INTRODUCTION TO MICROLOCAL ANALYSIS

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§ 0. INTRODUCTION

0.1. In this lecture, we explain the micro-local point of view (i.e. the consideration on the cotangent bundle) for the study of systems of linear differential equations.

0.2. The importance of the cotangent bundle in analysis has been recognized for a long time, although implicitly, for example by the following consideration.

We consider a linear differential operator

$$P(x, \partial) = \sum_{\alpha \in \mathbf{N}^n} a_\alpha(x)\partial^\alpha \quad \text{with} \quad \partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$$

for $\alpha = (\alpha_1, \dots, \alpha_n)$, and try to find a solution to $P(x, \partial)u(x) = 0$. If we suppose that $u(x)$ has a singularity along a hypersurface $f(x) = 0$, then the simplest possible form of $u(x)$ is

$$u(x) = c_0(x)f(x)^s + c_1(x)f(x)^{s+1} + \dots$$

Then setting $P_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ we have

$$\begin{aligned} (0.1.1) \qquad P(x, \partial)u(x) &= s(s-1)\dots(s-m+1)c_0(x)P_m(x, df)f(x)^{s-m} + \dots \\ &+ (s+j)\dots(s+j-m+1)c_j(x)P_m(x, df) \\ &+ (\text{terms in } c_0, \dots, c_{j-1})f(x)^{s-j} + \dots \end{aligned}$$

Therefore $P_m(x, df)$ must be a multiple of $f(x)$ (i.e. $P_m(x, df) = 0$ on $f^{-1}(0)$). In this case, $f^{-1}(0)$ is called characteristic.

Thus the hypersurface $f^{-1}(0)$ is not arbitrary and the singularity of the solution to $Pu(x) = 0$ has a very special form.

[1]) Survey lectures given at the University of Bern in June 1984 under the sponsorship of the International Mathematical Union.

0.3. If $P_m(x, \xi) \neq 0$ for any non-zero real vector ξ, then P is called an elliptic operator. In this case, one can easily solve $P(x, \partial)u(x) = f(x)$ for any $f(x)$, at least locally. We start from the plane wave decomposition of the δ-function.

$$\delta(x) = \text{const.} \int_{S^{n-1}} \frac{\omega(\xi)}{<x, \xi>^n} \tag{0.3.1}$$

where $\omega(\xi)$ is the invariant volume element of the sphere S^{n-1}.

By formula (0.1.1), we can solve

$$P(x, \partial)K(x, y) = \frac{1}{(<x, \xi> - <y, \xi>)^n},$$

by setting $K(x, y) = \sum c_j(<x, \xi> - <y, \xi>)^{m-n+j}$ and determining c_j recursively. Then $K(x, y) = \text{const} \int K(x, y, \xi)\omega(\xi)$ satisfies

$$P(x, \partial)K(x, y) = \delta(x-y)$$

by (0.3.1).

If we set $u(x) = \int K(x, y)f(y)dy$ then $u(x)$ satisfies $P(x, \partial)u(x) = f(x)$. In fact

$$P(x, \partial)u(x) = \int P(x, \partial)K(x, y)f(y)dy = \int \delta(x-y)f(y)dy = f(x).$$

0.4. By these considerations, M. Sato recognized explicitly the importance of the cotangent bundle by introducing the singular spectrum of functions and microfunctions [Sato]. For a real analytic manifold M, let $\mathscr{A}_M$ be the sheaf of real analytic functions and $\mathscr{B}_M$ the sheaf of hyperfunctions. Let $\pi: T^*M \to M$ be the cotangent bundle of M. Then he constructed the sheaf $\mathscr{C}_M$ of microfunctions and an exact sequence

$$0 \to \mathscr{A}_M \to \mathscr{B}_M \overset{\text{sp}}{\to} \pi_*\mathscr{C}_M \to 0.$$

The action of a differential operator $P(x, \partial)$ on $\mathscr{B}_M$ extends to the action on $\mathscr{C}_M$.

Moreover $P: \mathscr{C}_M \to \mathscr{C}_M$ is an isomorphism outside

$$\{(x, \xi) \in T^*M ; P_m(x, \xi) = 0\}.$$

0.5. In the situation of § 0.2, $u(x) = c_0(x)f(x)^s + \ldots$ satisfies supp sp$(u(x))$ $= \{\pm df(x)\}$. Therefore $P_m(x, df)$ must be zero if $P(x, \partial)u(x) = 0$. In fact otherwise the bijectivity of $P: \mathscr{C}_M \to \mathscr{C}_M$ implies sp$(u) = 0$.

0.6. Such a method of studying functions or differential equations locally on the cotangent bundle is called microlocal analysis. After Sato's discovery of microfunctions, microlocal analysis was studied intensively in Sato-Kawai-Kashiwara [SKK].

Also L. Hörmander [H] worked in the C^∞-case. Since then, microlocal analysis has been one of the most fundamental tools in the theory of linear partial differential equations.

§ 1. Systems of Differential Equations (See [O], [Bj])

1.1. Let X be a complex manifold. A system of linear differential equations can be written in the form

$$\sum_{j=1}^{N_0} P_{ij}(x, \partial)u_j = 0\,, \quad i = 1, 2, \ldots, N_1\,. \tag{1.1.1}$$

Here $u_1, \ldots, u_{N_0}$ denote unknown functions and the $P_{ij}(x, \partial)$ are differential operators on X. The holomorphic function solutions of (1.1.1) are simply the kernel of the homomorphism

$$P: \mathcal{O}_X^{N_0} \to \mathcal{O}_X^{N_1} \tag{1.1.2}$$

which assigns $(v_1, \ldots, v_{N_1})$ to $(u_1, \ldots, u_{N_0})$, where $v_i = \sum_j P_{ij}(x, \partial)u_j$.

Let us denote by $\mathscr{D}_X$ the ring of differential operators with holomorphic coefficients. Then

$$P: \mathscr{D}_X^{N_1} \to \mathscr{D}_X^{N_0} \tag{1.1.3}$$

given by $(Q_1, \ldots, Q_{N_1})$ to $(\Sigma Q_i P_{i1}, \ldots, \Sigma Q_i P_{iN_0})$ is a left $\mathscr{D}_X$-linear homomorphism. If we denote by $\mathscr{M}$ the cokernel of (1.1.3), then $\mathscr{M}$ becomes a left $\mathscr{D}_X$-module and $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_X)$ is the kernel of (1.1.2). This means that the set of holomorphic solutions to $Pu = 0$ depends only on $\mathscr{M}$.

For this reason we mean by a system of linear differential equations a left $\mathscr{D}_X$-module.

1.2. Let us take a local coordinate system $(x_1, \ldots, x_n)$ of X. Then any differential operator P can be written in the form

$$P(x, \partial) = \sum_{\alpha \in \mathbf{N}^n} a_\alpha(x) \partial^\alpha \tag{1.2.1}$$

where $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and the $a_\alpha(x)$ are holomorphic functions. For $j \in \mathbf{N}$, we set

$$P_j(x, \xi) = \sum_{|\alpha| = j} a_\alpha(x) \xi^\alpha ,$$

where $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$, and we call $\{P_j(x, \xi)\}$ the *total symbol* of P. The largest m such that $P_m \neq 0$ is called the *order* of P and P_m is called the *principal symbol* of P and denoted by $\sigma(P)$.

Let us denote by T^*X the cotangent bundle of X, and let

$$(x_1, \ldots, x_n; \xi_1, \ldots, \xi_n)$$

be the associated coordinates of T^*X. It is a classical result that if we consider $\sigma(P)$ as a function on T^*X, then this does not depend on our choice of the local coordinate system $(x_1, \ldots, x_n)$.

1.3. Let M be a real analytic manifold, and X its complexification, e.g., $M = \mathbf{R}^n \subset X = \mathbf{C}^n$. Let P be a differential operator on X. When $\sigma(P)(x, \xi) \neq 0$ for $(x, \xi) \in \mathbf{R}^n \times (\mathbf{R}^n \backslash \{0\})$, P is called an elliptic differential operator. In this case, we have the following result.

Proposition 1.3.1. *If u is a hyperfunction (or distribution) on M and Pu is real analytic, then u is real analytic. More precisely if we denote by $\mathscr{A}$ the sheaf of real analytic functions on M and by $\mathscr{B}$ (resp. $\mathscr{D}b$) the sheaf of hyperfunctions (resp. distributions) on M, then $P : \mathscr{B}/\mathscr{A} \to \mathscr{B}/\mathscr{A}$ (resp. $P : \mathscr{D}b/\mathscr{A} \to \mathscr{D}b/\mathscr{A}$) is a sheaf isomorphism.*

This suggests that if $\sigma(P)(x, \xi) \neq 0$, we can consider the inverse P^{-1} in a certain sense. Since (x, ξ) is a point of the cotangent bundle, P^{-1} is attached to the cotangent bundle.

In fact, as we shall see in the sequel, we can construct a sheaf of rings $\mathscr{E}_X$ on T^*X such that $\mathscr{D}_X \subset \pi_* \mathscr{E}_X$, where π is the canonical projection $T^*X \to X$. Moreover if $P \in \mathscr{D}_X$ satisfies $\sigma(P)(x, \xi) \neq 0$ at a point $(x, \xi) \in T^*X$, then P^{-1} exists as a section of $\mathscr{E}_X$ on a neighborhood of (x, ξ).

This can be compared to the analogous phenomena for polynomial rings, as shown in the following table.

$\mathbf{C}[x_1, \dots, x_n]$	$\mathscr{D}_X$
$\mathbf{C}^n$	T^*X
the sheaf $\mathcal{O}_{\mathbf{C}^n}$ of holomorphic functions	$\mathscr{E}_X$

§ 2. Micro-differential Operators (See [SKK], [Bj], [S], [K2])

2.1. Let X be an n-dimensional complex manifold and let $\pi_X : T^*X \to X$ be the cotangent bundle of X. Let us take a local coordinate system $(x_1, \dots, x_n)$ of X and the associated coordinates $(x_1, \dots, x_n; \xi_1, \dots, \xi_n)$ of T^*X. For a differential operator P, let $\{P_j(x, \xi)\}$ be the total symbol of P as in § 1.2. We sometimes write $P = \Sigma P_j(x, \partial)$.

Let $Q = \Sigma Q_j(x, \partial)$ be another differential operator. Set $S = P + Q$ and $R = PQ$. Then the total symbols $\{S_j\}$ and $\{R_j\}$ of R and S are given explicitly by

$$S_j = P_j + Q_j \tag{2.1.1}$$

$$R_l = \sum_{\substack{l = j + k - |\alpha| \\ \alpha \in \mathbf{N}^n}} \frac{1}{\alpha !} (\partial_\xi^\alpha P_j)(\partial_x^\alpha Q_k) \tag{2.1.2}$$

where $\partial_\xi^\alpha = (\partial/\partial\xi_1)^{\alpha_1} \dots (\partial/\partial\xi_n)^{\alpha_n}$ and $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$.

The total symbol $\{P_j(x, \xi)\}$ of a differential operator behaves as follows under coordinate transformations. Let $(x_1, \dots, x_n)$ and $(\tilde{x}_1, \dots, \tilde{x}_n)$ be two local coordinate systems. Let $(\xi_1, \dots, \xi_n)$ and $(\tilde{\xi}_1, \dots, \tilde{\xi}_n)$ be related by

$$\xi_k = \sum_j \tilde{\xi}_j \cdot \frac{\partial \tilde{x}_j}{\partial x_k}$$

i.e. $(x_1, \dots, x_n; \xi_1, \dots, \xi_n)$ and $(\tilde{x}_1, \dots, \tilde{x}_n; \tilde{\xi}_1, \dots, \tilde{\xi}_n)$ are the associated local coordinate systems of the cotangent bundle T^*X. Let P be a differential operator on X and let $\{P_j(x, \xi)\}$ and $\{\tilde{P}_j(\tilde{x}, \tilde{\xi})\}$ be the total symbols of P with respect to the local coordinate systems $(x_1, \dots, x_n)$ and $(\tilde{x}_1, \dots, \tilde{x}_n)$, respectively. Then one has

$$\tilde{P}_l(\tilde{x}, \tilde{\xi}) = \sum_{\nu, \alpha_1, \dots, \alpha_\nu} \frac{1}{\nu ! \, \alpha_1 ! \dots \alpha_\nu !} \langle \tilde{\xi}, \partial_x^{\alpha_1} \tilde{x} \rangle \dots \langle \tilde{\xi}, \partial_x^{\alpha_\nu} \tilde{x} \rangle \, \partial_\xi^{\alpha_1 + \dots + \alpha_\nu} P_j(x, \xi) . \tag{2.1.3}$$

Here the indices run over $j \in \mathbf{Z}$, $\nu \in \mathbf{N}$, $\alpha_1, \ldots, \alpha_\nu \in \mathbf{N}^n$ such that $|\alpha_1|, \ldots, |\alpha_\nu| \geqslant 2$ and $l = j + \nu - |\alpha_1| - \ldots - |\alpha_\nu|$. For $\beta \in \mathbf{N}^n$, $<\tilde{\xi}, \partial_x^\beta \tilde{x}>$ denotes $\sum_j \tilde{\xi}_j \partial^\beta \tilde{x}_j$.

2.2. The total symbol $\{P_j(x, \xi)\}$ of a differential operator is a polynomial in ξ. We shall define microdifferential operators by admitting P_j to be holomorphic in ξ.

For $\lambda \in \mathbf{C}$, let $\mathcal{O}_{T^*X}(\lambda)$ be the sheaf of homogeneous holomorphic functions of degree λ on T^*X, i.e., holomorphic functions $f(x, \xi)$ satisfying

$$(\Sigma \xi_j \partial / \partial \xi_j - \lambda) f(x, \xi) = 0 .$$

Definition 2.2.1. For $\lambda \in \mathbf{C}$ we define the sheaf $\mathscr{E}_X(\lambda)$ on T^*X by

$$\Omega \mapsto \{(P_{\lambda-j}(x, \xi))_{j \in \mathbf{N}} ; \quad P_{\lambda-j} \in \Gamma(\Omega; \mathcal{O}_{T^*X}(\lambda - j))$$
and satisfies the following conditions (2.2.1)$\}$

(2.2.1) for any compact subset K of Ω, there exists a $C_K > 0$ such that

$$\sup_K |P_{\lambda-j}| \leqslant C_K^{-j}(j!) \qquad \text{for all} \quad j > 0 .$$

Remark. The growth condition (2.2.1) can be explained as follows. For a differential operator $P = \Sigma P_j(x, \partial)$, we have

$$P(x, \partial)(<x, \xi> + p)^\mu = \Sigma P_j(x, \xi) \frac{\Gamma(\mu)}{\Gamma(\mu - j + 1)} (<x, \xi> + p)^{\mu - j} .$$

For $P = (P_{\lambda-j}(x, \xi)) \in \mathscr{E}(\lambda)$ we set, by analogy

$$P(<x, \xi> + p)^\mu = \sum_j P_{\lambda-j}(x, \xi) \frac{\Gamma(\mu)}{\Gamma(\mu - \lambda + j + 1)} (<x, \xi> + p)^{\mu - \lambda + j} .$$

Then the growth condition (2.2.1) is simply the condition that the right hand side converges when $0 < |<x, \xi> + p| \ll 1$.

Now, we have the following

Proposition 2.2.2 ([SKK], Chap. II, § 1, [Bj] Chap. IV, § 1).

(0) *$\mathscr{E}_X(\lambda)$ contains $\mathscr{E}_X(\lambda - m)$ as a subsheaf for $m \in \mathbf{N}$.*

(1) *Patching by rule* (2.1.3) *under coordinate transformations, $\mathscr{E}_X(\lambda)$ becomes a sheaf defined globally on T^*X.*

(2) *By rule* (2.1.1), $\mathscr{E}_X(\lambda)$ *is a sheaf of* **C**-*vector space on* T^*X.

(3) *By rule* (2.1.2), *we can define the* "product" *homomorphism:*

$$\mathscr{E}_X(\lambda) \underset{\mathbf{C}}{\otimes} \mathscr{E}_X(\mu) \to \mathscr{E}_X(\lambda+\mu),$$

which satisfies the associative law.

(4) *In particular,* $\mathscr{E}_X(0)$ *and* $\mathscr{E}_X = \bigcup_{m\in\mathbf{Z}} \mathscr{E}_X(m)$ *become sheaves of (non commutative) rings on* T^*X, *with a unit.*

The unit is given by $(P_j(x, \xi))$ *with* $P_j = 1$ *for* $j = 0$ *and* $P_j = 0$ *for* $j \neq 0$.

We define the homomorphism

$$\sigma_\lambda : \mathscr{E}_X(\lambda) \to \mathscr{O}_{T^*X}(\lambda)$$

by

$$(P_{\lambda-j}) \mapsto P_\lambda .$$

Then, σ_λ is a well-defined homomorphism on T^*X (i.e. compatible with coordinate transformation) and we have an exact sequence

$$0 \to \mathscr{E}_X(\lambda-1) \to \mathscr{E}_X(\lambda) \xrightarrow{\sigma_\lambda} \mathscr{O}_{T^*X}(\lambda) \to 0 .$$

Now we have the following proposition, which says that the ring $\mathscr{E}_X$ is a kind of localization of $\mathscr{D}_X$.

PROPOSITION 2.2.3.

(1) *For* $P \in \mathscr{E}(\lambda)$ *and* $Q \in \mathscr{E}(\mu)$, *we have* $\sigma_{\lambda+\mu}(PQ) = \sigma_\lambda(P)\sigma_\mu(Q)$.

(2) ([SKK] Chap. II, Thm. 2.1.1) *If* $P \in \mathscr{E}(\lambda)$ *satisfies* $\sigma_\lambda(P)(q) \neq 0$ *at* $q \in T^*X$, *then there exists* $Q \in \mathscr{E}(-\lambda)$ *such that* $PQ = QP = 1$.

The relations between $\mathscr{E}_X$ and $\mathscr{D}_X$ are summarized in the following theorem.

THEOREM 2.2.4 ([SKK], Chap. II, § 3).

(i) $\mathscr{E}_X$ *contains* $\pi^{-1}\mathscr{D}_X$ *as a subring and is flat over* $\pi^{-1}\mathscr{D}_X$.

(ii) $\mathscr{E}_X|_{T^*_XX} \simeq \mathscr{D}_X$, *where* T^*_XX *is the zero section of* T^*X.

(iii) *For a coherent* $\mathscr{D}_X$-*module* $\mathscr{M}$, *the characteristic variety of* $\mathscr{M}$ *coincides with the support of* $\mathscr{E}_X \underset{\pi_X^{-1}\mathscr{D}_X}{\otimes} \pi_X^{-1}\mathscr{M}$.

§ 3. The Algebraic Properties of $\mathscr{E}$ (See [SKK], [Bj])

3.1. In the preceding section, we introduced the notion of micro-differential operators. The ring $\mathscr{E}$ of micro-differential operators has nice algebraic properties similar to those of the ring of holomorphic functions.

Let us recall some definitions of finiteness properties.

Definition 3.1.1. Let $\mathscr{A}$ be a sheaf of rings on a topological space S.

(1) An $\mathscr{A}$-module $\mathscr{M}$ is called *of finite type* (resp. *of finite presentation*) if for any point $x \in X$ there exists a neighborhood U and an exact sequence $0 \leftarrow \mathscr{M}|_U \leftarrow \mathscr{A}^p|_U$ (resp. $0 \leftarrow \mathscr{M}|_U \leftarrow \mathscr{A}^p|_U \leftarrow \mathscr{A}^q|_U$).

(2) $\mathscr{M}$ is called *pseudo-coherent*, if any submodule of finite type defined on an open subset is of finite presentation. If $\mathscr{M}$ is pseudo-coherent and of finite type, then $\mathscr{M}$ is called *coherent.*

(3) $\mathscr{M}$ is called *Noetherian* if $\mathscr{M}$ satisfies the following properties:

   (a) $\mathscr{M}$ is coherent.

   (b) For any $x \in X$, $\mathscr{M}_x$ is a Noetherian $\mathscr{A}_x$-module (i.e. any increasing sequence of $\mathscr{A}_x$-submodules is stationary).

   (c) For any open subset U, any increasing sequence of coherent $(\mathscr{A}|_U)$-submodules of $\mathscr{M}|_U$ is locally stationary.

As for the sheaf of holomorphic functions, we have

Theorem 3.1.1 ([SKK] Chap. II, Thm. 3.4.1, Prop. 3.2.7). *Let* $\mathring{T}^*X$ *denote the complement of the zero section in* T^*X.

(1) $\mathscr{E}_X$ *and* $\mathscr{E}_X(0)$ *are Noetherian rings on* T^*X.

(2) $\mathscr{E}_X$ *is flat over* $\pi^{-1}\mathscr{D}_X$.

(3) $\mathscr{E}_X(\lambda)|_{\mathring{T}^*X}$ *is a Noetherian* $\mathscr{E}_X(0)|_{\mathring{T}^*X}$*-module.*

(4) *For* $p \in T^*X$, $\mathscr{E}_X(0)_p$ *is a local ring with the residual field* $\mathbf{C}$.

(5) *A coherent* $\mathscr{E}_X$*-module is pseudo-coherent over* $\mathscr{E}_X(0)$.

§ 4. Variants of $\mathscr{E}$ (See [SKK], [Bj], [S])

4.1. We have defined the sheaf of rings $\mathscr{E}$. However we can introduce other sheaves of rings, similar to $\mathscr{E}$, which makes the theory transparent.

4.2. The sheet $\hat{\mathscr{E}} = \varprojlim_{m \in \mathbf{N}} \mathscr{E}/\mathscr{E}(-m)$ is called the sheaf of *formal micro-differential operators.* This is nothing but the sheaf similar to $\mathscr{E}$, obtained by dropping the growth condition (2.2.1).

4.3. We can define the sheaf $\mathscr{E}^{\infty}$ of micro-differential operators of infinite order ([SKK]). For an open $\Omega \subset \mathbf{C}^n$, we set

$$\Gamma(\Omega; \mathscr{E}^{\infty}) = \{(p_j)_{j \in \mathbf{Z}};\quad p_j \in \Gamma(\Omega; \mathscr{O}_{T^*X}(j))$$

satisfying the following conditions (4.3.1) and (4.3.2)}.

(4.3.1) For any compact set $K \subset \Omega$, there is a $C_K > 0$ such that $\sup_K |p_j| \leqslant C_K^{-j}(-j)!$ for $j < 0$.

(4.3.2) For any compact set $K \subset \Omega$ and any $\varepsilon > 0$, there exists a $C_{K,\varepsilon} > 0$ such that

$$\sup_K |p_j| \leqslant C_{K,\varepsilon} \frac{\varepsilon^j}{j!} \quad \text{for} \quad j \geqslant 1.$$

4.4. We can also define the sheaf $\mathscr{E}^{\mathbf{R}}$ on T^*X by $\mathscr{H}^n(\mu_\Delta(\mathscr{O}_{X \times X}^{(0,n)}))$. (See [KS] Chap. II, [SKK]). Here $n = \dim X$, $\mathscr{O}_{X \times X}^{(0,n)}$ is the sheaf of holomorphic forms on $X \times X$ which are n-forms with respect to the second variable, and μ_Δ is the micro-localization with respect to the diagonal set of $X \times X$ (See [SKK] Chap. II for the details).

4.5. We have $\mathscr{E}_X \subset \mathscr{E}_X^{\infty} \subset \mathscr{E}_X^{\mathbf{R}}$, $\mathscr{E}_X \subset \hat{\mathscr{E}}_X$. Moreover, $\mathscr{E}_X^{\infty}$, $\mathscr{E}_X^{\mathbf{R}}$ and $\hat{\mathscr{E}}_X$ are faithfully flat over $\mathscr{E}_X$. The sheaf $\hat{\mathscr{E}}_X$ is Noetherian. The sheaf $\mathscr{E}_X^{\mathbf{R}}$ contains $\mathscr{E}_X(\lambda)$'s compatible with the multiplication.

4.6. If we denote by γ the projection map $T^*X \to T^*X/\mathbf{C}^*$, then $R^j\gamma_*\mathscr{E}^{\mathbf{R}} = 0$ for $j \neq 0$ and $\mathscr{E}^{\infty} = \gamma^{-1}\gamma_*\mathscr{E}^{\mathbf{R}}$.

4.7. In [SKK], $\mathscr{E}$, $\hat{\mathscr{E}}$, and $\mathscr{E}^{\infty}$ are denoted by $\mathscr{P}^f$, $\hat{\mathscr{P}}$ and $\mathscr{P}$.

4.8. To explain the differences between $\mathscr{E}$, $\mathscr{E}^{\infty}$, $\mathscr{E}^{\mathbf{R}}$ and $\hat{\mathscr{E}}$, we shall take the following example. Let X be a complex manifold and Y a hypersurface of X. We shall take local coordinates $(x_1, \ldots, x_n)$ of X such that Y is given by $x_1 = 0$. The $\mathscr{D}_X$-module $\mathscr{D}_X/\mathscr{D}_X x_1 + \sum_{j>1} \mathscr{D}_X \partial_j$ is denoted by $\mathscr{B}_{Y|X}$. Set

$$\mathscr{C}_{Y|X} = \mathscr{E}_X \underset{\pi^{-1}\mathscr{D}_X}{\otimes} \mathscr{B}_{Y|X}, \hat{\mathscr{C}}_{Y|X} = \hat{\mathscr{E}}_X \underset{\mathscr{E}_X}{\otimes} \mathscr{C}_{Y|X},$$

$$\mathscr{C}^{\infty}_{Y|X} = \mathscr{E}^{\infty}_X \underset{\mathscr{E}_X}{\otimes} \mathscr{C}_{Y|X} \quad \text{and} \quad \mathscr{C}^{\mathbf{R}}_{Y|X} = \mathscr{E}^{\mathbf{R}}_X \underset{\mathscr{E}_X}{\otimes} \mathscr{C}_{Y|X}.$$

Then we have, setting $p = (0, dx_1)$, $x_0 = 0$

$$\begin{aligned}\mathscr{C}_{Y|X,p} &= \{a + b \log x_1 ; a \in \mathscr{O}_{X,x_0}[1/x_1], b \in \mathscr{O}_{X,x_0}\}/\mathscr{O}_{X,x_0} \\ &\cong (\mathscr{O}_{X,x_0}[1/x_1]/\mathscr{O}_{X,x_0}) \oplus \mathscr{O}_{X,x_0} \\ \hat{\mathscr{C}}_{Y|X,p} &= \{a + b \log x_1 ; a \in \mathscr{O}_{X,x_0}[1/x_1], b \in \hat{\mathscr{O}}_{X|Y,x_0}\}/\mathscr{O}_{X,x_0} \\ &\cong (\mathscr{O}_{X,x_0}[1/x_1]/\mathscr{O}_{X,x_0}) \oplus \hat{\mathscr{O}}_{X|Y,x_0}.\end{aligned}$$

Here $\hat{\mathscr{O}}_{X|Y} = \varprojlim \mathscr{O}_X/x_1^m\mathscr{O}_X$ is the sheaf of formal power series in the x_1-direction.

$$\mathscr{C}^{\infty}_{Y|X,p} = \{a + b \log x_1 ; a \in (j_*j^{-1}\mathscr{O}_X)_{x_0}, \quad b \in \mathscr{O}_{X,x_0}\}/\mathscr{O}_{X,x_0}$$

where j is the open embedding $X \backslash Y \hookrightarrow X$.

$$\mathscr{C}^{\mathbf{R}}_{Y|X,p} = \varinjlim_{U} \mathscr{O}(U)/\mathscr{O}_{X,x_0}.$$

Here U ranges over the set of open subsets of the form

$$\{x \in X ; |x| < \varepsilon, \operatorname{Re} x_1 < \varepsilon \operatorname{Im} x_1\}.$$

4.8. If we use $\mathscr{E}^{\infty}_X$, the structure of $\mathscr{E}$-modules becomes simpler. We just mention two theorems in this direction.

THEOREM 4.8.1 ([KK] Thm. 5.2.1). *Let $\mathscr{M}$ be a holonomic $\mathscr{E}_X$-module. Then there exists a (unique) regular holonomic $\mathscr{E}_X$-module $\mathscr{M}_{\text{reg}}$ such that*

$$\mathscr{E}^{\infty} \underset{\mathscr{E}_X}{\otimes} \mathscr{M} \cong \mathscr{E}^{\infty} \underset{\mathscr{E}_X}{\otimes} \mathscr{M}_{\text{reg}}.$$

THEOREM 4.8.2 ([SKK] Chap. II, Thm. 5.3.1). *Let X and Y be complex manifolds and let $T^*_Y Y$ be the zero section of T^*Y. If $\mathscr{M}$ is an $\mathscr{E}_{X\times Y}$-module whose support is contained in $T^*X \times T^*_Y Y$, then there exists a (locally) coherent $\mathscr{E}_X$-module $\mathscr{L}$ such that*

$$\mathscr{E}^{\infty}_{X\times Y} \underset{\mathscr{E}_{X\times Y}}{\otimes} \mathscr{M} \cong \mathscr{E}^{\infty}_{X\times Y} \underset{\mathscr{E}_{X\times Y}}{\otimes} (\mathscr{L} \hat{\otimes} \mathscr{O}_Y).$$

Here $\hat{\otimes}$ denotes the exterior tensor product. (See § 8).

§ 5. The Vanishing Cycle Sheaf

5.1. Let M be a real manifold and $f: M \to \mathbf{R}$ a continuous map. For a sheaf $\mathscr{F}$ on M, $\mathscr{H}^j_{f^{-1}(\mathbf{R}^+)}(\mathscr{F})|_{f^{-1}(0)}$ is called the (j-th) *vanishing cycle sheaf* of $\mathscr{F}$. Here $\mathbf{R}^+ = \{t \in \mathbf{R}; t \geqslant 0\}$. This measures how the cohomology groups of $\mathscr{F}$ change across the fibers of f. Its algebro-geometric version is studied by Grothendieck-Deligne ([D]).

5.2. Let $(X, \mathscr{O}_X)$ be a complex manifold. Let $f: X \to \mathbf{R}$ be a C^∞-map and consider the vanishing cycle sheaf $\mathscr{H}^j_{f^{-1}(\mathbf{R}^+)}(\mathscr{O}_X)|_{f^{-1}(0)}$. Let s be the section of $f^{-1}(0) \to T^*X$ given by df. Then we have

Proposition 5.2.1 ([KS1] § 3, [K2] § 4.2). *$\mathscr{H}^j_{f^{-1}(\mathbf{R}^+)}(\mathscr{O}_X)|_{f^{-1}(0)}$ has a structure of an $s^{-1}\mathscr{E}_X$-module.*

Let P be a differential operator. If $\sigma(P)$ does not vanish on $s(f^{-1}(0))$, then P has an inverse in $s^{-1}\mathscr{E}_X$ by Proposition 2.2.3. Therefore we obtain

Corollary 5.2.2. *If $\sigma(P)|_{sf^{-1}(0)} \neq 0$, then*

$$P: \mathscr{H}^j_{f^{-1}\mathbf{R}^+}(\mathscr{O}_X)|_{f^{-1}(0)} \to \mathscr{H}^j_{f^{-1}(\mathbf{R}^+)}(\mathscr{O}_X)|_{f^{-1}(0)}$$

is bijective.

5.3. More generally, let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, and set

$$\mathscr{F}^\cdot = \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X).$$

Then the preceding corollary shows that

$$\mathbf{R}\Gamma_{f^{-1}\mathbf{R}^+}(\mathscr{F}^\cdot)|_{f^{-1}(0)} = 0 \quad \text{if} \quad s(f^{-1}(0)) \cap \mathrm{Ch}(\mathscr{M}) = \emptyset.$$

Here $\mathrm{Ch}\mathscr{M}$ denotes the characteristic variety of $\mathscr{M}$.

5.4. To consider vanishing cycle sheaves is very near to the "microlocal" consideration. In this direction, see [K-S2].

§ 6. Micro-Differential Operators and the Symplectic Structure on the Cotangent Bundle

6.1. The ring $\mathscr{E}_X$ is a non-commutative ring. This fact gives rise to new phenomena which are not shared by commutative rings such as the ring of holomorphic functions. They are also closely related to the symplectic structure of the cotangent bundle.

6.2. Let us recall the symplectic structure on the cotangent bundle.

Let θ_X denote the canonical 1-form on the cotangent bundle T^*X of a complex manifold. Then $d\theta_X$ gives the symplectic structure on T^*X. The Hamiltonian map $H: T^*(T^*X) \tilde{\to} T(T^*X)$ is given by

(6.1.1) $\langle \eta, v \rangle = \langle d\theta_X, v \wedge H(\eta) \rangle$ for $\eta \in T^*(T^*X)$ and $v \in T(T^*X)$.

For a function f on T^*X, $H(df)$ is denoted by H_f and the Poisson bracket $\{f, g\}$ is defined as $H_f(g)$. If we denote by $\mathscr{X}$ the Euler vector field (i.e. the infinitesimal action of $\mathbf{C}^*$ on T^*X), then we have

$$\mathscr{X} = H(-\theta_X).$$

With a local coordinate system $(x_1, ..., x_n)$ of X and the associated local coordinate system $(x_1, ..., x_n; \xi_1, ..., \xi_n)$ of T^*X, we have

$$\theta_X = \sum \xi_j dx_j,$$

$$d\theta_X = \sum d\xi_j dx_j,$$

$$H: d\xi_j \mapsto \partial/\partial x_j,\ dx_j \mapsto -\partial/\partial \xi_j$$

$$\mathscr{X} = \sum \xi_j \frac{\partial}{\partial \xi_j},$$

$$\{f, g\} = \sum \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial \xi_j} \frac{\partial f}{\partial x_j} \right).$$

6.3. This structure is deeply related to the ring of micro-differential operators. The first relation between them appears in the following

PROPOSITION 6.3.1. *For* $P \in \mathscr{E}(\lambda)$ *and* $Q \in \mathscr{E}(\mu)$, *set*

$$[P, Q] = PQ - QP \in \mathscr{E}(\lambda + \mu - 1).$$

Then

$$\sigma_{\lambda+\mu-1}([P, Q]) = \{\sigma_\lambda(P), \sigma_\mu(Q)\}.$$

An analytic subset V of T^*X is called *involutive* if $f|_V = g|_V = 0$ implies $\{f, g\}|_V = 0$.

The following theorem exhibits a phenomenon which has no analogue in the commutative case.

THEOREM 6.3.2 ([G]). *Let* $\mathcal{M}$ *be a coherent* $\mathcal{E}_X$*-module defined on an open subset* Ω *of* $\mathring{T}^*X$ *and let* $\mathcal{L}$ *be a* $\mathcal{E}_X(0)|_\Omega$*-module which is a union of coherent* $\mathcal{E}_X(0)$*-modules. Then* $V = \{p \in \Omega;\, \mathcal{L}$ is not coherent over $\mathcal{E}_X(0)$ on any neighborhood of $p\}$ *is an involutive analytic subset of* Ω.

COROLLARY 6.3.3 ([SKK] Chap. II, Theorem 5.3.2, [M]). *For any coherent* $\mathcal{E}_X$*-module* $\mathcal{M}$, Supp $\mathcal{M}$ *is involutive.*

Since any involutive subset has codimension less than or equal to dim X, we have

COROLLARY 6.3.4. *The support of a coherent* $\mathcal{E}_X$*-module has codimension* $\leqslant$ dim X.

After some algebraic calculation, this implies

THEOREM 6.3.5 ([SKK] Chap. II, Theorem 5.3.5). *For any point* $p \in T^*X$, $\mathcal{E}_{X,p}$ *has a global cohomological dimension* dim X.

6.4. An analytic subset Λ of T^*X is called *Lagrangean* if Λ is involutive and dim Λ = dim X. A coherent $\mathcal{E}_X$-module is called *holonomic* if its support is Lagrangean.

§ 7. QUANTIZED CONTACT TRANSFORMATIONS

7.1. In the previous section, we saw that the symplectic structure of T^*X is closely related to micro-differential operators via the relation of commutator and Poisson bracket. In this section, we shall explain another relation.

Definition 7.2.1. Let X and Y be complex manifolds of the same dimension. A morphism φ from an open subset U of T^*X to T^*Y is called a *homogeneous symplectic transformation* if $\varphi^*\theta_Y = \theta_X$.

We can easily see the following

(7.2.1) If φ is a homogeneous symplectic transformation, then φ is a local isomorphism and is compatible with the action of $\mathbf{C}^*$.

(7.2.2) Assume $Y = \mathbf{C}^n$ and let $(y_1, ..., y_n; \eta_1, ..., \eta_n)$ be the coordinates of T^*Y, so that $\theta_Y = \sum \eta_j dy_j$.

Set $p_j = \eta_j \circ \varphi$ and $q_j = y_j \circ \varphi$. Then we have

(7.2.3.1) $\{p_j, p_k\} = \{q_j, q_k\} = 0, \{p_j, q_k\} = \delta_{j,k}$ for $j, k = 1, ..., n$.

(7.2.3.2) p_j is homogeneous of degree 1 and q_j is homogeneous of degree 0 with respect to the fiber coordinates.

(7.2.4) Conversely assume that functions $\{q_1, ..., q_n, p_1, ..., p_n\}$ on $U \subset T^*X$ satisfy (7.2.3.1) and (7.2.3.2). Then the map $\varphi: U \to T^*Y$, given by

$$U \ni x \mapsto (q_1(x), ..., q_n(x); p_1(x), ..., p_n(x)) \in T^*Y ,$$

is a homogeneous symplectic transformation. We call $(q_1, ..., q_n; p_1, ..., p_n)$ a *homogeneous symplectic coordinate system.*

THEOREM 7.2.2 ([SKK] Chap. II § 3.2, [K2] § 2.4, [Bj] Chap. 4 § 6). *Let* $\varphi: T^*X \supset U \to T^*Y$ *be a homogeneous symplectic transformation, let* p_X *be a point of* U *and set* $p_Y = \varphi(p_X)$. *Then we have*

(a) *There exists an open neighborhood* U' *of* p_X *and a* **C**-*algebra isomorphism* $\Phi: \varphi^{-1}\mathscr{E}_Y|_{U'} \xrightarrow{\sim} \mathscr{E}_X|_{U'}$ *(we call* (φ, Φ) *a quantized contact transformation).*

(b) *If* $\Phi: \varphi^{-1}\mathscr{E}_Y \to \mathscr{E}_X|_U$ *is a* **C**-*algebra homomorphism then for any* m, Φ *gives an isomorphism* $\varphi^{-1}\mathscr{E}_Y(m) \xrightarrow{\sim} \mathscr{E}_X(m)|_U$. *Moreover the following diagram commutes:*

$$\begin{array}{ccc} \varphi^{-1}\mathscr{E}_Y(m) & \xrightarrow{\Phi} & \mathscr{E}_X(m)|_U \\ \downarrow \sigma_m & & \downarrow \sigma_m \\ \varphi^{-1}\mathscr{O}_{T^*Y}(m) & \xrightarrow{\varphi^*} & \mathscr{O}_{T^*Y}(m)|_U \end{array}$$

(c) *Let* Φ *and* Φ' *be two* **C**-*algebra homomorphisms* $\varphi^{-1}\mathscr{E}_Y \to \mathscr{E}_X|_U$.

Then there exist $\lambda \in \mathbf{C}$, *a neighborhood* U' *of* p_X *and* $P \in \Gamma(U; \mathscr{E}_X(\lambda))$ *such that* $\sigma_\lambda(P)$ *is invertible and*

$$\Phi'(Q) = P\Phi(Q)P^{-1} \quad \textit{for} \quad Q \in \varphi^{-1}\mathscr{E}_Y|_{U'}.$$

Moreover λ *is unique and* P *is unique up to constant multiple.*

(d) *Let* $Y = \mathbf{C}^n$ *and let* U *be an open subset of* T^*X.

If $P_j \in \Gamma(U; \mathscr{E}_X(1))$ *and* $Q_j \in \Gamma(U; \mathscr{E}_X(0))$ $(1 \leqslant j \leqslant n)$ *satisfy*

(7.2.5)
$$[P_j, P_k] = [Q_j, Q_k] = 0$$
$$[P_j, Q_k] = \delta_{jk}$$

then there exists a unique quantized contact transformation (φ, Φ) *such that*

$$\varphi(p) = (\sigma_0(Q_1)(p), \dots, \sigma_0(Q_n)(p), \sigma_1(P_1)(p), \dots, \sigma_1(P_n)(p)),$$

and $\Phi(y_j) = Q_j, \Phi(\partial_{y_j}) = P_j$.

We call $\{Q_1, \dots, Q_n, P_1, \dots, P_n\}$ quantized canonical coordinates.

7.3. We shall give several examples of quantized contact transformations.

Example 7.3.1. If $P(\partial)$ is a constant coefficient micro-differential operator of order 1, then

$$(x_1 + [P, x_1], x_2 + [P, x_2], \dots, x_n + [P, x_n], \partial_{x_1}, \dots, \partial_{x_n})$$

gives quantized canonical coordinates.

Example 7.3.2. More generally if P is a micro-differential operator of order 1 and $\exp tH_{\sigma_1(P)}$ exists, then $\exp tP$ gives a quantized contact transformation Φ_t, by solving the equation $\frac{d}{dt}\Phi_t(Q) = [P, \Phi_t(Q)]$ with the initial condition $\Phi_t(Q) = Q$ for $t = 0$.

Example 7.3.3. (Paraboloidal transformation [K2] p. 36). Set $X = \mathbf{C}^{1+n} = \{(t, x) \in \mathbf{C} \times \mathbf{C}^n\}$,

$$\Omega = \{(t, x; \tau, \xi) \in T^*X; \tau \neq 0\}, G = \mathrm{Sp}(n; \mathbf{C})$$
$$= \{g \in \mathrm{GL}(2n; \mathbf{C}); {}^t gJg = J\} \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$, let Ψ_g be the quantized contact transformation given by

$$\partial_x \mapsto \alpha\partial_x - \beta x\partial_t$$

$$x \mapsto \gamma\partial_x\partial_t^{-1} + \delta x$$

$$\partial_t \mapsto \partial_t$$

$$t \mapsto t + \frac{1}{2}\{<\partial_x, {}^t\gamma\alpha\partial_x> \partial_t^{-2} + <\partial_x, {}^t\gamma\beta x> \partial_t^{-1}$$

$$+ <{}^t\gamma\beta x, \partial_x> \partial_t^{-1} + <x, {}^t\delta\beta x>\}.$$

Then we have $\Psi_{g_1}\Psi_{g_2} = \Psi_{g_1 g_2}$.

§ 8. Functorial Properties of Micro-Differential Modules (See [SKK])

8.1. External Tensor Product.

Let X and Y be complex manifolds and let p_1 and p_2 be the projections $T^*(X \times Y) \to T^*X$ and $T^*(X \times Y) \to T^*Y$, respectively. Then $\mathscr{E}_{X \times Y}$ contains $p_1^{-1}\mathscr{E}_X \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathscr{E}_Y$ as a subring. For an $\mathscr{E}_X$-module $\mathscr{M}$ and an $\mathscr{E}_Y$-module $\mathscr{N}$, we define the $\mathscr{E}_{X \times Y}$-module $\mathscr{M} \hat{\otimes} \mathscr{N}$ by

$$\mathscr{M} \hat{\otimes} \mathscr{N} = \mathscr{E}_{X \times Y} \underset{p_1^{-1}\mathscr{E}_X \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathscr{E}_Y}{\otimes} (p_1^{-1}\mathscr{M} \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathscr{N}). \tag{8.1.1}$$

Then one can easily see

Proposition 8.1.1.

(i) $\mathscr{M} \hat{\otimes} \mathscr{N}$ *is an exact functor in* $\mathscr{M}$ *and in* $\mathscr{N}$ *and* $\operatorname{Supp}(\mathscr{M} \hat{\otimes} \mathscr{N}) = \operatorname{Supp} \mathscr{M} \times \operatorname{Supp} \mathscr{N}$.

(ii) *If* $\mathscr{M}$ *is* $\mathscr{E}_X$*-coherent and* $\mathscr{N}$ *is* $\mathscr{E}_Y$*-coherent, then* $\mathscr{M} \hat{\otimes} \mathscr{N}$ *is* $\mathscr{E}_{X \times Y}$*-coherent.*

8.2. For a complex submanifold Y of a complex manifold X of codimension l, the sheaf $\varinjlim_m \mathscr{E}xt^l_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{J}^m, \mathscr{O}_X)$ has a natural structure of $\mathscr{D}_X$-module, which is denoted by $\mathscr{B}_{Y|X}$. Here $\mathscr{J}$ is the defining ideal of Y. The homomorphism $\mathscr{O}_Y \to \mathscr{E}xt^l_{\mathscr{O}_X}(\mathscr{O}_Y, \Omega^l_X) \to \Omega^l_X \underset{\mathscr{O}_X}{\otimes} \mathscr{B}_{Y|X}$ gives the canonical section $c(Y, X)$ of $\Omega^l_X \underset{\mathscr{O}_X}{\otimes} \mathscr{B}_{Y|X}$. If we take local coordinates $(x_1, \ldots, x_n)$ of X such that Y is defined by $x_1 = \ldots = x_l = 0$, then we have

$$\mathscr{B}_{Y|X} \cong \mathscr{D}_X / \sum_{j \leq l} \mathscr{D}_X x_j + \sum_{j > l} \mathscr{D}_X \partial_j .$$

If we denote by δ the canonical generator of the left hand side, then $c(Y, X)$ corresponds to $dx_1 \wedge \ldots \wedge dx_l \otimes \delta$. We set

$$\mathscr{C}_{Y|X} = \mathscr{E}_X \underset{\pi^{-1}\mathscr{D}_X}{\otimes} \pi^{-1}\mathscr{B}_{Y|X} .$$

Therefore locally we have

$$\mathscr{C}_{Y|X} \cong \mathscr{E}_X / \sum_{j \leq d} \mathscr{E}_X x_j + \sum_{j > d} \mathscr{E}_X \partial_j .$$

Then $\mathscr{C}_{Y|X}$ is a coherent $\mathscr{E}_X$-module whose support is $T^*_Y X$.

8.3. For an invertible $\mathscr{O}_X$-module $\mathscr{L}$, $\mathscr{L} \underset{\mathscr{O}_X}{\otimes} \mathscr{E}_X \underset{\mathscr{O}_X}{\otimes} \mathscr{L}^{\otimes -1}$ has a natural structure of sheaves of rings, by the composition rule

$$(s \otimes P \otimes s^{\otimes -1}) \circ (s \otimes Q \otimes s^{\otimes -1}) = s \otimes PQ \otimes s^{\otimes -1}$$

for an invertible section s of $\mathscr{L}$ and $P, Q \in \mathscr{E}_X$.

Then the category $\mathrm{Mod}(\mathscr{E}_X)$ of left $\mathscr{E}_X$-modules and the category $\mathrm{Mod}(\mathscr{L} \underset{\mathscr{O}_X}{\otimes} \mathscr{E}_X \underset{\mathscr{O}_X}{\otimes} \mathscr{L}^{\otimes -1})$ of left $(\mathscr{L} \underset{\mathscr{O}_X}{\otimes} \mathscr{E}_X \underset{\mathscr{O}_X}{\otimes} \mathscr{L}^{\otimes -1})$-modules are equivalent by the functor

$$\mathrm{Mod}(\mathscr{E}_X) \ni \mathscr{M} \mapsto \mathscr{L} \underset{\mathscr{O}_X}{\otimes} \mathscr{M} \in \mathrm{Mod}(\mathscr{L} \underset{\mathscr{O}_X}{\otimes} \mathscr{E}_X \underset{\mathscr{O}_X}{\otimes} \mathscr{L}^{\otimes -1}) .$$

8.4. Let ω_X be the canonical sheaf on X, i.e. the sheaf of differential forms with top degree. Let a be the antipodal map of T^*X, i.e. the multiplication by -1. Then we have the anti-ring isomorphism.

(8.4.1) $$\omega_X \underset{\mathscr{O}_X}{\otimes} \mathscr{E}_X \underset{\mathscr{O}_X}{\otimes} \omega_X^{\otimes -1} \xrightarrow{\sim} a^{-1}\mathscr{E}_X .$$

This homomorphism is given by using a local coordinate system $(x_1, \ldots, x_n)$ as follows. For $P = \sum P_j(x, \partial) \in \mathscr{E}_X$ we define $P^* = \sum P_j^*(x, \partial)$, called the formal adjoint of P ([SKK] Chap. II, Th. 1.5.1), by

(8.4.2) $$P_l^*(x, -\xi) = \sum_{\substack{j = l - |\alpha| \\ \alpha \in \mathbf{N}^n}} \frac{(-1)^{|\alpha|}}{\alpha !} \partial_\xi^\alpha \partial_x^\alpha P_j(x, \xi) .$$

This is well-defined and satisfies

(8.4.3) $$(P^*)^* = P$$

(8.4.4) $$(PQ)^* = Q^*P^* .$$

Then the isomorphism (8.4.1) is given by

(8.4.5) $$dx \otimes P \otimes (dx)^{\otimes -1} \mapsto P^*$$

where $dx = dx_1 \wedge \ldots \wedge dx_n \in \omega_X$. This is independent of coordinate transformations.

8.5. The isomorphism (8.4.1) can be explained as follows. Let Δ_X be the diagonal set of $X \times X$, and let p_j be the j-th projection from $T^*_{\Delta_X}(X \times X)$ to T^*X for $j = 1, 2$. Then the p_j are isomorphisms and $p_2 \circ p_1^{-1} = a$. Let q_j be the j-th projection from $T^*(X \times X)$ to $X (j = 1, 2)$. Then $c(\Delta_X, X \times X)$ gives the canonical section of $q_2^{-1}\omega_X \underset{q_2^{-1}\mathcal{O}_X}{\otimes} \mathscr{C}_{\Delta_X|X \times X}$. Since $\mathscr{C}_{\Delta_X|X \times X}$ is a $p_1^{-1}\mathscr{E}_X$-module, this section gives a homomorphism

$$p_1^{-1}\mathscr{E}_X \to q_2^{-1}\omega_X \underset{q_2^{-1}\mathcal{O}_X}{\otimes} \mathscr{C}_{\Delta_X|X \times X} .$$

It turns out that this is an isomorphism and the right multiplication of $\mathcal{O}_X$ on $\mathscr{E}_X$ corresponds to the $\mathcal{O}_X$-module structure of $q_2^{-1}\omega_X \underset{q_2^{-1}\mathcal{O}_X}{\otimes} \mathscr{C}_{\Delta_X|X \times X}$ via q_2. Thus we obtain

$$p_1^{-1}(\omega_X \underset{\mathcal{O}_X}{\otimes} \mathscr{E}_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1}) \xrightarrow{\sim} q_1^{-1}\omega_X \underset{q_1^{-1}\mathcal{O}_X}{\otimes} \mathscr{C}_{\Delta_X|X \times X} .$$

This last being isomorphic to $p_2^{-1}\mathscr{E}_X$, we obtain

$$\omega_X \underset{\mathcal{O}_X}{\otimes} \mathscr{E}_X \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1} \xrightarrow{\sim} p_1 p_2^{-1}\mathscr{E}_X \simeq a^{-1}\mathscr{E}_X .$$

8.6. By 8.3 and 8.4, if $\mathscr{M}$ is a left $\mathscr{E}_{X|U}$-module for an open set U of T^*X, then $\omega_X \underset{\mathcal{O}_X}{\otimes} a^{-1}\mathscr{M}$ is a right $(\mathscr{E}_{X|aU})$-module.

8.7. For a left coherent $\mathscr{E}_X$-module $\mathscr{M}$, $\mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{E}_X)$ is a right coherent $\mathscr{E}_X$-module. Therefore $\mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{E}_X) \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1}$ is a left $\mathscr{E}_X$-module by § 8.6. If $\mathscr{M}$ is holonomic then $\mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{E}_X) = 0$ for $j \neq n = \dim X$ (See [SKK], [Kl]). Set $\mathscr{M}^* = \mathscr{E}xt^n_{\mathscr{E}_X}(\mathscr{M}, \mathscr{E}_X) \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1}$. Then $\mathscr{M}^*$ is also a holonomic $\mathscr{E}_X$-module.

We call $\mathscr{M}^*$ the *dual system* of $\mathscr{M}$. We have $\mathscr{M}^{**} = \mathscr{M}$, and $\mathscr{M} \mapsto \mathscr{M}^*$ is an exact contravariant functor on the category of holonomic $\mathscr{E}_X$-modules.

8.8. Let X and Y be complex manifolds, and let $p_1\colon T^*(X \times Y) \to T^*X$ and $p_2\colon T^*(X \times Y) \to T^*Y$ be the canonical projections. Let p_2^a denote $p_2 \circ a$. Let $\mathscr{K}$ be a left $\mathscr{E}_{X \times Y}$-module defined on an open subset Ω of $T^*(X \times Y)$. Then, by § 8.6, $\omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{K}$ has a structure of $(p_1^{-1}\mathscr{E}_X, p_2^{a-1}\mathscr{E}_Y)$-bi-module. For an $\mathscr{E}_Y$-module $\mathscr{N}$,

$$\mathscr{M} = p_{1*}((\omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{K}) \underset{p_2^{a-1}\mathscr{E}_Y}{\otimes} p_2^{a-1}\mathscr{N})$$

has a structure of $\mathscr{E}_X$-module. We have the following

THEOREM 8.8.1. *Let* Ω, U_X *and* U_Y *be open subsets of* $T^*(X \times Y)$, T^*X *and* T^*Y, *respectively. Let* $\mathscr{K}$ *be a coherent* $(\mathscr{E}_{X \times Y}|_\Omega)$*-module and* $\mathscr{N}$ *a coherent* $(\mathscr{E}_Y|_{U_Y})$*-module. Assume*

(i) $p_1\colon p_1^{-1}U_X \cap \operatorname{Supp}\mathscr{K} \cap p_2^{a-1}\operatorname{Supp}\mathscr{N} \to U_X$ *is a finite morphism.*
Then we have

(a) $\mathscr{T}or_j^{p_2^{a-1}\mathscr{E}_Y}(\omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{K}, p_2^{a-1}\mathscr{N})|_{p_1^{-1}U_X} = 0$ *for* $j \neq 0$.

(b) $\mathscr{M} = p_{1*}((\omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{K}) \underset{p_2^{a-1}\mathscr{E}_Y}{\otimes} p_2^{a-1}\mathscr{N})|_{U_X}$ is a coherent $\mathscr{E}_X$-module.

(c) $\operatorname{Supp}\mathscr{M} = U_X \cap p_1(\operatorname{Supp}\mathscr{K} \cap p_2^{a-1}\operatorname{Supp}\mathscr{N})$.

We denote $p_{1*}((\omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{K}) \underset{p_2^{a-1}\mathscr{E}_Y}{\otimes} p_2^{a-1}\mathscr{N})$ by $\int_Y \mathscr{K} \circ \mathscr{N}$.

8.9. Let $f\colon X \to Y$ be a holomorphic map and let Δ_f be the graph of f, i.e. $\{(x, f(x)) \in X \times Y; x \in X\}$, then $\mathscr{K} = \mathscr{C}_{\Delta_f|X \times Y}$ is a coherent $\mathscr{E}_{X \times Y}$-module whose support is $T^*_{\Delta_f}(X \times Y)$. Now let $\tilde{\omega}$ be the canonical map $X \underset{Y}{\times} T^*Y \to T^*X$ and ρ the projection $X \underset{Y}{\times} T^*Y$. Then we have the following diagram

$$\begin{array}{ccccc} T^*X & \overset{\tilde{\omega}_f}{\leftarrow} & X \underset{Y}{\times} T^*Y & \overset{\rho_f}{\rightarrow} & T^*Y \\ \text{id} \parallel & & \wr & & \parallel \text{id}. \\ T^*X & \underset{p_1}{\leftarrow} & T^*_{\Delta_f}(X \times Y) & \underset{p_2}{\overset{a}{\rightarrow}} & T^*Y \end{array} \tag{8.9.1}$$

We set $\mathscr{E}_{X\to Y} = \omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{C}_{\Delta_f|X\times Y}$ and consider this as a sheaf on $X \underset{Y}{\times} T^*Y$ by the above isomorphism. Then $\mathscr{E}_{X\to Y}$ is a $(\tilde{\omega}^{-1}\mathscr{E}_X,\ \rho^{-1}\mathscr{E}_Y)$-bi-module. For an $\mathscr{E}_Y$-module $\mathscr{N}$,

$$\int \mathscr{K} \circ \mathscr{N} = \mathbf{R}\tilde{\omega}_*\rho^{-1}(\mathscr{E}_{X\to Y} \underset{\rho^{-1}\mathscr{E}_Y}{\otimes} \rho^{-1}\mathscr{N}).$$

We shall denote this by $f^*\mathscr{N}$ and call it the pull-back of $\mathscr{N}$. Then Theorem 8.8.1 reads as follows.

THEOREM 8.9.1. *Let* U_X *and* U_Y *be open subsets of* T^*X *and* T^*Y, *respectively. Let* $\mathscr{N}$ *be a coherent* $(\mathscr{E}_Y|_U)$*-module. Assume*

(i) $\rho_f^{-1}(\mathrm{Supp}\,\mathscr{N}) \cap \tilde{\omega}_f^{-1}(U_X) \to U_X$ *is a finite morphism.*

Then we have

(a) $\mathscr{T}or_j^{\rho_f^{-1}\mathscr{E}_Y}(\mathscr{E}_{X\to Y}, \mathscr{N}) = 0$ *for* $j \neq 0$.

(b) $\mathscr{M} = \tilde{\omega}_{f*}(\mathscr{E}_{X\to Y} \underset{\rho_f^{-1}\mathscr{E}_Y}{\otimes} \rho_f^{-1}\mathscr{N})|_{U_X}$ *is a coherent* $\mathscr{E}_X$*-module.*

(c) $\mathrm{Supp}\, M = \tilde{\omega}_f\rho_f^{-1}\,\mathrm{Supp}\,\mathscr{N} \cap U_X$.

8.10. Similarly let $g\colon Y \to X$ be a holomorphic map and let Δ_g be the graph of g, i.e. $\{(g(y), y) \in X \times Y;\, y \in Y\}$. Then we have the isomorphisms

$$\begin{array}{ccccc} T^*X & \overset{\rho_g}{\leftarrow} & Y \times T^*X & \overset{\tilde{\omega}_g}{\rightarrow} & T^*Y \\ \| & & \wr\! \| & & \|\ \mathrm{id}\,. \\ T^*X & \underset{p_1}{\leftarrow} & T^*_{\Delta_g}(X\times Y) & \underset{p_2}{\overset{a}{\rightarrow}} & T^*Y \end{array} \tag{8.10.1}$$

We set $\mathscr{E}_{X\leftarrow Y} = \omega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{C}_{\Delta_g|X\times Y}$ and regard this as a sheaf on $Y \underset{X}{\times} T^*X$. Then $\mathscr{E}_{X\leftarrow Y}$ is a $(\rho^{-1}\mathscr{E}_X, \tilde{\omega}^{-1}\mathscr{E}_Y)$-bi-module. For an $\mathscr{E}_Y$-module $\mathscr{N}$ we have

$$\int \mathscr{C}_{\Delta_g|X\times Y} \circ \mathscr{N} = \mathbf{R}\rho_*\tilde{\omega}^{-1}(\mathscr{E}_{X\leftarrow Y} \underset{\tilde{\omega}^{-1}\mathscr{E}_Y}{\otimes} \tilde{\omega}^{-1}\mathscr{N}).$$

We shall denote this by $\int_g \mathscr{N}$. Then Theorem 8.8.1 applies to this case and we have

THEOREM 8.10.1. *Let* U_X *and* U_Y *be open subsets of* T^*X *and* T^*Y, *respectively. Let* $\mathscr{N}$ *be a coherent* $(\mathscr{E}_Y|_{U_Y})$*-module. Assume*

(i) $\rho_g\colon \tilde{\omega}_g^{-1}(\operatorname{Supp}\mathscr{N}) \cap \rho_g^{-1}(U_X) \to U_X$ *is a finite morphism.*

Then we have

(a) $\mathscr{T}or_j^{\tilde{\omega}_g^{-1}\mathscr{E}_Y}(\mathscr{E}_{X\leftarrow Y}, \tilde{\omega}_g^{-1}\mathscr{N}) = 0$ for $j \neq 0$.

(b) $\mathscr{M} = \rho_{g*}(\mathscr{E}_{X\leftarrow Y} \underset{\tilde{\omega}_g^{-1}\mathscr{E}_Y}{\otimes} \tilde{\omega}_g^{-1}\mathscr{N})|_{U_X}$ *is a coherent* $\mathscr{E}_X|_{U_X}$*-module.*

(c) $\operatorname{Supp}\mathscr{M} = \rho_g(\tilde{\omega}_g^{-1}\operatorname{Supp}\mathscr{N} \cap U_X)$.

§ 9. REGULARITY CONDITIONS (See [KK], [K-O])

9.1. Let us recall the notion of regular singularity of ordinary differential equations. Let $P(x, \partial) = \sum_{j\leq m} a_j(x)\partial^j$ be a linear differential operator in one variable x. We assume that the $a_j(x)$ are holomorphic on a neighborhood of $x = 0$. Then we say that the origin 0 is a regular singularity of $Pu = 0$ if

$$(*) \qquad \operatorname{ord}_{x=0} a_j(x) \geq \operatorname{ord}_{x=0} a_m(x) - (m-j)\,.$$

Here $\operatorname{ord}_{x=0}$ means the order of the zero. In this case, the local structure of the equation is very simple. In fact, the $\mathscr{D}_X$-module $\mathscr{D}_X/\mathscr{D}_X P$ is a direct sum of copies of the following modules:

$$\mathscr{O}_X = \mathscr{D}_X/\mathscr{D}_X\partial,\ \mathscr{B}_{\{0\}|X} = \mathscr{D}_X/\mathscr{D}_X x,\ \mathscr{D}_X/\mathscr{D}_X(x\partial-\lambda)^{m+1} \quad (\lambda\in\mathbf{C},\ m\in\mathbf{N}),$$
$$\mathscr{D}_X/\mathscr{D}_X(x\partial)^{m+1}x \quad (m\in\mathbf{N}),\ \mathscr{D}_X/\mathscr{D}_X\partial(x\partial)^{m+1} \quad (m\in\mathbf{N})\,.$$

If we denote by u the canonical generator, then we have $Pu = 0$. By multiplying either a power of ∂ or a power of x, we obtain

$$\sum_{j=0}^{N} b_j(x)\,(x\partial)^j u = 0$$

with $b_N(x) = 1$. Hence $\mathscr{F} = \sum_{j=0}^{\infty} \mathscr{O}(x\partial)^j u = \sum_{j=0}^{N-1} \mathscr{O}(x\partial)^j u$ is a coherent $\mathscr{O}$-submodule of $\mathscr{M}$ which satisfies $(x\partial)\mathscr{F} \subset \mathscr{F}$. We shall generalize this property to the case of several variables.

9.2. Let X be a complex manifold, Ω an open subset of $\mathring{T}^*X$ and V a closed involutive complex submanifold of Ω. Let us define

$$\mathscr{J}_V = \{u \in \mathscr{E}(1)|_\Omega;\ \sigma_1(P)|_V = 0\}$$

and let $\mathscr{E}_V$ be the subring of $\mathscr{E}_X|_\Omega$ generated by $\mathscr{J}_V$. For a coherent $\mathscr{E}_X$-module $\mathscr{M}$, a coherent sub-$\mathscr{E}_X(0)$-module $\mathscr{L}$ of $\mathscr{M}$ is called a *lattice* of $\mathscr{M}$ if $\mathscr{M} = \mathscr{E}_X\mathscr{L}$. The following proposition is easily derived from the fact that $\mathscr{E}(0)$ is a Noetherian ring.

PROPOSITION 9.2.1 ([K-O] Theorem 1.4.7). *Let $\mathscr{M}$ be a coherent $\mathscr{E}_X|_\Omega$-module. Then the following conditions are equivalent.*

(1) *For any point $p \in \Omega$, there is a lattice $\mathscr{M}_0$ of $\mathscr{M}$ on a neighborhood of p such that $\mathscr{J}_V\mathscr{M}_0 = \mathscr{M}_0$.*

(2) *For any open subset U of Ω and for any coherent $\mathscr{E}(0)$-submodule $\mathscr{L}$ of $\mathscr{M}|_U$, $\mathscr{E}_V\mathscr{L}$ is coherent over $\mathscr{E}(0)|_U$.*

Definition 9.2.2. If the equivalent conditions of the preceding proposition are satisfied, then we say that $\mathscr{M}$ has *regular singularities* along V.

Remark that if $\mathscr{M}$ has regular singularity along V, then the support of $\mathscr{M}$ is contained in V. Let us denote by $IR_V(\mathscr{M})$ the set of points p such that $\mathscr{M}$ has no regular singularities along V on any neighborhood of p.

The following theorem is an immediate consequence of Gabber's Theorem 6.3.2.

THEOREM 9.2.3. *$IR_V(\mathscr{M})$ is an involutive analytic subset of $\mathscr{M}$.*

In fact, if we take a lattice $\mathscr{M}_0$ of $\mathscr{M}$, then $T^*X \backslash IR_V(\mathscr{M})$ is the largest open subset on which $\mathscr{E}_V\mathscr{M}_0$ is coherent over $\mathscr{E}(0)$.

9.3. If an $\mathscr{E}$-module $\mathscr{M}$ has regular singularities along an involutive submanifold V then $\mathscr{M}$ is, roughly speaking, constant along the bicharacteristics of V. More precisely, let Y and Z be complex manifolds and $X = Y \times Z$. Let $z_0 \in Z$ and let j be the inclusion map $Y \hookrightarrow X$ by $y \mapsto (y, z_0)$. Then we have

THEOREM 9.3.1. *Let $\mathscr{M}$ be a coherent $\mathscr{E}_X$-module. Assume that $\mathscr{M}$ has regular singularities along $T^*Y \times T^*_ZZ$. Then $\mathscr{M}$ is isomorphic to $j^*\mathscr{M} \hat{\otimes} \mathscr{O}_Z$.*

Note that any involutive submanifold V of T^*X with $\theta_X|_V \neq 0$ is transformed by a homogeneous symplectic transformation to the form $T^*Y \times T^*_ZZ$.

9.4. Noting that any nowhere dense closed analytic subset of a Lagrangean variety is never involutive, Theorem 9.2.3 implies the following theorem.

THEOREM 9.4.1. *Let* $\mathcal{M}$ *be a holonomic* $\mathscr{E}_X$*-module. Then the following conditions are equivalent.*

(i) *There exists a Lagrangean subvariety* Λ *such that* $\mathcal{M}$ *has regular singularities along* Λ.

(ii) *For any involutive subvariety* Λ *which contains* $\operatorname{Supp}\mathcal{M}$, $\mathcal{M}$ *has regular singularities along* Λ.

(iii) *There exists an open dense subset* Ω *of* $\operatorname{Supp}\mathcal{M}$ *such that* $\mathcal{M}$ *has regular singularities along* $\operatorname{Supp}\mathcal{M}$ *on* Ω.

If these equivalent conditions are satisfied, we say that $\mathcal{M}$ is a *regular holonomic* $\mathscr{E}_X$-module.

The following properties are almost immediate.

THEOREM 9.4.2.

(i) *Let* $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ *be an exact sequence of three coherent* $\mathscr{E}_X$*-modules. If two of them are regular holonomic then so is the third.*

(ii) *If* $\mathcal{M}$ *is regular holonomic, its dual* $\mathcal{M}^*$ *is also regular holonomic.*

We just mention another analytic property of regular holonomic modules, which generalizes the fact that a formal solution of an ordinary differential equation with regular singularity converges.

THEOREM 9.4.3 ([KK] Theorem 6.1.3). *If* $\mathcal{M}$ *and* $\mathcal{N}$ *are regular holonomic* $\mathscr{E}_X$*-modules, then* $\mathscr{E}xt^j_{\mathscr{E}_X}(\mathcal{M}, \mathcal{N}) \to \mathscr{E}xt^j_{\mathscr{E}_X}(\mathcal{M}, \hat{\mathscr{E}}_X \underset{\mathscr{E}_X}{\otimes} \mathcal{N})$ *and* $\mathscr{E}xt^j_{\mathscr{E}_X}(\mathcal{M}, \mathcal{N}) \to \mathscr{E}xt^j_{\mathscr{E}_X}(\mathcal{M}, \mathscr{E}^\infty_X \underset{\mathscr{E}_X}{\otimes} \mathcal{N})$ *are isomorphisms.*

§ 10. STRUCTURE OF REGULAR HOLONOMIC $\mathscr{E}$-MODULES (See [SKK], [KK])

10.1. Let Λ be a Lagrangean submanifold of T^*X. We define $\mathscr{I}_\Lambda$ and $\mathscr{E}_\Lambda$ as in § 9.2.

Then $\mathscr{E}_\Lambda(-1) = \mathscr{E}_\Lambda \cdot \mathscr{E}(-1)$ is a two-sided ideal of $\mathscr{E}_\Lambda$ and $\mathscr{E}_\Lambda/\mathscr{E}_\Lambda(-1)$ is a sheaf of rings which contains $\mathcal{O}_\Lambda(0) = \mathscr{E}(0)/\mathscr{I}_\Lambda(-1)$, the sheaf of homogeneous functions on Λ.

Let us take an invertible $\mathcal{O}_\Lambda$-module $\mathcal{L}$ such that $\mathcal{L}^{\otimes 2} \cong \omega_\Lambda \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1}$. Such an $\mathcal{L}$ exists at least locally. For $P = P_1(x, \partial) + P_0(x, \partial) + \ldots \in \mathcal{J}$ we define, for $\varphi \in \mathcal{O}_\Lambda$ and an invertible section s of $\mathcal{L}$,

$$L(P)(\varphi s) = \left\{ H_{P_1}(\varphi) + \frac{1}{2}\varphi \frac{L_{H_{P_1}}(s^{\otimes 2} \otimes dx)}{s^{\otimes 2} \otimes dx} + \left(P_0 - \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2 P_1}{\partial x_i \partial \xi_i}\right)\varphi \right\} s .$$

Here $dx = dx_1 \wedge \ldots \wedge dx_n \in \omega_X$ and $s^{\otimes 2} \otimes dx$ is regarded as a section of ω_Λ. The Lie derivative $L_{H_{P_1}}$ of H_{P_1} operates on ω_Λ as the first order differential operators so that $L_{H_{P_1}}(s^{\otimes 2} \otimes dx)$ is a section of ω_Λ and $L_{H_{P_1}}(s^{\otimes 2} \otimes dx)/s^{\otimes 2} \otimes dx$ is a function on Λ.

We thus obtain $L\colon \mathcal{J}_\Lambda \to \mathcal{E}nd_{\mathbf{C}}(\mathcal{L})$. Then this does not depend on the choice of local coordinate system and moreover it extends to the ring homomorphism $L\colon \mathcal{E}_\Lambda \to \mathcal{E}nd_{\mathbf{C}}(\mathcal{L})$. Since the image is contained in the differential endomorphism of $\mathcal{L}$, we obtain the ring homomorphism $L\colon \mathcal{E}_\Lambda \to \mathcal{L} \underset{\mathcal{O}_\Lambda}{\otimes} \mathcal{D}_\Lambda \underset{\mathcal{O}_\Lambda}{\otimes} \mathcal{L}^{\otimes -1}$.

PROPOSITION 10.1.1. *By* L, $\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)$ *coincides with the subsheaf of* $\mathcal{L} \underset{\mathcal{O}_\Lambda}{\otimes} \mathcal{D}_\Lambda \underset{\mathcal{O}_\Lambda}{\otimes} \mathcal{L}^{\otimes -1}$ *consisting of differential endomorphisms of* $\mathcal{L}$ *homogeneous of degree* 0.

If we take

$$\mathcal{J}_\Lambda \ni \vartheta = \vartheta_1(x, \partial) + \vartheta_0(x, \partial) + \ldots$$

such that $d\vartheta_1 \equiv -\theta_X \bmod I_\Lambda \Omega^1$ and

$$\frac{1}{2}\sum \frac{\partial^2 \vartheta_1}{\partial x_i \partial \xi_i} \equiv \vartheta_0(x, \xi) \bmod \mathcal{J}_\Lambda$$

then $L(\vartheta)$ gives the Euler operator of $\mathcal{L}$. Such a ϑ is unique modulo $\mathcal{J}_\Lambda^2(-1) = \mathcal{E}_\Lambda(-1) \cap \mathcal{E}_X(1)$.

10.2. Let $\mathcal{M}$ be a regular holonomic $\mathcal{E}_X$-module whose support is Λ. Let $\mathcal{M}_0$ be a coherent sub-$\mathcal{E}_\Lambda$-module of M which generates $\mathcal{M}$. Such an $\mathcal{M}_0$ is called a *saturated lattice* of $\mathcal{M}$. Then $\bar{\mathcal{M}} = \mathcal{M}_0/\mathcal{E}(-1)\mathcal{M}_0$ is an $\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)$-module, which is coherent over $\mathcal{O}_\Lambda(0)$.

Since a coherent sheaf with integrable connection is locally free, we have

LEMMA 10.2.1. $\bar{\mathcal{M}}$ *is a locally free* $\mathcal{O}_\Lambda(0)$*-module of finite rank.*

Since ϑ belongs to the center of $\mathscr{E}_\Lambda/\mathscr{E}_\Lambda(-1)$, ϑ can be considered as an endomorphism of $\mathscr{H}om_{\mathscr{E}_\Lambda/\mathscr{E}_\Lambda(-1)}(\bar{\mathscr{M}}, \mathscr{L})$, which is a locally constant sheaf on Λ. Its eigenvalues are called the *order* of $\mathscr{M}$ with respect to $\mathscr{M}_0$.

10.3. Let us take a section $G \subset \mathbf{C}$ of $\mathbf{C} \to \mathbf{C}/\mathbf{Z}$. Then there exists a unique saturated lattice $\mathscr{M}_0$ such that the orders of $\mathscr{M}$ with respect to $\mathscr{M}_0$ are contained in G (See [K4]). Then

$$\mathscr{F} = \mathscr{H}om_{\mathscr{E}_\Lambda/\mathscr{E}_\Lambda(-1)}(\bar{\mathscr{M}}, \mathscr{L})$$

and

$$M = \exp 2\pi i\vartheta \in \mathscr{A}ut(\mathscr{F})$$

does not depend on the choice of G.

THEOREM 10.3.1 ([KK] Chapter I, § 3). *Assume that there exists an invertible $\mathscr{O}_\Lambda$-module $\mathscr{L}$ such that $\mathscr{L}^{\otimes 2} = \omega_\Lambda \otimes \omega_X^{\otimes -1}$. Then the category of regular holonomic $\mathscr{E}_X$-modules with support in Λ is equivalent to the category of $(\mathscr{F}, M)$'s where $\mathscr{F}$ is a locally constant $\mathbf{C}_\Lambda$-module and $M \in \mathscr{A}ut_{\mathbf{C}}(\mathscr{F})$.*

10.4. If $u \in \mathscr{M}$, then the solution to $L(P)\varphi = 0$ for $P \in \mathscr{E}_\Lambda$ with $Pu = 0$ is called a principal symbol of u and denoted by $\sigma(u)$. The homogeneous degree of $\sigma(u)$ is called the order of u. In the terminology of § 10.2, the principal symbol is a section of $\mathscr{H}om_{\mathscr{E}_\Lambda/\mathscr{E}_\Lambda(-1)}(\mathscr{E}_\Lambda u/\mathscr{E}_\Lambda(-1)u, \mathscr{L})$ and the order is the eigenvalue of ϑ in $\mathscr{H}om_{\mathscr{E}_\Lambda/\mathscr{E}_\Lambda(-1)}(\mathscr{E}_\Lambda u/\mathscr{E}_\Lambda(-1)u, \mathscr{L})$.

10.4. When the characteristic variety is not smooth, we don't know much about the structure of holonomic systems. In this direction, we have

THEOREM 10.4.1 ([K-K] Theorem 1.2.2). *Let Z be a closed analytic subset of an open subset Ω of T^*X, $n = \dim X$, and let $\mathscr{M}$ and $\mathscr{N}$ be holonomic $\mathscr{E}_X|_\Omega$-modules.*

(i) *If* $\dim Z \leqslant n-1$, *then*

$$\Gamma(\Omega; \mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{N})) \to \Gamma(\Omega \backslash Z, \mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{N}))$$

is injective.

(ii) *If* $\dim Z \leqslant n-2$, *then*

$$\Gamma(\Omega; \mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{N})) \to \Gamma(\Omega \backslash Z; \mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{N}))$$

is an isomorphism.

In particular if $\operatorname{Supp}\mathcal{M} \subset \Lambda_1 \cup \Lambda_2$ *and if* $\dim(\Lambda_1 \cap \Lambda_2) \leqslant n-2$, *then* $\mathcal{M}$ *is a direct sum of two holonomic* $\mathcal{E}_X$*-modules supported on* Λ_1 *and* Λ_2, *respectively.*

Here is another type of theorem.

THEOREM 10.4.3 ([SKKO]). *Let* $\mathcal{M} = \mathcal{E}u = \mathcal{E}/\mathcal{J}$ *be a holonomic* $\mathcal{E}$*-module defined on a neighborhood of* $p \in T^*X$. *Assume* $\operatorname{Supp}\mathcal{M} = \Lambda_1 \cup \Lambda_2$ *and*

(i) Λ_1, Λ_2 *and* $\Lambda_1 \cap \Lambda_2$ *are non-singular and* $\dim \Lambda_1 = \dim \Lambda_2 = n$, $\dim(\Lambda_1 \cap \Lambda_2) = n-1$.

(ii) $T_{p'}\Lambda_1 \cap T_{p'}\Lambda_2 = T_{p'}(\Lambda_1 \cap \Lambda_2)$ *for any* p' *in a neighborhood of* p *in* $\Lambda_1 \cap \Lambda_2$.

(iii) *The symbol ideal of* $\mathcal{J}$ *coincides with the ideal of functions vanishing on* $\Lambda_1 \cup \Lambda_2$.

Setting $k = \operatorname{ord}_{\Lambda_1}u - \operatorname{ord}_{\Lambda_2}u - 1/2$, *we have*

(a) $\mathcal{M}$ *has a non-zero quotient supported on* $\Lambda_1 \Leftrightarrow \mathcal{M}$ *has a non-zero submodule supported on* $\Lambda_2 \Leftrightarrow k \in \mathbf{Z}$.

(b) $\mathcal{M}_p$ *is a simple* $\mathcal{E}_p$*-module* $\Leftrightarrow k \notin \mathbf{Z}$.

Sketch of the proof. By a quantized contact transformation, we can transform p, Λ_1, Λ_2 and $\mathcal{J}$ as follows:

$$p = (0, dx_1)$$

$$\Lambda_1 = \{(x, \xi); x_1 = \xi_2 = \dots = \xi_n = 0\}$$

$$\Lambda_2 = \{(x, \xi); x_1 = x_2 = \xi_3 = \dots = \xi_n = 0\}$$

$$\mathcal{J} = \mathcal{E}(x_1\partial_1 - \lambda) + \mathcal{E}(x_2\partial_2 - \mu) + \sum_{j>2} \mathcal{E}\partial_j$$

In this case, we can easily check the theorem.

§ 11. APPLICATION TO THE b-FUNCTION (see [SKKO])

11.1. As one of the most successful application of microlocal analysis, we shall sketch here how to calculate the b-function of a function under certain conditions.

11.2. Let f be a holomorphic function on a complex manifold X. Then, it is proved ([Bj], [Be] [K1]) that there exist (locally) a non zero polynomial $b(s)$ and $P(s) \in \mathscr{D}[s] \underset{\text{def}}{=} \mathscr{D} \underset{\mathbf{C}}{\otimes} \mathbf{C}[s]$ such that $P(s)f(x)^{s+1} = b(s)f(x)^s$ for any $s \in \mathbf{N}$. Such a polynomial $b(s)$ of smallest degree is called the b-function of $f(x)$ and is denoted by $b_f(s)$. For the relations between the b-function and the local monodromy see [M1], [K3].

11.3. Set $\mathscr{J} = \{P(s) \in \mathscr{D}[s]\,;\ P(s)f^s = 0 \text{ for } s \in \mathbf{N}\}$ and $\mathscr{N} = \mathscr{D}[s]/\mathscr{J}$. We shall denote the canonical generator of $\mathscr{N}$ by f^s. Then $t : \mathscr{N} \ni P(s)f^s \to P(s+1)f \cdot f^s \in \mathscr{N}$ gives a $\mathscr{D}$-endomorphism of $\mathscr{N}$ and $t\mathscr{N} = \mathscr{D}[s]f^{s+1}$. Here $f^{s+1} = f \cdot f^s \in \mathscr{N}$. In this terminology $b_f(s)$ is the minimal polynomial of $s \in \mathscr{E}nd_{\mathscr{D}}(\mathscr{N}/t\mathscr{N})$.

For $\lambda \in \mathbf{C}$, we set $\mathscr{M}_\lambda = \mathscr{D}[s]/(\mathscr{J} + \mathscr{D}[s]\,(s-\lambda))$ and denote by f^λ the canonical generator of $\mathscr{M}_\lambda$. Then $f^{\lambda+1} \mapsto f\, f^\lambda$ defines a $\mathscr{D}$-linear homomorphism $\mathscr{M}_{\lambda+1} \to \mathscr{M}_\lambda$.

11.4. Let W be the closure of

$$\{(s, x, \xi) \in \mathbf{C} \times T^*X\,;\ \xi = s d \log f(x),\ f(x) \neq 0\}$$

in $\mathbf{C} \times T^*X$. Set $W_0 = W \cap \{s=0\} \subset T^*X$. Then we can prove

PROPOSITION 11.4.1 ([K1]).

(i) *N is a coherent $\mathscr{D}_X$-module and* $\mathrm{Ch}\,(\mathscr{N}) = p(W)$, *where* p *is the projection from* $\mathbf{C} \times T^*X$ *to* T^*X.

(ii) *For any* $\lambda \in \mathbf{C}$, $\mathscr{M}_\lambda$ *is a regular holonomic* $\mathscr{D}_X$-*module and* $\mathrm{Ch}\,(\mathscr{M}_\lambda) = W_0$.

(iii) $\mathscr{N}/t\mathscr{N}$ *is a regular holonomic* $\mathscr{D}_X$-*module and* $\mathrm{Ch}\,(\mathscr{N}/t\mathscr{N}) = W_0 \cap (\pi \circ f)^{-1}(0)$.

11.5. In the sequel, for the sake of simplicity, we assume that there exists a vector field v such that $v(f) = f$. Therefore we have $v^k(f^s) = s^k f^s$. Hence $\mathscr{N}$ is a $\mathscr{D}$-module generated by f^s. If we set $\tilde{\mathscr{J}} = \mathscr{D} \cap \mathscr{J}$, then $\mathscr{N} \cong \mathscr{D}/\mathscr{J}$ and $\mathscr{J} = \mathscr{D}[s]\,(s-v) + \mathscr{D}[s]\tilde{\mathscr{J}}$.

11.6. The following lemma is almost obvious but affords a fundamental tool to calculate the b-function.

LEMMA 11.6.1. *Let* $\mathscr{L}$ *be an* $\mathscr{E}_X$*-module and* w *a non-zero section of* $\mathscr{L}$. *For* $\lambda \in \mathbf{C}$, *we assume*

(i) $v(w) = \lambda w$

(ii) $\tilde{\mathscr{J}} w = 0$

(iii) $fw = 0$.

Then we have $b_f(\lambda) = 0$.

Proof. There is a $P \in \mathscr{D}$ such that $b_f(s) f^s = P f^{s+1}$. Hence $\big(b_f(v) - Pf\big) f^s = 0$, which implies $b_f(v) - Pf \in \tilde{\mathscr{J}}$. Since $b_f(v) w = b_f(\lambda) w$ we have

$$0 = \big(b_f(v) - Pf\big) w = b_f(\lambda) w .$$

This implies $b_f(\lambda) = 0$.

11.7. Let $\bar{\mathscr{J}}$ be the symbol ideal of $\tilde{\mathscr{J}}$. Then the zero set of $\bar{\mathscr{J}}$ is W, and the zero of $\bar{\mathscr{J}} + \mathscr{O}\sigma(v)$ is W_0. Let Λ be an irreducible component of W_0. If $\bar{\mathscr{J}} + \mathscr{O}_{T*X}\sigma(v)$ is a reduced ideal at a generic point p of Λ then we call Λ a *good Lagrangean.*

If Λ is a good Lagrangean, then W is non-singular on a neighborhood of a generic point p of Λ and $\sigma = \sigma(s)|_W$ has non zero-differential. Let $p\colon W \to X$ denote the projection. We define $m(\Lambda) \in \mathbf{N}$ as the degree of zero of $f \circ p$ along Λ, and set $f_\Lambda = (f \circ p / \sigma^{m(\Lambda)})|_\Lambda$. Let ω be the non-vanishing n-form on X. Then $(p^*\omega) \wedge d\sigma$ is an $(n+1)$-form on W. Let $\mu(\Lambda)$ be the degree of zeros of $(p^*\omega) \wedge d\sigma$ along Λ, and let η be the n-form on Λ given by

$$\left.\frac{p^*\omega \wedge d\sigma}{\sigma^{\mu(\Lambda)}}\right|_\Lambda = \eta \wedge d\sigma .$$

If we set $\kappa_\Lambda = \eta \otimes \omega^{\otimes -1} \in \omega_\Lambda \otimes \omega_X^{\otimes -1}$, then this is independent of the choice of ω. We have

PROPOSITION 11.7.1 ([SKKO]). *If* Λ *is a good Lagrangean, then for any* $\lambda \in \mathbf{C}$, $\mathscr{M}_\lambda$ *is a simple holonomic system on a neighborhood of a generic point* p *of* Λ *and we have*

(i) $\sigma(f^\lambda) = f_\Lambda^\lambda \sqrt{\kappa_\Lambda}$.

In particular

$$\operatorname{ord} f^\lambda = -m(\Lambda)\lambda - \mu(\Lambda)/2 .$$

(ii) *There exists a monic polynomial* $b_\Lambda(s)$ *of degree* $m(\Lambda)$ *and an invertible micro-differential operator* P_Λ *of order* $m(\Lambda)$ *such that*

$$b_\Lambda(s)f^s = P_\Lambda f \cdot f^s \quad \textit{in} \quad \mathscr{E} \underset{\mathscr{D}}{\otimes} \mathscr{N}$$

and
$$\sigma(P_\Lambda)|_\Lambda = f_\Lambda^{-1}.$$

Remark that f_Λ and ω_Λ are homogeneous of degree $-m(\Lambda)$ and $-\mu(\Lambda)$, respectively.

Remark also that the minimal polynomial of $s \in \mathscr{E}nd_{\mathscr{E}}(\mathscr{E} \underset{D}{\otimes} \mathscr{N}/t\mathscr{N})|_\Lambda$ is $b_\Lambda(s)$. In fact, if $Pf^{s+1} = b(s)f^s$ in $\mathscr{E} \otimes \mathscr{N}$, then $\big(P \cdot P_\Lambda^{-1} b_\Lambda(s) - b(s)\big)f^s = 0$. This implies that $P \cdot P_\Lambda^{-1} b_\Lambda(v) - b(v) \in \mathscr{E}\tilde{\mathscr{J}}$. Hence

$$\sigma\big(P \cdot P_\Lambda^{-1} b_\Lambda(v) - b(v)\big)|_W = 0\,.$$

If $\operatorname{ord} P \cdot P_\Lambda^{-1} b_\Lambda(v) = \operatorname{ord} P > \deg b$, then $\sigma(P)|_W = 0$. Therefore $P = P' + P''$ with $P'' \in \mathscr{E}\tilde{\mathscr{J}}$ and $\sigma(P') < \sigma(P)$. Hence $P' f^{s+1} = b(s)f^s$. Thus, we may assume $\operatorname{ord} P \leqslant \deg b$. Then

$$0 = \sigma\big(b(v) - P \cdot P_\Lambda^{-1} b_\Lambda(v)\big)|_W = b(\sigma) - \big(\sigma(P)|_W f_\Lambda b_\Lambda(\sigma)\big)\,.$$

This shows that $b(s)$ is a multiple of $b_\Lambda(s)$.

Corollary 11.7.2. *If every irreducible component of* W_0 *is good Lagrangean, then* $b_f(s)$ *is the least common multiple of the* $b_\Lambda(s)$.

11.8. Let Λ_1 and Λ_2 be two good Lagrangeans. We assume the following conditions for a point $p \in \Lambda_1 \cap \Lambda_2$:

(11.8.1) $\dim_p \Lambda_1 \cap \Lambda_2 = n-1$ and Λ_1, Λ_2 and $\Lambda_1 \cap \Lambda_2$ are non singular on a neighborhood of p.

(11.8.2) For any point p' on a neighborhood of p in $\Lambda_1 \cap \Lambda_2$, we have $T_{p'}\Lambda_1 \cap T_{p'}\Lambda_2 = T_{p'}(\Lambda_1 \cap \Lambda_2)$.

(11.8.3) $\tilde{\mathscr{J}} + \mathscr{O}\sigma(v)$ coincides with the defining ideal of $\Lambda_1 \cup \Lambda_2$ with the reduced structure.

In this case we say that Λ_1 and Λ_2 have a *good intersection.*

We have the following theorem.

THEOREM 11.7.3. *Let* Λ_1 *and* Λ_2 *be good Lagrangeans with a good intersection. If* $m(\Lambda_1) \geqslant m(\Lambda_2)$, *then*

$$\prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1} \left(\mathrm{ord}_{\Lambda_2} f^s - \mathrm{ord}_{\Lambda_1} f^s + \frac{1}{2} + k\right) \mid b_f(s)\,.$$

In order to prove this let us take $\lambda \in \mathbf{C}$ such that

$$(11.8.4)\qquad \begin{aligned} k &= \mathrm{ord}_{\Lambda_1} f^\lambda - \mathrm{ord}_{\Lambda_2} f^\lambda - 1/2 \in \mathbf{N} \quad \text{and} \\ k' &= \mathrm{ord}_{\Lambda_1} f^{\lambda+1} - \mathrm{ord}_{\Lambda_2} f^{\lambda+1} - 1/2 \in \mathbf{N}\,. \end{aligned}$$

Recall that

$$k = (m(\Lambda_2) - m(\Lambda_1))\lambda - \frac{1}{2}(\mu(\Lambda_2) - \mu(\Lambda_1) - 1/2)$$

and $k' = k + (m(\Lambda_2) - m(\Lambda_1))$. Then by Theorem 10.4.3, $\mathcal{M}_\lambda$ has a non-zero quotient $\mathcal{L}$ whose support is Λ_1. Let $w \in \mathcal{L}$ be the image of $f^\lambda \in \mathcal{M}_\lambda$.

Let $\alpha : \mathcal{M}_\lambda \to \mathcal{L}$ be the canonical homomorphism and $\beta : \mathcal{M}_{\lambda+1} \to \mathcal{M}_\lambda$ be the homomorphism given by $f^{\lambda+1} \mapsto f \cdot f^\lambda$. Then, since $k' \notin \mathbf{N}$, $\mathcal{M}_{\lambda+1}$ has no non-zero quotient supported in Λ_1. Hence $\alpha \circ \beta = 0$. Therefore $fw = \alpha\beta(f^{\lambda+1}) = 0$. Thus we can apply Lemma 11.6.1 to conclude that $b_f(\lambda) = 0$. If $k \in \mathbf{Z}$ with $0 \leqslant k < m(\Lambda_1) - m(\Lambda_2)$ then

$$\lambda = \frac{1}{m(\Lambda_1) - m(\Lambda_2)}\left(k + \frac{1}{2}(\mu(\Lambda_1) - \mu(\Lambda_2) - 1)\right)$$

satisfies (11.8.4). This shows that $b_f(s)$ is a multiple of

$$\begin{aligned} &\prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1} \left((m(\Lambda_1) - m(\Lambda_2))s - \frac{1}{2}(\mu(\Lambda_1) - \mu(\Lambda_2) - 1) + k\right) \\ &= \text{const.} \prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1} \left(\mathrm{ord}_{\Lambda_2} f^s - \mathrm{ord}_{\Lambda_1} f^s + \frac{1}{2} + k\right). \end{aligned}$$

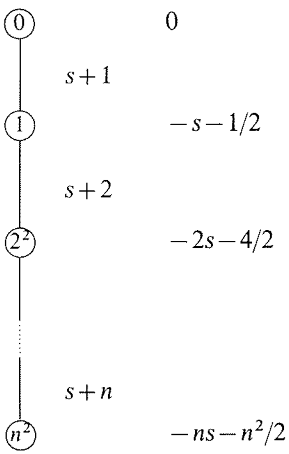
If we refine this argument, we can prove

THEOREM 11.8.2 ([SKKO]). *If* Λ_1 *and* Λ_2 *are good Lagrangeans with a good intersection and if* $m(\Lambda_1) \geqslant m(\Lambda_2)$ *then*

$$\frac{b_{\Lambda_1}(s)}{b_{\Lambda_2}(s)} = \text{const.} \prod_{k=0}^{m(\Lambda_1)-m(\Lambda_2)-1} \left(\mathrm{ord}_{\Lambda_2} f^s - \mathrm{ord}_{\Lambda_1} f^s + \frac{1}{2} + k\right).$$

Example 11.8.3.

(i) $X = M_n(\mathbf{C}) = \mathbf{C}^{n^2}$ and $f(x) = \det x$.

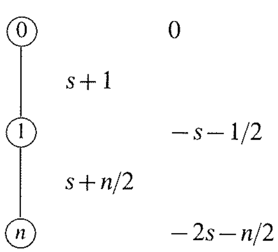


$$b_f(s) = \prod_{j=1}^{n} (s+j).$$

Here ⓐ means a good Lagrangean which is the conormal bundle to an a-codimensional submanifold. ○—○ means that the two corresponding good Lagrangeans have a good intersection.

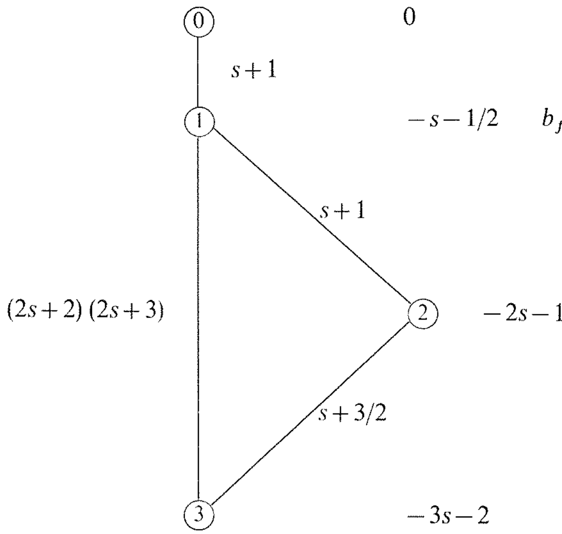
The polynomial attached to the intersection is the ratio of the corresponding b_Λ-functions, calculated by Theorem 11.8.2. The polynomial attached to the circle is the order of f^λ.

(ii) $X = \mathbf{C}^n$, $f(x) = x_1^2 + \dots + x_n^2$



$$b(s) = (s+1)(s+n/2)$$

(iii) $X = \mathbf{C}^3 \quad f = x^2y + z^2$



$$b_f(s) = (s+1)^2(s+3/2)$$

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