-\frac{1}{\mu}\right)+\mu\xi_1^{\mu-1}\varphi_2(p)\right)\frac{\partial}{\partial\xi_1}.$$

Then $\alpha=\left(-\varphi_1(p)\left(\frac{1}{\mu+\nu}-\frac{1}{\mu}\right)+\mu\xi_1^{\mu-1}\varphi_2(p)\right)$ is the eigen-value of A_{v_ψ}. If $\mu>1$, we have $\alpha=\mu a\left(\frac{1}{\mu+\nu}-\frac{1}{\mu}\right)$, i.e., $\frac{\alpha+a}{a}=\frac{\mu}{\mu+\nu}$. Since α and a can be calculated, we can determine μ and ν under the condition of $(\mu,\nu)=1$. If $\mu=1$, then α depends on not only a but ψ. Consider the case when $\Lambda_0=\overline{G(x_0,y_0)}$, $\Lambda_1=\overline{G(x_1,y_1)}$, $S=\overline{G(x_0,y_1)}$ and $p=(x_0,y_1)$.

Then $\mathscr{J}$ is generated by $\{\langle d\rho(A)x,y\rangle; A\in\mathfrak{g}\}$ and $\bar{\mathscr{J}}=J_{\Lambda_0\cup\Lambda_1}$ (i.e., the defining ideal of $\Lambda_0\cup\Lambda_1$). Since

$$H_{\langle d\rho(A)x,y\rangle}=\langle d\rho(A)x,D_x\rangle+\langle d\rho^*(A)y,D_y\rangle,$$

we have $H_{\langle d\rho(\tilde{A})x, y\rangle} = \langle y, D_y\rangle_p$ for $\tilde{A} \in \mathfrak{g}$ satisfying $d\rho(\tilde{A})x_0 = 0$ and $d\rho(\tilde{A})y_0 = y_0$. This implies that $a = 1$ for $\psi = \langle d\rho(\tilde{A})x, y\rangle$.

Since $A_{H_\psi - \langle y, D_y\rangle} = A_{\langle d\rho(\tilde{A})x, D_x\rangle + \langle d\rho^*(\tilde{A})y - y, D_y\rangle}$ acts on $V \times V^*$ as

$$(x, y) \mapsto (d\rho(\tilde{A})x, d\rho^*(\tilde{A})y - y),$$

the action of $A_{H_\psi} - \langle y, D_y\rangle$ on $T_p\Lambda_0/T_pS \cong V^*_{x_0}/d\rho_{x_0}(\mathfrak{g}_{x_0})y_1$ is the induced one from $d\rho^*(\tilde{A}) - 1$. Let β be the eigenvalue of $d\rho^*(\tilde{A})$ on $V^*_{x_0}/d\rho_{x_0}(\mathfrak{g}_{x_0})y_1$. Then we have $\alpha = \beta - 1$, $a = 1$, $\dfrac{\alpha + a}{a} = \dfrac{\mu}{\mu + \nu}$ and hence $\beta = \dfrac{\mu}{\mu + \nu}$. If β depends on $\tilde{A}$, then $\mu = 1$ and hence, by convention, $\nu = 1$. In this case, Λ_0 and Λ_1 intersect regularly. Q.E.D.

Let $\Lambda = T(\rho(G)x)^\perp$ be a conormal bundle of a G-orbit. We shall consider some sufficient conditions to be $\Lambda \subset W$, i.e., $\Lambda \subset W_0$. If x is a generic point, then $\Lambda = V \times \{0\}$ is clearly contained in W. If $x = 0$, then $\Lambda = \{0\} \times V^*$ is contained in W when and only when (G, ρ, V) is a regular P.V. (See Proposition 4.6).

Proposition 6.6. *Let Λ_0 and Λ_1 be two conormal bundles of some G-orbits. Assume that* $\dim \mathfrak{g}_0 p = n - 1$ *for some $p = (x_0, y_0) \in \Lambda_0 \cap \Lambda_1$ where $\mathfrak{g}_0 = \{A \in \mathfrak{g};\ \delta\chi(A) = 0\}$. Assume that Λ_0(or Λ_1)$\subset W$. Then we have $\Lambda_0 \cup \Lambda_1 \subset W$. Moreover W is non-singular and $W = W'$ near p, where $W' = \{(x, y) \in V \times V^*;\ \langle d\rho(A)x, y\rangle = 0$ for all $A \in \mathfrak{g}_0\}$.*

Proof. Since $p \in W \subset W'$ and $\dim W = n + 1$, we have $\dim W' \geqq n + 1$ near p. On the other hand, by assumption, there exists $B_1, \dots, B_{n-1} \in \mathfrak{g}_0$ such that

$$\langle d\rho(B_j)x_0,\ d\rho^*(B_j)y_0\rangle \qquad (1 \leqq j \leqq n-1)$$

are independent, i.e.,

$$d\langle d\rho(B_j)x, y\rangle|_p = \langle d\rho(B_j)x_0, dy\rangle - \langle d\rho^*(B_j)y_0, dx\rangle \qquad (1 \leqq j \leqq n-1)$$

are independent. This shows that W' is non-singular, $\dim W' = n + 1$, and $W' = W$ near p by the same argument as the proof of Proposition 4.7. Since $\Lambda_0 \cup \Lambda_1 \subset W'$, we get $\Lambda_0 \cup \Lambda_1 \subset W$ near p. Therefore $\Lambda_0 \cup \Lambda_1 \subset W$ because Λ_0 and Λ_1 are irreducible. Q.E.D.

Definition 6.7. Let $f(x)$ be the relative invariant of an irreducible regular P.V. (G, ρ, V). We define the localization f_{x_0} of f at $x_0 \in V$ by

$$f(x_0 + \varepsilon x') = \varepsilon^k f_{x_0}(x') + \varepsilon^{k+1} \sum_{j=0}^{\infty} \varepsilon^j f_j(x')$$

where $\varepsilon \in \mathbb{C}$, $x' \in V$, and $f_{x_0}(x')$ is not identically zero.

Lemma 6.8. (1) *$f_{x_0}(\rho(g)x') = \chi(g) f_{x_0}(x')$ for $g \in G_{x_0}$.*

(2) *If $x \equiv x' \bmod d\rho(\mathfrak{g})x_0$, then $f_{x_0}(x) = f_{x_0}(x')$.*

Proof. (1) Since $\rho(g)x=x_0+\varepsilon\rho(g)x'$ for $g\in G_{x_0}$ and $x=x_0+\varepsilon x'\in V$, we have

$$f(\rho(g)x)=\varepsilon^k f_{x_0}(\rho(g)x')+\varepsilon^{k+1}\sum_{j=0}^{\infty}\varepsilon^j f_j(\rho(g)x')$$
$$=\chi(g)\cdot f(x)=\varepsilon^k\chi(g)f_{x_0}(x')+\varepsilon^{k+1}\chi(g)\sum_{j=0}^{\infty}\varepsilon^j f_j(x')$$

and hence $f_{x_0}(\rho(g)x')=\chi(g)f_{x_0}(x')$.

(2) Since $\rho(\exp\varepsilon A)x=\exp\varepsilon d\rho(A)(x_0+\varepsilon x')=x_0+\varepsilon(x'+d\rho(A)x_0)+$(higher term of ε) for $A\in\mathfrak{g}$, we have $f_{x_0}(x'+d\rho(A)x_0)=f_{x_0}(x')$ by comparison of the coefficients of ε^k of the equality: $f(\rho(\exp\varepsilon A)x)=\exp\varepsilon\delta\chi(A)f(x_0+\varepsilon x')$. Q.E.D.

By this lemma, $f_{x_0}(x')$ can be regarded as a relative invariant function of the normal vector space $V_{x_0}=V/d\rho(\mathfrak{g})x_0$ on which G_{x_0} acts. Assume that $(G_{x_0}, \tilde{\rho}, V_{x_0})$ is a P.V. where $\tilde{\rho}$ is induced from ρ. Note that this is the dual P.V. of the colocalization $(G_{x_0}, \rho_{x_0}, V_{x_0}^*)$ of (G, ρ, V) at x_0 (See Definition 4.4). Let S_{x_0} be the singular set of $(G_{x_0}, \tilde{\rho}, V_{x_0})$.

Proposition 6.9. *If* $\operatorname{grad}_{x'}\log f_{x_0}\colon V_{x_0}-S_{x_0}\to V_{x_0}^*$ *is generically surjective, then* $\Lambda_0=T(\rho(G)x_0)^{\perp}\subset W$, *i.e.,* Λ_0 *is a good holonomic variety.*

Proof. Take $x'\in V$ such that $x=x_0+\varepsilon x'\in V-S$ and $x' \bmod d\rho(\mathfrak{g})x_0\in V_{x_0}-S_{x_0}$. Since

$$\operatorname{grad}_x\log f(x)=\frac{1}{\varepsilon}\operatorname{grad}_{x'}\log f(x_0+\varepsilon x')$$
$$=\frac{1}{\varepsilon}\operatorname{grad}_{x'}\log\varepsilon^k\cdot f_{x_0}(x')\left\{1+\sum_{j\geqq 1}\varepsilon^j h_j(x')\right\}$$
$$=\frac{1}{\varepsilon}\operatorname{grad}_{x'}\log f_{x_0}(x')+\frac{1}{\varepsilon}\operatorname{grad}_{x'}\log\left\{1+\sum_{j\geqq 1}\varepsilon^j h_j(x')\right\}$$

and hence

$$(x_0+\varepsilon x', \operatorname{grad}_{x'}\log f_{x_0}(x')+\operatorname{grad}_{x'}\log\{1+\textstyle\sum\varepsilon^j h_j\})$$
$$=(x, \varepsilon\operatorname{grad}\log f(x))\in W.$$

Since W is closed, we have $(x_0, \operatorname{grad}_{x'}\log f_{x_0}(x'))\in W$ by $\varepsilon\to 0$. Since $\operatorname{grad}_{x'}\log f_{x_0}$ is generically surjective, we have $(x_0, V_{x_0}^*)\in W$. Since W is G-admissible and $\rho^*(g)V_{x_0}^*=V_{\rho(g)x_0}^*$, we have $\Lambda_0=T(\rho(G)x_0)^{\perp}\subset W$. Q.E.D.

Corollary 6.10. *Assume that the colocalization* $(G_{x_0}, \rho_{x_0}, V_{x_0}^*)$ *of* (G, ρ, V) *at* $x_0(\in V)$ *is a regular P.V. If* $\delta x|\mathfrak{g}_{x_0}$ *is a non-degenerate element (See Definition 1.2), then the conormal bundle of the orbit* $\rho(G)x_0$ *is a good holonomic variety.*

Proof. By Proposition 1.4, we have $\operatorname{grad}_{x'}\log f_{x_0}$ is generically surjective, and hence we obtain our assertion. Q.E.D.

Corollary 6.11. *Assume that the colocalization* $(G_{x_0}, \rho_{x_0}, V_{x_0}{}^*)$ *of* (G, ρ, V) *at* x_0 *is an irreducible regular P.V. Then* $\Lambda_0=T(\rho(G)x_0)^{\perp}$ *is a good holonomic variety.*

§7. Calculation of Local *b*-functions

In this section, we shall calculate the ratio $b_{\Lambda_0}(s)/b_{\Lambda_1}(s)$ of local b-functions $b_{\Lambda_0}(s)$ and $b_{\Lambda_1}(s)$ when Λ_0 and Λ_1 have an intersection of codimension one under some assumption. Then since we have $b_{V\times\{0\}}(s)=1$ and $b_{\{0\}\times V^*}(s)=b(s)$, we can expect to obtain $b(s)$ by using a holonomy diagram. Note that $b_\Lambda(s)$ is uniquely determined up to a constant multiple. Therefore if we assume that $b_\Lambda(s)$ is monic, then it is unique.

Let Λ_0 and Λ_1 be holonomic varieties such that codim $\Lambda_0\cap\Lambda_1=1$, $(\mu\colon\nu)$ its intersection exponent (See Definition 6.4). Let $\mathscr{M}=\mathscr{E}u$ be a simple holonomic system with support $\Lambda_0\cup\Lambda_1$. The following theorem will be proved in §8 by using Theorem 6.3. The assumptions are same as those of Theorem 6.3.

Theorem 7.1. (1) *The principal symbol* $\sigma_{\Lambda_0}(u)$ *has zeros of order*

$$\left(\frac{\nu+\mu}{\nu+1}(\operatorname{ord}_{\Lambda_0}u-\operatorname{ord}_{\Lambda_1}u)-\frac{\mu}{2}\right)\qquad \text{at } S=\Lambda_0\cap\Lambda_1.$$

(2) *There exists a quotient of* $\mathscr{M}$ *with support* Λ_0, *i.e., there exists a submodule of* $\mathscr{M}$ *with support* Λ_1, *if and only if*

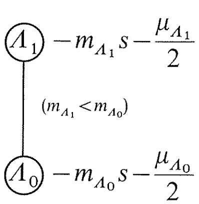
$$\text{(7.1)}\qquad \begin{aligned}&\frac{\nu+\mu}{\nu+1}(\operatorname{ord}_{\Lambda_0}u-\operatorname{ord}_{\Lambda_1}u)-\frac{\mu}{2}=l'\in\mathbb{Z}_+=\{0,1,2,\dots\}\\ &\textit{and}\quad l'\equiv 0,1,\dots,\nu \bmod(\mu+\nu).\end{aligned}$$

(3) *There exists a submodule of* $\mathscr{M}$ *with support* Λ_0, *i.e., there exists a quotient of* $\mathscr{M}$ *with support* Λ_1, *if and only if*

$$\text{(7.2)}\qquad \begin{aligned}&\frac{\nu+\mu}{\nu+1}(\operatorname{ord}_{\Lambda_1}u-\operatorname{ord}_{\Lambda_0}u)-\frac{\mu}{2}=-l'-\mu\in\mathbb{Z}_+=\{0,1,2,\dots\}\quad \textit{and}\\ &-l'-\mu\equiv 0,1,\dots,\nu \bmod(\mu+\nu).\end{aligned}$$

Assume that $\mathscr{M}=\mathscr{E}f^s$ is a simple holonomic system with support $\Lambda_0\cup\Lambda_1$, i.e., the symbol ideal of $\mathscr{J}(\mathscr{M}=\mathscr{E}/\mathscr{J})$ is a reduced ideal on $\Lambda_0\cup\Lambda_1$. We have $\operatorname{ord}_\Lambda f^s=-m_\Lambda s-\frac{\mu_\Lambda}{2}$ (See Proposition 4.14).

Proposition 7.2. *Assume that* $m_{\Lambda_1}<m_{\Lambda_0}$. *Then we have* $b_{\Lambda_1}(s)|b_{\Lambda_0}(s)$,



i.e., $b_{\Lambda_0}(s)/b_{\Lambda_1}(s)$ *is a polynomial.*

Fig. 7.1

Proof. To prove this proposition, it is sufficient to show that $d(s)(s-\alpha)|b_{\Lambda_0}(s)$ whenever $d(s)|b_{\Lambda_0}(s)$ and $d(s)(s-\alpha)|b_{\Lambda_1}(s)$. By Lemma 5.12 and 5.15, there exists a micro-differential operator G defined on $\Lambda_0 \cup \Lambda_1$ satisfying $fu_s = d(s)Gu_s$. Since $fu_s = b_{\Lambda_i}(s)P_{\Lambda_i}u_s$ on Λ_i, we have $Gu_s = (b_{\Lambda_i}(s)/d(s))P_{\Lambda_i}u_s$ on $\Lambda_i (i=0,1)$. Assume that $d(s)(s-\alpha)\nmid b_{\Lambda_0}(s)$. Then we have $Gu_\alpha|_{\Lambda_0} \neq 0$ and $Gu_\alpha|_{\Lambda_1} = 0$. Then $\mathscr{E}Gu_\alpha$ is a submodule of $\mathscr{M}_\alpha = \mathscr{E}u_\alpha$ with support Λ_0, and hence by (7.2), we have

$$\frac{v+\mu}{v+1}\left\{(m_{\Lambda_0}-m_{\Lambda_1})\alpha + \frac{1}{2}(\mu_{\Lambda_0}-\mu_{\Lambda_1})\right\} - \frac{\mu}{2} \in \mathbb{Z}_+ = \{0,1,2,\ldots\}.$$

On the other hand, since Gu_α satisfies the equations of $u_{\alpha+1}$ with support Λ_0, by (7.1), we have

$$-\frac{v+\mu}{v+1}\left\{(m_{\Lambda_0}-m_{\Lambda_1})(\alpha+1) + \frac{1}{2}(\mu_{\Lambda_0}-\mu_{\Lambda_1})\right\} - \frac{\mu}{2} \in \mathbb{Z}_+ = \{0,1,2,\ldots\}$$

and hence

$$-\frac{v+\mu}{v+1}(m_{\Lambda_0}-m_{\Lambda_1}) - \mu \in \mathbb{Z}_+ = \{0,1,2,\ldots\}.$$

Since $m_{\Lambda_0}-m_{\Lambda_1}>0$, $\mu \geqq 1$, $v \geqq 0$, this is a contradiction. Q.E.D.

Proposition 7.3. *Assume that* $m_{\Lambda_1} < m_{\Lambda_0}$ *and* $\gamma \in \mathbb{C}$ *satisfies the following two conditions:*

$$(1) \quad l' = -\frac{v+\mu}{v+1}\left(\gamma + \frac{1}{2}(\mu_{\Lambda_0}-\mu_{\Lambda_1})\right) - \frac{\mu}{2} \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$$

and $l' \equiv 0, 1, \ldots, v \bmod (\mu+v)$

$$(2) \quad l' - \frac{v+\mu}{v+1}(m_{\Lambda_0}-m_{\Lambda_1}) \notin \mathbb{Z}_+, \text{ or } l' - \frac{v+\mu}{v+1}(m_{\Lambda_0}-m_{\Lambda_1}) \not\equiv 0,\ 1,\ \ldots,\ v \bmod (\mu+v).$$

Then $\{(m_{\Lambda_0}-m_{\Lambda_1})s-\gamma\}$ *is a factor of the polynomial* $b_{\Lambda_0}(s)/b_{\Lambda_1}(s)$. *Note that* $\frac{v+\mu}{v+1}(m_{\Lambda_0}-m_{\Lambda_1}) \in \mathbb{Z}$ *(See* (1) *of Theorem* 7.1).

Proof. Take $\alpha = \gamma/(m_{\Lambda_0}-m_{\Lambda_1})$. By the first condition, there exists a quotient $\mathscr{N}$ of $\mathscr{M}_\alpha = \mathscr{E}u_\alpha$ with support Λ_0: $\mathscr{M}_\alpha \xrightarrow{\varphi_\alpha} \mathscr{N} \to 0$ (exact) (See (2) of Theorem 7.1). Let v be the image of u_α, i.e., $v = \varphi_\alpha(u_\alpha)$. Since $b_{\Lambda_1}(s)|b_{\Lambda_0}(s)$, by Lemma 5.12 and 5.15, we have $fu_s = b_{\Lambda_1}(s)Gu_s$ where G is defined on $\Lambda_0 \cup \Lambda_1$. Define a map $\varphi_{\alpha+1}$: $\mathscr{M}_{\alpha+1} \to \mathscr{M}_\alpha$ by $\varphi_{\alpha+1}(u_{\alpha+1}) = Gu_\alpha$ and a map $\varphi_{\alpha+1}(\mathscr{M}_{\alpha+1}) = \mathscr{E}Gu_\alpha \to \mathscr{E}Gv$ which is induced from φ_α. By composing these maps, we have $\mathscr{M}_{\alpha+1} \to \mathscr{E}Gv \to 0$ (exact) and $\operatorname{supp}(\mathscr{E}Gv) \subset \Lambda_0$. By the second condition, $\mathscr{M}_{\alpha+1}$ has no quotient with support Λ_0, and hence $Gv=0$. Since $\mathscr{M}_\alpha \cong \mathscr{N}$ on Λ_0, we have $Gu_\alpha|_{\Lambda_0}=0$. Since $fu_s = b_{\Lambda_1}(s)Gu_s = b_{\Lambda_0}(s)P_{\Lambda_0}u_s$ on Λ_0, we have $Gu_s = (b_{\Lambda_0}(s)/b_{\Lambda_1}(s))P_{\Lambda_0}u_s$ on Λ_0. Note that G is defined also on Λ_0. Since P_{Λ_0} is invertible and u_α is a generator of $\mathscr{M}_\alpha = \mathscr{E}u_\alpha$, we have $h(\alpha)=0$ where $h(s) = b_{\Lambda_0}(s)/b_{\Lambda_1}(s)$ is a polynomial by Proposition 7.2, and hence $\{(m_{\Lambda_0}-m_{\Lambda_1})s-\gamma\}|h(s)$. Q.E.D.

Proposition 7.4. *Assume that* $m_{A_1} < m_{A_0}$.

(1) $\frac{\nu+\mu}{\nu+1}(m_{A_0}-m_{A_1}) \equiv 0 \bmod (\nu+\mu)$, *i.e.*, $(m_{A_0}-m_{A_1}) \equiv 0 \bmod (\nu+1)$.

(2) $l' = k + j(\mu+\nu) \left(k=0,\dots,\nu; j=0,1,\dots,\frac{m_{A_0}-m_{A_1}}{\nu+1}-1\right)$ *satisfies the conditions in Proposition* 7.3.

Proof. (1) Assume that $\frac{\nu+\mu}{\nu+1}(m_{A_0}-m_{A_1}) \equiv c' \bmod (\mu+\nu)$ where $0 < c' < \mu+\nu$. Then there exist l_1 and l_2 such that $c' = l_1 + l_2$, $0 \leqq l_1 \leqq \nu$ and $0 \leqq l_2 \leqq \mu - 1$. Since $l_1 \equiv 0, 1, \dots, \nu \bmod (\mu+\nu)$ and

$$l_1 - \frac{\nu+\mu}{\nu+1}(m_{A_0}-m_{A_1}) \equiv -l_2 \not\equiv 0, 1, \dots, \nu \bmod (\mu+\nu),$$

$l_1 + t(\mu+\nu)$ satisfies the conditions in Proposition 7.3 for any $t \geqq 0$. Since $\deg b_{A_0}(s)/b_{A_1}(s) < +\infty$, this is a contradiction by Proposition 7.3.

(2) $l' \in \mathbb{Z}_+$ and $l' \equiv 0, 1, \dots, \nu \bmod (\mu+\nu)$ are obvious. Since $0 \leqq j \leqq \frac{1}{\nu+1}(m_{A_0} - m_{A_1}) - 1$, we have $l' - \frac{\nu+\mu}{\nu+1}(m_{A_0}-m_{A_1}) = k - t(\mu+\nu) \notin \mathbb{Z}_+ \ (t \geqq 1, 0 \leqq k \leqq \nu)$. Q.E.D.

Theorem 7.5. *Let* Λ_0 *and* Λ_1 *be good holonomic varieties whose intersection is of codimension one with the intersection exponent* $(\mu\colon \nu)$. *Assume that* $\mathscr{M} = \mathscr{E} f^s$ *is a simple holonomic system with support* $\Lambda_0 \cup \Lambda_1$. *Assume that* $m_{A_0} > m_{A_1}$ *where* $\operatorname{ord}_A f^s = -m_A s - \frac{\mu_A}{2}$. *Then we have, up to a constant multiple,*

$$b_{A_0}(s)/b_{A_1}(s) = \prod_{k=0}^{\nu} \left[\frac{1}{\nu+1}(\operatorname{ord}_{A_1} f^s - \operatorname{ord}_{A_0} f^s) + \frac{\mu+2k}{2(\nu+\mu)}\right]^{\frac{m_{A_0}-m_{A_1}}{\nu+1}} \tag{7.3}$$

where $[\alpha]^k = \alpha(\alpha+1)\dots(\alpha+k-1)$.

Proof. By Proposition 7.3 and 7.4, $b_{A_0}(s)/b_{A_1}(s)$ has a factor $\{(m_{A_0}-m_{A_1})s-\gamma\}$ where $k + j(\mu+\nu) = -\frac{\nu+\mu}{\nu+1}\left(\gamma + \frac{1}{2}(\mu_{A_0}-\mu_{A_1})\right) - \frac{\mu}{2}$, i.e.,

$$\gamma = -\frac{\nu+1}{\nu+\mu}\left\{k + j(\mu+\nu) + \frac{\mu}{2}\right\} - \frac{1}{2}(\mu_{A_0}-\mu_{A_1})$$

with $k = 0, 1, \dots, \nu; j = 0, 1, \dots, \frac{m_{A_0}-m_{A_1}}{\nu+1} - 1$. Namely $b_{A_0}(s)/b_{A_1}(s)$ has a factor

$$\frac{1}{\nu+1}\{(m_{A_0}-m_{A_1})s-\gamma\} = \frac{1}{\nu+1}(\operatorname{ord}_{A_1} f^s - \operatorname{ord}_{A_0} f^s) + \frac{\mu+2k}{2(\nu+\mu)} + j$$

with

$$k = 0, 1, \dots, \nu; j = 0, 1, \dots, \frac{m_{A_0}-m_{A_1}}{\nu+1} - 1.$$

Therefore

$$\prod_{k=0}^{\nu}\left[\frac{1}{\nu+1}(\mathrm{ord}_{\Lambda_1}f^s-\mathrm{ord}_{\Lambda_0}f^s)+\frac{\mu+2k}{2(\nu+\mu)}\right]^{\frac{m_{\Lambda_0}-m_{\Lambda_1}}{\nu+1}}$$

divides a polynomial $b_{\Lambda_0}(s)/b_{\Lambda_1}(s)$. Since both are of degree $(m_{\Lambda_0}-m_{\Lambda_1})$, they coincide up to a constant multiple. Q.E.D.

Corollary 7.6. *If Λ_0 and Λ_1 intersect regularly (i.e., $\mu=1$ and $\nu=0$) with codimension one, we have*

$$b_{\Lambda_0}(s)/b_{\Lambda_1}(s)=[\mathrm{ord}_{\Lambda_1}f^s-\mathrm{ord}_{\Lambda_0}f^s+\tfrac{1}{2}]^{(m_{\Lambda_0}-m_{\Lambda_1})},\ \textit{i.e.},$$

$$b_{\Lambda_0}(s)=b_{\Lambda_1}(s)\cdot\prod_{k=1}^{m_{\Lambda_0}-m_{\Lambda_1}}\left((m_{\Lambda_0}-m_{\Lambda_1})s+\frac{\mu_{\Lambda_0}-\mu_{\Lambda_1}-1}{2}+k\right).$$

Here $\mathrm{ord}_{\Lambda}f^s=-m_{\Lambda}s-\dfrac{\mu_{\Lambda}}{2}$ *and* $m_{\Lambda_0}>m_{\Lambda_1}$.

Proof. This follows from (7.3) by putting $\mu=1$ and $\nu=0$. Q.E.D.

Remark 7.7. Theorem 7.5 shows that if we calculate $\mathrm{ord}_{\Lambda_i}f^s (i=0,1)$ and the intersection exponent $(\mu\colon\nu)$, which are given by Proposition 4.14 and 6.5 respectively, then we get the ratio $b_{\Lambda_0}(s)/b_{\Lambda_1}(s)$ under some conditions. Next we shall show another way to obtain $b_{\Lambda_0}(s)/b_{\Lambda_1}(s)$ by investigating the colocalization $(G_{x_0},\rho_{x_0},V^*_{x_0})$ of (G,ρ,V) at x_0. These different methods are useful to check the calculation of each other.

Assume that the colocalization $(G_{x_0},\rho_{x_0},V^*_{x_0})$ of an irreducible regular P.V. (G,ρ,V) at $x_0 (x_0\in V)$ is a P.V., i.e., the conormal bundle of $\rho(G)x_0$ is G-prehomogeneous. Let $g_1,\dots,g_t$ be algebraically independent relative invariant irreducible polynomials of $(G_{x_0},\rho_{x_0},V^*_{x_0})$ corresponding to the characters $\rho_1,\dots,\rho_t$ respectively (See §1). Assume that $\overline{\{y\in V^*_{x_0};\, g_1(y)=0\}}$ is G_{x_0}-prehomogeneous, i.e., $\overline{\{y\in V^*_{x_0};\, g_1(y)=0\}}=\overline{\rho_{x_0}(G_{x_0})y_1}$ for some $y_1\in V^*_{x_0}(\subset V^*)$. Assume also that the localization $(G_{y_1},\rho^*_{y_1},V_{y_1})$ of the dual (G,ρ^*,V^*) is a P.V., and let x_1 be its generic point: $x_1\in V_{y_1}\subset V$. By Remark 6.2, we have $\Lambda_0=\overline{G(x_0,y_0)}$, $\Lambda_1=\overline{G(x_1,y_1)}$, codim $\Lambda_0\cap\Lambda_1=1$ and $p=(x_0,y_1)\in\Lambda_0\cap\Lambda_1$. We assume that $\Lambda_0,\Lambda_1\subset W_0$, i.e., Λ_0 and Λ_1 are good holonomic varieties. Let $(\mu\colon\nu)$ be their intersection exponent (Definition 6.4). Now $f_{\Lambda_0}(x_0,y)$ (See Definition 4.11) is a non-zero relative invariant of $(G_{x_0},\rho_{x_0},V^*_{x_0})$ corresponding to $\chi|_{G_{x_0}}$ where χ is the character of the relative invariant $f(x)$, of (G,ρ,V), and hence we have $\chi|^{-1}_{G_{x_0}}=\rho_1^{c_1}\dots\rho_t^{c_t}$ for some $(c_1,\dots,c_t)\in\mathbb{Z}^t$, i.e., $-\delta\chi=c_1\,\delta\rho_1+\dots+c_t\,\delta\rho_t$. On the other hand, $\varphi(t,\tau)$ is a relative invariant corresponding to the infinitesimal character $-2\,tr_{V^*_{x_0}}\,d\rho_{x_0}(A)$ where $\omega_\Lambda=\varphi(t,\tau)\,d\tau_1\dots d\tau_r\,dt_{r+1}\dots dt_n$ (See the proof of Proposition 4.14). Therefore we have

$$(7.4)\qquad \begin{cases}-\delta\chi=c_1\,\delta\rho_1+c_2\,\delta\rho_2+\dots+c_t\,\delta\rho_t\\ tr_{V^*_{x_0}}=a_1\,\delta\rho_1+a_2\,\delta\rho_2+\dots+a_t\,\delta\rho_t\end{cases}$$

$$\text{where}\quad c_i\in\mathbb{Z},\ 2a_i\in\mathbb{Z}\qquad (i=1,\dots,t).$$

This shows that $f^s_{A_0}(x_0, y) = g_1(y)^{-c_1 s} \dots g_t(y)^{-c_t s}$ for $y \in V^*_{x_0}$, and $\omega_{A_0} = g_1(y)^{-2a_1}$ $\times$ {non-zero regular part at y_1}. Thus the order of zeros of $\sigma_{A_0}(f^s) = f^s_{A_0}\sqrt{\omega_{A_0}}$ at $p = (x_0, y_1)$ is given by $-c_1 s - a_1$, and hence by (1) of Theorem 7.1, we have

$$\text{(7.5)} \qquad \frac{\nu+\mu}{\nu+1}(\operatorname{ord}_{A_0} f^s - \operatorname{ord}_{A_1} f^s) - \frac{\mu}{2} = -c_1 s - a_1.$$

In particular, we have $c_1 = \dfrac{\nu+\mu}{\nu+1}(m_{A_0} - m_{A_1})$.

By (7.3), we have

$$\text{(7.6)} \qquad b_{A_0}(s)/b_{A_1}(s) = \prod_{k=0}^{\nu}\left[c s + \frac{a_1 + k}{\nu+\mu}\right]^c \quad \text{with } c = \frac{c_1}{\nu+\mu}.$$

Since $\operatorname{ord}_{A_i} f^s (i = 0, 1)$ and c_1, a_1 can be calculated independently, the relation (7.5) is useful to check the actual calculation.

§8. Structure of Simple Holonomic Systems

The main purpose of this section is to give a proof of Theorem 6.3 and Theorem 7.1. But here we shall prove them in a generalized form.

Let X be a complex manifold of dimension n, T^*X the cotangent vector bundle of X. Let $(z_1, \dots, z_n)$ be a local coordinate system of X and $(z_1, \dots, z_n, \xi_1, \dots, \xi_n)$ the corresponding local coordinate system of T^*X so that $\omega_X = \sum_{i=1}^{n} \xi_i dz_i$ is the canonical 1-form on T^*X. A map φ of an open subset of T^*X to T^*X is said to be *a homogeneous canonical transformation* or *a contact transformation* if $\varphi^*\omega_X = \omega_X$. This equals to say $\omega_X = \sum_{i=1}^{n} \xi'_i dz'_i$ by denoting

$$(z'_1, \dots, z'_n, \xi'_1, \dots, \xi'_n) = \varphi((z_1, \dots, z_n, \xi_1, \dots, \xi_n)).$$

In this case φ is a local isomorphism and z'_i (resp. ξ'_i) are homogeneous functions of degree 0 (resp. 1) with respect to $(\xi_1, \dots, \xi_n)$, that is, φ is compatible with the $\mathbb{C}^\times$-action on T^*X. Moreover for a function f on T^*X, we have

$$\sum_{i=1}^{n}\left(\frac{\partial f}{\partial \xi_i}\cdot\frac{\partial}{\partial z_i} - \frac{\partial f}{\partial z_i}\cdot\frac{\partial}{\partial \xi_i}\right) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial \xi'_i}\cdot\frac{\partial}{\partial z'_i} - \frac{\partial f}{\partial z'_i}\cdot\frac{\partial}{\partial \xi'_i}\right).$$

This vector field is called the Hamilton vector field of f and denoted by H_f. A subset V of T^*X is called *homogeneous* if $p \in V$ implies $c p \in V$ for $c \in \mathbb{C}^\times$. Hence a contact transformation transforms homogeneous involutory (resp. homogeneous holonomic) varieties into varieties in the same kind.

Any local coordinate system $(z_1, \dots, z_n, \xi_1, \dots, \xi_n)$ of T^*X so that $\omega_X = \sum_{i=1}^{n} \xi_i \, dz_i$ is in general called *a homogeneous canonical coordinate system* of T^*X. A non-singular homogeneous involutory submanifold V of T^*X is said to be *regular* if $\omega_X|_V$ nowhere vanishes on V. The classical theory of analytical dynamics says the following fundamental theorem.

Theorem 8.1 (*cf.* Carathéodory [11]). *Let V be a regular homogeneous involutory submanifold of codimension d (resp. a homogeneous holonomic submanifold) and let p be a point of V which does not belong to the zero sections of T^*X. Then we can choose a suitable homogeneous canonical coordinate system (z, ξ) around p so that*

$$V = \{(z, \xi) \in T^*X;\ \xi_{n-d+1} = \xi_{n-d+2} = \dots = \xi_n = 0\}$$
$$(resp.\ \{(z, \xi) \in T^*X;\ x_1 = \xi_2 = \xi_3 = \dots = \xi_n = 0\}).$$

Let $\mathscr{E}_X$ be the sheaf of micro-differential operators on T^*X. Then we can find *a quantized contact transformation* Φ of $\mathscr{E}_X$ associated with a local contact transformation φ of T^*X (§4–3 Chap. II in [2]). This means that Φ is a ring isomorphism from $\varphi^{-1}(\mathscr{E}_X)$ onto $\mathscr{E}_X$ which satisfies the following conditions:

(8.1) $\Phi(\varphi^{-1}(\mathscr{E}_X(m)))$ equals $\mathscr{E}_X(m)$ for any integer m.

(8.2) The diagram

$$\begin{array}{ccc} \varphi^{-1}(\mathscr{E}_X(m)) & \xrightarrow{\ \Phi\ } & \mathscr{E}_X(m) \\ \sigma_m \downarrow & \circlearrowleft & \downarrow \sigma_m \\ \varphi^{-1}(\mathscr{O}(m)) & \xrightarrow{\ \varphi_*\ } & \mathscr{O}(m) \end{array}$$

is commutative.

(8.3) We also denote by Φ the isomorphism between the systems of micro-differential equations on T^*X which is induced by the quantized contact transformation Φ. Then if $\mathscr{M} = \mathscr{E}_X u$ is a simple holonomic system with one unknown function u, the system $\Phi(\mathscr{M}) = \mathscr{E}_X \Phi(u)$ is also a simple holonomic system satisfying $\operatorname{ord} u = \operatorname{ord} \Phi(u)$.

Here for a system of micro-differential equations

$$\mathscr{N}: \sum_{j=1}^{l} P_{ij}(z, D)\, v_j = 0 \qquad (i = 1, \dots, k)$$

on T^*X, we define the system $\Phi(\mathscr{N})$ by

$$\Phi(\mathscr{N}): \sum_{j=1}^{l} \Phi(P_{ij}(z, D))\, \Phi(v_j) = 0 \qquad (i = 1, \dots, k).$$

Such a Φ is called a *quantized contact transformation* associated with φ. Then we can "quantize" Theorem 8.1:

Theorem 8.2 (§4.2 and §5.1 Chap. II in [2]). *Let $\mathscr{M} = \mathscr{E}_X u$ be a system of micro-differential equations with one unknown function u.*

(1) *If the characteristic variety of $\mathscr{M}$ is a regular involutory submanifold of codimension d and the symbol Ideal of $\mathscr{M}$ is reduced, then $\mathscr{M}$ is micro-locally*

isomorphic to the system

$$D_{n-d+1}v = D_{n-d+2}v = \dots = D_n v = 0$$

under a suitable quantized contact transformation.

(2) *If $\mathcal{M}$ is a simple holonomic system whose characteristic variety is non-singular, then $\mathcal{M}$ is micro-locally isomorphic to the system*

$$(x_1 D_1 - \alpha) v = D_2 v = \dots = D_n v = 0$$

under a suitable quantized contact transformation. Here $\alpha = -\frac{1}{2} - \operatorname{ord} u$.

Now we shall investigate the structure of simple holonomic systems whose characteristic variety consists of several irreducible components. First we give a structure theorem of homogeneous holonomic varieties.

Theorem 8.3. *Let p be a point of the cotangent bundle T^*X of a complex manifold X of dimension n and assume p does not belong to the zero section of T^*X. Let Λ be the germ of a homogeneous holonomic variety at p and let $\Lambda_0, \dots, \Lambda_l$ be its irreducible components. Assume that there exists an $(n+1)$-dimensional non-singular homogeneous variety which contains Λ. Then there exist a homogeneous canonical coordinate system $(z_1, \dots, z_n, \xi_1, \dots, \xi_n)$ and positive integers μ and ν with $(\mu, \nu) = 1$ such that p corresponds to the point*

$$(z_1, \dots, z_n, \xi_1, \xi_2, \xi_3, \dots, \xi_n) = (0, \dots, 0, 0, 1, 0, \dots, 0)$$

and that

$$\Lambda_i = \left\{ (z, \xi);\ z_2 + \frac{\mu}{\nu + \mu} z_1 \frac{\xi_1}{\xi_2} = A_i \left(\frac{\xi_1}{\xi_2} \right)^{\mu} + B_i z_1^{\nu} = \xi_3 = \dots = \xi_n = 0 \right\}$$

for $i = 0, \dots, l$ with pairs (A_i, B_i) in $\mathbb{C}^2 - \{(0,0)\}$.

Remark 8.4. (1) The homogeneous canonical transformation

$$\begin{aligned} &(z_1, z_2, z_3, \dots, z_n, \xi_1, \xi_2, \xi_3, \dots, \xi_n) \\ &\mapsto \left(-\frac{\xi_1}{\xi_2}, z_2 + z_1 \frac{\xi_1}{\xi_2}, z_3, \dots, z_n, z_1 \xi_2, \xi_2, \xi_3, \dots, \xi_n \right) \end{aligned}$$

transforms (μ, ν, A_i, B_i) to $(\nu, \mu, B_i, (-1)^{\mu} A_i)$ in Theorem 8.3.

(2) If Λ_0 is non-singular, the local coordinate system (z, ξ) in Theorem 8.3 can be taken so that $A_0 = 0$, that is,

$$\Lambda_0 = \{(z, \xi);\ z_1 = z_2 = \xi_3 = \dots = \xi_n = 0\}.$$

Moreover if Λ_0 and Λ_1 are non-singular and $T_p\Lambda_0 \neq T_p\Lambda_1$ we can choose $A_0 = 0$ and $B_1 = 0$ in Theorem 8.3, namely

$$\Lambda_1 = \{(z, \xi);\ \xi_1 = z_2 = \xi_3 = \dots = \xi_n = 0\}.$$

These are easily proved by Theorem 8.3 as follows. The first remark shows that if Λ_0 is non-singular, we may assume either $\nu = 1$ and $B_0 \neq 0$ or $A_0 = 0$. We have $A_0=0$ also in the case $\nu=1$ and $B_0\neq 0$ by the homogeneous canonical transformation

$$(z_1, z_2, z_3, \ldots, z_n, \xi_1, \ldots, \xi_n)$$
$$\mapsto \left(z_1+\frac{A_0}{B_0}\left(\frac{\xi_1}{\xi_2}\right)^{\mu}, \quad z_2-\frac{A_0\mu}{B_0(\mu+1)}\left(\frac{\xi_1}{\xi_2}\right)^{\mu+1}, z_3, \ldots, z_n, \xi_1, \ldots, \xi_n\right).$$

Hence in the latter claim, we may assume $A_0=0$ and that either $\mu=1$ and $A_1\neq 0$ or $B_1=0$. When $\mu=1$ and $A_1\neq 0$, we apply the homogeneous canonical transformation

$$(z_1, z_2, z_3, \ldots, z_n, \xi_1, \xi_2, \ldots, \xi_n)$$
$$\mapsto \left(z_1, z_2-\frac{B_1}{A_1(\nu+1)}z_1^{\nu+1}, z_3, \ldots, z_n, \xi_1+\frac{B_1}{A_1}z_1^{\nu}\xi_2, \xi_2, \ldots, \xi_n\right)$$

to the above situation in order to have $A_0=B_1=0$.

Proof of Theorem 8.3. Assuming Theorem 8.3 in the case $n=2$, we shall prove Theorem 8.3 by the induction on n. Hence we suppose $n\geqq 3$. Let $I(\Lambda)$ be the ideal of $\mathcal{O}_p$ generated by the functions which vanish on Λ. The assumption of Theorem 8.3 implies the existence of the function f in $I(\Lambda)$ such that df and ω are linearly independent at p and that f is homogeneous with respect to the $\mathbb{C}^{\times}$-action on T^*X. Then under a suitable homogeneous canonical coordinate system (z, ξ) we have $z_n(p)=0$ and $f=c\xi_n$ with a non-vanishing function c (cf. Theorem 8.1). We put $(z', \xi')=(z_1, \ldots, z_{n-1}, \xi_1, \ldots, \xi_{n-1})$. Then we can choose functions $g_1(z, \xi'), \ldots, g_{n-1}(z, \xi')$ so that $g_1, \ldots, g_{n-1}$ and ξ_n generate $I(\Lambda)$. Since Λ is involutory, $\{\xi_n, g_j\}\in I(\Lambda)$ for $j=1, \ldots, n-1$, which equals to say

$$\frac{\partial}{\partial z_n}(g_1, \ldots, g_{n-1})=(g_1, \ldots, g_{n-1})\,A(z, \xi')$$

with a suitable matrix-valued function A of (z, ξ'). Using the solution $U(z, \xi')$ of the equation

$$\begin{cases} \dfrac{\partial U}{\partial z_n}+A(z, \xi')\,U=0, \\ U|_{z_n=0}=I_{n-1}, \end{cases}$$

we put $(h_1, \ldots, h_{n-1})=(g_1, \ldots, g_{n-1})\,U(z, \xi')$. Then

$$\frac{\partial}{\partial z_n}(h_1, \ldots, h_{n-1})=\left\{\frac{\partial}{\partial z_n}(g_1, \ldots, g_{n-1})\right\}U+(g_1, \ldots, g_{n-1})\frac{\partial U}{\partial z_n}=0.$$

Hence $h_i (1\leqq i\leqq n-1)$ are functions of (z', ξ') and $\{h_1(z', \xi'), \ldots, h_{n-1}(z', \xi'), \xi_n\}$ is a system of generators of $I(\Lambda)$. Considering the space of the $(2n-2)$-variables (z', ξ') with the canonical 1-form $\sum_{i=1}^{n-1} \xi_i dz_i$ and its subvariety Λ' defined by $h_1=\ldots=h_{n-1}=0$, we have Theorem 8.3 by the hypothesis of induction.

Thus we may assume $n=2$. We denote by (z_1, z_2, ξ_1, ξ_2) a suitable homogeneous coordinate system of T^*X such that p corresponds to $(0,0,0,1)$. Let f_i and g be functions that generate the ideal $I(\Lambda_i)$ of $\mathcal{O}_p$ and satisfy $(dg)_p \neq 0$ ($i=0,\dots,l$). If

(8.4) $$l=0 \quad \text{and } \Lambda_0 \text{ is non-singular},$$

we may suppose dg and ω_X are linearly independent at p and therefore that $g=\xi_1$ by the same argument in the induction. Then $\Lambda=\{(z,\xi);\ \xi_1=z_2=0\}$ because $(\xi_1\, dz_1+\xi_2\, dz_2)|\Lambda=0$ (cf. Theorem 8.1). In general if ω and dg are linearly independent at p, we may suppose $g=\xi_1$ and therefore Λ_i must be $\{(z,\xi);\ \xi_1=z_2=0\}$ by the same arguments as above. Hence if (8.4) does not hold, $(dg)_p=c\,\omega_p(=c(dz_2)_p)$ with a complex number c and therefore we may assume

(8.5) $$g=z_2-h(z_1,\xi_1/\xi_2) \quad \text{and} \quad f_i=f_i(z_1,\xi_1/\xi_2),$$

where h is a function of $(z_1,\xi_1/\xi_2)$ satisfying $(dh)_p=0$.

Next consider the case

(8.6) $$l=1, \Lambda_0 \text{ and } \Lambda_1 \text{ are non-singular and } T_p\Lambda_0 \neq T_p\Lambda_1.$$

Since $\{f_0, f_1\}(p) \neq 0$, we may assume $f_0=\xi_1/\xi_2$ and $f_1=z_1+k(\xi_1/\xi_2)$ with a function k of ξ_1/ξ_2. Then by the homogeneous canonical transformation

$$(z_1,z_2,\xi_1,\xi_2) \mapsto \Big(z_1+k(\xi_1/\xi_2),\ z_2-\int_0^{\xi_1/\xi_2} t\frac{dk(t)}{dt}dt,\ \xi_1,\ \xi_2\Big)$$

we may moreover assume $k=0$. Hence $g|\Lambda=z_2|\Lambda$ because $\omega_X|\Lambda_0=\omega_X|\Lambda_1=0$, which implies $\Lambda_0=\{(z,\xi);\ \xi_1=z_2=0\}$ and $\Lambda_1=\{(z,\xi);\ z_1=z_2=0\}$.

We have proved Theorem 8.3 in the cases (8.4) and (8.6). Thus we consider the other cases. Noting that $v=H_{\xi_2 g}$ can be regarded as a vector field on the space Y of the two variables $(z_1,\xi_1/\xi_2)$ and satisfies $vf_i|_{f_i=0}=0$, we prepare the following lemma.

Lemma 8.5. *Let $v=a(x,y)\,\partial/\partial x+b(x,y)\,\partial/\partial y$ be the germ of a vector field at the origin of $\mathbb{C}^2$, where (x,y) is its local coordinate system. Assume there exist $(l+1)$ germs $V_i (i=0,\dots,l)$ of irreducible analytic curves through the origin such that V_i are integral curves of v (i.e. V_i is the zeros of an irreducible analytic function f_i satisfying $(vf_i)|_{V_i}=0$). Moreover assume one of the following conditions:*

(8.7)

V_0 is singular at the origin,

(8.8)

$l=1$, V_0 and V_1 are non-singular and tangent at the origin,

(8.9)

$l \geqq 2$.

Then if the matrix $M=\begin{pmatrix}\partial a/\partial x & \partial a/\partial y\\ \partial b/\partial x & \partial b/\partial y\end{pmatrix}(0)$ is not nilpotent, we can choose a local coordinate system (x,y) so that $v=c(x\partial/\partial x+r\,y\partial/\partial y)$ with a suitable complex number c and a suitable rational number r satisfying $r \geqq 1$.

We shall continue the proof of Theorem 8.3 and the proof of this lemma is given after that. In our situation we have

$$v=\frac{\partial h}{\partial y}\frac{\partial}{\partial x}+\left(y-\frac{\partial h}{\partial x}\right)\frac{\partial}{\partial y} \tag{8.10}$$

by denoting $(x, y)=(z_1, -\xi_1/\xi_2)$. Hence the trace of M in Lemma 8.5 equals 1. Therefore Lemma 8.5 implies the existence of coordinate functions $x'(x, y)$ and $y'(x, y)$ so that $v=c(x'\partial/\partial x'+ry'\partial/\partial y')$ under the local coordinate system (x', y') of $Y(c\in\mathbb{C}, 0<r\in Q)$. Since the coordinate transformation does not change the trace of M, we have

$$v=\frac{\mu}{\mu+\nu}x'\frac{\partial}{\partial x'}+\frac{\nu}{\mu+\nu}y'\frac{\partial}{\partial y'} \tag{8.11}$$

with suitable positive integers μ and ν satisfying $(\mu, \nu)=1$. Since $\{x, y\}=1/\xi_2$, we have $\{x', \xi_2 y'\}=\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y}-\frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x}$. Hence, we have $\{x', \xi_2 y'\}(p)\neq 0$. Replacing x', by $x'/\{x', \xi_2 y'\}(p)$, we may moreover assume $\{x', \xi_2 y'\}(p)=1$. By the Jacobi's identity

$$\{\xi_2 g, \{x', \xi_2 y'\}\}+\{x', \{\xi_2 y', \xi_2 g\}\}+\{\xi_2 y', \{\xi_2 g, x'\}\}=0$$

we have

$$\begin{aligned} v(\{x', \xi_2 y'\}) &= \{x', \xi_2 v(y')-\xi_2 y'\}+\{v(x'), \xi_2 y'\} \\ &= \left\{x', \xi_2\left(\frac{\nu}{\mu+\nu}-1\right)y'\right\}+\left\{\frac{\mu}{\mu+\nu}x', \xi_2 y'\right\} \\ &= 0. \end{aligned}$$

This equation (cf. (8.11)) and the Taylor expansion of the function $\{x', \xi_2 y'\}$ with respect to the variables x' and y' easily prove that the function is constant. Hence $\{x', \xi_2 y'\}=1$, which is equal to

$$\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y}-\frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x}=1$$

and therefore to $dy'\wedge dx'=dy\wedge dx$. This assures the existence of a function $w(x, y)$ such that $dw=y'dx'-y\,dx$ and $w(0)=0$. Then we have $\xi_1 dz_1+\xi_2 dz_2=-\xi_2 y'dx'+\xi_2 d(z_2+w)$. Considering the homogeneous canonical coordinate transformation $(z_1, z_2, \xi_1, \xi_2)\mapsto(x', z_2+w, -\xi_2 y', \xi_2)$, we can assume $(x', y')=(z_1 x, -\xi_1/\xi_2)=(x, y)$. Hence comparing (8.10) and (8.11) we have $\frac{\partial h}{\partial y}=\frac{\mu}{\mu+\nu}x$ and $\frac{\partial h}{\partial x}=\frac{\mu}{\mu+\nu}y$, which shows $h=\frac{\mu}{\mu+\nu}xy$ and $g=z_2+\frac{\mu}{\mu+\nu}z_1\left(\frac{\xi_1}{\xi_2}\right)$. Moreover any integral curve of v in Y through the origin is defined by $Ax^\nu+By^\mu=0$ for a suitable pair (A, B) in $\mathbb{C}\times\mathbb{C}-\{(0,0)\}$. This proves Theorem 8.3.

Proof of Lemma 8.5. The assumption implies that v vanishes at the origin. Hence the eigenvalues of M do not depend on the choice of local coordinate systems and at least one of them is not zero. We shall show that the ratio r of the two eigenvalues is a positive rational number.

First consider the case (8.7). We may assume f_0 satisfies $(D_x^i D_j^j f_0)=0$ for $i+j<k$ and $(D_y^k f_0)(0)\neq 0$ for a suitable positive integer k. Then by Weierstrass' preparation theorem we may moreover assume $f_0=y^k+g_1(x)y^{k-1}+g_2(x)y^{k-2}+\dots+g_k(x)$ with analytic functions $g_i(x)$ at the origin satisfying $(D_x^j g_i)(0)=0$ for $j<i$ $(i=1,\dots,k)$. Therefore we can define V_0 by the Puiseux series $y=\sum_{i=0}^{\infty} C_i x^{1+\frac{i}{m}}$, where m is a suitable positive integer and $\sum_{i=0}^{\infty} C_i (x^{\frac{1}{m}})^{m+i}$ is a convergent power series of $x^{\frac{1}{m}}$. Since V_0 is singular, there exists a positive integer n such that $n/m\notin\mathbb{Z}$, $C_n\neq 0$ and that $C_i=0$ if $i/m\notin\mathbb{Z}$ and $0<i<n$. Replacing y by $y-\sum_{i=0}^{n-1} C_i x^{1+\frac{i}{m}}$ we have

$$y=\sum_{i=n}^{\infty} C_i x^{1+\frac{i}{m}}, \tag{8.12}$$

where $C_n\neq 0$ *and* $n/m\notin\mathbb{Z}$. Since $v(y-\sum_{i=n}^{\infty} C_i x^{1+\frac{i}{m}})$ is an analytic function of $(x^{\frac{1}{m}}, y)$ and vanishes if we put $y=\sum_{i=n}^{\infty} C_i x^{1+\frac{i}{m}}$, we have

$$v\left(y-\sum_{i=n}^{\infty} C_i x^{1+\frac{i}{m}}\right)=h(x^{\frac{1}{m}}, y)\left(y-\sum_{i=n}^{\infty} C_i x^{1+\frac{i}{m}}\right)$$

with a suitable analytic function h of $(x^{\frac{1}{m}}, y)$. Hence

$$-a(x,y)\sum_{i=n}^{\infty}\left(1+\frac{i}{m}\right) C_i x^{\frac{i}{m}}+b(x,y)=h(x^{\frac{1}{m}}, y)\left(y-\sum_{i=n}^{\infty} C_i x^{1+\frac{i}{m}}\right). \tag{8.13}$$

Comparing the coefficients of x, y and $x^{1+\frac{n}{m}}$ in the expansions of the both sides of (8.13) into power series of $(x^{\frac{1}{m}}, y)$, we have $(\partial b/\partial x)(0)=0$, $(\partial b/\partial y)(0)=h(0)$ and $-\left(1+\frac{n}{m}\right) C_n(\partial a/\partial x)(0)=-C_n h(0)$. Hence the ratio r equals $1+\dfrac{n}{m}$.

Next consider the case (8.8). We may assume $f_0=y$ and $f_1=y-x^{m+1}$ with a suitable positive integer m. Since $v(f_0)|_{V_0}=0$, v has the form $a(x,y)\partial/\partial x+b'(x,y)y\partial/\partial y$. Then the equation $v(f_1)|_{V_1}=0$ shows

$$\begin{aligned}0&=(-a(x,y)(m+1)x^m+b'(x,y)y)|_{V_1}\\&=(-(m+1)a(x,x^{m+1})+b'(x,x^{m+1})x)x^m.\end{aligned}$$

Hence we have $-(m+1)(\partial a/\partial x)(0)+b'(0)=0$ and thus $r=m+1$.

By the above results in the cases (8.7) and (8.8), we may assume $f_0=y, f_1=x$ and $f_2=y-x$ in the remaining case (8.9). Then v has the form $a'(x,y)x\partial/\partial x$

$+b'(x, y)\, y\partial/\partial y$ because $v(f_i)|_{V_i}=0$ for $i=0$ and 1. Since $v(f_2)|_{V_2}=0$, we have $0=-a'(x, x)\,x+b'(x, x)\,x$, which shows $a'(0)=b'(0)$ and $r=1$.

Thus we have proved that the eigenvalues of M are c and cr under the notation in Lemma 8.5 for some complex number c and some rational number $r \geqq 1$. Then we can choose a local coordinate system (x, y) so that

$$v=c\left(x\frac{\partial}{\partial x}+(\lambda x^r+r\,y)\frac{\partial}{\partial y}\right), \tag{8.14}$$

where $\lambda \in \mathbb{C}$ (resp. $\lambda=0$) if r is an integer (resp. not an integer). This follows from Remark 1.9 in [12], but we can easily construct the coordinate functions x and y by solving the equation $(v-c)(x)=(v-c\,r)(y)-\lambda x^r=0$. If $\lambda \neq 0$, there is no analytic integral curve of v through the origin except the curve $\{x=0\}$. Hence λ must be zero. Q.E.D.

Now we shall "quantize" Theorem 8.3:

Theorem 8.6. *Let $\mathcal{M}=\mathcal{E}_X u$ be a simple holonomic system defined near a point p of the cotangent vector bundle T^*X of an n-dimensional complex manifold X. Assume the characteristic variety Λ of $\mathcal{M}$ is contained in an $(n+1)$-dimensional non-singular variety. Then by a quantized contact transformation, $\mathcal{M}$ can be transformed to one of the following systems $\mathcal{N}_m (m=1, 2, 3)$ defined near the point $(z_1, \ldots, z_n, \xi_1, \xi_2, \xi_3, \ldots, \xi_n)=(0, \ldots, 0, 0, 1, 0, \ldots, 0)$:*

$$\mathcal{N}_1:\begin{cases}\left(\dfrac{1}{\nu+\mu}z_1 D_1+\dfrac{1}{\mu}z_2 D_2+\lambda_0\right)v=0,\\[2ex] \Bigg[\displaystyle\prod_{i=0}^{l}(D_1^{\mu}+C_i z_1^{\nu} D_2^{\mu})+\sum_{(\mu+\nu)k-\mu(l+1)\leqq j\leqq \nu k-1} \\ \quad\cdot\lambda_{jk}\, z_1^j\, D_1^{j-(\mu+\nu)k+\mu(l+1)}\, D_2^{\mu k}\Bigg]\,v=0,\\[2ex] D_3 v=\ldots=D_n v=0.\end{cases}$$

$$\mathcal{N}_2:\begin{cases}\left(\dfrac{1}{\nu+\mu}z_1 D_1+\dfrac{1}{\mu}z_2 D_2+\lambda_0\right)v=0,\\[2ex] \Bigg[z_1\displaystyle\prod_{i=1}^{l}(D_1^{\mu}+C_i z_1^{\nu} D_2^{\mu})+\sum_{(\mu+\nu)k-\mu l+1\leqq j\leqq \nu k} \\ \quad\cdot\lambda_{jk}\, z_1^j\, D_1^{j-(\mu+\nu)k+\mu l-1}\, D_2^{\mu k}\Bigg]\,v=0,\\[2ex] D_3 v=\ldots=D_n v=0.\end{cases}$$

$$\mathcal{N}_3:\begin{cases}\left(\dfrac{1}{\nu+\mu}z_1 D_1+\dfrac{1}{\mu}z_2 D_2+\lambda_0\right)v=0,\\[2ex] \Bigg[z_1 D_1\displaystyle\prod_{i=2}^{l}(D_1^{\mu}+C_i z_1^{\nu} D_2^{\mu})+\sum_{(\mu+\nu)k-\mu(l-1)\leqq j\leqq \nu k} \\ \quad\cdot\lambda_{jk}\, z_1^j\, D_1^{j-(\mu+\nu)k+\mu(l-1)}\, D_2^{\mu k}\Bigg]\,v=0,\\[2ex] D_3 v=\ldots=D_n v=0.\end{cases}$$

Here j and k are non-negative integers and μ and ν (resp. C_i, λ_0 and λ_{jk}) are suitable positive integers (resp. complex numbers) satisfying $(\mu, \nu)=1$, $C_i \neq 0$ and $C_i \neq C_{i'}$ if $i \neq i'$.

Remark 8.7. Let l' be the number of the parameters λ_0, λ_{jk} in Theorem 8.6. These parameters are not determined by the characteristic variety Λ of $\mathscr{M}$. On the other hand the order of a simple holonomic system is invariant under quantized contact transformations. Hence if Λ has $\tilde{l}$ irreducible components, $\mathscr{M}$ has $\tilde{l}$ invariants, that is, the orders of $\mathscr{M}$ at the irreducible components. Therefore $l'-\tilde{l}$ is the number of the parameters of simple holonomic systems with support Λ that cannot be determined by the structure of the systems at the non-singular points of Λ.

By Theorem 8.6 we can give the following necessary and sufficient condition for $l'=\tilde{l}$: The number of singular irreducible components of Λ is one or zero and $\tilde{l} \leqq 3$ and if $\tilde{l}=3$, there are two non-singular irreducible components Λ_0 and Λ_1 of Λ such that $T_p\Lambda_0 \neq T_p\Lambda_1$.

Proof of Theorem 8.6. By a quantized contact transformation, $\mathscr{M}$ can be transformed to a system $\mathscr{N}=\mathscr{E}_X v=\mathscr{E}_X/\mathscr{J}$ whose characteristic variety has the form given in Theorem 8.3. Using the notation in Theorem 8.3, Remark 8.4 (1) assures that we may assume (1) $A_i \neq 0$ and $B_i \neq 0$ for $i=0, \ldots, l$ or (2) $A_0=0$ and $B_i \neq 0$ for $i=0, \ldots, l$ or (3) $A_0=0$ and $B_1=0$. Therefore by putting $C_i=B_i/A_i$ for the i satisfying $A_i \neq 0$, the symbol ideal $\mathscr{J}$ of $\mathscr{N}$ equals that of $\mathscr{N}_1$ or $\mathscr{N}_2$ or $\mathscr{N}_3$ in Theorem 8.6, respectively. Since the proof of the theorem in the cases (1) and (2) are the same as that in the case (3), we shall prove only in the case (3).

Now we quote the following results:

Lemma 8.7. (*a special case of Lemma* 3.6 *in* [13]). *Let $\mathscr{J}$ be a coherent ideal of $\mathscr{E}_X$ containing D_i $(r \leqq i \leqq n)$. Then there are micro-differential operators Q_j $(1 \leqq j \leqq N)$ such that $\mathscr{J}=\mathscr{E}_X Q_1+\ldots+\mathscr{E}_X Q_N+\mathscr{E}_X D_r+\ldots+\mathscr{E}_X D_n$ and Q_j commutes with x_i and D_i $(r \leqq i \leqq n,\ 1 \leqq j \leqq N)$.*

Lemma 8.8 (*a special case of Theorem* 3.1 *in* [12]). *Let $P(z_1, z_2, D_1, D_2)$ be a micro-differential operator defined near $(0; dz_2)$. If $\sigma(P)=c z_1\xi_1+z_2\xi_2$ with a number c in $\mathbb{C}-\{t \in \mathbb{R};\ t \leqq 0$ or $t \geqq 1\}$, there exists an invertible micro-differential operator $U(z_1, z_2, D_1, D_2)$ of order 0 defined near $(0; dz_2)$ such that $UPU^{-1}=c z_1 D_1+z_2 D_2+\lambda$ with a suitable complex number λ.*

Applying Proposition 2.2, Lemma 8.7 and Lemma 8.8 to the system $\mathscr{N}$ in the case (3), we may assume

$$\mathscr{N}:\begin{cases}\left(\dfrac{1}{\nu+\mu} z_1 D_1+\dfrac{1}{\mu} z_2 D_2+\lambda_0\right) v=0, \\ \left[z_1 D_1 \prod_{i=2}^{l}\left(D_1^{\mu}+C_i z_1^{\nu} D_2^{\mu}\right)+R(z_1, z_2, D_1, D_2)\right] v=0, \\ D_3 v=\ldots=D_n v=0,\end{cases}$$

where R is a micro-differential operator of order $\leqq \mu(l-1)$ and commutes with $z_3, \ldots, z_n, D_3, \ldots, D_n$. By the Späth-type theorem and the Weierstrass preparation-type theorem for micro-differential operators (cf. §2.2 Chap. II in [2]), we may moreover assume R is of the form

$$
\begin{aligned}
R &= R_0(D_1, D_2) + z_1 \sum_{\alpha=0}^{\mu(l-1)} R_{\alpha+1}(z_1, D_2) D_1^\alpha \\
&= \sum_{\substack{\alpha+\beta\leqq\mu(l-1)\\ 0\leqq\alpha}} \lambda_{0\alpha\beta} D_1^\alpha D_2^\beta + \sum_{\substack{\alpha+\beta\leqq\mu(l-1)\\ 1\leqq j,\ 0\leqq\alpha\leqq\mu(l-1)}} \lambda_{j\alpha\beta} z_1^j D_1^\alpha D_2^\beta
\end{aligned}
\tag{8.15}
$$

where $\lambda_{j\alpha\beta}\in\mathbb{C}$ and $j, \alpha, \beta\in\mathbb{Z}$. We put

$$
\begin{aligned}
Q = \Big[& z_1 D_1 \prod_{i=2}^{l} (D_1^\mu + C_i z_1^\nu D_2^\mu) + R, \mu z_1 D_1 \\
& + (\nu+\mu) z_2 D_2 + \lambda_0 \mu(\nu+\mu)\Big] - \mu^2(l-1)\Big(z_1 D_1 \prod_{i=2}^{l} (D_1^\mu + C_i z_1^\nu D_2^\mu) + R\Big).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\Big[z_1 D_1 \prod_{i=2}^{l} (D_1^\mu + C_i z_1^\nu D_2^\mu), \mu z_1 D_1 + (\nu+\mu) z_2 D_2\Big] \\
&= \mu^2(l-1) z_1 D_1 \prod_{i=2}^{l} (D_1^\mu + C_i z_1^\nu D_2^\mu),
\end{aligned}
$$

we have

$$
\begin{aligned}
Q &= [R, \mu z_1 D_1 + (\nu+\mu) z_2 D_2] - \mu^2(l-1) R \\
&= \sum_{\substack{j\geqq 0,\ \alpha\geqq 0\\ \alpha+\beta\leqq\mu(l-1)\\ j\alpha\leqq j\mu(l-1)}} (\mu(\alpha-j) + (\mu+\nu)\beta - \mu^2(l-1)) \lambda_{j\alpha\beta} z_1^j D_1^\alpha D_2^\beta.
\end{aligned}
$$

Hence if $Q\in\mathscr{E}_X(m)$, $\sigma_m(Q)$ is of the form

$$
\begin{aligned}
&\xi_2^m(r_0(\xi_1/\xi_2) + z_1 r_1(z_1) + (\xi_1/\xi_2) z_1 r_2(z_1) + \dots \\
&\quad + (\xi_1/\xi_2)^{\mu(l-1)} z_1 r_{\mu(l-1)+1}(z_1))
\end{aligned}
$$

with analytic functions r_j of one variable defined near the origin. Therefore $\sigma_m(Q)=0$ because $\sigma_m(Q)\in\mathscr{J}$, which implies $Q=0$. This entails that if $\lambda_{j\alpha\beta}\neq 0$, the triplet (j, α, β) satisfies $\mu(\alpha-j)+(\mu+\nu)\beta=\mu^2(l-1)$.

Suppose $\lambda_{j\alpha\beta}\neq 0$. Then $\alpha=j+\mu(l-1)-\beta(\mu+\nu)/\mu$, which shows

$$
\alpha = j-(\mu+\nu)k+\mu(l-1) \text{ and } \beta=\mu k \text{ with } k \text{ in } \mathbb{Z}
\tag{8.16}
$$

because $(\mu, \nu)=1$. The relations $j\geqq 0$, $\alpha\geqq 0$ and $\alpha+\beta\leqq\mu(l-1)$ equal

$$
(\mu+\nu)k-\mu(l-1)\leqq j\leqq k\nu \quad\text{and}\quad j\geqq 0.
\tag{8.17}
$$

On the other hand (8.17) implies $k\geqq 0$ and $j\alpha\leqq j\mu(l-1)$. Thus $\mathscr{N}$ equals $\mathscr{N}_3$ with $\lambda_{jk}=\lambda_{j,\, j-(\mu+\nu)k+\mu(l-1),\, \mu k}$. Q.E.D.

Now we shall consider the simple holonomic systems treated in §6 and §7. Let Λ_0 and Λ_1 be irreducible holonomic varieties defined near a point p in the cotangent bundle of a complex manifold of dimension n. Assume that $\Lambda_0 \cap \Lambda_1 \ni p$, Λ_0 is non-singular and $\Lambda_0 \cup \Lambda_1$ is contained in an $(n+1)$-dimensional non-singular variety. Then Theorem 8.6 says that any simple holonomic system $\mathcal{M} = \mathcal{E}u$ defined near p with support $\Lambda_0 \cup \Lambda_1$ can be transformed to the following form by a quantized contact transformation, which is the claim of Theorem 6.3:

$$\mathcal{M}_{\alpha,\beta}: \begin{cases} \left(\frac{\mu}{\nu+\mu} z_1 D_1 + z_2 D_2 + \alpha\right) u = 0 \\ [z_1 (D_1^\mu - z_1^\nu D_2^\mu) + (\beta+\mu) D_1^{\mu-1}] u = 0 \\ D_3 u = \ldots = D_n u = 0 \end{cases}$$

$$\Lambda_0 = \{(z, \xi); z_1 = z_2 = \xi_3 = \xi_4 = \ldots = \xi_n = 0\},$$

$$\Lambda_1 = \left\{(z, \xi); z_2 + \frac{\mu}{\nu+\mu} z_1 \frac{\xi_1}{\xi_2} = \left(\frac{\xi_1}{\xi_2}\right)^\mu - z_1^\nu = \xi_3 = \ldots = \xi_n = 0\right\}.$$

In fact, Remark 8.4 says that $\mathcal{M}$ can be transformed to the system $\mathcal{N}_2$ in Theorem 8.6 with $l=1$ and that if $\mu=1$, ν can be taken arbitrarily. Moreover we can choose $C_1 = -1$ by a coordinate transformation $z_1 \mapsto C z_1$ with C in $\mathbb{C}$. The complex numbers α and β are determined by the orders of u at Λ_0 and Λ_1 (cf. Remark 8.7) as is given in the following proof.

Proof of Theorem 7.1. Since $\mathcal{M}$ is isomorphic to $\mathcal{M}_{\alpha\beta}$, we may assume $\mathcal{M} = \mathcal{M}_{\alpha\beta}$. First we shall calculate the principal symbol of u. We put

$$\sigma_{\Lambda_i}(u) = \varphi_i(\xi_1, \xi_2, z_3, \ldots, z_n) \sqrt{d\xi_1\, d\xi_2\, dz_3 \ldots dz_n} / \sqrt{dz_1 \ldots dz_n}.$$

Since

$$L_{\frac{\mu}{\nu+\mu} z_1 D_1 + z_2 D_2 + \alpha}$$
$$= \frac{\mu}{\nu+\mu} z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} - \frac{\mu}{\nu+\mu} \xi_1 \frac{\partial}{\partial \xi_1} - \xi_2 \frac{\partial}{\partial \xi_2} - \frac{1}{2}\left(\frac{\mu}{\nu+\mu} + 1\right) + \alpha,$$

$$L_{z_1(D_1^\mu - z_1^\nu D_2^\mu) + (\beta+\mu) D_1^{\mu-1}}$$
$$= \mu z_1 \xi_1^{\mu-1} \frac{\partial}{\partial z_1} - \mu z_1^{\nu+1} \xi_2^{\mu-1} \frac{\partial}{\partial z_2}$$
$$- (\xi_1^\mu - (\nu+1) z_1^\nu \xi_2^\mu) \frac{\partial}{\partial \xi_1} - \frac{\mu}{2} \xi_1^{\mu-1} + (\beta+\mu) \xi_1^{\mu-1},$$

$$L_{D_j} = \frac{\partial}{\partial x_j} \quad \text{for} \quad j = 3, \ldots, n,$$

φ_0 and φ_1 satisfy

$$\left(\frac{\mu}{\nu+\mu} \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2} + \frac{\mu}{\nu+\mu} + 1 - \alpha\right) \varphi_i = 0 \quad \text{for} \quad i = 0, 1,$$

$$\left(\xi_1^\mu \frac{\partial}{\partial \xi_1} - \beta \xi_1^{\mu-1}\right) \varphi_0 = \left(\nu \xi_1^\mu \frac{\partial}{\partial \xi_1} + \left(\beta + \frac{\nu\mu+\mu}{2}\right) \xi_1^{\mu-1}\right) \varphi_1 = 0,$$

$$\frac{\partial}{\partial x_j} \varphi_i = 0 \quad \text{for} \quad i = 0, 1 \quad \text{and} \quad j = 3, \ldots, n.$$

Hence we have

$$(8.18)\qquad \sigma_{\Lambda_0}(u)=\xi_1^{\beta}\,\xi_2^{\alpha-\frac{\mu}{\nu+\mu}(\beta+1)-1}\sqrt{d\xi_1\,d\xi_2\,dz_3\ldots dz_n}/\sqrt{dz_1\ldots dz_n},$$

$$\operatorname{ord}_{\Lambda_0}u=\alpha+\frac{\nu\beta-\mu}{\nu+\mu},$$

$$(8.19)\qquad \sigma_{\Lambda_1}(u)=\xi_1^{-\frac{2\beta+\nu\mu+\mu}{2\nu}}\,\xi_2^{\alpha+\frac{(2\beta+\nu\mu-2\nu+\mu)\mu}{2\nu(\nu+\mu)}-1}\sqrt{d\xi_1\,d\xi_2\,dz_3\ldots dz_n}/\sqrt{dz_1\ldots dz_n},$$

$$\operatorname{ord}_{\Lambda_1}u=\alpha-\frac{2\beta+\nu\mu+3\mu}{2(\nu+\mu)},$$

and

$$(8.20)\qquad \alpha=\frac{1}{\nu+1}(\operatorname{ord}_{\Lambda_0}u+\nu\operatorname{ord}_{\Lambda_1}u)+\frac{(\nu+2)\mu}{2(\nu+\mu)},$$

$$\beta=\frac{\nu+\mu}{\nu+1}(\operatorname{ord}_{\Lambda_0}u-\operatorname{ord}_{\Lambda_1}u)-\frac{\mu}{2}.$$

The above calculation proves the statement (1) in Theorem 7.1.

To prove (2) we assume that $\mathscr{M}_{\alpha\beta}=\mathscr{E}u=\mathscr{E}/\mathscr{J}$ has a quotient $\mathscr{M}'=\mathscr{E}u'=\mathscr{E}/\mathscr{J}'$ (i.e. $\mathscr{J}'\supset\mathscr{J}$) with support Λ_0. Let $\bar{\mathscr{J}}$ and $\bar{\mathscr{J}}'$ be their symbol ideals, respectively. Since $\bar{\mathscr{J}}'\supset\bar{\mathscr{J}}$ is reduced, $(\bar{\mathscr{J}}')_q=(\bar{\mathscr{J}})_q$ for any point q in $\Lambda_0-\Lambda_1$, which implies $(\mathscr{M}_{\alpha\beta})_q=(\mathscr{M}')_q$ by the correspondence $u=u'$. Consider the simple holonomic system $\mathscr{N}=\mathscr{E}D_2^{\alpha+\frac{\nu\beta-\mu}{\nu+\mu}-1}\delta(z_1,z_2)$ with support Λ_0, that is,

$$\mathscr{N}:\begin{cases}\left(z_2D_2+\alpha+\dfrac{\nu\beta-\mu}{\nu+\mu}\right)v=0,\\ z_1v=D_3v=\ldots=D_nv=0,\end{cases}$$

by denoting $v=D_2^{\alpha+\frac{\nu\beta-\mu}{\nu+\mu}-1}\delta(z_1,z_2)$. We note that owing to the Späth-type theorem for micro-differential operators, any section of $\mathscr{N}$ has a unique expression $Q(z_3,\ldots,z_n,D_1,D_2)v$ with a micro-differential operator Q satisfying

$$(8.21)\qquad [D_1,Q]=[D_2,Q]=[z_3,Q]=\ldots=[z_n,Q]=0.$$

Since $\operatorname{ord}_{\Lambda_0}u'=\operatorname{ord}_{\Lambda_0}u=\operatorname{ord}_{\Lambda_0}v$ (cf. Example 3.21), for any $q\in\Lambda_0-\Lambda_1$ there exists an isomorphism ι of $(\mathscr{M}')_q$ to $\mathscr{N}_q$ defined by $u'=P(z_3,\ldots,z_n,D_1,D_2)v$ with an invertible micro-differential operator P of order 0 defined near q. (Theorem 4.2.5 Chap II. in [2] says that the isomorphism is uniquely determined up to a constant multiple.) In view of $0=D_iu'=D_iPv=[D_i,P]v$ for $i=3,\ldots,n$, we have $[D_i,P]=0$ for $i=3,\ldots,n$ because $[D_i,P]$ also satisfies the condition (8.21). Hence P is of the form $P(D_1,D_2)$. In the same way by the relation

$$0=\left(z_2D_2+\frac{\mu}{\nu+\mu}z_1D_1+\alpha\right)Pv-P\left(z_2D_2+\alpha+\frac{\nu\beta-\mu}{\nu+\mu}+\frac{\mu}{\nu+\mu}(z_1D_1+1)\right)v$$

$$=\left(z_2D_2+\frac{\mu}{\nu+\mu}z_1D_1\right)Pv-PD_2^{\frac{\nu\beta}{\nu+\mu}}\left(z_2D_2+\frac{\mu}{\nu+\mu}z_1D_1\right)D_2^{-\frac{\nu\beta}{\nu+\mu}}v$$

$$=\left[z_2D_2+\frac{\mu}{\nu+\mu}z_1D_1,PD_2^{\frac{\nu\beta}{\nu+\mu}}\right]D_2^{-\frac{\nu\beta}{\nu+\mu}}v$$

we have $\left[z_2 D_2+\frac{\mu}{\nu+\mu} z_1 D_1, P(D_1, D_2) D_2^{\frac{\nu\beta}{\nu+\mu}}\right]=0$. Therefore by denoting $t=D_1^{-\frac{\nu+\mu}{\nu}} D_2^{\frac{\mu}{\nu}}$, P is of the form

$$P=\sum_{j\geqq 0}^{\infty} a_j t^{-\frac{\nu\beta}{\nu+\mu}+j} D_2^{-\frac{\nu\beta}{\nu+\mu}}$$

with $a_j\in\mathbb{C}$. We put $F(t)=\sum_{j\geqq 0}^{\infty} a_j t^j$ and we shall examine the equation that the formal power series $F(t)$ satisfies. Using the relation

$$[D_1 z_1, t^\lambda]=\frac{\nu+\mu}{\nu}\lambda t^\lambda=\frac{\nu+\mu}{\nu} t\frac{d}{dt} t^\lambda,$$

we have

$$\begin{aligned}
0&=\{z_1(D_1^\mu - z_1^\nu D_2^\mu)+(\beta+\mu) D_1^{\mu-1}\} P v\\
&=(D_1^\mu z_1 - D_2^\mu D_1^{-\nu-1} D_1^{\nu+1} z_1^{\nu+1}+D_1^{\mu-1}\beta) P v\\
&=D_1^{\mu-1}\left(D_1 z_1 - t^\nu \prod_{i=0}^{\nu}(D_1 z_1+i)+\beta\right) F(t)\, t^{-\frac{\nu\beta}{\nu+\mu}} D_2^{\alpha-\frac{\mu}{\nu+\mu}-1}\delta(z_1, z_2)\\
&=D_1^{\mu-1} D_2^{\alpha-\frac{\mu}{\nu+\mu}-1} t^{-\frac{\nu\beta}{\nu+\mu}}\left[\left\{\frac{\nu+\mu}{\nu}\left(t\frac{d}{dt}-\frac{\nu\beta}{\nu+\mu}\right)\right.\right.\\
&\quad\left.\left. - t^\nu \prod_{i=0}^{\nu}\left(\frac{\nu+\mu}{\nu}\left(t\frac{d}{dt}-\frac{\nu\beta}{\nu+\mu}\right)+i\right)+\beta\right\} F(t)\right]\delta(z_1, z_2).
\end{aligned}$$

This equation shows $\frac{\nu+\mu}{\nu} t\frac{dF}{dt}=t^\nu \prod_{i=0}^{\nu}\left(\frac{\nu+\mu}{\nu} t\frac{d}{dt}-\beta+i\right) F$, which is equivalent to

$$\frac{\nu+\mu}{\nu} j a_j=\prod_{i=0}^{\nu}\left(\frac{\nu+\mu}{\nu}(j-\nu)-\beta+i\right) a_{j-\nu}$$

for any $j\in\mathbb{Z}_+=\{0,1,2,\dots\}$ with the convention $a_j=0$ for any $j<0$. Hence by induction we get $a_j=0$ whenever $j\not\equiv 0 \bmod \nu$ and

$$(\nu+\mu) k a_{\nu k}=\prod_{i=0}^{\nu}((\nu+\mu)(k-1)-\beta+i) a_{\nu(k-1)}$$

for $k\in\mathbb{Z}$. Therefore we have

$$F(t)=a_0\sum_{k=0}^{\infty}\frac{\prod_{j=0}^{k-1}[(\nu+\mu)j-\beta]^{\nu+1}}{(\nu+\mu)^k k!} t^{\nu k}, \tag{8.22}$$

where $[r]^{l+1}=\prod_{i=0}^{l}(r+i)$ for any $r\in\mathbb{C}$ and $l\in\mathbb{Z}_+$.

Moreover since the characteristic variety of $\mathcal{M}'$ equals Λ_0, there exists $R\in\mathcal{J}'$ defined near $(0; dz_2)$ such that $\sigma(R)=z_1^m$ with a sufficiently large positive integer

m. By Lemma 8.7 and the Späth-type and Weierstrass preparation-type theorems for micro-differential operators, we may assume R has the form

$$R = z_1^m - \sum_{\substack{i \geqq 0, i+j<0 \\ 1 \leqq l \leqq m}} C_{ij}^l D_1^i D_2^j z_1^{m-l}$$

with $C_{ij}^k \in \mathbb{C}$. In the same way as above, we have

$$\begin{aligned} 0 &= D_1^m R P v \\ &= (D_1^m z_1^m - \sum_{\substack{i \geqq 0, i+j<0 \\ 1 \leqq l \leqq m}} C_{ij}^l D_1^{i+l} D_2^j D_1^{m-l} z_1^{m-l}) P v \\ &= D_2^{\alpha - \frac{\mu}{\nu+\mu} - 1} t^{-\frac{\nu\beta}{\nu+\mu}} \left\{ \left(\left[\frac{\nu+\mu}{\nu} t \frac{d}{dt} - \beta \right]^m - \sum_{\substack{i \geqq 0, i+j<0 \\ 1 \leqq l \leqq m}} C_{ij}^l t^{-\frac{\nu}{\nu+\mu}(i+l)} \right. \right. \\ &\quad \left. \left. \cdot D_2^{\frac{\mu}{\nu+\mu}(i+l)+j} \left[\frac{\nu+\mu}{\nu} t \frac{d}{dt} - \beta \right]^{m-l} \right) F(t) \right\} \delta(z_1, z_2), \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\left(\left[\frac{\nu+\mu}{\nu} t \frac{d}{dt} - \beta \right]^m - \sum_{\substack{i \geqq 0, i+j<0 \\ 1 \leqq l \leqq m}} C_{ij}^l t^{-\frac{\nu}{\nu+\mu}(i+l)} \right. \\ &\quad \left. \cdot D_2^{\frac{\mu}{\nu+\mu}(i+l)+j} \left[\frac{\nu+\mu}{\nu} t \frac{d}{dt} - \beta \right]^{m-l} \right) F(t) = 0. \end{aligned}$$

Here we remark that if we put $i+l = (\nu+\mu) r$ and $j = -\mu r$, the conditions $i \geqq 0$ and $i+j<0$ are equal to $\dfrac{l}{\nu+\mu} \leqq r < \dfrac{l}{\nu}$. Hence by the coefficient of $t^{\nu k}$ in the expansion of the above equation into the power series of (t, D_2) we have

$$[(\nu+\mu) k - \beta]^m a_{\nu k} = \sum_{\substack{1 \leqq l \leqq m \\ \frac{l}{\nu+\mu} \leqq r < \frac{l}{\nu}}} C_r^l [(\nu+\mu) k - \beta]^{m-l} a_{\nu(k+r)} \tag{8.23}$$

where $C_r^l = C_{(\nu+\mu) r - l, -\mu r}^l$ and r is a positive integer.

Suppose $a_{\nu k} \neq 0$ for any $k \in \mathbb{Z}_+$. Then if $0 < r\nu < l$, (8.22) implies

$$\begin{aligned} \lim_{k \to \infty} \frac{[(\nu+\mu) k - \beta]^{m-l} a_{\nu(k+r)}}{[(\nu+\mu) k - \beta]^m a_{\nu k}} &= \lim_{k \to \infty} \frac{\prod_{i=1}^r [(\nu+\mu)(k+i-1) - \beta]^{\nu+1}}{[(\nu+\mu) k - \beta + m - l]^l (\nu+\mu)^r [k+1]^r} \\ &= \lim_{k \to \infty} \frac{1}{[(\nu+\mu) k - \beta + m - l]^{l - r\nu}} \\ &= 0. \end{aligned}$$

This contradicts to the equation (8.23). Hence there exists a positive integer N such that $a_{\nu N} = 0$, which equals the condition

$$\beta \geqq 0 \quad \text{and} \quad \beta \equiv 0, 1, \ldots, \nu \bmod (\nu+\mu). \tag{8.24}$$

Conversely we assume (8.24). Then the non-zero section

$$Pv=\sum_{k=0}^{\infty}\frac{\prod_{j=0}^{k-1}[(\nu+\mu)j-\beta]^{\nu+1}}{(\nu+\mu)^k k!}t^{\nu k}t^{-\frac{\nu\beta}{\nu+\mu}}D_2^{-\frac{\nu\beta}{\nu+\mu}}v$$

$$=\sum_{0\leqq k\leqq(\nu+\mu)\beta}\frac{\prod_{j=0}^{k-1}[(\nu+\mu)j-\beta]^{\nu+1}}{(\nu+\mu)^k k!}D_1^{\beta-(\nu+\mu)k}D_2^{\mu k-\beta}v$$

of $\mathscr{N}$ is defined near $\Lambda_0\cap\Lambda_1$ and it is a solution of $\mathscr{M}_{\alpha\beta}$ because we have proved

$$\left(\frac{\mu}{\nu+\mu}z_1D_1+z_2D_2+\alpha\right)Pv=\left[z_1(D_1^{\mu}-z_1^{\nu}D_2^{\mu})+(\beta+\mu)D_1^{\mu-1}\right]Pv$$

$$=D_3Pv=\ldots=D_nPv=0$$

in the above argument. Therefore the correspondence $u=Pv$ defines a surjective homomorphism of $\mathscr{M}_{\alpha\beta}$ to $\mathscr{N}$ and the system defined by its kernel has the support Λ_1, which is clear by the exact sequence of symbol ideals corresponding to the exact sequence of the holonomic systems.

In the same way if $\mathscr{M}_{\alpha\beta}$ has a submodule with support Λ_1, the quotient module by the submodule has the support Λ_0. Hence $\mathscr{M}$ has a quotient module (resp. a submodule) with the support Λ_0 (resp. Λ_1) if and only if the condition (8.24) holds.

Thus we have proved the statement (2). The statement (3) is easily proved by (2) considering the adjoint system of $\mathscr{M}_{\alpha\beta}$ as follows. Since $\mathscr{M}_{\alpha\beta}$ is holonomic, $\mathrm{Ext}^i_{\mathscr{E}}(\mathscr{M}_{\alpha\beta},\mathscr{E})=0$ for $i\neq n(=\dim X)$ and the adjoint system $\mathscr{M}^*_{\alpha\beta}=\mathscr{E}u^*$ is defined by $\mathscr{M}^*_{\alpha\beta}=(\mathrm{Ext}^n_{\mathscr{E}}(\mathscr{M}_{\alpha\beta},\mathscr{E}))'$. Here $'$ is the map between right $\mathscr{E}$-Modules and left $\mathscr{E}$-Modules defined by the correspondences $z_i\mapsto z_i$ and $D_i\mapsto -D_i$ $(i=1,\ldots,n)$. Then we have

$$\mathscr{M}^*_{\alpha\beta}\begin{cases}\left(\frac{\mu}{\nu+\mu}(-D_1)z_1+(-D_2)z_2+\alpha'\right)u^*=0,\\ [(-D_1)^{\mu}-(-D_2)^{\mu}z_1^{\nu})z_1+(\beta+\mu)(-D_1)^{\mu-1}]u^*=0,\\ (-D_3)u^*=\ldots=(-D_n)u^*=0\end{cases}$$

for some α' and therefore $\mathscr{M}^*_{\alpha\beta}=\mathscr{M}_{\frac{\nu+2\mu}{\nu+\mu}-\alpha',\,-\mu-\beta}$. If there exists an exact sequence of holonomic systems

$$0\to\mathscr{M}_1\to\mathscr{M}_{\alpha\beta}\to\mathscr{M}_2\to 0,$$

we have an adjoint exact sequence

$$0\to\mathscr{M}_2^*\to\mathscr{M}_{\frac{\nu+2\mu}{\nu+\mu}-\alpha',\,-\mu-\beta}\to\mathscr{M}_1^*\to 0$$

by applying the functor $(\mathbb{R}\,\mathrm{Hom}_{\mathscr{E}}(\cdot,\mathscr{E}))'$, where $\mathscr{M}_i^*=(\mathrm{Ext}^n_{\mathscr{E}}(\mathscr{M}_i,\mathscr{E}))'$ for $i=1,2$. This reduces the statement (3) to the statement (2). Q.E.D.

§ 9. Examples

We shall use the same notations as in [1].

Example 9.1. ($GL(2)$, $3\Lambda_1$, $V(4)$)

Although this space was intensively investigated by T. Shintani (See [5]), we shall investigate in view of micro-local structure. The representation space can be identified with all binary cubic forms $F_x(u,v)=\sum_{i=1}^{4} x_i u^{4-i} v^{i-1}$. Then the action $\rho=3\Lambda_1$ is given by $\rho(g)\cdot F_x(u,v)=F_x((u,v)g)$ for $g\in GL(2)$. By an isomorphism $F_x(u,v)\mapsto x={}^t(x_1,x_2,x_3,x_4)\in\mathbb{C}^4$, we identify V with $\mathbb{C}^4$. In this case, we have

$$(9.1)\quad d\rho(A)x=\begin{pmatrix} 3a & b & 0 & 0 \\ 3c & 2a+d & 2b & 0 \\ 0 & 2c & a+2d & 3b \\ 0 & 0 & c & 3d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{for } A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathfrak{gl}(2)$$

and the relative invariant $f(x)$ is given by

$$f(x)=x_2^2x_3^2+18x_1x_2x_3x_4-4x_1x_3^3-4x_2^3x_4-27x_1^2x_4^2.$$

First we shall do the orbital decomposition. Put

$$t_{12}(\lambda)=\rho\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \quad t_{21}(\lambda)=\rho\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}, \quad t_1(\mu)=\rho\begin{pmatrix} \mu & \\ & 1 \end{pmatrix}$$

and

$$t_2(\mu)=\rho\begin{pmatrix} 1 & \\ & \mu \end{pmatrix}\ (\mu\neq 0).$$

If $x\neq 0$, we may assume $x_1=1$ by $t_{12}(\lambda)$ and $t_1(\mu)$. Moreover, by $t_{21}(\lambda)$, we may assume $x_4=0$. If $x_2=x_3=0$, we have $x={}^t(1,0,0,0)=u^3$. If $x_2\neq 0$ or $x_3\neq 0$, we have $x={}^t(0,x_2,x_3,0)$ by $t_{12}(\lambda)$. Using $t_1(\mu)$ and $t_2(\mu)$, we have ${}^t(0,1,1,0)$, ${}^t(0,1,0,0)$ and ${}^t(0,0,1,0)$. Since $\rho\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}u^2v=uv^2$, ${}^t(0,1,0,0)$ and ${}^t(0,0,1,0)$ are $GL(2)$-equivalent. Hence there are four orbits $\rho(G)X_i (1\leqq i\leqq 4)$ where $X_1=uv(u+v)={}^t(0,1,1,0)$, $X_2=u^2v={}^t(0,1,0,0)$, $X_3=u^3={}^t(1,0,0,0)$ and $X_4={}^t(0,0,0,0)=0$. If we identify the dual V^* of V with $\mathbb{C}^4$ by $\langle x,y\rangle=\sum_{i=1}^{4} x_i y_i$, then we have

$$(9.2)\qquad d\rho^*(A)y=\begin{pmatrix} -3a & -3c & 0 & 0 \\ -b & -2a-d & -2c & 0 \\ 0 & -2b & -a-2d & -c \\ 0 & 0 & -3b & -3d \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$$\text{for } A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathfrak{gl}(2).$$

Similarly there exist four orbits $\rho^*(G)Y_i(1\leqq i\leqq 4)$ where $Y_1={}^t(0,0,0,0)=0$, $Y_2={}^t(0,0,0,1)$, $Y_3={}^t(0,0,1,0)$ and $Y_4={}^t(0,1,1,0)$. Let Λ_i be the conormal bundle of $\rho(G)X_i(1\leqq i\leqq 4)$. We shall show that $\Lambda_i=\overline{G(X_i,Y_i)}$. Since (G,ρ,V) is a regular P.V., $\Lambda_4=\{0\}\times V^*$ is a good holonomic variety. Put $A_4=\begin{pmatrix}-\frac{1}{3} & 0\\ 0 & -\frac{1}{3}\end{pmatrix}$. Then we have $d\rho(A_4)X_4=0$ and $d\rho^*(A_4)Y_4=Y_4$. Since

$$\delta\chi(A_4)=6(a+d)=-4,\qquad tr_{V^*_{X_4}}d\rho_{X_4}(A_4)=-6(a+d)=4$$

and $\dim V^*_{X_4}=4$, we have

$$\mathrm{ord}_{\Lambda_4}f^s=s\,\delta\chi(A_4)-tr_{V^*_{X_4}}d\rho_{X_4}(A_4)+\tfrac{1}{2}\dim V^*_{X_4}=-4s-\tfrac{4}{2}.$$

Since $d\rho(A)X_3={}^t(3a,3c,0,0)$, we have $V^*_{X_3}=\{{}^t(0,0,y_3,y_4)\}$ and $\mathfrak{g}_{X_3}$ acts on $V^*_{X_3}$ as

$$\begin{pmatrix}y_3\\ y_4\end{pmatrix}\mapsto\begin{pmatrix}-2d & 0\\ -3b & -3d\end{pmatrix}\begin{pmatrix}y_3\\ y_4\end{pmatrix}.$$

Since Y_3 is its generic point, we have $\Lambda_3=\overline{G(X_3,Y_3)}$. Since the orbit of Y_3 in $V^*_{X_4}=V^*$ is of codimension one, we have $\mathrm{codim}\,\Lambda_3\cap\Lambda_4=1$. Moreover, since $\Lambda_3\cap\Lambda_4$ is $SL(2)$-prehomogeneous, Λ_3 is a good holonomic variety by Proposition 6.6. Put $A_3=\begin{pmatrix}0 & 0\\ 0 & -\frac{1}{2}\end{pmatrix}$. Then $d\rho(A_3)X_3=0$ and $d\rho^*(A_3)Y_3=Y_3$. Since $\delta\chi(A_3)=-3$, $tr_{V^*_{X_3}}d\rho_{X_3}(A_3)=-5d=\frac{5}{2}$ and $\dim V^*_{X_3}=2$, we have $\mathrm{ord}_{\Lambda_3}f^s=-3s-\frac{3}{2}$. Since $d\rho(A)X_2={}^t(b,2a+d,2c,0)$, we have $V^*_{X_2}=\{{}^t(0,0,0,y_4)\}$ and $\mathfrak{g}_{X_2}$ acts on $V^*_{X_2}$ as $d\rho_{X_2}(A')y_4=6ay_4$ for $A'=\begin{pmatrix}a & 0\\ 0 & -2a\end{pmatrix}\in\mathfrak{g}_{X_2}$. Since Y_2 is its generic point, we have $\Lambda_2=\overline{G(X_2,Y_2)}$. Since $\mathrm{codim}\,\rho(GL(2))X_2=\mathrm{codim}\,\rho(SL(2))X_2=1$, we have $\mathrm{codim}\,\Lambda_1\cap\Lambda_2=1$ and Λ_2 is a good holonomic variety. Put $A_2=\begin{pmatrix}\frac{1}{6} & 0\\ 0 & -\frac{1}{3}\end{pmatrix}$. Then $d\rho(A_2)X_2=0$ and $d\rho^*(A_2)Y_2=0$. Since $\delta\chi(A_2)=-1$, $tr_{V^*_{X_2}}d\rho_{X_2}(A_2)=1$ and $\dim V^*_{X_2}=1$, we have $\mathrm{ord}_{\Lambda_2}f^s=-s-\frac{1}{2}$. Since the orbit of Y_2 in $V^*_{X_3}$ is of codimension one, we have $\mathrm{codim}\,\Lambda_2\cap\Lambda_3=1$. Finally, we shall calculate the intersection exponent of Λ_i and $\Lambda_{i+1}(1\leqq i\leqq 3)$. Put $A'_4=\begin{pmatrix}-1-2d & 0\\ 0 & d\end{pmatrix}$. Then $d\rho(A'_4)X_4=0$ and $d\rho^*(A'_4)Y_3=Y_3$. Since the trace of A'_4 on $V^*_{X_4}$ mod $d\rho^*(\mathfrak{g})Y_3$ is equal to $3+6d$, we have $\mu=1$ and $\nu=0$ by Proposition 6.5, i.e., Λ_3 and Λ_4 intersect regularly. Since $(GL(2),\rho,V)\cong(GL(2),\rho^*,V^*)$, Λ_1 and Λ_2 intersect regularly. If $d\rho(A'_3)X_3=0$ and $d\rho^*(A'_3)Y_2=Y_2$, we have $A'_3=\begin{pmatrix}0 & b\\ 0 & -\frac{1}{3}\end{pmatrix}$ and hence the trace of A'_3 in $V^*_{X_3}$ modulo $d\rho_{X_3}(\mathfrak{g}_{X_3})Y_2$ is equal to $\frac{2}{3}$, and hence the intersection exponent $(\mu:\nu)$ is $(2:1)$. By Theorem 7.5, we have $b_{\Lambda_3}(s)/b_{\Lambda_2}(s)=(s+\frac{5}{6})(s+\frac{7}{6})$, $b_{\Lambda_2}(s)/b_{\Lambda_1}(s)=b_{\Lambda_4}(s)/b_{\Lambda_3}(s)=(s+1)$, and hence $b(s)=(s+1)^2(s+\frac{5}{6})\cdot(s+\frac{7}{6})$. We denote Ⓐ (Λ) by (μ) if Λ is the conormal bundle of an μ-codimensional orbit. We have $(\Lambda_1)=(0)$, $(\Lambda_2)=(1)$, $(\Lambda_3)=(2)$ and $(\Lambda_4)=(4)$. The holonomy diagram is given in Fig. 9.1.

In general, when conormal bundles intersect regularly, we omit the intersection exponent in the holonomy diagram.

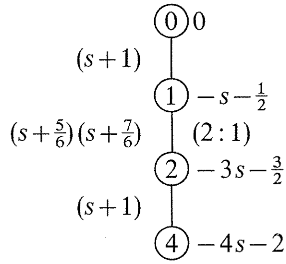


Fig. 9.1. Holonomy diagram of $(GL(2), 3\Lambda_1, V(4))$

Example 9.2. $(SO(n)\times GL(m),\ \Lambda_1\otimes\Lambda_1,\ V(n)\otimes V(m))$ with $n\geqq 3$, $\frac{n}{2}\geqq m\geqq 1$. The representation space can be identified with all $n\times m$ matrices $V=M(n,m)$. Then the action $\rho=\Lambda_1\otimes\Lambda_1$ is given by $\rho(g)X=g_1X^tg_2$ for $g=(g_1,g_2)\in SO(n)\times GL(m)$, $X\in M(n,m)$. First of all, we do the orbital decomposition. For $X\in V$, clearly rank X and rank ${}^tXX$ is invariant under the action ρ of G. Now assume that rank $X=\nu$ and rank ${}^tXX=\mu$ with $0\leqq\mu\leqq\nu\leqq m$. Then by the action of $GL(m)$, we may assume that $X=[\tilde{e}_1,\dots,\tilde{e}_\nu,0,\dots,0]$ where $\tilde{e}_i\in\mathbb{C}^n$ for $1\leqq i\leqq\nu$, and $0={}^t(0,\dots,0)$. Since ρ induces ${}^tXX\mapsto g_2({}^tXX){}^tg_2$ and rank ${}^tXX=\mu$, we may assume that ${}^tXX=\left(\begin{array}{c|c} I_\mu & 0\\ \hline 0 & 0\end{array}\right)$, i.e., $(\tilde{e}_i,\tilde{e}_j)=\delta_{ij}$ for $1\leqq i,\ j\leqq\mu$ and $(\tilde{e}_i,\tilde{e}_j)=0$ otherwise. Now put $e_i={}^t(0,\dots,0,\overset{\overset{i}{\vee}}{1},0,\dots,0)$ for $1\leqq i\leqq\mu$ and $e_j=(0,\dots,0,\overset{\overset{j}{\vee}}{1},0,\dots,0,\overset{\overset{j+\nu-\mu}{\vee}}{\sqrt{-1}},0,\dots,0)$ for $\mu+1\leqq j\leqq\nu$. Note that it is possible since $n\geqq 2m$. Then we have $(e_i,e_j)=\delta_{ij}$ for $1\leqq i,j\leqq\mu$ and $(e_i,e_j)=0$ otherwise, i.e., $(e_i,e_j)=(\tilde{e}_i,\tilde{e}_j)$ for $\forall i,j$, and hence there exists $g_1\in O(n)$ satisfying $g_1X=[e_1,\dots,e_\nu,0,\dots,0]$. By the action of $GL(m)$, we may assume that $g_1\in SO(n)$. This implies that $S_{\nu,\mu}=\{X\in M(n,m);\ \text{rank } X=\nu,\ \text{rank } {}^tXX=\mu\}$ $(0\leqq\mu\leqq\nu\leqq m)$ consists of a single G-orbit, and we obtain the orbital decomposition. Put $X_{\nu,\mu}=[e_1,\dots,e_\nu,0,\dots,0]$, i.e.,

$$(9.3)\qquad X_{\nu,\mu}=\left(\begin{array}{c|c|c} I_\mu & 0 & 0\\ \hline 0 & I_{\nu-\mu} & 0\\ \hline 0 & \sqrt{-1}I_{\nu-\mu} & 0\\ \hline 0 & 0 & \underbrace{0}_{m-\nu}\end{array}\right)\}\,n-2\nu+\mu,$$

and

$$\tilde{A}=A\oplus B=\left(\begin{array}{c|c|c|c} A_1 & A_{12} & A_{13} & A_{14}\\ \hline -{}^tA_{12} & A_2 & A_{23} & A_{24}\\ \hline -{}^tA_{13} & -{}^tA_{23} & A_3 & A_{34}\\ \hline \underbrace{-{}^tA_{14}}_{\mu} & \underbrace{-{}^tA_{24}}_{\nu-\mu} & \underbrace{-{}^tA_{34}}_{\nu-\mu} & \underbrace{A_4}_{n-2\nu+\mu}\end{array}\right)\oplus\left(\begin{array}{c|c|c} B_1 & B_{12} & B_{13}\\ \hline B_{21} & B_2 & B_{23}\\ \hline \underbrace{B_{31}}_{\mu} & \underbrace{B_{32}}_{\nu-\mu} & \underbrace{B_3}_{m-\nu}\end{array}\right)\in\mathfrak{g},$$

where ${}^tA_i = -A_i (1 \leqq i \leqq 4)$. Then we have

$$(9.4)\quad d\rho(\tilde{A})X_{\nu,\mu} = AX_{\nu,\mu} + X_{\nu,\mu}{}^tB$$

$$= \left(\begin{array}{c|c|c} A_1 + {}^tB_1 & \begin{array}{c} A_{12} + \sqrt{-1}A_{13} \\ + {}^tB_{21} \end{array} & {}^tB_{31} \\ \hline {}^tB_{12} - {}^tA_{12} & \begin{array}{c} A_2 + \sqrt{-1}A_{23} \\ + {}^tB_2 \end{array} & {}^tB_{32} \\ \hline \sqrt{-1}\,{}^tB_{12} - {}^tA_{13} & \begin{array}{c} -{}^tA_{23} + \sqrt{-1}A_3 \\ + \sqrt{-1}\,{}^tB_2 \end{array} & \sqrt{-1}\,{}^tB_{32} \\ \hline -{}^tA_{14} & -{}^tA_{24} - \sqrt{-1}\,{}^tA_{34} & 0 \end{array}\right)$$

and hence the isotropy subalgebra $\mathfrak{g}_{X_{\nu,\mu}}$ at $X_{\nu,\mu}$ is given by

$$(9.5)\quad \mathfrak{g}_{X_{\nu,\mu}} = \left\{ \left(\begin{array}{c|c|c|c} A_1 & A_{12} & \sqrt{-1}A_{12} & 0 \\ \hline -{}^tA_{12} & A_2 & A_{23} & A_{24} \\ \hline -\sqrt{-1}\,{}^tA_{12} & -{}^tA_{23} & A_2 + \sqrt{-1}(A_{23} - {}^tA_{23}) & \sqrt{-1}A_{24} \\ \hline 0 & -{}^tA_{24} & -\sqrt{-1}\,{}^tA_{24} & A_4 \end{array}\right) \right.$$

$$\left. \oplus \left(\begin{array}{c|c|c} A_1 & A_{12} & B_{13} \\ \hline 0 & -A_2 - \sqrt{-1}A_{23} & B_{23} \\ \hline 0 & 0 & B_3 \end{array}\right) \right\}$$

If we identify the dual V^* of V with $V = M(n, m)$ by $\langle X, Y \rangle = tr\, X^t Y$, we get the conormal vector space $V^*_{X_{\nu,\mu}}$ by (9.4) as follows.

$$(9.6)\quad V^*_{X_{\nu,\mu}} = \left\{ \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & -\sqrt{-1}X & -\sqrt{-1}Y \\ \hline 0 & X & Y \\ \hline 0 & 0 & Z \end{array}\right) \in V^* \left| \begin{array}{c} {}^tX = X, X \in M(\nu - \mu), Y \in M(\nu - \mu, m - \nu) \\ Z \in M(n - 2\nu + \mu, m - \nu) \end{array} \right. \right\}$$

$$\cong \left\{ \left(\begin{array}{c|c} X & Y \\ \hline & Z \end{array}\right);\ {}^tX = X \right\}.$$

In particular, we have $\operatorname{codim} S_{\nu,\mu} = \dim V^*_{X_{\nu,\mu}} = (m-\nu)(n-\nu) + \frac{1}{2}(\nu-\mu)(\nu-\mu+1)$. Since $d\rho^*(A \oplus B)Y = -{}^tAY - YB$ for $A \oplus B \in \mathfrak{o}(n) \oplus \mathfrak{gl}(m)$, $Y \in V^*$, the action $\rho_{X_{\nu,\mu}}$ of $\mathfrak{g}_{X_{\nu,\mu}}$ on $V^*_{X_{\nu,\mu}}$ is given by

(9.7)
$$\left(\begin{array}{c|c} X & Y \\ \hline & Z \end{array}\right) \mapsto \left(\begin{array}{c|c} A_2+\sqrt{-1}A_{23} & \sqrt{-1}A_{24} \\ \hline 0 & A_4 \end{array}\right)\left(\begin{array}{c|c} X & Y \\ \hline & Z \end{array}\right) + \left(\begin{array}{c|c} X & Y \\ \hline & Z \end{array}\right)\left(\begin{array}{c|c} {}^t(A_2+\sqrt{-1}A_{23}) & -B_{23} \\ \hline & -B_3 \end{array}\right).$$

Note that the action on X-space (resp. Z-space) induced by $\rho_{X_{\nu,\mu}}$ is isomorphic to $(GL(\nu-\mu),\ 2\Lambda_1,\ V(\frac{1}{2}(\nu-\mu)(\nu-\mu+1)))$ (resp. $(SO(n-2\nu+\mu)\times GL(m-\nu),\ \Lambda_1\otimes V_1,\ V(n-2\nu+\mu)\otimes V(m-\nu))$). Since $n\geqq 2m$, we have $n-2\nu+\mu\geqq m-\nu$. Since $SO(n)\times GL(m)$ is reductive, the dual (G,ρ^*,V^*) has the same orbital decomposition $S^*_{\nu,\mu}=\{Y\in V^*;\ \mathrm{rank}\,Y=\nu,\ \mathrm{rank}\,{}^tYY=\mu\}$ $(0\leqq\mu\leqq\nu\leqq m)$. Put

(9.8)
$$X^*_{\nu,\mu}=\left(\begin{array}{c|c} I_{\nu-\mu} & 0 \\ \hline & 0 \\ 0 & I_{m-\nu} \end{array}\right),\quad X^{*(1)}_{\nu,\mu}=\left(\begin{array}{c|c} \begin{smallmatrix}0 & \\ & I_{\nu-\mu-1}\end{smallmatrix} & 0 \\ \hline & 0 \\ 0 & I_{m-\nu} \end{array}\right)$$

$$X^{*(2)}_{\nu,\mu}=\left(\begin{array}{c|c} I_{\nu-\mu} & 0 \\ \hline & \sqrt{-1} \\ & \begin{smallmatrix}0 & 0\end{smallmatrix} \\ 0 & I_{\nu-\mu} \end{array}\right),$$

and denote by $Y_{m-\mu,m-\nu}$(resp. $Y_{m-\mu-1,m-\nu}$, $Y_{m-\mu,m-\nu-1}$) the point of $V^*_{X_{\nu,\mu}}$ corresponding to $X^*_{\nu,\mu}$(resp. $X^{*(1)}_{\nu,\mu}$, $X^{*(2)}_{\nu,\mu}$). Clearly $Y_{m-\mu,m-\nu}$ is a generic point of $V^*_{X_{\nu,\mu}}$ and $Y_{m-\mu,m-\nu}\in S^*_{m-\mu,m-\nu}$, i.e., $\Lambda_{\nu,\mu}=\overline{G(X_{\nu,\mu},Y_{m-\mu,m-\nu})}$ where $\Lambda_{\nu,\mu}$ denotes the conormal bundle of $S_{\nu,\mu}$. Clearly we have $Y_{m-\mu-1,m-\nu}\in S^*_{m-\mu-1,m-\nu}$ and $Y_{m-\mu,m-\nu-1}\in S^*_{m-\mu,m-\nu-1}$, and hence their conormal bundle coincide with $\Lambda_{\nu,\mu+1}$ and $\Lambda_{\nu+1,\mu}$ respectively. Since they are points of one-codimensional orbits in $V^*_{X_\nu}$, we have $\mathrm{codim}\,\Lambda_{\nu,\mu}\cap\Lambda_{\nu,\mu+1}=\mathrm{codim}\,\Lambda_{\nu,\mu}\cap\Lambda_{\nu+1,\mu}=1$. Since $\Lambda_{\nu,\mu}\cap\Lambda_{\nu,\mu+1}$ and $\Lambda_{\nu,\mu}\cap\Lambda_{\nu+1,\mu}$ are $SO(n)\times SL(m)$-prehomogeneous, these conormal bundles $\Lambda_{\nu,\mu}$ are good holonomic varieties by Proposition 6.6 since $\Lambda_{X_{m,m}}=V\times\{0\}\subset W$. We shall calculate the order $\mathrm{ord}_{\Lambda_{\nu,\mu}}f^s$ at $\Lambda_{\nu,\mu}$ where $f(X)=\det{}^tXX(X\in V=M(n,m))$ is the relative invariant of this P.V. (G,ρ,V). Let A_0 be the matrix in (9.5) with $A_{23}=\dfrac{1}{2\sqrt{-1}}I_{\nu-\mu}$, $B_3=-I_{m-\nu}$, all remaining parts zero. Then $d\rho(A_0)X_{\nu,\mu}=0$, $d\rho^*(A_0)Y_{m-\mu,m-\nu}=Y_{m-\mu,m-\nu}$ and $\delta\chi(A_0)=2\,tr B=2\cdot\{-\frac{1}{2}(\nu-\mu)-(m-\nu)\}=-(2m-\nu-\mu)$. By (9.7), we have $tr\,d\rho_{X_{\nu,\mu}}(A_0)=\dfrac{(\nu-\mu)(\nu-\mu+1)}{2}+\dfrac{3}{2}(m-\nu)(\nu-\mu)+(m-\nu)(n-2\nu+\mu)$ and hence $\mathrm{ord}_{\Lambda_{\nu,\mu}}f^s=s\,\delta\chi(A_0)-tr\,d\rho_{X_{\nu,\mu}}(A_0)+\frac{1}{2}\dim\,V^*_{X_{\nu,\mu}}=-(2m-\nu-\mu)s-\frac{1}{4}(\nu-\mu)\,(\nu-\mu+1)-\frac{1}{2}(m-\nu)\,(n-\mu)$. Next we shall show that $\Lambda_{\nu,\mu}$ and $\Lambda_{\nu,\mu+1}$ (or $\Lambda_{\nu+1,\mu}$) intersect regularly, i.e., $\tilde{\mu}=1$, $\tilde{\nu}=0$ in Proposition 6.5. Let $\tilde{A}_\beta$ be an element in (9.5) such that $B_3=-I_{m-\nu}$, $A_{23}=-\dfrac{\sqrt{-1}}{2}\begin{pmatrix}\beta & \\ & I_{\nu-\mu-1}\end{pmatrix}$, all remaining parts zero, where $\beta\in\mathbb{C}$. Then by (9.7) we have $d\rho(\tilde{A}_\beta)X_{\nu,\mu}=0$, $d\rho^*(\tilde{A}_\beta)Y_{m-\mu-1,m-\nu}=Y_{m-\mu-1,m-\nu}$, and $tr\tilde{A}_\beta=\beta$ where tr denotes the trace in $V^*_{X_{\nu,\mu}}\bmod d\rho_{X_{\nu,\mu}}(Y_{m-\mu-1,m-\nu})$. Note that $Y_{m-\mu-1,m-\nu}$ denotes the point of $V^*_{X_{\nu,\mu}}$ corresponding to $X^{*(1)}_{\nu,\mu}$ in (9.8). This implies that $\Lambda_{\nu,\mu}$ and $\Lambda_{\nu,\mu+1}$ intersect regularly by Proposition 6.5. If $\tilde{A}\in\mathfrak{g}_{X_{\nu,\mu}}$ satisfies that $d\rho^*(\tilde{A})\,Y_{m-\mu,m-\nu-1}=Y_{m-\mu,m-\nu-1}$, then $d\rho_{X_{\nu,\mu}}(\tilde{A})$ induces the identity on $V^*_{X_{\nu,\mu}}\bmod d\rho^*(\mathfrak{g}_{X_{\nu,\mu}})Y_{m-\mu,m-\nu-1}$ by (9.7)

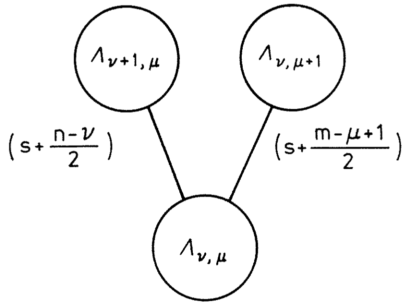


Fig. 9.2

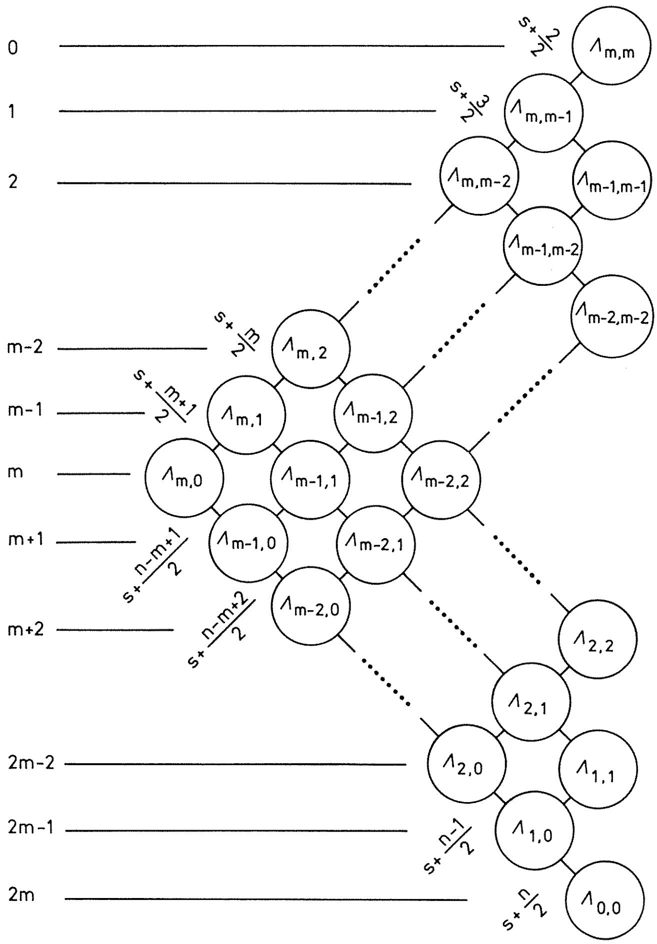


Fig. 9.3. Holonomy diagram of $(SO(n)\times GL(m),\ \Lambda_1\otimes\Lambda_1,\ V(n)\otimes V(m))$ with $n\geqq 2m$.

where $Y_{m-\mu,m-\nu-1}$ is the point corresponding to $X^{*(2)}_{\nu,\mu}$ in (9.8). Therefore we have $\tilde{\mu}=1$ and $\tilde{\nu}=0$ by Proposition 6.5 where $(\tilde{\mu}:\tilde{\nu})$ denotes the intersection exponent. Since

$$(9.9)\qquad \begin{aligned} &\operatorname{ord}_{\Lambda_{\nu,\mu+1}} f^s - \operatorname{ord}_{\Lambda_{\nu,\mu}} f^s + \frac{1}{2} = s + \frac{m-\mu+1}{2} \\ &\operatorname{ord}_{\Lambda_{\nu+1,\mu}} f^s - \operatorname{ord}_{\Lambda_{\nu,\mu}} f^s + \frac{1}{2} = s + \frac{n-\nu}{2}, \end{aligned}$$

we have Fig. 9.2 by Corollary 7.6. Thus we obtain the holonomy diagram (Fig. 9.3), and the b-function

$$b(s) = \prod_{k=1}^{m} \left(s + \frac{k+1}{2}\right) \cdot \prod_{l=1}^{m} \left(s + \frac{n-l+1}{2}\right).$$

Appendix

In this Appendix, we shall give the proof of (2) in Theorem 4.3 in a generalized form.

A.1. Let X be a complex manifold and $f(x)$ a holomorphic function on X which is not identically zero. Let $\mathscr{J}$ be the Ideal of $\mathscr{D}[s] = \mathscr{D} \underset{\mathbb{C}}{\otimes} \mathbb{C}[s]$ consisting of $P(s) \in \mathscr{D}[s]$ such that

(A.1.1) $f^{N-s}P(s)(f^s)=0$ for a sufficiently large integer N.

Note that, for $N \gg 0$, $Q(s) = f^{N-s}P(s)f^s$ belongs to $\mathscr{D}[s]$ and (A.1.1) means that $Q(s)(1)=0$ as a section of $\mathscr{O}[s] = \mathscr{O} \underset{\mathbb{C}}{\otimes} \mathbb{C}[s]$. The condition (A.1.1) is equivalent to the following condition;

(A.1.2) $P(s)f^s(x)=0$ for any $s \in \mathbb{C}$ and any x with $f(x) \neq 0$.

Let W be the closure of the set $\{(s; d\log f(x)^s) \in \mathbb{C} \times T^*X;\ s \in \mathbb{C}, f(x) \neq 0\}$ and we shall denote by W_0 the intersection $W \cap \{s=0\}$ identified with the subset of T^*X. For any $\alpha \in \mathbb{C}$, we define $\mathscr{J}_\alpha = \mathscr{D} \cap (\mathscr{J} + (s-\alpha)\mathscr{D}[s]) = \{P(\alpha);\ P(s) \in \mathscr{J}\}$. We shall denote $\mathscr{N}$ by $\mathscr{D}[s]/\mathscr{J}$ and $\mathscr{N}_\alpha = \mathscr{D}/\mathscr{J}_\alpha$, and u the generator $(1 \bmod \mathscr{J})$ of $\mathscr{N}$ and u_α the generator $(1 \bmod \mathscr{J}_\alpha)$ of $\mathscr{N}_\alpha$. We have $\mathscr{N}/(s-\alpha)\mathscr{N} = \mathscr{N}_\alpha$. We shall prove the following theorem which is a generalization of (2) in Theorem 4.3.

Theorem A.1. *The characteristic variety of* $\mathscr{N}_\alpha$ *is* W_0 *for any* $\alpha \in \mathbb{C}$.

In [7], it is proved that the characteristic variety of $\mathscr{N}$ coincides with W and that of $\mathscr{N}_\alpha$ is contained in W_0. In order to prove the converse inclusion relation, we shall use the theory of matrices of micro-differential operators developped in [10].

A.2. Let us recall the notion of determinants of matrices of micro-differential operators introduced in [10]. For an $N \times N$ matrix $P=(P_{ij})$ of micro-differential operators, we can define the homogeneous holomorphic function $\det P$ satisfying the following properties;

(A.2.1) If P and Q are $N\times N$ matrices of micro-differential operators, then we have $\det(PQ)=(\det P)\cdot(\det Q)$.

(A.2.2) Let $P=(P_{ij})$ be an $N\times N$ matrix of micro-differential operators and m_i $(1\leqq i\leqq N)$ the sequence of integers. Suppose that P_{ij} is of order $\leqq m_i-m_j$ and that the determinant φ of the $N\times N$ matrix $(\sigma_{m_i-m_j}(P_{ij}))$ is not identically zero. Then we have $\det P=\varphi$.

(A.2.3) $\det\begin{pmatrix}P & Q\\ 0 & R\end{pmatrix}=(\det P)\cdot(\det R)$.

(A.2.4) A matrix P is invertible if and only if $\det P$ is nowhere vanishing.

(A.2.5) "det" is invariant under quantized contact transformations, i.e., if Φ is a homogeneous symplectic transformation from an open set Ω_0 to an open set Ω_1 and if $\Psi\colon \Phi^{-1}\mathscr{E}_X\xrightarrow{\sim}\mathscr{E}_X$ is a $\mathbb{C}$-Algebra isomorphism, then we have $(\det P)\circ\Phi=\det\Psi(P)$.

A.3. Let $\mathscr{L}$ be a locally free $\mathscr{E}_X$-Module of finite rank and φ an $\mathscr{E}_X$-endomorphism of $\mathscr{L}$. Let $u_1,\dots,u_N$ be a basis of $\mathscr{L}$. Then $\varphi(u_i)=\sum_j P_{ij}u_j$ for some $P_{ij}\in\mathscr{E}_X$ $(i,j=1,\dots,N)$. We define $\det(\varphi;\mathscr{L})$ by $\det(P_{ij})$. This definition does not depend on the choice of bases. In fact, if $v_1,\dots,v_N$ is another basis, then there exists an invertible matrix $Q=(Q_{ij})$ such that $v_i=\sum_j Q_{ij}u_j$ $(i=1,\dots,N)$ and hence $u_i=\sum_j(Q^{-1})_{ij}v_j$. Therefore we have

$$\begin{aligned}\varphi(v_i)&=\varphi\Big(\sum_j Q_{ij}v_j\Big)=\sum_j Q_{ij}\,\varphi(v_j)=\sum_{j,k}Q_{ij}P_{jk}u_k\\&=\sum_{j,k,l}Q_{ij}P_{jk}(Q^{-1})_{kl}v_l=\sum_l(QPQ^{-1})_{il}v_l.\end{aligned}$$

By (A.2.1), we obtain $\det(QPQ^{-1})=\det Q\cdot\det P\cdot\det Q^{-1}=\det P$. Thus, $\det(\varphi;\mathscr{L})$ is a well-defined holomorphic function. If $\mathscr{L}_j$ is a locally free $\mathscr{E}_X$-Module and φ_j is an $\mathscr{E}_X$-linear endomorphism $(j=1,2)$, one can see easily

(A.3.1) $\det(\varphi_1\oplus\varphi_2;\mathscr{L}_1\oplus\mathscr{L}_2)=\det(\varphi_1;\mathscr{L}_1)\cdot\det(\varphi_2;\mathscr{L}_2)$.

A.4. Any coherent $\mathscr{E}_X$-Module is locally free at a generic point of T^*X. More precisely, we have the following lemma.

Lemma A.2. *Let $\mathscr{M}$ be a coherent $\mathscr{E}_X$-Module defined on a neighborhood of a point p of T^*X. Then, there exist an integer N and an injective homomorphism φ: $\mathscr{E}_X^N\to\mathscr{M}$ on a neighborhood of p such that the cokernel of φ has the support of codimension at least one.*

Proof. Let L be a maximal free submodule of $\mathscr{M}_p$, and $\mathscr{L}$ a coherent $\mathscr{E}_X$-sub-Module of $\mathscr{M}$ defined on a neighborhood of p such that $\mathscr{L}_p=L$. Then $\mathscr{L}$ is free on a neighborhood of p. Set $\mathscr{N}=\mathscr{M}/\mathscr{L}$. If the support of $\mathscr{N}$ has codimension at least one, then there is nothing to prove. Suppose that the support of $\mathscr{N}$ is T^*X. Then there is a section s of $\mathscr{N}$ whose support is T^*X. Then the Ideal of annihilators of s is $\{0\}$. Let s' be a section of $\mathscr{M}$ such that $s' \bmod \mathscr{L}$ is equal to s.

Then $\mathscr{L} \oplus \mathscr{E}_X \to \mathscr{M}$ defined by $(u, P) \mapsto u + Ps'$ is injective, which contradicts the choice of $\mathscr{L}$. Q.E.D.

Let $\mathscr{M}$ be a locally projective coherent $\mathscr{E}_X$-Module (i.e., for any $p \in T^*X$, $\mathscr{M}_p$ is a projective $\mathscr{E}_p$-module) and φ an $\mathscr{E}_X$-endomorphism of $\mathscr{M}$. Then there exists locally a coherent $\mathscr{E}_X$-sub-Module $\mathscr{N}$ such that $\mathscr{M} \oplus \mathscr{N}$ is a locally free $\mathscr{E}_X$-Module of finite rank. We shall define $\det(\varphi; \mathscr{M})$ by $\det(\varphi \oplus 1_{\mathscr{N}}; \mathscr{M} \oplus \mathscr{N})$. This definition does not depend on the choice of $\mathscr{N}$. In fact, let $\mathscr{N}'$ be another $\mathscr{E}_X$-Module such that $\mathscr{M} \oplus \mathscr{N}'$ is locally free. Then, by Lemma A.2, there exists locally an analytic set Z of codimension $\geqq 1$ such that $\mathscr{L}$, $\mathscr{N}$ and $\mathscr{N}'$ are locally free outside Z, and hence (A.3.1) implies that

$$\begin{aligned} &\det(\varphi \oplus 1_{\mathscr{N}}; \mathscr{M} \oplus \mathscr{N})|_{T^*X - Z} \\ &\quad = \det(\varphi; \mathscr{M})|_{T^*X - Z} = \det(\varphi \oplus 1_{\mathscr{N}'}; \mathscr{M} \oplus \mathscr{N}')|_{T^*X - Z}. \end{aligned}$$

Since $\det(\varphi \oplus 1_{\mathscr{N}}; \mathscr{M} \oplus \mathscr{N})$ and $\det(\varphi \oplus 1_{\mathscr{N}'}; \mathscr{M} \oplus \mathscr{N}')$ are holomorphic functions, the equality $\det(\varphi \oplus 1_{\mathscr{N}}; \mathscr{M} \oplus \mathscr{N}) = \det(\varphi \oplus 1_{\mathscr{N}'}; \mathscr{M} \oplus \mathscr{N}')$ holds everywhere. Therefore, $\det(\varphi; \mathscr{M})$ is well-defined when $\mathscr{M}$ is a locally projective coherent $\mathscr{E}_X$-Module.

A.5. Let $\mathscr{M}$ be a coherent $\mathscr{E}_X$-Module defined on an open subset Ω of T^*X and φ an $\mathscr{E}_X$-endomorphism of $\mathscr{M}$. Since $\mathscr{M}$ is projective outside $Y = \bigcup_{j=1}^{n} \operatorname{Supp} \mathscr{E}xt^j(\mathscr{M}, \mathscr{E})$, $\det(\varphi; \mathscr{M})$ is defined outside Y ([7]). We know that Y is an analytic set of codimension $\geqq 1$ (e.g. Lemma A.2).

Proposition A.3. $\det(\varphi; \mathscr{M})$ *is prolonged to a holomorphic function defined on* Ω.

Proof. Set $\mathscr{M}' = \{s \in \mathscr{M};\ \operatorname{codim} \operatorname{supp} s \geqq 1\}$. Then by [7], $\mathscr{M}'$ is a coherent $\mathscr{E}_X$-sub-Module of $\mathscr{M}$. We have $\det(\varphi; \mathscr{M}) = \det(\varphi; \mathscr{M}/\mathscr{M}')$ outside Y because $\mathscr{M}' = 0$ outside Y. Any section of $\mathscr{M}/\mathscr{M}'$ whose support is codim $\geqq 1$, is zero. Hence by replacing $\mathscr{M}$ by $\mathscr{M}/\mathscr{M}'$, we may assume from the beginning that any section of $\mathscr{M}$ whose support is codimension $\geqq 1$ is zero. Then, by using the notion in [7], we have $T^0_{0,n}(\mathscr{M}) = 0$. In order to prove the proposition A.3, it is enough to show that $\operatorname{codim} Y \geqq 2$. Since $\operatorname{codim} \operatorname{Supp} \mathscr{E}xt^j(\mathscr{M}, \mathscr{E}) \geqq 2$ for $j \geqq 2$ and since the support of $T^0_{0,1}(\mathscr{M}) = \mathscr{E}xt^1(\mathscr{E}xt^1(\mathscr{M}, \mathscr{E}), \mathscr{E})$ has the same irreducible components of codimension one as those of $\mathscr{E}xt^1(\mathscr{M}, \mathscr{E})$, it is enough to show that $\operatorname{codim} \operatorname{Supp} T^0_{0,1}(\mathscr{M}) \geqq 2$. We shall prove that $\operatorname{codim} \operatorname{Supp} T^0_{0,i}(\mathscr{M}) \geqq 2$ for $i \geqq 1$ by the descending induction on i. If $i = n$, then this is true because $T^0_{0,n}(\mathscr{M}) = 0$. Suppose that $1 \leqq i < n$ and $\operatorname{codim} \operatorname{Supp} T^0_{0,i+1}(\mathscr{M}) \geqq 2$. We have the exact sequence (see Proposition 2.8 (0) in [7]) $T^0_{0,i+1}(\mathscr{M}) \to T^0_{0,i}(\mathscr{M}) \to T^0_{i,i+1}(\mathscr{M})$. By Proposition 2.9 in [7], we have $\operatorname{codim} \operatorname{Supp} T^0_{i,i+1}(\mathscr{M}) \geqq i > 1$, which implies $\operatorname{codim} \operatorname{Supp} T^0_{0,i}(\mathscr{M}) \geqq 2$. Q.E.D.

By the preceding discussion, we can define $\det(\varphi; \mathscr{M})$ as a homogeneous holomorphic function for any coherent $\mathscr{E}_X$-Module $\mathscr{M}$ and any $\mathscr{E}_X$-endomorphism φ of $\mathscr{M}$. The following properties are obvious by definition.

(A.5.1) $\det(1_{\mathscr{M}}; \mathscr{M}) = 1$.

(A.5.2) If φ and ψ are endomorphisms of $\mathscr{M}$, then we have

$$\det(\varphi \circ \psi; \mathscr{M}) = \det(\varphi; \mathscr{M}) \cdot \det(\psi; \mathscr{M}).$$

(A.5.3) Let $0 \to \mathscr{M}' \xrightarrow{\psi'} \mathscr{M} \xrightarrow{\psi} \mathscr{M}'' \to 0$ be an exact sequence of $\mathscr{E}_X$-Modules and let φ, φ' and φ'' be $\mathscr{E}_X$-endomorphisms of $\mathscr{M}, \mathscr{M}'$ and $\mathscr{M}''$ respectively such that $\varphi \circ \psi' = \psi' \circ \varphi'$ and $\varphi'' \circ \psi = \psi \circ \varphi$. Then we have $\det(\varphi; \mathscr{M}) = \det(\varphi'; \mathscr{M}') \cdot \det(\varphi''; \mathscr{M}'')$.

(A.5.4) $\det \varphi$ is invariant under quantized contact transformations.

A.6. Let $\mathscr{M}$ be a coherent $\mathscr{E}_X$-Module and let V be the support of $\mathscr{M}$. Assume that $\mathscr{M}$ has multiplicity 1 along each irreducible component of V. Suppose that $\mathscr{M}$ is generated by a section u.

Proposition A.4. *Let φ be an $\mathscr{E}_X$-endomorphism of $\mathscr{M}$ and P a micro-differential operator such that $\varphi(u) = Pu$. Suppose that $\sigma(P)$ is not identically zero on each irreducible component of V. Then the support of $\mathscr{M}/\varphi(\mathscr{M})$ coincides with $\sigma(P)^{-1}(0) \cap V$.*

Proof. Outside $\sigma(P)^{-1}(0)$, φ is surjective, and hence the support of the cokernel of φ is contained in $V \cap \sigma(P)^{-1}(0)$. We shall prove the converse inclusion relation. Let $\mathscr{J}$ be the Ideal of annihilators of u and $\bar{\mathscr{J}}$ the symbol Ideal of $\mathscr{J}$. By the condition, $\bar{\mathscr{J}}$ coincides with the defining ideal J_V of V outside an analytic subset Z of V such that $\operatorname{codim}_V Z \geqq 1$. We may assume that Z contains the singular locus of V. For any homogeneous function g in J_V, there exists locally on $V - Z$, a section G of $\mathscr{J}$ such that $\sigma(G) = g$. Since $0 = \varphi(Gu) = G\varphi(u) = GPu = [G, P]u$, we have $\sigma([G, P])|_{V-Z} = 0$. Therefore we have $\{g, \sigma(P)\}|_V = 0$ outside Z, and hence $\{J_V, \sigma(P)\}|_V = 0$. By Proposition 12 in [8], $W = \sigma(P)^{-1}(0) \cap V$ is an involutory subset.

Since it is enough to show that $\operatorname{Supp} \operatorname{Coker} \varphi \supset W$ at a generic point of W, we may assume that W is non-singular. We shall prove Proposition A.4 by reduction to absurdity. If $\operatorname{Supp} \operatorname{Coker} \varphi \not\supset W$, then $\operatorname{Supp} \operatorname{Coker} \varphi \cap W$ is an analytic subset of W of codimension $\geqq 1$. Since $\operatorname{codim}_V W = 1$ and $\operatorname{codim}_V Z \geqq 1$, $\overline{Z - W} \cap W$ is also an analytic subset of W of codimension $\geqq 1$.

Let us take a point p of $W - \operatorname{Supp} \operatorname{Coker} \varphi - \overline{Z - W}$. Then, $V - W$ is non-singular and φ is surjective on a neighborhood of p. This implies that φ is an isomorphism at p since any surjective endomorphism of a coherent $\mathscr{E}_X$-Module is an isomorphism. In fact, if $\operatorname{Ker} \varphi \neq \{0\}$, then $\{\operatorname{Ker} \varphi^i\}_{i=1,2,\dots}$ is a really increasing series of finitely generated $\mathscr{E}_X$-submodules of $\mathscr{M}$. This contradicts that $\mathscr{E}_X$ is noetherian (See [2] Chap. II, Theorem 3.4.1). Let $l+1$ be the codimension of W in T^*X, then $V - W$ is of codimension l. Since W is involutory, we have $0 \leqq l < n$.

Let $C_p(V)$ be the tangent cone of V at p.

Lemma A.5. *Let (A, E) be a symplectic vector space of dimension $2n$, i.e., A is a vector space of dimension $2n$ and E is a non-degenerate skew-symmetric form on A. Let l be an integer such that $0 \leqq l \leqq n$ and, V a homogeneous analytic subset of A of codimension $\geqq l$. Then, there is an isotropic vector subspace μ of dimension l such that $V \cap \mu \subset \{0\}$.*

We shall prove this lemma by the induction on l. If $l = 0$, the lemma is trivial. Suppose $l > 0$. Let $\{V_i\}_{i=1,\dots,N}$ be the set of irreducible components of V. If $V \subset \{0\}$, then there is nothing to prove. Hence, we may assume that $\dim V_i \geqq 1$

for any i. Let W_i be the linear subspace generated by V_i. Then $W_i \neq \{0\}$, and hence $W_i^\perp \neq A$. Therefore $\bigcup_{i=1}^{N} W_i^\perp \cup V \neq A$. Let x be a point of $A - \left(\bigcup_{i=1}^{N} W_i \cup V\right)$, and set $\mu_0 = \mathbb{C}x$. Then, by the condition, the function $f(y) = E(x, y)$ is not identically zero on any V_i and hence $\mu_0^\perp \cap V = V \cap f^{-1}(0)$ has codimension $\geqq l+1$. Since $\mu_0 \cap V \subset \{0\}$, the map $\psi: V \cap \mu_0^\perp \to \mu_0^\perp/\mu_0$ is a finite map. Let V' be the image of ψ. Then, $\operatorname{codim}_{\mu_0^\perp/\mu_0} V' \geqq l-1$, and hence, there is an isotropic subspace μ' of $\mu_0^\perp/\mu_0$ of dimension $l-1$ such that $V' \cap \mu' \subset \{0\}$ by the hypothesis of induction. Let μ be the subspace of $\mu_0^\perp$ such that $\mu' = \mu/\mu_0$. Then, μ is an isotropic subspace of dimension l. By the choice of μ', we have $V \cap \mu \subset \mu_0$. Since $V \cap \mu_0 \subset \{0\}$, we obtain $V \cap \mu \subset \{0\}$. Q.E.D.

We shall resume the proof of Proposition A.4. By the preceding lemma, there is an l-dimensional vector space μ such that $\mu \cap C_p(V) \subset \{0\}$. By a quantized contact transformation, we may assume that μ is spanned by $H_{x_1}, \ldots, H_{x_l}$ and $p = (0, dx_n)$. The choice of μ implies that p is an isolated point of $V \cap \{(x, \xi); x = 0, \xi_{l+1} = \ldots = \xi_{n-1} = 0, \xi_n = 1\}$. We have

(A.6.1) the map $V \to \mathbb{C}^{2n-l}$ defined by $(x, \xi) \mapsto (x, \xi_{l+1}, \ldots, \xi_n)$ is a finite map. Moreover, we have $\mu \cap T_p W = 0$, and hence $\mu \cap (T_p W)^\perp = 0$. Therefore, $dx_1|_W, \ldots, dx_l|_W$ are linearly independent.

Thus we obtain

(A.6.2) the map $W \to \mathbb{C}^l$ defined by $x_1, \ldots, x_l$ is a smooth map.

Let Y be the submanifold of X defined by $x_j = 0$ $(1 \leqq j \leqq l)$ and ρ the canonical projection from $Y \times_X T^*X$ onto T^*Y. Then by (A.6.1), $Y \times_X V \xrightarrow{\rho} T^*Y$ is a finite map, and hence Y is non-characteristic with respect to $\mathscr{M}$; moreover by (A.6.2), W is transversal to $Y \times_X T^*X$ and $Y \times_X W \xrightarrow{\rho} T^*Y$ is an embedding.

We shall show that $V - W$ is also transversal to $Y \times_X T^*X$ on a neighborhood of p. Otherwise, there exists a sequence $\{p_n\}$ of $Y \times_X (V - W)$ such that $V - W$ is not transversal to $Y \times_X T^*X$. Hence $dx_1|_{T_{p_n}V}, \ldots, dx_l|_{T_{p_n}V}$ are linearly dependent. We may assume that $T_{p_n}V$ converges to a linear subspace τ of $T_p(T^*X)$. Then $dx_1|\tau, \ldots, dx_l|\tau$ are linearly dependent. Since $\tau \supset T_p W$ and $dx_1|_{T_p W}, \ldots, dx_l|_{T_p W}$ are linearly independent, we have the contradiction. Thus, $V - W$ is transversal to $Y \times_X T^*X$ on a neighborhood of p.

Note that $\dim Y \times_X V = 2n - 2l = \dim T^*Y$. Hence, there are an open neighborhood Ω of p, an open neighborhood Ω' of $q = \rho(p)$ and a closed analytic subset G of Ω' of dimension less than $2(n-l)$ such that $\rho^{-1}(\Omega' - G) \cap V \cap \Omega \to \Omega' - G$ is a finite covering space. Let q' be a point of $\Omega' - G$, and let $\Omega'' (\subset \Omega' - G)$ be a sufficiently small connected open neighborhood of q'. Then, $\rho^{-1}(\Omega'') \cap V \cap \Omega$ is a finite union of its connected components V_ν $(\nu = 1, \ldots, N)$ and $V_\nu \xrightarrow{\sim} \Omega''$. Set

$$\mathscr{N} = \mathscr{M}|_Y = \rho_*(\mathscr{E}_{Y \to X} \bigotimes_{\mathscr{E}_X} \mathscr{M}|_{V \cap \Omega \cap \rho^{-1}(\Omega')})$$

(See [2]). Then $\mathscr{N}$ is a coherent $\mathscr{E}_Y$-Module. We have $\mathscr{N}=\bigoplus_{\nu=1}^{N}\mathscr{N}_\nu$ on Ω'' where $\mathscr{N}_\nu=\rho_*(\mathscr{E}_{Y\to X}\underset{\mathscr{E}_X}{\otimes}\mathscr{M}|_{V_\nu})$, and φ induces the endomorphism $\tilde{\varphi}$ (resp. $\tilde{\varphi}_\nu$) on $\mathscr{N}$ (resp. $\mathscr{N}_\nu$). We have

(A.6.3.) $$\det(\varphi;\mathscr{N})|_{\Omega''}=\prod_\nu \det(\tilde{\varphi}_\nu;\mathscr{N}_\nu).$$

We shall prove the following (A.6.4) later.

(A.6.4) $$\det(\varphi_\nu;\ \mathscr{N}_\nu)=\sigma(P)|_{V_\nu}.$$

By admitting (A.6.4), we shall prove Proposition A.4. By (A.6.3) and (A.6.4), we obtain $\det(\tilde{\varphi};\mathscr{N})=\prod_\nu \sigma(P)|_{V_\nu}$ or equivalently $\det(\tilde{\varphi};\mathscr{N})(q')=\prod\sigma(P)(p')$ for $q'\in\Omega'-G$ and $p'\in\rho^{-1}(q')\cap V\cap\Omega$. Since $\sigma(P)|_W=0$, this implies that

(A.6.5) $$\det(\tilde{\varphi};\mathscr{N})|_{\rho(W)}=0.$$

Since φ is an isomorphism on a neighborhood of p, φ has an inverse ψ. Hence ψ induces an $\mathscr{E}_X$-linear endomorphism $\tilde{\psi}$ on $\mathscr{N}$ which is an inverse of $\tilde{\varphi}$. Hence $1=\det(\tilde{\varphi}\circ\tilde{\psi};\mathscr{N})=\det(\tilde{\varphi};\mathscr{N})\det(\tilde{\psi};\mathscr{N})$, which contradicts (A.6.5).

Now, it only remains to prove (A.6.4). This is an easy consequence of the following lemma.

Lemma A.6. *Let Y be an l-codimensional submanifold of X, V an involutory submanifold of codimension l, $\mathscr{M}$ a coherent $\mathscr{E}_X$-Module generated by a section u, $\mathscr{J}$ the Ideal of annihilators of u and φ an $\mathscr{E}_X$-endomorphism of $\mathscr{M}$. Suppose the following conditions;*

(A.6.6) *The symbol Ideal $\bar{\mathscr{J}}$ of $\mathscr{J}$ coincides with the Ideal of functions vanishing on V.*

(A.6.7) *V is transversal to $Y\underset{X}{\times}T^*X$.*

(A.6.8) *There is a micro-differential operator P such that $\varphi(u)=Pu$ and $\sigma(P)|_{Y\underset{X}{\times}V}\not\equiv 0$.*

Then, the determinant of the $\mathscr{E}_Y$-linear endomorphism $\tilde{\varphi}$ of $\mathscr{N}=\mathscr{M}|_Y$ coincides with $\sigma(P)|_{Y\underset{X}{\times}V}$.

Proof. The condition (A.6.7) assures that by a contact transformation, $Y\underset{X}{\times}V$ and V are transformed to $\{(x,\xi);\ x_1=\dots=x_l=0\}$ and $\{(x,\xi);\ \xi_1=\dots=\xi_l=0\}$ respectively. Hence we may assume from the beginning that $Y=\{x;\ x_1=\dots=x_l=0\}$ and $V=\{(x,\xi)\in T^*X;\ \xi_1=\dots=\xi_l=0\}$. By (A.6.6), there are micro-differential operators $P_\nu(x,D'')=P_\nu(x,D_{l+1},\dots,D_n)$ of order $\leqq 0$ such that $D_\nu u=P_\nu(x,D'')u$ ($\nu=1,\dots,l$). Then, we can express $P=P_0(x,D'')+\sum_{\nu=1}^{l}Q_\nu(x,D)(D_\nu-P_\nu)$ where $P_0(x,D'')$ is a differential operator which does not contain $D_{l+1},\dots,D_n$ and $\operatorname{ord}P\geqq\operatorname{ord}P_0$, $\operatorname{ord}Q_\nu$. Therefore, we have $Pu=P_0u$ and $\sigma(P)|_V=\sigma(P_0)|_V$. The $\mathscr{E}_Y$-Module $\mathscr{N}=\rho_*(\mathscr{E}_{Y\to X}\underset{\mathscr{E}_X}{\otimes}\mathscr{M})$ is a free $\mathscr{E}_Y$-Module generated by $\tilde{u}=1_{Y\to X}\otimes u$. On the other hand we have

$$\begin{aligned}\tilde{\varphi}(\tilde{u}) &= 1_{Y\to X}\otimes\varphi(u) = 1_{Y\to X}\otimes Pu\\ &= 1_{Y\to X}\otimes P_0(x, D'')\,u\\ &= P_0(0, x'', D'')\,\tilde{u}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\det(\tilde{\varphi}; \tilde{\mathcal{N}}) &= \sigma(P_0(0, x'', D''))\\ &= \sigma(P_0)|_{Y\times_X V} = \sigma(P)|_{Y\times_X V}.\end{aligned}$$

This completes the proof of Lemma A.6.

A.7. Now let us prove Theorem A.1. Suppose first that

(A.7.1) there is a vector field v such that $v(f)=f$. Set

$$\tilde{W} = \overline{\{s\,d\log f(x)\in T^*X;\ s\in\mathbb{C};\ f(x)\neq 0\}}\subset T^*X.$$

Then we have $W_0 = W\cap\sigma(v)^{-1}(0)$, because we have $s=\sigma(v)(x,\xi)$ for $(s,x,\xi)\in W$. We know that the characteristic variety of $\mathcal{N}=\mathcal{D}[s]\,u=\mathcal{D}u$ is $\tilde{W}$. $\mathcal{N}$ has multiplicity 1 at a generic point of W. Set $\varphi=s-\alpha$: $\mathcal{N}\to\mathcal{N}$. We have $\varphi u=(v-\alpha)\,u$ and we can apply Proposition A.4, and we obtain

$$\operatorname{Supp}\operatorname{Cok}\operatorname{Ker}\varphi = W\cap\sigma(v-\alpha)^{-1}(0) = W_0.$$

Thus we obtain Theorem A.1 under the condition (A.7.1).

Now, we shall prove Theorem A.1 in a general case. Set $X'=\mathbb{C}\times X$ and let f' be the holomorphic function $yf(x)$ on X'. Then we have $vf'=f'$ for $v=y\dfrac{\partial}{\partial y}$. Define $\mathcal{N}'=\mathcal{D}_{X'}[s]f'^s$, $\tilde{W}'=$ the closure of $\{s\,d\log f'\in T^*X;\ f'(y,X)\neq 0,\ s\in\mathbb{C}\}$ and $W_0'=W'\cap\{s=0\}\subset T^*X'$. Then by the preceding argument, we have $SS(\mathcal{N}'/(s-\alpha)\mathcal{N}')=W_0'$ for any $\alpha\in\mathbb{C}$. If we identify T^*X' with $T^*\mathbb{C}\times T^*X=\mathbb{C}\times\mathbb{C}\times T^*X$, we have $\tilde{W}'\cap\{y\neq 0\}=$ the closure of $\left\{\left(y,\dfrac{s}{y}, d\log f(x)^s\right)\in\mathbb{C}\times\mathbb{C}\times T^*X;\ y\neq 0,\ f(x)\neq 0,\ s\in\mathbb{C}\right\}=\left\{\left(y,\dfrac{s}{y},p\right)\in\mathbb{C}^\times\times\mathbb{C}\times T^*X;\ (s,p)\in W\right\}$. Since $\sigma(v)\left(y,\dfrac{s}{y},p\right)=s$, we have $W_0'\cap\{y\neq 0\}=\mathbb{C}^\times\times\{0\}\times W_0$. We shall prove $SS(\mathcal{N}/(s-\alpha)\mathcal{N})\supset W_0$. If $p\notin SS(\mathcal{N}/(s-\alpha)\mathcal{N})$, then there is a differential operator $P(x,D)\in\mathcal{J}_\alpha$ such that $\sigma(P)(p)\neq 0$. Let $Q(s)$ be a section of $\mathcal{J}$ such that $P(x,D)=Q(\alpha)$. Let $\tilde{P}$ be the differential operator P considered as a differential operator on X', and $\tilde{Q}(s)$ the section $Q(s)$ considered as a section of $\mathcal{D}_{X'}[s]$. Then, we have $\tilde{Q}(s)f'^s=0$ and hence $\tilde{P}=\tilde{Q}(\alpha)$ annihilates $f'^s \bmod (s-\alpha)\mathcal{N}'$. Hence $\sigma(\tilde{P})|_{W_0'}=0$. In particular $\sigma(\tilde{P})(y,0,p)=0$. Since $\sigma(\tilde{P})(y,0,p)=\sigma(P)(p)$, we obtain $\sigma(P)(p)=0$, which is a contradiction. This shows the desired result; $SS(\mathcal{N}/(s-\alpha)\mathcal{N})\supset W_0$.

References

1. Sato, M., Kimura, T.: A classification of irreducible prehomogeneous vector spaces and their relative invariants. Nagoya Math. J., **65**, 1–155 (1977)
2. Sato, M., Kawai, T., Kashiwara, M.: Hyperfunctions and Pseudo-differential Equations. Lecture Notes in Mathematics 287, Berlin Heidelberg New York: Springer-Verlag
3. Sato, M., Shintani, T.: On zeta-functions associated with prehomogeneous vector spaces. Ann. of Math., **100**, 131–170 (1974)
4. Sato, M.: Theory of prehomogeneous vector spaces, (noted by T. Shintani in Japanese). Sugaku no ayumi, **15**, 85–157 (1970)
5. Shintani, T.: On Dirichlet series whose coefficients are class-numbers of integral binary cubic forms. J. Math. Soc. Japan, **24**, 132–188 (1972)
6. Servedio, F.: Affine Open Orbits, Reductive Isotropy Groups, and Dominant Gradient Morphisms; A Theorem of Mikio Sato. Pacific J. of Math., **72**, 537–545 (1977)
7. Kashiwara, M.: B-Functions and Holonomic Systems (Rationality of b-functions). Inv. **38**, 33–53 (1976)
8. Kashiwara, M., Kawai, T.: On holonomic systems for $\prod_{l=1}^{N} (f_l+\sqrt{-1}\,0)^{\lambda_l}$, RIMS, to appear
9. Kashiwara, M., Kawai, T.: Holonomic systems of micro-differential equations III, to appear
10. Sato, M., Kashiwara, M.: The determinant of matrices of pseudo-differential operators. Proc. Japan Acad. **51**, 17–19 (1975)
11. Carathéodory, C.: Calculus of Variations and Partial Differential Equations of the First Order, Part 1, Amsterdam: Holden-Day 1965, Translated from the German original, 1935
12. Oshima, T.: Singularities in contact geometry and degenerate pseudo-differential equations. J. Fac. Sci. Univ. Tokyo Sec. IA, **21**, 43–83 (1974)
13. Kashiwara, M., Oshima, T.: Systems of differential equations with regular singularities and their boundary value problems. Ann. of Math., **106**, 145–200 (1977)
14. Bernstein, I.N.: The analytic continuation of generalized functions with respect to a parameter. Functional Anal. Appl. **6**, 26–40 (1972)
15. Kimura, T.: The b-functions and holonomy diagrams of irreducible regular prehomogeneous vector spaces, to appear
16. Bijork, J.E.: Rings of differential operators, Amsterdam: North-Holland 1979
17. Boutet de Monvel-Kree: Pseudo-differential operators and Gevrey classes. Ann. Inst. Fourier **17**, 295–323 (1967)

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