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On the Segal-Shale-Weil Representations and Harmonic Polynomials

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In this paper, we give the answer to the following two intimately related problems.

(a) To decompose the tensor products of the harmonic representations into irreducible components to get a series of new unitary irreducible representations with highest weight vectors of the group $G = Mp(n)$, two-sheeted covering group of the symplectic group, or $G = \mathrm{U}(p, q)$.

(b) To describe the representations of the group $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{O}(k, \mathbb{C})$ (resp. $\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C})$) in the space of pluriharmonic polynomials on the space $\mathrm{M}(n, k; \mathbb{C})$ of $n \times k$ complex matrices (resp. $\mathrm{M}(p, k; \mathbb{C}) \times \mathrm{M}(q, k; \mathbb{C})$).

The second problem arises when we construct an intertwining operator from the tensor product of the harmonic representation into a space of vector-valued holomorphic functions on the associated hermitian symmetric space G/K, or equivalently when we consider highest weight vectors in the tensor products.

Some of our motivations are the following:

1) Apart from special cases the unitary dual $\hat{G}$ of a real semi-simple Lie group is not known. There exist isolated points in $\hat{G}$ which are not members of discrete or "mock-discrete" series, (for exemple for $\mathrm{Sp}(2, \mathbb{C})$ where the unitary dual has been computed by M. Duflo [18], there are two isolated points in $\hat{G}$, the trivial representation and the odd component of the Segal-Shale-Weil representation) and we are interested to produce series of such representations.

2) We are extending to matrix spaces classical results for harmonic polynomials on $\mathbb{R}^n$.

(0.2) Let us now describe with more details our methods and results. Let G be $Mp(n)$ or $\mathrm{U}(p, q)$. There is some interesting "minimal" representation in $\hat{G}$:

(α) The consideration of $\mathrm{Sp}(n, \mathbb{R})$ as a group of automorphisms of the commutation relations (i.e. Heisenberg group) leads to the definition of the Segal-Shale-Weil representation of the metaplectic group $Mp(n)$ in $L^2(\mathbb{R}^n)$. We call this unitary representation L the harmonic representation of $Mp(n)$. L is the

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sum of two irreductible representations L_+ and L_- both having highest weight vectors.

(β) Let h be a hermitian form of signature (p, q) on $\mathbb{C}^{p+q}$; as an element of $U(p, q)$ leaves h stable and a fortiori the skew-symmetric form $\operatorname{Im} h$, $U(p, q)$ is naturally embedded in $Sp(p+q, \mathbb{R})$. The restriction of the harmonic representation to $U(p, q)$ will be also denoted by L and called the harmonic representation: L breaks into a discrete sum of irreducible representations L_n ($n \in \mathbb{Z}$) refered in the Physics literature as ladder representations.

(The representations L_+, L_-, L_n are in a sense we will not discuss here, associated to the minimal orbits of the co-adjacent representations [7, 13].)

(0.3) We consider the tensor product $L_k = \overset{k}{\bigotimes} L$. This tensor product decomposes into a discrete sum of irreducible representations of G having highest weight vectors (see I for precise definitions). We will describe the components.

For the small values of k ($k < 2n$, in case α) we get a series of new irreducible unitary representations of G.

Let us explain here the decomposition of L_k for the group $Mp(n) = G$. Let $M_{n,k}$ be the space of all $n \times k$ real matrices. We realize L_k in $L^2(M_{n,k})$ by the formulas

$$\left[L_k \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} f\right](x) = (\det a)^{k/2} f({}^t a x), \qquad a \in GL(n, \mathbb{R}),$$

$$(F) \quad \left[L_k \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f\right](x) = e^{\frac{i}{2} \operatorname{Tr} {}^t x b x} f(x), \qquad b = {}^t b,$$

$$(L_k(\sigma) f)(x) = \left(\frac{i}{2\pi}\right)^{\frac{nk}{2}} \int_{M_{n,k}} e^{i \operatorname{Tr} {}^t x y} f(y)\, dy, \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The orthogonal group $O(k)$ acts on $L^2(M_{n,k})$ by (*cf.*) $(x) = f(xc)$, $c \in O(k)$ and this action commutes with L_k. Let (V_λ, λ) be an irreducible unitary representation of $O(k)$ and $L^2(M_{n,k}; \lambda)$ be the space of all V_λ-valued square-integrable functions $f(x)$ satisfying $f(xc) = \lambda(c)^{-1} f(x)$ ($x \in M_{n,k}$, $c \in O(k)$). We denote by $L_k(\lambda)$ the representation of G in $L^2(M_{n,k}; \lambda)$ given by the same formula (F).

Let Σ be the set of all $\lambda \in O(k)^\wedge$ such that $L^2(M_{n,k}; \lambda) \neq 0$. Then, we have: $L^2(M_{n,k}) = \bigoplus_{\lambda \in \Sigma} L^2(M_{n,k}; \lambda) \otimes V'_\lambda$ as a representation of $G \times O(k)$, where V'_λ is the dual vector space of V_λ.

(0.4) We prove:

(1) For each $\lambda \in \Sigma$, $L_k(\lambda)$ is an irreducible unitary representation of G having a highest weight vector.

(2) $L_k(\lambda)$ appears in L_k(dim V_λ)-times, i.e. $L_k = \bigoplus_{\lambda \in \Sigma} (\dim V_\lambda) L_k(\lambda)$ is the decomposition of L_k into irreducible representations.

(0.5) We will describe further the representations $L_k(\lambda)$. Let (V_τ, τ) be an irreducible unitary representation of K, v_τ its highest weight vector. We also denote by τ the highest weight of τ. We consider $D = G/K$ and realize it as the Siegel upper half plane

$$\{z \in M(n, n; \mathbb{C});\ z = {}^t z,\ \operatorname{Im} z > 0\}.$$

We consider the space $\mathcal{O}(D, V_\tau)$ of all V_τ-valued holomorphic functions on D and the representation $T(\tau)(g)$ of G in $\mathcal{O}(D, V_\tau)$ given by

$$(T(\tau)(g) f)(z) = \tau({}^t(c z + d)) f((a z + b)(c z + d)^{-1}) \qquad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

It is easy to see that any unitary subrepresentation of $T(\tau)$ (if there is one) is irreducible, and has a highest weight vector $\phi_\tau(z) = \tau(z+i)\, v_\tau$ of weight τ.

We will observe now how $L^2(\mathrm{M}_{n,k}; \lambda)$ is embedded in some $\mathcal{O}(D, V_\tau)$:

(0.6) First let us look at a simple case in which λ is the trivial representation λ_0 of $\mathrm{O}(k)$.

Let f be a function in $L^2(\mathrm{M}_{n,k}; \lambda_0)$; since we have $f(xc) = f(x)$, $f(x)$ is a function of $\xi = x\,{}^t x$.

We consider the map

(F-1) $\displaystyle (\mathcal{F} f) = \int_{M_{n,k}} e^{i \operatorname{Tr} x^t x z} f(x)\, dx.$

It is clear that $\mathcal{F}$ is an injective map from $L^2(\mathrm{M}_{n,k}; \lambda_0)$ in $\mathcal{O}(D)$. Let us see $\mathcal{F}$ intertwines the representation $\mathrm{L}_k(\lambda_0)$ with $T(\det^{-k/2})$. The commutation relations are obvious to check on the formula for the elements $\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ of G.

We check the action of σ; as $\mathrm{L}_k(\sigma)$ is the Fourier transform $\hat{f}$ of f, we have

$$\begin{aligned}(\mathcal{F} L_k(\sigma) f)(z) &= \text{const.} \int e^{i \operatorname{Tr} x^t x z} f(x)\hat{}\, dx \\ &= \text{const.} \int (e^{i \operatorname{Tr} x^t x z})\hat{}\, f(x)\, dx\end{aligned}$$

But, we know that

(F-2) $(e^{i \operatorname{Tr} x^t x z})\hat{} = \text{const.} (\det z)^{-k/2}\, e^{-i \operatorname{Tr} x^t x z^{-1}}$

and hence we obtain the necessary commutation relations. Therefore, $L^2(M_{n,k}; \lambda_0)$ is irreducible having the highest weight vector $(\mathcal{F}^{-1} \psi_\tau)(x) = e^{-\operatorname{Tr} x^t x}$ of weight $(-k/2, \ldots, -k/2)$.

Furthermore we have imbedded $L^2(\mathrm{M}_{n,k}; \lambda_0)$ as a subspace of $\mathcal{O}(D)$ or via boundary values in a subspace of a principal series representation. Let us observe on this exemple how the image of $L^2(\mathrm{M}_{n,k}; \lambda_0)$ in $\mathcal{O}(D)$ varies in function of k. Let C be the cone of positive definite $n \times n$ symmetric matrices. If $k \geqq n$ the image of $\mathrm{M}_{n,k}$ under the map $x \mapsto \xi = x^t x$ is the solid cone $\bar{C}$. Hence the image of $L^2(\mathrm{M}_{n,k}; \lambda_0)$ is dense in $\mathcal{O}(D)$ (however the representation $T(\det^{-k/2})$ is in the holomorphic discrete series only when $k > 2n$). At the contrary when $k < n$ the image of $M_{n,k}$ under the map $\xi = x^t x$ is the set O_k of $\bar{C} - C$ of all positive semi-definite matrices of rank less or equal to k. Hence $\mathcal{F} f$ is the Fourier-Laplace transform of the measure on O_k derived from $f(x)\,dx$. Hence the holomorphic functions in the space $\mathscr{F}(L^2(M_{n,k};\lambda_0))$ will satisfy the differential equations corresponding to the equations of O_k. It is a difficult and interesting question to describe the G-invariant norm directly on this space of solutions or, otherwise stated, to grasp directly the existence of this small unitary subspace of the representation $T(\det^{-k/2})$.

(0.7) Now let us consider any λ in $\Sigma\subset O(k)\hat{}$. We want to find an irreducible unitary representation (V_τ,τ) of K and an intertwining operator $\mathscr{F}_\lambda$ from $L_k(\lambda)$ to T_τ. Suggested by (F-1), we shall assume that the intertwining operator $\mathscr{F}_\lambda$ is given by

(F-3) $$(\mathscr{F}_\lambda f)(z)=\int_{M_{n,k}} e^{i\operatorname{Tr}x{}^txz}\,P(x)^*\,f(x)\,dx$$

Here $P(x)$ is a $\operatorname{Hom}_{\mathbb{C}}(V_\tau, V_\lambda)$-valued polynomial on $M_{n,k}$. We may suppose evidently

(F-4) $P(xc)=\lambda(c)^{-1}\,P(x)$.

The commutation relations with $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$ are trivially satisfied and the commutation relations with $\begin{pmatrix}a & 0\\ 0 & {}^ta^{-1}\end{pmatrix}$ give us the relation

(F-5) $P(a^{-1}x)=P(x)\,(\tau\otimes\det^{k/2})(a)$ for $a\in GL(n,\mathbb{R})$.

Let us consider the commutation relation with σ. In the same way as when $\lambda=\lambda_0$

$$\begin{aligned}(\mathscr{F}_\lambda L_k(\sigma) f)(z)&=\text{const.}\int e^{i\operatorname{Tr}x{}^txz}\,P(x)^*\,\hat{f}(x)\,dx\\ &=\text{const.}\int (e^{i\operatorname{Tr}x{}^txz}\,P(x)^*)\hat{}\,f(x)\,dx.\end{aligned}$$

Therefore, we need the following relation similar to (F-2)

(F-6) $(e^{i\operatorname{Tr}x{}^txz}\,P(x)^*)\hat{}=\text{const.}\,\tau(z)\,e^{-i\operatorname{Tr}x{}^txz^{-1}}\,P(x)^*$

and this relation is verified when and only when $P(x)^*$ (and hence $P(x)$) is a harmonic polynomial. Therefore, $P(x)$ should be a harmonic polynomial satisfying (F-4) and (F-5). Since $P(ax)$ is harmonic for all a, $P(x)$ satisfies the equations

(F-7) $\Delta_{ij}P(x)=0 \quad (1\leqq i\leqq j\leqq n)$

where

$$\Delta_{ij}=\sum_{\nu=1}^{k}\frac{\partial^2}{\partial x_{i\nu}\,\partial x_{j\nu}}.$$

This consideration leads us to Problem (b).

(0.8) We call a polynomial satisfying the equations (F-7) pluriharmonic and denote by $\mathfrak{H}$ the space of all pluriharmonic polynomials. $\mathfrak{H}$ is a representation space of $GL(n,\mathbb{R})\times O(k)$ by the action $(g,c)\colon f(x)\mapsto f(g^{-1}xc)$.

We define $\mathfrak{H}(\lambda)=\{P;\ V_\lambda$-valued pluriharmonic polynomials on $M_{n,k}, P(xc)=\lambda(c)^{-1}P(x)$ for any $c\in \mathrm{O}(k)\}$.

The group $\mathrm{GL}(n,\mathbb{R})$ acts on $\mathfrak{H}(\lambda)$ by left translation; we denote this representation by $\tau(\lambda)$.

(0.9) We prove

a) $\lambda\in\Sigma \Leftrightarrow \mathfrak{H}(\lambda)\neq 0$.

b) If $\lambda\in\Sigma$, $\tau(\lambda)$ is an irreducible finite-dimensional representation of $\mathrm{GL}(n,\mathbb{R})$.

c) $\lambda\mapsto\tau(\lambda)$ is an injective map.

Hence, as a representation space of $\mathrm{GL}(n,\mathbb{R})\times \mathrm{O}(k)$, we have $\mathfrak{H}=\bigoplus_{\lambda\in\Sigma}\tau(\lambda)\otimes\lambda'$.

d) We explicit Σ and the map $\lambda\mapsto\tau(\lambda)$ in terms of highest weights; we give also explicitly the pluriharmonic polynomial with highest weight $\tau(\lambda)\otimes\lambda'$.

Defining the $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{H}(\lambda), V_\lambda)$-valued polynomial $P_\lambda(x)$ by $P_\lambda(x)f=f(x)$ for $f\in\mathfrak{H}(\lambda)$, we get the intertwining operator from $L^2(M_{n,k};\lambda)$ into $T(\tau(\lambda)\otimes\det^{-k/2})$ by the formula (F-3). Thus, $L^2(M_{n,k};\lambda)$ is an irreducible unitary representation with highest weight $\tau(\lambda)\otimes\det^{-k/2}$.

(0.10) Let us say something about the plan of our article.

Chapter I is a paraphrase of results of Harish-Chandra.

Chapter II deals with the case $\mathrm{Mp}(n)$; Results on pluri-harmonic polynomials are in (II.5) and (II.6) and can be read independently.

Chapter III deals with $\mathrm{U}(p,q)$; we follow a similar line of arguments to those for $\mathrm{Mp}(n)$ and we will only give a sketch of our arguments when it is enough. Again, (III.5) and (III.6) on pluri-harmonic polynomials can be read independently.

(0.11) Let us say that results on the decomposition of L_k were obtained by several authors. For $k\geqq 2n$, by Gelbart [2] for $\mathrm{Sp}(n,\mathbb{R})$, Gross and Kunze [4] for $\mathrm{Sp}(n,\mathbb{R})$, $U(n,n)$ and $\mathrm{O}^*(2n)$. Saito investigated the case $k\geqq n$ for $\mathrm{Sp}(n,\mathbb{R})$ ([20]).

In these cases the representations L_k breaks into representations of the holomorphic discrete series, or of some limit points. We are particularly thankful to Gross and Kunze for discussions on these topics.

Results on the decomposition of pluri-harmonic polynomials are obtained (mainly also with the same restriction $k\geqq 2n$) in [2, 8, 14, 15], often with applications to the analysis on the Stiefel manifolds. We are thankful to Stein for discussions on the construction of theses intertwining operators.

Howe [19] proved an abstract double commutant theorem in a more general context of graded Lie algebras leading to independant proofs of (0.4) and (0.9) a) b) c).

As we said at the beginning, the harmonic representation is of interest in physics [1, 6, 11–13]. In [6] the components L_n of the representation L of $U(2,2)$ are identified with Hilbert spaces of distribution on the Minkowski space which are solutions of the Dirac and Maxwell equations. As a consequence of our results here we can describe the set P of all positive energy representations of the group $U(2,2)$ determined by Mack [9] using different methods.

More generally, unitarisability of representations having a highest weight vector τ is a problem originated in Harish-Chandra [5]; the first exemples of such representations which are not in a discrete series are in Gross and Kunze [4]. Complete results when τ is a one dimensional representation of K are obtained by Wallach [16], Rossi and Vergne [10] and Gindikin [3].

In view of our results, it is natural to pose here a conjecture: For $G = Mp(n)$ or $U(p,q)$, any irreducible unitary representation with highest weight appears in the tensor product L_k for some k.

If it is true, it will be an interesting phenomenon when we compare it with the fact that any irreducible finite-dimensional representation of $SL(n;\mathbb{C})$ (or $SO(n,\mathbb{C})$ appears in the tensor product of the fundamental representation.

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I. Review of Unitary Representations with Highest Weight Vectors

1. Notations

Let $\mathfrak{g}$ be a simple Lie algebra over the reals $\mathbb{R}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition for $\mathfrak{g}$. We shall suppose that $\mathfrak{k}$ has a non-zero center $\mathfrak{z}$; then $\mathfrak{z} = \mathbb{R}Z$ where the eigenvalues of the adjoint action of Z on $\mathfrak{p}^{\mathbb{C}}$ are $\pm i$.

Let

$$\mathfrak{p}^+ = \{x \in \mathfrak{p}^{\mathbb{C}};\ [Z,x] = ix\},$$
$$\mathfrak{p}^- = \{x \in \mathfrak{p}^{\mathbb{C}};\ [Z,x] = -ix\}.$$

Now let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{k}$, then $\mathfrak{h} = \mathfrak{h} \cap [\mathfrak{k},\mathfrak{k}] \oplus \mathbb{R}Z$; we shall let $x \mapsto \bar{x}$ denote the conjugation in $\mathfrak{g}^{\mathbb{C}}$ relative to the real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$, then $\mathfrak{p}^+ = \overline{\mathfrak{p}^-}$. Let $\mathfrak{U}$ denote the enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$ and $u \mapsto u^*$ the antilinear automorphism of $\mathfrak{U}$ which extends the map $x \to -\bar{x}$ on $\mathfrak{g}^{\mathbb{C}}$.

Let Δ denote the system of roots of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$; these roots take purely imaginary values on $\mathfrak{h}$. We have $\Delta = \Delta_{\mathfrak{k}} \cup \Delta_{\mathfrak{p}}$, where

$$\Delta_{\mathfrak{k}} = \{\gamma \in \Delta;\ (\mathfrak{g}^{\mathbb{C}})^{\gamma} \subset \mathfrak{k}^{\mathbb{C}}\},$$
$$\Delta_{\mathfrak{p}} = \{\gamma \in \Delta;\ (\mathfrak{g}^{\mathbb{C}})^{\gamma} \subset \mathfrak{p}^{\mathbb{C}}\}.$$

Choose an ordering on the roots so that $\mathfrak{p}^+ = \sum_{\alpha \in \Delta_{\mathfrak{p}}^+} (\mathfrak{g}^{\mathbb{C}})^{\gamma}$. Let $\mathfrak{g}^+ = \sum_{\alpha \in \Delta^+} (\mathfrak{g}^{\mathbb{C}})^{\alpha}$. If $\gamma \in \Delta$, let H_{γ} be the unique element of $i\mathfrak{h} \cap [(\mathfrak{g}^{\mathbb{C}})^{\gamma}, (\mathfrak{g}^{\mathbb{C}})^{-\gamma}]$ such that $\gamma(H_{\gamma}) = 2$. If $\gamma \in \Delta_{\mathfrak{p}}^+$, we shall choose $E_{\gamma} \in (\mathfrak{g}^{\mathbb{C}})^{\gamma}$ so that $[E_{\gamma}, \bar{E}_{\gamma}] = H_{\gamma}$. If $\alpha \in \Delta_{\mathfrak{k}}^+$, we shall choose $E_{\alpha} \in (\mathfrak{g}^{\mathbb{C}})^{\alpha}$ so that $[E_{\alpha}, \bar{E}_{\alpha}] = -H_{\alpha}$.

2. Modules with Highest Weight

Let Λ be a linear form on $\mathfrak{h}^{\mathbb{C}}$. Let us consider the Verma module $V(\Lambda)$ of highest weight Λ that is, for I_{Λ} the left ideal generated by $\mathfrak{g}^+$ and $H - \Lambda(H)$ $(H \in \mathfrak{h}^{\mathbb{C}})$, $V(\Lambda) = \mathfrak{U}/I_{\Lambda}$. Let 1_{Λ} denote the image of 1 in $V(\Lambda)$. The $\mathfrak{U}$-module $V(\Lambda)$ has a unique maximal submodule. We will denote by $W(\Lambda)$ the unique simple quotient of $V(\Lambda)$.

Let $B(m, m')$ be a sesquilinear form on a $\mathfrak{U}$-module M; We will say that B is $\mathfrak{g}$-invariant if $B(u m, m') = B(m, u^* m')$ for any u in $\mathfrak{U}$ and m, m' in M. M is called unitarisable if there exists a hermitian positive definite and $\mathfrak{g}$-invariant form on M. We denote by p the projection of $\mathfrak{U}$ onto $\mathfrak{U}(\mathfrak{h}^{\mathbb{C}})$ according to the decomposition $\mathfrak{U} = (\mathfrak{U}\,\mathfrak{g}^+ + \mathfrak{g}^-\,\mathfrak{U}) \oplus \mathfrak{U}(\mathfrak{h}^{\mathbb{C}})$.

(2.1) **Lemma.** a) $B_\Lambda(u.1_\Lambda, v.1_\Lambda) = \langle \Lambda, p(u^* v)\rangle$ *defines a* $\mathfrak{g}$-*invariant sesquilinear form on* $V(\Lambda)$.

b) *Any other* $\mathfrak{g}$-*invariant sesquilinear form on* $V(\Lambda)$ *is proportional to* B_Λ.

c) B_Λ *defines a* $\mathfrak{g}$-*invariant and non-degenerate form on* $W(\Lambda)$ *(still denoted by* B_Λ*).*

d) B_Λ *is hermitian, if and only if* Λ *is real on* $i\mathfrak{h}$.

All these assumptions are known and easy to prove [5]. IV. We will consider for the cases $\mathrm{Sp}(n, \mathbb{R})$ and $U(p, q)$ the following problem originated in Harish-Chandra [5] IV: for which Λ, the module $W(\Lambda)$ is unitarisable, i.e. for which Λ there exists a positive definite $\mathfrak{g}$-invariant form on $W(\Lambda)$. By the preceding lemma, this form has to be B_Λ. We will denote

$$P = \{\Lambda;\ \text{real linear form on } i\mathfrak{h} \text{ such that, } \forall u \in \mathfrak{U},\ \langle \Lambda, p(u^* u)\rangle \geqq 0\}.$$

Hence $\Lambda \in P$ if and only if $W(\Lambda)$ is unitarisable. The following properties of P are easily verified [5]. IV

(2.2) If $\Lambda \in P$ then

(2.2a) $\forall \alpha \in \Delta_{\mathfrak{k}}^+$, $\Lambda(H_\alpha)$ is a non-negative integer,

(2.2b) $\forall \gamma \in \Delta_{\mathfrak{p}}^+$, $\Lambda(H_\gamma) \leqq 0$.

3. *Spaces of Holomorphic Functions on G/K*

Let Λ be a real linear form on $i\mathfrak{h}$ satisfying only the condition 2.2.a). We will realize the $\mathfrak{U}$-module $W(\Lambda)$ as a module of holomorphic functions on the hermitian symmetric domain associated to the pair $(\mathfrak{g}, \mathfrak{k})$.

Let $\tilde{G}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$ (for a change, the center Z of $\tilde{G}$ is infinite). For $x \in \mathfrak{g}$ and f a differentiable function on $\tilde{G}$, we shall let $r(x) f$ denote the function $(r(x) f)(g) = \frac{d}{dt} f(g \exp tx)|_{t=0}$ and $l(x) f$ denote the function $(l(x) f)(g) = \frac{d}{dt} f(\exp(-tx)\, g)|_{t=0}$.

Let $\tilde{K}$ be the analytic subgroup of $\tilde{G}$ with Lie algebra $\mathfrak{k}$, then $\tilde{G}/\tilde{K}$ is a hermitian symmetric space. The holomorphic functions on $\tilde{G}/\tilde{K}$ will be identified as the space of functions on $\tilde{G}$ annihilated by all the vector fields $r(x)$ with $x \in \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$.

Let $G_{\mathbb{C}}$ be the simply connected group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and G, K, $K_{\mathbb{C}}$, P_+, P_-, the connected subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{p}^+$, $\mathfrak{p}^-$ respectively. Every element of $P_+ K_{\mathbb{C}} P_-$ can be written in a unique way $g = \exp \zeta(g)\, k(g) \exp \zeta'(g)$

with $\zeta(g)\in\mathfrak{p}^+$, $k(g)\in K_{\mathbb{C}}$ and $\zeta'(g)\in\mathfrak{p}^-$. We have $G\subset P^+ K_{\mathbb{C}} P^-$ and the map $g\mapsto k(g)$ lifts to a map, denoted $\tilde{k}(g)$, of $\tilde{G}$ into $\tilde{K}_{\mathbb{C}}$ the universal cover of $K_{\mathbb{C}}$.

The group $K_{\mathbb{C}} P_-$ is a maximal parabolic subgroup of $G_{\mathbb{C}}$, we denote by e_0 the image of 1 in $G_{\mathbb{C}}/K_{\mathbb{C}} P_-$; the map $g\mapsto g.e_0$ induces a biholomorphism of the complex manifold G/K into an open subset D of the complex manifold $G_{\mathbb{C}}/K_{\mathbb{C}} P_-$.

Let Λ be a real linear form on $i\mathfrak{h}$ satisfying (2.2) a), i.e. Λ is a dominant integral form with respect to $\Delta_{\mathfrak{k}}^+$. Hence there exists a unique holomorphic representation τ_Λ of $\tilde{K}_{\mathbb{C}}$ on a finite dimensional vector space V_Λ of highest weight Λ. Let v_Λ be the highest weight vector of V_Λ; we choose a scalar product on V_Λ such that $\tau_\Lambda(k)$ for $k\in\tilde{K}$ is unitary and normalize it by $\langle v_\Lambda, v_\Lambda\rangle=1$.

(3.1). We define $\mathcal{O}(\Lambda)=\{f;\ C^\infty$-functions on $\tilde{G}$, V_Λ-valued, such that $f(gk)=\tau_\Lambda(k)^{-1} f(g)$, $g\in\tilde{G}$, $k\in\tilde{K}$; $r(x)\cdot f=0$ for $x\in\mathfrak{p}^-\}$. The group $\tilde{G}$ acts on $\mathcal{O}(\Lambda)$ by left translations. We consider $\mathcal{O}(\Lambda)$ as a $\mathfrak{U}$-module by $x\cdot\phi=l(x)\,\phi$. A function f in $\mathcal{O}(\Lambda)$ is a real analytic function on $\tilde{G}$ (as the space $\mathcal{O}(\Lambda)$ consists of holomorphic sections of the vector bundle $\tilde{G}\underset{\tilde{K}}{\times} V_\Lambda\to\tilde{G}/\tilde{K}$). We consider the $\mathfrak{U}$-submodule $\mathcal{O}_0(\Lambda)$ of $\mathcal{O}(\Lambda)$ consisting of $\tilde{K}$-finite functions.

(3.2). Let us consider the function $\psi_\Lambda(g)=\tau_\Lambda(k(g))^{-1}\cdot v_\Lambda$, then obviously $\psi_\Lambda\in\mathcal{O}_0(\Lambda)$ (ψ_Λ is of type τ_Λ). It is clear that $l(x)\cdot\psi_\Lambda=0$ for every $x\in\mathfrak{g}^+$, hence ψ_Λ is a highest weight vector of the module $\mathcal{O}_0(\Lambda)$ of highest weight Λ.

Conversely, let $\phi\in\mathcal{O}(\Lambda)$ and satisfying $l(x)\,\phi=0$ for every $x\in\mathfrak{g}^+$ then ϕ is proportional to ψ_Λ (ϕ being analytic, to determine ϕ it is sufficient to know the derivatives of all orders of ϕ at the origin 1 of $\tilde{G}$; we have $(l(u)\,\phi)(1)=0$ for every $u\in\mathfrak{U}\mathfrak{g}^+$ by the hypothesis and this implies $\phi(1)=c\,v_\Lambda$; as $\phi\in O(\Lambda)$ we have $(l(x)\,l(u)\,\phi)(1)=0$ for $x\in\mathfrak{g}$ and $u\in\mathfrak{U}$, the condition of covariance with respect to $\mathfrak{k}^{\mathbb{C}}$ determines. then ϕ completely.)

On $\mathcal{O}_0(\Lambda)$ the compact Cartan subalgebra $\mathfrak{h}$ acts semi-simply. We have

(3.3) **Lemma.** *The weights μ of $\mathfrak{h}^{\mathbb{C}}$ on $\mathcal{O}_0(\Lambda)$ are on the form $\mu=\Lambda-\sum_{\alpha\in\Delta^+} n_\alpha\,\alpha$, where n_α are non-negative integers.*

Proof. Let $u\in\mathfrak{U}$ and $f\in\mathcal{O}_0(\Lambda)$, we define $(u,f)=\langle(u\cdot f)(1), v_\Lambda\rangle$. Then if $(u,f)=0$ for every $u\in\mathfrak{U}$, $f=0$. If u belongs to the right ideal generated by $\mathfrak{g}^-$ and $H-\Lambda(H)$ ($H\in\mathfrak{h}^{\mathbb{C}}$) we have $(u,f)=0$. Let $f\in\mathcal{O}_0(\Lambda)$ of weight μ, hence there exists an element u of $\mathfrak{U}(\mathfrak{g}^+)$ of weight $\psi=\sum_{\alpha\geqq 0} n_\alpha\,\alpha$ such that $(u,f)\neq 0$. Let $H\in h^{\mathbb{C}}$, we have

$$\begin{aligned}(H\,u-u\,H, f)&=\psi(H)\,(u,f)\\ &=\Lambda(H)\,(u,f)-\mu(H)\,(u,f).\end{aligned}$$

It follows that $\mu=\Lambda-\Sigma\,n_\alpha\,\alpha$.

Let $\mathrm{L}(\Lambda)$ be the $\mathfrak{U}$-module generated by the function ψ_Λ.

(3.4) **Corollary.** *Every non-zero $\mathfrak{U}$-submodule of $\mathcal{O}_0(\Lambda)$ contains $L(\Lambda)$.*

Proof. The weights of $\mathcal{O}_0(\Lambda)$ being bounded from above, each submodule has a vector ψ of highest weight, hence contains ψ_Λ.

It is clear then that $\mathrm{L}(\Lambda)=\mathfrak{U}\cdot\psi_\Lambda$ is an irreducible module of highest weight Λ, hence is the module $W(\Lambda)$.

We are concerned with the existence of unitarisable submodules M of $\mathcal{O}_0(\Lambda)$.

(3.5) **Lemma.** *If* M *is a non-zero unitarisable submodule of* $\mathcal{O}_0(\Lambda)$, *then* $\mathrm{M}=\mathrm{L}(\Lambda)$, *hence* M *is irreducible.*

Proof. By Corollary 1.8, M contains $\mathrm{L}(\Lambda)$ but the orthogonal of $\mathrm{L}(\Lambda)$ in M is a submodule disjoint from $\mathrm{L}(\Lambda)$ so is reduced to 0.

(3.6) Let us consider a unitary irreducible representation of $\tilde{G}$ inside a Hilbert space $\mathscr{H}$. Let M′ be the irreducible $\mathfrak{U}$-module consisting of $\tilde{K}$-finite vectors; let us suppose that M′ is a module with a highest weight vector v_Λ with respect to $\mathfrak{g}^+$ of weight Λ; Hence $\mathrm{M}'=W(\Lambda)=\mathrm{L}(\Lambda)$. Let $f\in\mathscr{H}$ and $m\in\mathrm{M}'$, then as m is an analytic vector, the function $\psi(g)=\langle g^{-1} f, m\rangle_{\mathscr{H}}$ is an analytic function on G and we have $(r(u)\psi)(g)=\langle g^{-1}\cdot f, \bar{u}.m\rangle$. Let $V_\Lambda\subset\mathrm{M}'$ be the irreducible unitary representation of $\tilde{K}_{\mathbb{C}}$ generated by m_Λ, then if $m\in V_\Lambda$ and $x\in\mathfrak{p}^+$, $x.m=0$. Let $f\in\mathscr{H}$ and $m\in V_\Lambda$, we define the V_Λ-valued function $\tau(f)$ on $\tilde{G}$ by $\langle\tau(f)(g), m\rangle$ $=\langle g^{-1}\cdot f, m\rangle$, hence if $x\in\mathfrak{p}^-$, $r(x)\,\tau(f)=0$. We remark also that if $f\in\mathrm{M}'$, $\tau(x\cdot f)$ $=l(x)\,\tau(f)$. These observations are reformulated in the

(3.7) **Lemma.** *The map* $f\mapsto\tau(f)$ *is an imbedding of* $\mathscr{H}$ *into* $\mathcal{O}(\Lambda)$; *the image of* M′ *under this imbedding is* $\mathrm{L}(\Lambda)$.

Hence, if we know the highest weight vector, we can embed $\mathscr{H}$ in $\mathcal{O}(\Lambda)$.

(3.8) Let us suppose that $\mathscr{H}$ is a non-zero Hilbert space contained in $\mathcal{O}(\Lambda)$ and where G acts unitarily by left translations; we suppose also that if $f\in\mathscr{H}$ and $g\in G$ the evaluation map at g, $f\mapsto f(g)$ is continuous from $\mathscr{H}$ to $\mathbb{C}$.

We consider the infinitesimal $\mathfrak{U}$-module M of the $\tilde{K}$-finite vectors on $\mathscr{H}$, then M is a $\mathfrak{U}$-submodule of $\mathcal{O}_0(\Lambda)$, so by (3.5), $\mathrm{M}=\mathrm{L}(\Lambda)$ hence $\Lambda\in P$, and the representation of $\tilde{G}$ in $\mathscr{H}$ is irreducible.

Remark. We have that if $f\in\mathscr{H}$, $\langle f(1), v_\Lambda\rangle_{V_\Lambda}=\text{const.}\,\langle f, \psi_\Lambda\rangle_{\mathscr{H}}$ (by hypothesis, there exists a ψ in $\mathscr{H}$ such that $\langle f(1), v_\Lambda\rangle_{V_\Lambda}=\langle f, \psi\rangle_{\mathscr{H}}$; we see easily that ψ is annihilated by $\mathfrak{k}^+\oplus\mathfrak{p}^+$, and hence ψ is proportional to ψ_Λ).

We are interested in determining for which Λ, there is such a $\mathscr{H}$. Naturally, if $\mathscr{H}(\Lambda)=\{f\in\mathcal{O}(\Lambda);\ \int_{\tilde{G}/Z}\|f(g)\|^2_{V_\Lambda}\,dg<\infty\}$ is not reduced to zero, then $\tilde{G}$ acts unitarily in $\mathscr{H}(\Lambda)$ by left translations, so $\Lambda\in P$. It is also clear that the corresponding representation is a member of the relative discrete series of $\tilde{G}$. We then denote by D the set of Λ such that $\mathscr{H}(\Lambda)\neq 0$.

(3.9) This set D has been determined by Harish-Chandra. Let γ_1 the highest non-compact root and let us denote by $\rho=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$, then $\Lambda\in D$ if and only if $\langle\Lambda+\rho, H_{\gamma_1}\rangle<0$.

II. Tensor Products of the Harmonic Representation of Sp$(n,\mathbb{R})$

1. Description of the Harmonic Representation

(2.1) We shall discuss in this section the harmonic representation (or the Segal-Shale-Weil representation) of Sp$(n,\mathbb{R})$ and its tensor products.

The harmonic representation of the symplectic group $\mathrm{Sp}(n, \mathbb{R})$ can be, besides other ways, introduced via intertwining operators of irreducible unitary representations of the Heisenberg group.

Take a $2n$-dimensional vector space with a non-degenerate skew-symmetric form E. We can identify this space with $\mathbb{R}^{2n}$ and denote a point of this space by $\begin{pmatrix} x \\ y \end{pmatrix}$ with $x, y \in \mathbb{R}^n$ such that $E\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = (x_1, y_2) - (x_2, y_1)$. Consider the Heisenberg group $H = \mathbb{R}^{2n} \times \mathbb{R}$ endowed with the multiplication law

$$(w_1, t_1) \circ (w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \tfrac{1}{2} E(w_1, w_2)).$$

The unique class of unitary representation τ of H such that $\tau(0, t) = e^{-it}\,\mathrm{Id}$ can be realized as acting on $L^2(\mathbb{R}^n)$ by

$$\left(T\left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, t\right) f\right)(x) = e^{-i((y_1, x) + t - \frac{1}{2}(y_1, y_2))} f(x - y_2).$$

The symplectic group $\mathrm{Sp}(n, \mathbb{R}) = G$ can be considered as the group of automorphisms of $\mathbb{R}^{2n}$ which leave E invariant.

$\mathrm{Sp}(n, \mathbb{R})$ acts on H by $g \cdot (w, t) = (g w, t)$, and by Von-Neumann theorem, the irreducible unitary representation $T^g(w, t) = T(g w, t)$ is equivalent to T for any g in G. Therefore, there exists a unitary intertwining operator $L(g)$ (unique up to a constant multiple) acting on $L^2(\mathbb{R}^n)$ such that $L(g)\, T(w, t) = T(g \cdot (w, t))\, L(g)$ for any $(w, t) \in H$, and $L(g_1, g_2) = c(g_1, g_2)\, L(g_1)\, L(g_2)$ for $c(g_1, g_2)$ a scalar of modulus one. However, if we choose $L(g)$ suitably, $g \mapsto L(g)$ becomes a unitary representation of the two-sheeted covering group $Mp(n) = G_2$ (called the metaplectic group) of $\mathrm{Sp}(n, \mathbb{R})$.

(1.2) We denote by $\mathrm{M}_n(\mathbb{R})$ (resp. $\mathrm{M}_n(\mathbb{C})$) the space of all $n \times n$ real (resp. complex) matrices. We denote by $\mathrm{M}_{n,k} = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^k, \mathbb{R}^n)$ (resp. $\mathrm{M}_{n,k}(\mathbb{C})$) the space of all $n \times k$ real (resp. complex) matrices, $x \mapsto {}^t x$ the transposition; we consider $\mathrm{M}_{n,k}$ with the scalar product $\mathrm{Tr}\, x\, {}^t x$. We denote by $S(n)$ the space of all symmetric real matrices of size n.

We define the following elements of $\mathrm{Sp}(n, \mathbb{R})$; (we write any $2n \times 2n$ matrices $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ where x_{ij} are $n \times n$ matrices).

$$g(a) = \begin{pmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{pmatrix} \quad \text{for } a \in GL(n, \mathbb{R}),$$

$$t(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{for } b \in S(n),$$

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

These elements generate $\mathrm{Sp}(n, \mathbb{R})$.

Since $\{t(b);\, b \in S(n)\}$ is simply connected, $t(b)$ can be identified as an element in G_2 so that $t(0)$ is the identity of G_2.

For each $a \in GL(n, \mathbb{R})$, we choose a determination of $(\det a)^{\frac{1}{2}}$. For this choice, we still denote the element $(a, (\det a)^{\frac{1}{2}})$ by $g(a)$. We will identify in (3) $g(a)$ as an element of G_2. We will also define $(\sigma, i^{\frac{1}{2}})$ in G_2 above σ according to the choice of $i^{\frac{1}{2}}$.

We will see that the following choice of $L(g)$ determines a representation of G_2.

$$(L(g(a))f)(x) = (\det a)^{\frac{1}{2}} f({}^t a x),$$

$$(L(t(b))f)(x) = e^{-\frac{i}{2}(bx,x)} f(x),$$

$$(L(\sigma)f)(x) = \left(\frac{i}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{i(x,y)} f(y)\,dy,$$

i.e. $L(\sigma)$ is proportional to the Fourier transform.

(1.3) Therefore the k-th tensor product L_k of L is given as follows:

Let $L^2(\mathrm{M}_{n,k})$ be the Hilbert space of all square-integrable functions on $\mathrm{M}_{n,k}$. L_k acts on $L^2(\mathrm{M}_{n,k})$ as follows

$$(L_k(g(a))f)(x) = (\det a)^{k/2} f({}^t a x),$$

$$(L_k(t(b))f)(x) = e^{-\frac{i}{2}\mathrm{Tr}\,{}^t x b x} f(x),$$

$$(L_k(\sigma)f)(x) = \left(\frac{i}{2\pi}\right)^{nk/2} \int_{M_{n,k}} e^{i\,\mathrm{Tr}\,x\,{}^t y} f(y)\,dy.$$

2. *The Action of the Orthogonal Group* O(k)

On $L^2(\mathrm{M}_{n,k})$ the action of the orthogonal group $\mathrm{O}(k)$ given by $(h \cdot f)(x) = f(xh)$ commutes with the representation L_k and hence we can decompose $L^2(\mathrm{M}_{n,k})$ under $\mathrm{O}(k)$. For any representation space M of $\mathrm{O}(k)$ and any λ an irreducible unitary representation of $\mathrm{O}(k)$, we denote by M_λ the isotypic component of M of type λ. If we denote $M(\lambda) = \mathrm{Hom}_{\mathrm{O}(k)}(\lambda', M)$, we have $M_\lambda = V_\lambda \otimes_{\mathbb{C}} M(\lambda')$. Here λ' is the contragredient representation of λ. We have $L^2(\mathrm{M}_{n,k}) = \bigoplus_{\lambda \in \mathrm{O}(k)^\wedge} L^2(\mathrm{M}_{n,k})_\lambda$ and each $L^2(\mathrm{M}_{n,k})$ is stable under G_2.

(2.1) We denote by $L^2(\mathrm{M}_{n,k}; \lambda)$ the space of all square-integrable V_λ-valued functions $f(x)$ with the covariance relation $f(xh) = \lambda(h)^{-1} f(x)$ for any h in $\mathrm{O}(k)$. Here (V_λ, λ) is an irreducible unitary representation of $\mathrm{O}(k)$. The group G_2 acts on $L^2(M_{n,k}; \lambda)$ by the same formulas as L_k (1.3). We denote by $\mathrm{L}_k(\lambda)$ the corresponding representation.

Let λ' be the contragredient representation of λ on the dual space V'_λ and let us denote by (x, f) the canonical bilinear pairing on $V_\lambda \times V'_\lambda$. $L^2(M_{n,k})_\lambda$ is isomorphic to $L^2(M_{n,k}; \lambda') \otimes V_\lambda$ by $(\phi \otimes f)(x) = (\phi(x), f)$ for ϕ in $L^2(M_{n,k}; \lambda')$ and f in V_λ. Therefore we get

$$L^2(\mathrm{M}_{n,k}) = \bigoplus_{\lambda \in \mathrm{O}(k)^\wedge} L^2(\mathrm{M}_{n,k}; \lambda') \otimes V_\lambda$$

as a representation of $G_2 \times O(k)$, and $L_k = \bigoplus (\dim V_\lambda) L_k(\lambda)$ as a representation of G_2.

In the sequel, we shall prove that $L_k(\lambda)$ is an irreducible representation of G_2 with highest weight vector of type $\tau(\lambda)$. We will identify $L^2(M_{n,k}; \lambda)$ with a Hilbert space of holomorphic functions on the Siegel upper half-plane D. We shall see also that $\lambda \mapsto \tau(\lambda)$ is injective and hence $L_k(\lambda)$ appears in $L^2(M_{n,k})$ with multiplicity $\dim V_\lambda$.

3. Representation on the Space of Holomorphic Functions on Siegel Domain

(3.1) We shall apply the discussion of I.3 to our case $Sp(n, \mathbb{R}) = G$ and use the notations of I. Let X be the complex manifold of all Lagrangian planes in the symplectic vector space $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^{2n}$, i.e. all n-dimensional isotropic vector spaces λ. The complexification $G_{\mathbb{C}} = Sp(n, \mathbb{C})$ of the group $Sp(n, \mathbb{R})$ acts on X homogeneously in the obvious way. Let us denote by D the open subset of X consisting of all λ's such that the hermitian form $\frac{1}{i} E(x, \bar{y})$ is positive definite on λ. D is identified with the Siegel upper half-plane $\{z \in M_n(\mathbb{C});\ z = {}^t z,\ \operatorname{Im} z \gg 0\}$ by $z \leftrightarrow \lambda = \left\{ \begin{pmatrix} x \\ y \end{pmatrix}; x = zy \right\}$. The group G acts homogeneously on D by $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $z \mapsto (az+b)(cz+d)^{-1}$. The isotropy subgroup K at $z = i$ is given by

$$K = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix}; a, b \in M_n(\mathbb{R}); a + ib \in U(n) \right\}$$

and K is a maximal compact subgroup of G. The Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of $Sp(n, \mathbb{C})$ consists of all $2n \times 2n$ complex matrices $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ with $x_{22} = -{}^t x_{11}$, $x_{12} = {}^t x_{12}$, $x_{21} = {}^t x_{21}$. Let $\mathbf{c} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ be the Cayley transform; it is an element of $Sp(n, \mathbb{C})$, and

$$\mathbf{c}^{-1} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mathbf{c} = \begin{pmatrix} a+ib & 0 \\ 0 & a-ib \end{pmatrix}.$$

Therefore

$$\operatorname{Ad} \mathbf{c}^{-1}(K_{\mathbb{C}}) = \left\{ \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix}; g \in GL(n, \mathbb{C}) \right\}.$$

Then we choose

$$\mathfrak{p}^+ = (\operatorname{Ad} \mathbf{c}) \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}; x \in M_n(\mathbb{C}), x = {}^t x \right\},$$

$$\mathfrak{p}^- = (\operatorname{Ad} \mathbf{c}) \left\{ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}; y \in M_n(\mathbb{C}), y = {}^t y \right\},$$

$$\mathfrak{h}^{\mathbb{C}} = \operatorname{Ad} \mathbf{c} \left\{ \left(\begin{array}{ccc|ccc} a_1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & a_n & & & \\ \hline & & & -a_1 & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & -a_n \end{array} \right); a_i \in \mathbb{C} \right\}$$

and the system Δ^+ of positive roots such that

$$\mathfrak{k}^+ = \operatorname{Ad}(\mathbf{c}) \left\{ \begin{pmatrix} x_{11} & 0 \\ 0 & -{}^t x_{11} \end{pmatrix}; (x_{11})_{ij} = 0 \quad \text{for } i \geqq j \right\}.$$

Let γ_1 be the highest non-compact root; then

$$H_{\gamma_1} = \operatorname{Ad} \mathbf{c} \left(\begin{array}{cc|cc} 1 & \overset{\overset{n-1}{\leftrightarrow}}{0} & & \\ 0 & 0 & \multicolumn{2}{c}{0} \\ \hline & & -1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 0 \end{array} \right) \quad \text{and} \quad \rho(H_{\gamma_1}) = n.$$

(3.2) Let (V_τ, τ) be a representation of K in a finite dimensional vector space V_τ; we extend τ as a holomorphic representation of $K_{\mathbb{C}}$ identified with $GL(n, \mathbb{C})$ by $a \mapsto \mathbf{c} \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \mathbf{c}^{-1}$. Let us denote by x_0 the element $\lambda = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix}; y \in \mathbb{C}^n \right\}$ of X. We have $\mathbf{c}(x_0) = i = e_0$ and $K_{\mathbb{C}} P_-$ is the stabilizer of e in $G_{\mathbb{C}}$. We define $G_u = \{g \in G_{\mathbb{C}}; g \cdot e_0 \in D\}$, i.e. $G_u = G K_{\mathbb{C}} P_-$. Let $z \in D$, then, by the definition, $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot x_0 = z$, hence every element of G_u can be written uniquely

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mathbf{c}^{-1}$$

with $z \in D$ $a \in GL(n, \mathbb{C})$, $x = {}^t x \in M_n(\mathbb{C})$. Recall that $\mathcal{O}(\tau)$ is the space of holomorphic sections of $G \underset{K}{\times} V_\tau$. A function in $\mathcal{O}(\tau)$ can obviously be prolonged to $G_u = G K_{\mathbb{C}} P_-$ via $f(gkp) = \tau(k)^{-1} f(g)$ $(g \in G,\ k \in K_{\mathbb{C}},\ p \in P_-)$. We identify f as a V_τ-valued holomorphic function on D by $f(z) = f\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mathbf{c}^{-1} \right)$.

For $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we have:

$$g^{-1} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (az+b)(cz+d)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^t(cz+d)^{-1} & 0 \\ c & (cz+d) \end{pmatrix}.$$

Hence we have

$$(T(g)f(z)=f\left(g^{-1}\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\mathbf{c}^{-1}\right)$$
$$=f\left(\begin{pmatrix}1 & (az+b)(cz+d)^{-1}\\ 0 & 1\end{pmatrix}\mathbf{c}^{-1}\cdot\operatorname{Ad}\mathbf{c}\begin{pmatrix}{}^t(cz+d)^{-1} & 0\\ c & (cz+d)\end{pmatrix}\right)$$
$$=\tau({}^t(cz+d))\,f((az+b)(cz+d)^{-1}).$$

Thus we have

(3.3) **Proposition.** *$\mathcal{O}(\tau)$ is isomorphic to the space $\mathcal{O}(D, V_\tau)$ of all V_τ-valued holomorphic functions $f(z)$ on D, on which G acts by*

$$(T(\tau)(g)f)(z)=\tau({}^t(cz+d))\,f((az+b)(cz+d)^{-1})$$

for $z\in D$ and $g^{-1}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$.

(3.4) Let us assume here that τ is irreducible. Let v_τ be the highest weight vector of τ, then under this identification the highest weight vector ψ_τ (I.3.2) becomes (up to a constant factor) the function $f_\tau(z)=\tau\left(\frac{z+i}{i}\right)\cdot v_t$

$$\left(\text{because if } \begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\mathbf{c}^{-1}=\mathbf{c}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & {}^ta^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}\mathbf{c}^{-1}\right.$$

$$\left.\text{then } {}^ta^{-1}=\frac{1}{\sqrt{2}}\left(\frac{z+i}{i}\right)\right).$$

(3.5) We will use for our purpose the following description of the universal covering group of G. For each $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in G$ we consider the function $z\mapsto d(g,z)=\det(cz+d)$ from D into $\mathbb{C}-\{0\}$. A determination on D of $\operatorname{Log}(\det(cz+d))$ is then completely determined by its value on $z=i$, as D is simply connected, which is a complex number defined modulo $2i\pi\mathbb{Z}$. We have $d(g_1\cdot g_2, z)=d(g_1, g_2\cdot z)\cdot d(g_2, z)$. We consider the group $\tilde{G}=\{(g, \operatorname{Log} d(g,z))\}$ endowed with the law

$$(g_1, \operatorname{Log} d(g_1,z))\cdot(g_2, \operatorname{Log} d(g_2,z))=(g_1g_2, \operatorname{Log} d(g_1, g_2\cdot z)+\operatorname{Log} d(g_2,z)).$$

It is clear that $\tilde{G}$ is a covering group of G, and it is in fact its universal covering. Let α be any real number, we can then define $\det(cz+d)^\alpha=e^{i\alpha \operatorname{Log} d(g,z)}$ on $\tilde{G}\times D$. In particular, we can consider the representations

$$(T(\tau,\alpha)(\tilde{g})f)(z)=\det(cz+d)^{-\alpha}\,\tau({}^t(cz+d))\,f((az+b)(cz+d)^{-1})$$

for $\tilde{g}^{-1}=(g, \operatorname{Log} d(g,z))$, $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ of $\tilde{G}$ inside the space of V_τ-valued holomorphic functions on D. This is equivalent to $\mathcal{O}(\tau\otimes\delta_{-\alpha})$ where δ_α is the representation of $\tilde{K}$ defined by $\det(cz+d)^\alpha$ at $z=i$. We will study in this article only the values $\alpha\in\mathbb{Z}/2$. Therefore we define $G_2=\{\tilde{g}=(g, (\det(cz+d))^{\frac{1}{2}})\}$ with the obvious law and the representations, for $k \in \mathbb{Z}$, of G_2 in $\mathcal{O}(D, V_\tau)$

$$(T(\tau, k)(\tilde{g}) f)(z) = \det(cz+d)^{-k/2}\, \tau({}^t(cz+d))\, f((az+b)(cz+d)^{-1})$$

for $(\tilde{g})^{-1} = (g, \det(cz+d)^{\frac{1}{2}})$ with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(3.6) Let $a \in \mathrm{GL}(n, \mathbb{R})$, suppose we have chosen $(\det a)^{\frac{1}{2}}$ one of the 2 determinations of $(\det a)^{\frac{1}{2}}$. Above $g(a) = \begin{pmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{pmatrix}$ we choose as determination of $\det(cz+d)^{\frac{1}{2}} = (\det(a^t)^{-1})^{\frac{1}{2}}$ the value $((\det a)^{\frac{1}{2}})^{-1}$, and we denote this element of G_2 by $g(a)$.

Above $\sigma^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we choose the determination of $(\det - z)^{\frac{1}{2}}$ such that it is equal to $(i^{\frac{1}{2}})^{-n}$ at $z = i$, according to the choice of $i^{\frac{1}{2}}$. we still denote this element by σ^{-1}.

These are the precise definitions of the elements of G_2 used in Formulas (1.2) and (1.3).

4. *Intertwining Operators*

(4.1) Let (V_τ, τ) be a holomorphic representation of $\mathrm{GL}(n, \mathbb{C})$ in a finite dimensional vector space V_τ and (V_λ, λ) an irreducible unitary representation of $\mathrm{O}(k)$. We choose a scalar product on V_τ such that $\tau(g^*) = \tau(g)^*$. We seek to construct an operator from $L^2(\mathrm{M}_{n,k}; \lambda)$ into $\mathcal{O}(D, V_\tau)$. Let $I(x)$ be a $\mathrm{Hom}_{\mathbb{C}}(V_\tau, V_\lambda)$-valued polynomial of $x \in \mathrm{M}_{n,k} = M(n, k; \mathbb{R})$. Consider the integral transform

(4.2) $$(\mathcal{F}_{\mathrm{I}} f)(z) = \int_{\mathrm{M}_{n,k}} e^{\frac{i}{2}\mathrm{Tr}\, {}^t x z x}\, I(x)^*\, f(x)\, dx$$

for $f \in L^2(\mathrm{M}_{n,k}; \lambda)$ and $z \in D \subset M_n(\mathbb{C})$.

(4.3) **Lemma.** *$\mathcal{F}_{\mathrm{I}}(f)$ is an absolutely convergent integral. The map $f \mapsto \mathcal{F}_{\mathrm{I}} f$ is continuous from $L^2(\mathrm{M}_{n,k}; \lambda)$ to $\mathcal{O}(D, V_\tau)$ with the topology of uniform convergence on compacta.*

Proof. Since $I(x)$ is a polynomial, we have

$$\|I(x)^* f(x)\| \leq (1 + \mathrm{tr}\, x\, {}^t x)^N \|f(x)\| \qquad \text{for some } N.$$

Therefore

$$\begin{aligned} \|(\mathcal{F}_{\mathrm{I}} f)(z)\|^2 &\leq \Big(\int e^{-\mathrm{Tr}\frac{{}^t x (\mathrm{Im}\, z) x}{2}} (1 + \mathrm{Tr}\, x\, {}^t x)^N \|f(x)\|\, dx\Big)^2 \\ &\leq \Big(\int e^{-\mathrm{Tr}\, {}^t x (\mathrm{Im}\, z) x} (1 + \mathrm{Tr}\, x\, {}^t x)^{2N} dx\Big)\Big(\int \|f(x)\|^2\, dx\Big). \end{aligned}$$

which implies immediately the desired result, as $e^{-\mathrm{Tr}\, {}^t x y x}(1 + \mathrm{Tr}\, x\, {}^t x)^{2N}$ is integrable for any positive definite matrix y.

A function $\phi(x)$ on $\mathrm{M}_{n,k}$ is called harmonic if it satisfies

$$\sum_{i=1}^{n}\sum_{\nu=1}^{k}\frac{\partial^2}{\partial x_{i\nu}\partial x_{i\nu}}\phi=0 \quad \text{where} \quad x=\begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & & \vdots \\ x_{n1} & & x_{nk}\end{pmatrix}\in \mathrm{Hom}(\mathbb{R}^k;\mathbb{R}^n).$$

(4.4) **Proposition.** *Suppose that I satisfies the following two conditions:*

a) $I(axh)=\lambda(h)^{-1}I(x)\tau(a)^{-1}$ *for* $a\in GL(n,\mathbb{R})$ *and* $h\in \mathrm{O}(k)$.

b) $I(x)$ *is a harmonic polynomial.*

Then the map $\mathscr{F}_I$: $L^2(\mathrm{M}_{n,k};\lambda)\to\mathcal{O}(D,V_\tau)$ *intertwines the representation* $L_k(\lambda)$ *with the representation* $T(\tau,k)$ (3.5) *of* G_2.

Proof. Since G_2 is generated by $g(a)$, $t(b)$ and σ, it is enough to show that $\mathscr{F}_I$ commutes with these actions. It is obvious to check on the formulas that $\mathscr{F}_I$ commutes with $g(a)$, $t(b)$; for example, for $g=g(a)$, we have

$$(L_k(g)f)(x)=(\det a)^{k/2}f({}^t a x)$$

and

$$(T(\tau,k)(g)f)(z)=(\det a)^{-k/2}\tau(a)f(a^{-1}z{}^ta^{-1}).$$

We compute

$$\begin{aligned}&(\mathscr{F}_I L_k(g)f)(z)\\ &=\int e^{\frac{i}{2}\mathrm{Tr}\,{}^txzx}I(x)^*(\det a)^{k/2}f({}^tax)\,dx\\ &=(\det a)^{\frac{k}{2}-k}\int e^{\frac{i}{2}\mathrm{Tr}\,{}^txa^{-1}z{}^ta^{-1}x}I({}^ta^{-1}x)^*f(x)\,dx\\ &=(\det a)^{-k/2}\tau(a)(\mathscr{F}_I f)(a^{-1}z{}^ta^{-1}).\end{aligned}$$

We shall check now the action of σ. We have

$$\begin{aligned}(\mathscr{F}_I L_k(\sigma)f)(z)&=\int e^{\frac{i}{2}\mathrm{Tr}\,{}^txzx}\left(\frac{i}{2\pi}\right)^{\frac{nk}{2}}I(x)^*\left(\int e^{i\,\mathrm{Tr}\,{}^txy}f(y)\,dy\right)dx\\ &=\left(\frac{i}{2\pi}\right)^{\frac{nk}{2}}\int\left(\int e^{i\,\mathrm{Tr}\,{}^txy}e^{\frac{i}{2}\mathrm{Tr}\,{}^txzx}I(x)^*\,dx\right)f(y)\,dy.\end{aligned}$$

by Plancherel formula.

Let us assume the

(4.5) **Lemma.** *Let* $f(x)$ *be a polynomial on* $\mathrm{M}_{n,k}$ *(that we extend as a complex polynomial on* $\mathrm{M}_{n,k}(\mathbb{C})$*) such that the function* $x\mapsto f(gx)$ *is harmonic for any* $g\in \mathrm{GL}(n,\mathbb{R})$, *then for* $z\in D$:

$$\int e^{i\,\mathrm{Tr}\,{}^txy}e^{\frac{i}{2}\mathrm{Tr}\,{}^txzx}f(x)\,dx=(2\pi)^{\frac{nk}{2}}\left(\det\frac{z}{i}\right)^{-k/2}e^{\frac{i}{2}\mathrm{Tr}\,{}^ty(-z^{-1})y}f(-z^{-1}y).$$

Here, $(\det z/i)^{-k/2}$ *takes a branch such that it equals* 1 *at* $z=i$.

As we have $I(ax)^* = \tau({}^t a^{-1})\, I(x)^*$ on $M_{n,k}$, $I(x)^*$ satisfies the condition of the lemma. If we extend $I(x)^*$ to a complex polynomial $J(x)$ on $M_{n,k}(\mathbb{C})$, we will have $J(ax) = \tau({}^t a^{-1})\, J(x)$ for $x \in M_{n,k}(\mathbb{C})$, $a \in GL(n, \mathbb{C})$. We apply the lemma and thus we have:

$$\begin{aligned}
&\int e^{i\operatorname{Tr}{}^t xy}\, e^{\frac{i}{2}\operatorname{Tr}{}^t xzx}\, I(x)^*\, dx \\
&\quad = (2\pi)^{nk/2} \left(\det \frac{z}{i}\right)^{-k/2} e^{\frac{i}{2}\operatorname{Tr}{}^t y(-z^{-1})y}\, J(-z^{-1}y) \\
&\quad = (2\pi)^{nk/2} \left(\det \frac{z}{i}\right)^{-k/2} e^{\frac{i}{2}\operatorname{Tr}{}^t y(-z^{-1})y}\, \tau(-z)\, I(y)^*.
\end{aligned}$$

And hence we get

$$\begin{aligned}
(\mathscr{F}_I L(\sigma) f)(z) &= (\det -z)^{-k/2}\, \tau(-z) (\mathscr{F}_I f)(-z^{-1}) \\
&= (T(\tau, k)(\sigma)(\mathscr{F}_I f))(z). \quad \text{q.e.d.}
\end{aligned}$$

It remains to prove (4.5); as both sides are holomorphic in z, it is enough to check it when $z = i\alpha^2$ with a positive definite real symmetric matrix α. The left hand side is equal to

$$\int e^{i\operatorname{Tr}{}^t xy}\, e^{-\frac{1}{2}\operatorname{Tr}{}^t(\alpha x)(\alpha x)} f(x)\, dx = (\det \alpha)^{-k} \int e^{i\operatorname{Tr}{}^t x\alpha^{-1}y}\, e^{-\frac{1}{2}\operatorname{Tr}{}^t xx} f(\alpha^{-1}x)\, dx.$$

Since $f(\alpha^{-1}x)$ is a harmonic polynomial, it follows from the mean-value formula, that this is equal to:

$$\begin{aligned}
&(\det\alpha)^{-k} (2\pi)^{\frac{nk}{2}}\, e^{-\frac{1}{2}\operatorname{Tr}{}^t(\alpha^{-1}y)\alpha^{-1}y} f(\alpha^{-1}(i\alpha^{-1}y)) \\
&\quad = (2\pi)^{\frac{nk}{2}} \left(\det \frac{z}{i}\right)^{-k/2} e^{-\frac{1}{2}\operatorname{Tr}{}^t y(-z^{-1})y} f(-z^{-1}y).
\end{aligned}$$

(4.6) Let us consider the map $x \mapsto \xi(x) = x\,{}^t x$. It is a proper map from $M_{n,k}$ into the space $S(n)$ of symmetric $n \times n$ real matrices, and we have $\xi(x) = \xi(x')$ if and only if $x = x'h$ with $h \in O(k)$. Let C be the cone of all positive definite symmetric matrices. If $k \geqq n$, the image of the map ξ consists of all the positive semi-definite symmetric matrices, i.e. is the closure $\bar{C}$ of the cone C. We denote by $d_k(\xi)$ the measure on $\bar{C}$ obtained from dx, i.e. $d_k(\xi) = \text{const.} (\det \xi)^{\frac{k-(n+1)}{2}}\, d\xi$.

If $k < n$, then the matrix $x\,{}^t x$ is of rank less than or equal to k; we will denote by $b_k(C)$ the subset of the boundary of C consisting of all the positive semi-definite matrices of rank less or equal to k, and by $d_k(\xi)$ the measure on $b_k(C)$ defined by

$$\int f(x\,{}^t x)\, dx = \int_{b_k(C)} f(\xi)\, d_k(\xi).$$

Let I verify the conditions of the Proposition (4.4) and $f \in L^2(M_{n,k}; \lambda)$; then the function $I(x)^* f(x)$ is invariant under right translations by the group $O(k)$, hence is a function $\phi(\xi)$ of $\xi = x\,{}^t x$. The integral transform $\mathscr{F}_I$ can be written

$$(\mathscr{F}_I f)(z) = \int e^{\frac{i\operatorname{Tr}\xi z}{2}} \phi(\xi)\, d_k(\xi)$$

i.e. $\mathscr{F}_I f$ is the Fourier Laplace transform of the measure $\phi(\xi)\, d_k(\xi)$ supported on the cone $\bar{C}$;

It follows that if $\mathscr{F}_I f=0$ then $\phi(\xi)$ is almost every where zero for $d_k(\xi)$ i.e. $I(x)^* f(x)$ must be zero almost everywhere.

5. *Pluriharmonic Polynomials*

(5.1) In order to use the Proposition (4.4), we have to investigate for a given λ what are the possible τ's for which a non zero I satisfying (4.4) a) b) will exist.

Since such an I verifies $I(ax)=I(x)\,\tau(a)^{-1}$, the function $x \mapsto I(ax)$ is also harmonic. It is easily observed that a function f is such that $f_a(x)=f(ax)$ is harmonic for any $a \in \mathrm{GL}(n, \mathbb{R})$, if and only if

(5.2) $(\Delta_{i,j} f)(x)=0 \quad 1 \leqq i \leqq j \leqq n.$

Here

$$\Delta_{i,j} = \sum_{\nu=1}^{k} \frac{\partial^2}{\partial x_{i\nu}\, \partial x_{j\nu}}.$$

Note that the ring of constant coefficient differential operators invariant by $\mathrm{O}(k)$ is generated by $\Delta_{i,j}$'s.

If a function f satisfies (5.2), then we will call f pluriharmonic. We consider indifferently a polynomial on $\mathrm{M}_{n,k}$ as a complex polynomial on $\mathrm{M}_{n,k}(\mathbb{C})$. We denote by $\mathbb{C}[\mathrm{M}_{n,k}]$ the ring of polynomials on $\mathrm{M}_{n,k}$ and $\mathfrak{H}$ the space of all pluriharmonic polynomials. Evidently $GL(n, \mathbb{C}) \times \mathrm{O}(k, \mathbb{C})$ acts on $\mathfrak{H}$ by (g, h): $f(x) \mapsto f(g^{-1} x h)$. $\mathbb{C}[M_{n,k}]$ has a positive definite hermetian form $\langle f, g\rangle = (f(D_x)\,\bar{g})(0)$. This scalar product satisfies $\langle a \cdot f, g\rangle = \langle f, a^* g\rangle$ for $a \in GL(n, \mathbb{C})$. If we denote by J the ideal of $\mathbb{C}[\mathrm{M}_{n,k}]$ generated by the coefficient $(x^t x)_{i,j}$ of $x^t x$, then it is obvious that $\mathfrak{H}$ is the orthogonal complement of J and hence we have $\mathbb{C}[\mathrm{M}_{n,k}] = \mathfrak{H} \oplus J$.

We denote by $\mathbb{C}[x^t x]$ the subring of $\mathbb{C}[\mathrm{M}_{n,k}]$ generated by $(x^t x)_{i,j}$ $(1 \leqq i \leqq j \leqq n)$ i.e. the ring of invariant polynomials under $\mathrm{O}(k)$.

(5.3) **Lemma.** $\mathbb{C}[\mathrm{M}_{n,k}] = \mathfrak{H} \cdot \mathbb{C}[x^t x]$.

Proof. We shall show that a homogeneous polynomial $f(x)$ is contained in $\mathfrak{H} \cdot \mathbb{C}[x^t x]$ by induction on the degree of f. Since $\mathbb{C}[\mathrm{M}_{n,k}] = \mathfrak{H} \oplus J$, f can be written on the form $f(x) = h(x) + \Sigma\, \phi_{i,j}(x)(x^t x)_{i,j}$ where $h \in \mathfrak{H}$. We may assume that h is of degree deg. f, and $\phi_{i,j}$ of degree deg. $f-2$, so by induction hypothesis $\phi_{i,j} \in \mathfrak{H} \cdot \mathbb{C}[x^t x]$ and hence f.

Remark. In general we have not $\mathbb{C}[\mathrm{M}_{n,k}] = \mathfrak{H} \otimes \mathbb{C}[x^t x]$; however, if $k \geqq 2n$ this is true [15].

(5.4) **Corollary.** *We denote by* $\mathbb{C}[\mathrm{M}_{n,k}](\lambda)$ (resp. $\mathfrak{H}(\lambda)$) *the space of all* V_λ*-valued polynomials* $f(x)$ (*resp. pluriharmonic polynomials* $f(x)$) *such that* $f(xh) = \lambda(h)^{-1} f(x)$, *then*

$$\mathbb{C}[M_{n,k}](\lambda) = \mathbb{C}[x^t x] \cdot \mathfrak{H}(\lambda).$$

Proof. We have $\mathfrak{H} = \oplus\, \mathfrak{H}_\lambda$, and $\mathfrak{H}_{\lambda'} = \mathfrak{H}(\lambda) \otimes V_{\lambda'}$ hence

$$(5.3)\quad \begin{aligned} \mathbb{C}[\mathrm{M}_{n,k}]_{\lambda'} &= \mathbb{C}[x^t x]\, \mathfrak{H}_{\lambda'} \\ &= \mathbb{C}[x^t x]\, \mathfrak{H}(\lambda) \otimes V_{\lambda'} \\ &= \mathbb{C}[M_{n,k}](\lambda) \otimes V_{\lambda'} \end{aligned}$$

and the corollary follows.

(5.5) **Lemma.** *Let P and Q be harmonic polynomials on $M_{n,k}$. Then*

$$\langle P, Q \rangle = (2\pi)^{-nk/2} \int e^{-\mathrm{Tr}\frac{x^t x}{2}}\, P(x)\, \overline{Q(x)}\, dx.$$

Proof. If Q is harmonic, so is any derivative of Q. By the mean-value formula for harmonic polynomials on $\mathbb{R}^{\frac{nk}{2}}$, we have:

$$\int e^{-\mathrm{Tr}\frac{x^t x}{2}} (P(D_x)\,\bar{Q})(x)\, dx = (2\pi)^{\frac{nk}{2}} (P(D_x)\,\bar{Q})(0) = (2\pi)^{\frac{nk}{2}} \langle P, Q \rangle.$$

Now the first hand side is also equal to

$$\int (P(-D_x)\, e^{-\mathrm{Tr}\frac{x^t x}{2}})\, \bar{Q}(x)\, dx.$$

Now as

$$e^{-\mathrm{Tr}\frac{x^t x}{2}} = (2\pi)^{\frac{-nk}{2}} \int e^{i\mathrm{Tr}\, x^t y}\, e^{-\mathrm{Tr}\frac{y^t y}{2}}\, dy,$$

$$\begin{aligned} P(-D_x)\, e^{-\mathrm{Tr}\frac{x^t x}{2}} &= (2\pi)^{\frac{-nk}{2}} \int e^{i\mathrm{Tr}\, x^t y}\, e^{-\mathrm{Tr}\frac{y^t y}{2}}\, P(-iy)\, dy \\ &= e^{-\mathrm{Tr}\frac{x^t x}{2}}\, P(x), \qquad \text{as } P \text{ is harmonic.} \end{aligned}$$

(5.6) We shall denote by Σ the subset of all $\lambda \in \mathrm{O}(k)^\wedge$ such that $L^2(\mathrm{M}_{n,k}; \lambda) \neq 0$. This condition is equivalent to $\mathbb{C}[\mathrm{M}_{n,k}](\lambda) \neq 0$, and hence $\mathfrak{H}(\lambda) \neq 0$ (5.4) as

$$\{e^{-\mathrm{Tr}\, x^t x} f(x);\, f(x) \in \mathbb{C}[M_{n,k}](\lambda)\} \quad \text{is dense in } L^2(\mathrm{M}_{n,k}; \lambda).$$

For $\lambda \in \Sigma$, we denote by $\tau(\lambda)$ the representation of $GL(n, \mathbb{C})$ by left translation on $\mathfrak{H}(\lambda)$.

(5.7) **Proposition.** *The representation $\tau(\lambda)$ is an irreducible representation of $GL(n, \mathbb{C})$.*

Proof. As the action of $\mathrm{GL}(n, \mathbb{C})$ conserves the degree of the polynomials, the representation $\tau(\lambda)$ is semi-simple; Let V_1 be an invariant subspace of $\mathfrak{H}(\lambda)$ and V_2 its orthogonal; thus we have by (5.5), if $P \in V_1$, $Q \in V_2$

$$\int e^{-\mathrm{Tr}\frac{x^t x}{2}} \langle (a \cdot P)(x), (a \cdot Q)(x) \rangle\, dx = 0 \qquad \text{for any } a \in \mathrm{GL}(n, \mathbb{R}),$$

hence

$$\int e^{-\frac{1}{2}\mathrm{Tr}\,{}^t a a x {}^t x} \langle P(x), Q(x)\rangle \, dx = 0.$$

As $P, Q \in \mathfrak{H}(\lambda)$, the function $\langle P(x), Q(x)\rangle$ is invariant under right translation of $\mathrm{O}(k)$, i.e. is a function of $\xi = x\,{}^t x$; as the set of functions $\{e^{-\mathrm{Tr}\, y x {}^t x}; y \in C\}$ is dense in this space, we obtain $\langle P(x), Q(x)\rangle \equiv 0$.

(5.8) Let us suppose $k \leqq n$; Let us consider the point $1_k = \begin{pmatrix} \mathbb{1}_k \\ 0 \end{pmatrix}$ of $\mathrm{M}_{n,k}$ where $\mathbb{1}_k$ is the $k \times k$ identity matrix. Let M_0 be the dense open subset of $M_{n,k}$ consisting of the x's which are injective maps from $\mathbb{R}^k$ to $\mathbb{R}^n$; M_0 is the orbit of 1_k under the action of $\mathrm{GL}(n, \mathbb{R})$, because if $x(e_1), \ldots, x(e_k)$ are linearly independent, they can be transformed in $e_1, e_2, \ldots, e_k$ by an invertible matrix. Let us embed $\mathrm{O}(k)$ in $GL(n, \mathbb{R})$ by $a \mapsto \tilde{a} = \left(\begin{array}{c|c} a & 0 \\ \hline 0 & 1 \end{array}\right)$. We have $\tilde{a} \cdot 1_k = 1_k\, a$.

Now if $V_1 \neq 0$, then exists a P in V_1 such that $P(1_k) \neq 0$ (translate by $\mathrm{GL}(n, \mathbb{R})$). Furthermore, the subset $\{P(1_k); P \in V_1\}$ of V_λ is invariant under λ, as

$$(\tilde{h} P)(1_k) = P(\tilde{h}^{-1} \cdot 1_k) = P(1_k h^{-1}) = \lambda(h) \cdot P(1_k) \qquad \text{for } h \in \mathrm{O}(k).$$

Since V_λ is an irreducible representation of $\mathrm{O}(k)$, $\{P(1_k); P \in V_1\}$ is equal to V_λ. Now it follows that $V_2 = 0$, as otherwise we could choose in V_2 a polynomial Q such that $P(1_k) = Q(1_k) \neq 0$, contradicting the fact that $\langle P(x), Q(x)\rangle \equiv 0$.

(5.9) Let us now suppose that $k \geqq n$. Let f be a function on $\mathrm{M}_{n,k}$; we consider f as a function $\tilde{f}$ on $\mathrm{M}_{k,k}$ by $\tilde{f}\begin{pmatrix} x \\ y \end{pmatrix} = f(x)$, where $x \in \mathrm{M}_{n,k}$, $y \in \mathrm{M}_{k-n,k}$.

It is clear that if $f \in \mathfrak{H}_{n,k}(\lambda)$, then $\tilde{f} \in \mathfrak{H}_{k,k}(\lambda)$.

Let us consider the Borel subalgebra $\mathfrak{b}_n = \begin{pmatrix} & 0 \\ \blacktriangle & \end{pmatrix}$ of $\mathfrak{gl}(n)$.

Let $f \in \mathfrak{H}_{n,k}(\lambda)$ be a highest weight for the representation $\tau(\lambda)$, with respect to $\mathfrak{b}_n$, i.e. there exists a character μ of the lower triangular subgroup B_n of $\mathrm{GL}(n; \mathbb{C})$ such that $f(b^{-1} x) = \mu(b) f(x)$ for $b \in B_n$. Let $\tilde{b}$ be an element

$$\tilde{b} = \left(\begin{array}{c|c} \overset{\overset{n}{\leftrightarrow}}{b_1} & 0 \\ \hline b_{12} & b_2 \end{array}\right) \Bigg| k$$

of the lower triangular subgroup of $\mathrm{GL}(k; \mathbb{C})$. As

$$\tilde{b}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1^{-1} & x \\ & * \end{pmatrix},$$

we have, setting $\tilde{\mu}(\tilde{b}) = \mu(b_1)$,

$$\tilde{f}\left(\tilde{b}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}\right) = f(b_1^{-1} x) = \mu(b_1) f(x) = \tilde{\mu}(\tilde{b}) \tilde{f}\begin{pmatrix} x \\ y \end{pmatrix}.$$

So $\tilde{f}$ is a highest weight vector of the irreducible representation $\tau_{k,k}(\lambda)$; it follows that $\tilde{f}\begin{pmatrix}x\\y\end{pmatrix}=f(x)$ is unique up to a constant multiple and so is f. This proves that $\tau_{n,k}(\lambda)$ is irreducible as well.

(5.10) Let $\lambda\in\Sigma$ and $\tau=\tau(\lambda)$ the irreducible representation of $\mathrm{GL}(n,\mathbb{C})$ in $V_\tau=\mathfrak{H}(\lambda)$. We define a $\mathrm{Hom}(\mathfrak{H}(\lambda), V_\lambda)$-valued polynomial $I_\lambda(x)$ as follows:

(5.11) $I_\lambda(x)\,P=P(x)$ for $P\in\mathfrak{H}(\lambda)$.

It is immediate that $I_\lambda(x)$ satisfies the conditions (4.4)a)b). Hence we get a map $\mathscr{F}_\lambda=\mathscr{F}_{I_\lambda}$ intertwining the representations $\mathrm{L}_k(\lambda)$ and $T(\tau(\lambda), k)$.

(5.12) **Proposition.** *The map* $\mathscr{F}_\lambda$ *is injective.*

Proof. Suppose that $f\in\mathrm{L}(\mathrm{M}_{n,k};\lambda)$ is such that $\mathscr{F}_\lambda f=0$, then, by (4.6), we have $I_\lambda(x)^*f(x)=0$ almost everywhere. Let $P\in\mathfrak{H}(\lambda)$, then $\langle I_\lambda(x)^*f(x), P\rangle=\langle f(x), P(x)\rangle=0$.

But since $\mathbb{C}[\mathrm{M}_{n,k}](\lambda)=\mathbb{C}[x{}^tx]\,\mathfrak{H}(\lambda)$ (5.4), this holds for any $\phi\in\mathbb{C}[\mathrm{M}_{n,k}](\lambda)$. Hence $e^{-\mathrm{Tr}x^tx}\langle f(x),\phi(x)\rangle=0$ for any $\phi\in\mathbb{C}[\mathrm{M}_{n,k}](\lambda)$; since $\{e^{-\mathrm{Tr}x^tx}\phi(x);\ \phi\in\mathbb{C}[\mathrm{M}_{n,k}](\lambda)\}$ is dense in $L^2(\mathrm{M}_{n,k};\lambda)$, we have $f=0$.

(5.13) We will describe in the next section the set Σ and the correspondence $\lambda\mapsto\tau(\lambda)$ in terms of the highest weight.

6. *Description of* $\mathfrak{H}(\lambda)$

(6.1) Let $\lambda\in\mathrm{O}(k)^\wedge$, it is known that $\lambda=\lambda'$.

(6.2) First we will study the case when k is odd. Then the matrix $-\mathrm{Id}_k$ is of determinant -1, hence $\mathrm{O}(k,\mathbb{R})=\mathrm{SO}(k,\mathbb{R})\times\mathbb{Z}_2$ (direct product) and a representation of $\mathrm{O}(k)$ is determined by two irreducible representations of $\mathrm{SO}(k)$ and of $\mathbb{Z}_2$.

We consider the group $\mathrm{O}(k\,\mathbb{C})$. We set $k=2l+1$; we write a $(2l+1)\times(2l+1)$ matrix by

$$\begin{matrix} & \overset{l}{\longleftrightarrow} & \overset{l}{\longleftrightarrow} & \overset{1}{\longleftrightarrow} \\ l\updownarrow & & & \\ l\updownarrow & & & \\ 1\updownarrow & & & \end{matrix}\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix};$$

we consider the symmetric matrix

$$J=\begin{pmatrix}0 & 1_l & 0\\ 1_l & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

and

$$\mathrm{O}(k;\mathbb{C})=\{g\in GL(k,\mathbb{C});\ {}^tgJg=J\},$$
$$\mathrm{SO}(k;\mathbb{C})=\{g\in\mathrm{O}(k,\mathbb{C});\ \det g=1\}.$$

The Lie algebra $\mathfrak{so}(k)$ of $SO(k, \mathbb{C})$ is then

$$\mathfrak{so}(k)=\left\{\begin{pmatrix} \alpha & \beta & \delta \\ \gamma & -{}^t\alpha & h \\ -{}^th & -{}^t\delta & 0 \end{pmatrix};\ \beta, \gamma \text{ skew symmetric}\right\}.$$

We consider the Borel subalgebra

$$\mathfrak{b}_k=\left\{\begin{pmatrix} \alpha & \beta & \delta \\ 0 & -{}^t\alpha & 0 \\ 0 & -{}^t\delta & 0 \end{pmatrix};\ \alpha \text{ upper triangular}\right\}$$

and the Cartan subalgebra

$$\mathfrak{h}_k=\left\{\left(\begin{array}{ccc|ccc|c} t_1 & & 0 & & & & \\ & \ddots & & & 0 & & 0 \\ 0 & & t_l & & & & \\ \hline & & & -t_1 & & 0 & \\ & 0 & & & \ddots & & 0 \\ & & & 0 & & -t_l & \\ \hline & 0 & & & 0 & & 0 \end{array}\right);\ t_i \in \mathbb{C}\right\}.$$

Then $SO(k)^\wedge$ is parametrized by the highest weight $(m_1, \dots, m_l)$, with $m_1 \geqq m_2 \geqq \cdots \geqq m_l \geqq 0$ $(m_j \in \mathbb{Z})$ ($(m_1, m_2, \dots, m_l)$ corresponds to $\Sigma m_j t_j$).

Thus $O(k)^\wedge$ is parametrized by $\lambda=(m_1, \dots, m_l; \varepsilon)$. This notation means that λ is a tensor product of $(m_1, \dots, m_l)\otimes\varepsilon$. Here ε is the 1-dimensional representation of $\mathbb{Z}_2$ trivial or nontrivial according to $\varepsilon=1$ or -1.

Therefore $\lambda|SO(k)$ is $(m_1, m_2, \dots, m_l)$ and $\lambda(-\mathbb{1}_k)=\varepsilon$.

We also take a Borel subalgebra $\mathfrak{b}_n=\begin{pmatrix} 0 \\ \diagdown\!\!\!\blacktriangle \end{pmatrix}$ of $\mathfrak{gl}(n)$ and a Cartan subalgebra

$$\mathfrak{h}_n=\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix}.$$

We write a member w of $M_{n,k}(\mathbb{C})$ $w=(\overset{l}{\leftrightarrow}{x}, \overset{l}{\leftrightarrow}{y}, \overset{1}{\leftrightarrow}{t})$; The ring of invariant polynomials on $M_{n,k}(\mathbb{C})$ by this form of $O(k, \mathbb{C})$ are the $(wJ\,{}^tw)_{i,j}$. So the ring of differential operators invariant by $O(k, \mathbb{C})$ is generated by

$$(6.3)\quad \Delta_{i,j}=\sum_{\nu=1}^{l}\left(\frac{\partial^2}{\partial x_{i\nu}\partial y_{j\nu}}+\frac{\partial^2}{\partial x_{j\nu}\partial y_{i\nu}}\right)+\frac{\partial^2}{\partial t_i \partial t_j} \qquad (1\leqq i,\ j\leqq n).$$

We set:

$$(6.4)\quad \Delta_j(x)=\det\begin{pmatrix} x_{11}, \dots, x_{1j} \\ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot \\ x_{j1}, \dots, x_{jj} \end{pmatrix} \qquad 1\leqq j\leqq n, l,$$

$$\tilde{\Delta}_j(x, y, t) = \det\begin{pmatrix} x_{11}, x_{12}, \dots, & x_{1,l}, y_{1,j+1}, \dots, & y_{1,l}, t_1 \\ x_{21}, x_{22}, \dots, & x_{2,l}, y_{2,j+1}, \dots, & y_{2,l}, t_2 \\ \dots & \dots & \dots \\ x_{2l-j+1,1}, x_{2l-j+1,2}, \dots, & x_{2l-j+1,l}, y_{2l-j+1,j+1}, \dots, & y_{2l-j+1,l}, t_{2l-j+1} \end{pmatrix}$$

$0 \leqq j \leqq l, \quad 2l-j+1 \leqq n.$

We consider the action of $GL(n, \mathbb{C}) \times O(k, \mathbb{C})$ in $\mathfrak{H}$ and we want to decompose $\mathfrak{H}$ into irreducible components under this action. By (5.7), as $\tau(\lambda) \otimes \lambda'$ is an irreducible representation of $GL(n, \mathbb{C}) \times O(k, \mathbb{C})$, this decomposition is

$$\mathfrak{H} = \bigoplus_{\lambda \in \Sigma} \mathfrak{H}(\lambda) \otimes V_{\lambda'} = \bigoplus_{\lambda \in \Sigma} \mathfrak{H}(\lambda) \otimes V_\lambda \quad ((6.1)).$$

(6.5) This means that for each representation $\lambda \in \Sigma \subset O(k)\hat{}$ and $v \in V_\lambda$ there exists a unique element f, up to scalar multiple, in $\mathfrak{H}$ such that

a) f transforms under right translations by $O(k, \mathbb{C})$ according to the action of λ on v.

b) f is highest weight vector with respect to $\mathfrak{b}_n$ (left action). The corresponding weight will be the highest weight of $\tau(\lambda)$ with respect to $\mathfrak{b}_n$.

The following proposition will then determine Σ and the map $\lambda \mapsto \tau(\lambda)$.

(6.6) **Proposition.** *Let $f \in \mathfrak{H}$ be a highest weight vector for the action of $GL(n, \mathbb{C}) \times O(k, \mathbb{C})$ on $\mathfrak{H}$. Then f is on the form:*

1) $\Delta_1(x)^{\alpha_1} \dots \Delta_j(x)^{\alpha_j}$ $(0 \leqq j \leqq n, l)$

$(\alpha_1, \dots, \alpha_j \in \mathbb{N} = \{0, 1, 2, \dots\})$.

or

2) $\Delta_1(x)^{\alpha_1} \dots \Delta_j(x)^{\alpha_j} \tilde{\Delta}_j(x, y, t)$

$(0 \leqq j \leqq n, l, 2l-j+1 \leqq n)$.

Proof. 1. The function f of the form 1) are obviously pluriharmonic and semi-invariant under $\mathfrak{b}_n \times \mathfrak{b}_k$.

2. We set $X_\nu = \begin{pmatrix} x_{1,\nu} \\ \vdots \\ x_{2l-j+1,\nu} \end{pmatrix}$ $Y_\nu = \begin{pmatrix} y_{1,\nu} \\ \vdots \\ x_{2l-j+1,\nu} \end{pmatrix}$ $T = \begin{pmatrix} t_1 \\ \vdots \\ t_{2l-j+1} \end{pmatrix}$.

Let e be the canonical non zero element of $\Lambda^{2l-j+1} \mathbb{R}^{2l-j+1}$. Then $\tilde{\Delta}_j(x,y,t) = X_1 \wedge \dots \wedge X_\nu \wedge \dots \wedge X_l \wedge Y_{j+1} \wedge \dots \wedge Y_\nu \wedge \dots \wedge Y_l \wedge T$.

It is easy to check on the form of the Lie algebra $\mathfrak{b}_k$ that $\tilde{\Delta}_j(x,y,t)$ is semi-invariant under $\mathfrak{b}_k$ and hence a function of the form 2) is semi-invariant under $\mathfrak{b}_n \times \mathfrak{b}_k$. Let us check that $f(x) = \Delta_1(x)^{\alpha_1} \cdot \dots \cdot \Delta_j(x)^{\alpha_j} \tilde{\Delta}_j(x,y,t)$ is pluriharmonic.

It is evident that

$$\frac{\partial^2}{\partial x_{i\nu}\,\partial y_{i'\nu}} f=0 \quad \text{if } \nu \leqq j$$

$$\frac{\partial^2}{\partial t_i\,\partial t_{i'}} f=0.$$

Therefore

$$\Delta_{i,i'} f=\sum_{\nu=j+1}^{l}\left(\frac{\partial^2}{\partial x_{i\nu}\,\partial y_{i'\nu}}+\frac{\partial^2}{\partial x_{i'\nu}\,\partial y_{i\nu}}\right) f=0$$

and as $\Delta_1(x), \ldots, \Delta_j(x)$ do not depend on $x_{i,\nu}$ for $\nu \geqq j+1$, we have

$$\Delta_{ii'} f=\Delta_1(x)^{\alpha_1}\ldots\Delta_j(x)^{\alpha_j}\sum_{\nu=j+1}^{l}\left(\frac{\partial^2}{\partial x_{i\nu}\,\partial y_{i'\nu}}+\frac{\partial^2}{\partial x_{i'\nu}\,\partial y_{i\nu}}\right)\tilde{\Delta}_j.$$

Let us fix $\nu \geqq j+1$, then

$$\tilde{\Delta}_j(x, y, t)\, e=\pm X_\nu \wedge Y_\nu \wedge X_1 \wedge \cdots \wedge \hat{X}_\nu \wedge \cdots \wedge X_l \wedge Y_{j+1} \wedge \cdots \wedge \hat{Y}_\nu \wedge \cdots \wedge Y_l \wedge T.$$

Each component of the vector $X_\nu \wedge Y_\nu$ in $\wedge^2 \mathbb{R}^{2l-j+1}$ is of the form

$$(x_{\alpha\nu}\, y_{\beta\nu}-x_{\beta\nu}\, y_{\alpha\nu}),$$

hence for any (i, i') we have

$$\left(\frac{\partial^2}{\partial x_{i\nu}\,\partial y_{i'\nu}}+\frac{\partial^2}{\partial x_{i'\nu}\,\partial y_{i\nu}}\right) X_\nu \wedge Y_\nu=0$$

and as the other elements $X_{\nu'}$, $Y_{\nu'}$ do not depend on $x_{\alpha\nu}, y_{\beta\nu}$, we have

$$\left(\frac{\partial^2}{\partial x_{i\nu}\,\partial y_{i'\nu}}+\frac{\partial^2}{\partial x_{i'\nu}\,\partial y_{i\nu}}\right)\tilde{\Delta}_j=0.$$

And finally $\Delta_{ii'} f=0$. Let us show now that each highest weight vector f appearing in $\mathfrak{H}$ is on the form 1) or 2). Let us suppose first that $k \leqq n$; we will show that for any $\lambda \in \mathrm{O}(k)\hat{}$ there exists an f in $\mathfrak{H}$ of the form 1) or 2) satisfying the condition (6.5) a). This will prove the result (and that $\Sigma=\mathrm{O}(k)\hat{}$).

Let $\lambda=(m_1, \ldots, m_l; \varepsilon)$. Suppose that $\varepsilon=(-1)^{m_1+\cdots+m_l}$, then

$$f=\Delta_1(x)^{m_1-m_2}\,\Delta_2(x)^{m_2-m_3}\ldots\Delta_{l-1}(x)^{m_{l-1}-m_l}\,\Delta_l(x)^{m_l}$$

corresponds to λ (under the transformation $x \mapsto -x$, f is multiplied by $(-1)^{(m_1-m_2)+2(m_2-m_3)+\cdots+(l-1)(m_{l-1}-m_l)+l m_l}=(-1)^{m_1+m_2+\cdots+m_l}$).

Suppose that $\varepsilon=(-1)^{m_1+m_2+\cdots+m_l+1}$. Take j such that $m_j \geqq 1$ and $m_{j+1}=0$ then $f=\Delta_1(x)^{m_1-m_2}\ldots\Delta_j(x)^{m_j-1}\,\tilde{\Delta}_j(x, y, t)$ if of type λ (we have always $2l-j+1 \leqq n$).

Let us suppose now that $k \geqq n$. Let us consider $\tilde{f}$ as in (5.9) $\tilde{f}$ is a function on $\mathrm{M}_{k,k}$; we know then that $\tilde{f}$ has the form 1) or 2) and so is f.

(6.7) **Corollary.** *$\Sigma=\{\lambda;\ \mathfrak{H}(\lambda)\neq 0\}$ is*

$$\{(m_1, m_2, \dots, m_l;\ \varepsilon);\ m_j=0 \text{ for } j>n \text{ and } \varepsilon=(-1)^{m_1+m_2+\cdots+m_l}\}$$
$$\cup\{(m_1, m_2, \dots, m_l;\ \varepsilon);\ \varepsilon=(-1)^{m_1+m_2+\cdots+m_l+1}\ k<2n\ m_j\neq 0 \text{ for } j\leqq k-n\}.$$

(6.8) We will now parametrise the representation τ of $\mathrm{GL}(n, \mathbb{C})$ by their highest weight with respect to the Borel subalgebra $\mathfrak{b}_n^+=\begin{pmatrix} & * \\ 0 & \end{pmatrix}$, i.e. $\tau=(m_1, m_2, \dots, m_n)$ with $m_1\geqq m_2\geqq\cdots\geqq m_n$ for any decreasing sequences of n integers, positive or negative. Then we express in this parametrisation the correspondence $\lambda\mapsto\tau(\lambda)$.

(6.9) **Theorem.** *Let $k=2l+1$.*
Let

$$\lambda\in\Sigma, \text{ then if } \lambda=(m_1, m_2, \dots, m_j, 0, \dots, 0;\ \varepsilon) \quad \text{with } j\leqq n$$

and

$$\varepsilon=(-1)^{m_1+m_2+\cdots+m_j}$$

then

$$\tau(\lambda)=(0, \dots, 0, -m_j, -m_{j-1}, \dots, -m_2, -m_1).$$

If

$$\lambda=(m_1, m_2, \dots, m_j, 0, \dots, 0;\ \varepsilon) \quad \text{with } m_j\neq 0 \quad \text{and} \quad k-n\leqq j\leqq l$$

and

$$\varepsilon=(-1)^{m_1+m_2+\cdots+m_j+1}$$

then

$$\tau(\lambda)=(0, \dots, 0, \underbrace{-1, -1, \dots, -1, -m_j, -m_{j-1}, \dots, -m_2, -m_1}_{k-j}).$$

(6.10) We shall discuss now the case when $k=2l$ is even. We take the form

$$\mathrm{O}(k)=\{g\in\mathrm{GL}(2l;\ \mathbb{C});\ {}^t g J g=J\} \text{ with } J=\begin{pmatrix} 0 & 1_l \\ 1_l & 0 \end{pmatrix}.$$

then

$$\mathfrak{so}(k)=\left\{\begin{pmatrix} \alpha & \beta \\ \gamma & -{}^t\alpha \end{pmatrix};\ \beta, \gamma \text{ skew-symmetric}\right\}.$$

We take a Borel subalgebra

$$\mathfrak{b}_k=\left\{\begin{pmatrix} \alpha & \beta \\ 0 & -{}^t\alpha \end{pmatrix};\ \alpha \text{ upper triangular};\ \beta \text{ skew symmetric}\right\}$$

and a Cartan subalgebra

$$\mathfrak{h}_k = \left\{ \left(\begin{array}{ccc|ccc} t_1 & & & & & \\ & \ddots & & & 0 & \\ & & t_l & & & \\ \hline & & & -t_1 & & \\ & 0 & & & \ddots & \\ & & & & & -t_l \end{array} \right); \ t_i \in \mathbb{C} \right\}.$$

An irreducible representation of $\mathrm{SO}(k)$ is parametrized by its highest weight with respect to $\mathfrak{b}_k$ i.e. $(m_1, m_2, \ldots, m_{l-1}, m_l)$ with $m_1 \geqq m_2 \geqq \cdots \geqq m_{l-1} \geqq |m_l|$, $m_j \in \mathbb{Z}$. $\mathrm{O}(k)$ is the semi-direct product of $\mathrm{SO}(k)$ by $\mathbb{Z}_2$.

Let

$$\sigma = \left(\begin{array}{ccccc|ccccc} 1 & & & & & 0 & & & & \\ & 1 & & & & & 0 & & & \\ & & 1 & & & & & 0 & & \\ & & & 1 & & & & & 0 & \\ & & & & 0 & & & & & 1 \\ \hline 0 & & & & & 1 & & & & \\ & 0 & & & & & 1 & & & \\ & & 0 & & & & & 1 & & \\ & & & 0 & & & & & 1 & \\ & & & & 1 & & & & & 0 \end{array} \right).$$

Then $\sigma \in \mathrm{O}(k) - \mathrm{SO}(k)$ and σ normalises the Borel subalgebra $\mathfrak{b}_k$: Hence we see easily that if λ is the representation of $\mathrm{SO}(k)$ of highest weight $(m_1, m_2, \ldots, m_{l-1}, m_l)$, then the representation $\sigma.\lambda$ has highest weight $(m_1, m_2, \ldots, m_{l-1}, -m_l)$. It follows that if $m_l \neq 0$ the stabiliser of the representation $(m_1, m_2, \ldots, m_l)$ of $\mathrm{SO}(k)$ is $\mathrm{SO}(k)$ itself, and the induced representation of $(m_1, m_2, \ldots, m_l)$ to $\mathrm{O}(k)$ is irreducible and its restriction to $\mathrm{SO}(k)$ is sum of $(m_1, m_2, \ldots, m_l)$ and $(m_1, m_2, \ldots, -m_l)$.

We denote this irreducible representation λ of $\mathrm{O}(k)$ by $(m_1, m_2, \ldots, m_{l-1}, |m_l|)$, $m_l \neq 0$. If $m_l = 0$, the representation $(m_1, m_2, \ldots, m_{l-1}, 0)$ extends into two different representation λ and $\lambda \otimes \det$ of $\mathrm{O}(k, \mathbb{C})$.

We denote a member of $M_{n,k}(\mathbb{C})$ by $(\overset{l}{\overleftrightarrow{x}}, \overset{l}{\overleftrightarrow{y}}) \updownarrow n$, in these coordinates we have:

$$\Delta_{ij} = \sum_{\nu=1}^{l} \frac{\partial^2}{\partial x_{i\nu} \partial y_{j\nu}} + \frac{\partial^2}{\partial x_{j\nu} \partial y_{i\nu}}.$$

We set

$$\Delta_j(x) = \det \begin{pmatrix} x_{11} \cdots x_{1j} \\ x_{j1} \cdots x_{jj} \end{pmatrix} \qquad 1 \leqq j \leqq l, n,$$

$$\tilde{\Delta}_j(x, y) = \det \begin{pmatrix} x_{1,1}, x_{1,2}, \ldots, x_{1,l}, & y_{1,j+1}, \ldots, & y_{1,l} \\ x_{2,1}, \ldots & & \vdots \\ \vdots & & \vdots \\ x_{2l-j,1}, x_{2l-j,2}, \ldots, & y_{2l-j,j+1}, \ldots, & y_{2l-j,l} \end{pmatrix}$$

$$2l - j \leqq n, \qquad 0 \leqq j \leqq l.$$

We denote (taking $n \geqq k$) for any decreasing sequence of non negative integers $(m_1, m_2, \dots, m_l)$ by $\lambda = (m_1, m_2, \dots, m_l)_+$ the irreducible representation of $O(k, \mathbb{C})$ generated by $\Delta_1(x)^{m_1 - m_2} \cdot \dots \cdot \Delta_{l-1}(x)^{m_{l-1} - m_l} \Delta_l(x)^{m_l}$ under right translation. Taking the integer j such that $m_j \neq 0$, $m_{j+1} = 0$, we denote by $\lambda = (m_1, m_2, \dots, m_l)$ the irreducible representation of $O(k, \mathbb{C})$ generated by

$$\Delta_1(x)^{m_1 - m_2} \cdot \dots \cdot \Delta_{j-1}(x)^{m_{j-1} - m_j} \Delta_j(x)^{m_j - 1} \tilde{\Delta}_j(x, y).$$

(if $m_l \neq 0$, $(m_1, \dots, m_l)_+ = (m_1, \dots, m_l)_-$).

(6.11) **Proposition.** *Let* $f \in \mathfrak{H}$ *be a highest weight vector for the action of* $GL(n, \mathbb{C}) \times O(k, \mathbb{C})$ *on* $\mathfrak{H}$, *then* f *is on the form*

1) $\Delta_1(x)^{\alpha_1} \cdot \dots \cdot \Delta_j(x)^{\alpha_j}$ $(j \leqq n, l)$ or
2) $\Delta_1(x)^{\alpha_1} \cdot \dots \cdot \Delta_j(x)^{\alpha_j} \tilde{\Delta}_j(x, y)$ $(2l - j \leqq n, 0 \leqq j \leqq l - 1)$.

(6.12) **Corollary.**

$$\begin{aligned} \Sigma = \{\lambda; \mathfrak{H}(\lambda) \neq 0\} &\ \textit{is} \\ = \{(m_1, m_2, \dots, m_l)_+;\ m_j = 0 &\ \textit{for } j > n\} \\ \cup \{(m_1, m_2, \dots, m_l)_-;\ k < 2n,\ m_j \neq 0 &\ \textit{for } j \leqq k - n\}. \end{aligned}$$

(6.13) **Theorem.** *Let* $k = 2l$; *Let* $\lambda \in \Sigma$, *then if*

$$\lambda = (m_1, m_2, \dots, m_j, 0, \dots, 0)_+ \quad \textit{with } j \leqq n, l,$$

then

$$\tau(\lambda) = (0, \dots, 0, -m_j, -m_{j-1}, \dots, -m_2, -m_1).$$

If

$$\lambda = (m_1, m_2, \dots, m_j, 0, \dots, 0)_- \quad \textit{with } m_j \neq 0$$

with

$$l \geqq j \geqq k - n, 0$$

then

$$\tau(\lambda) = (0, \dots, 0, \underbrace{-1, -1, \dots, -1, -m_j, -m_{j-1}, \dots, -m_2, -m_1}_{k-j}).$$

As these propositions are proved exactly in the same way as (6.6), (6.7), (6.9), we do not repeat this proof.

(6.14) **Proposition.** *For any* k, *the map* $\lambda \mapsto \tau(\lambda)$ *is injective.*
This is immediate from the explicit formula of $\tau(\lambda)$ given (6.9) and (6.13).

7. *Decomposition of* $L^2(M_{n,k})$

(7.1) We know that $L^2(M_{n,k}; \lambda)$ is embedded in $\mathcal{O}(\tau(\lambda) \otimes \delta_{-k/2})$ ((3.4), (4.4) and (5.12)). Since $\tau(\lambda) \otimes \delta_{k/2}$ is an irreducible representation ((5.7)); $L^2(M_{n,k}; \lambda)$ is an irreducible unitary representation of G_2 with highest weight $\tau(\lambda) \otimes \delta_{-k/2}$ ((3.5) in I). On the other hand, we have

$$L^2(\mathrm{M}_{n,k}) = \bigoplus_{\lambda\in\Sigma} L^2(\mathrm{M}_{n,k};\lambda)\otimes V_{\lambda'},$$

and hence

$$L^2(\mathrm{M}_{n,k}) = \bigoplus_{\lambda\in\Sigma} (\dim V_\lambda)\, L^2(\mathrm{M}_{n,k};\lambda).$$

Since $\lambda\mapsto\tau(\lambda)$ is injective ((6.14)), $L^2(\mathrm{M}_{n,k};\lambda)$ are not equivalent to one another. Hence we have

(7.2) **Theorem.** 1) *For any* $\lambda\in\Sigma$ *the representation* $\mathrm{L}_k(\lambda)$ *appears in the representation* $\mathrm{L}_k \dim V_\lambda$*-times, and we have*

$$\mathrm{L}_k = \bigoplus_{\lambda\in\Sigma} (\dim V_\lambda)\, L_k(\lambda).$$

2) *For* $\lambda\in\Sigma$, *the representation* $\mathrm{L}_k(\lambda)$ *is an irreducible unitary representation of* G_2 *of highest weight*

$$\tau(\lambda)\otimes\delta_{-k/2}\left(\delta_{-k/2} \text{ has weight } \left(-\frac{k}{2},\dots,-\frac{k}{2}\right)\right).$$

3) *The description of* Σ *and* $\tau(\lambda)$ *is given in* (6.7), (6.9) *and* (6.12), (6.13).

8. Unitary Representations of Sp$(n,\mathbb{R})$ *with Highest Weight*

(8.1) Let k be an integer. Let $\lambda\in\Sigma_k$ and $\tau=\tau(\lambda)$ the irreducible representation of $\mathrm{GL}(n,\mathbb{C})$ associated to λ. We can then construct the map $\mathscr{F}_\lambda: L^2(\mathrm{M}_{n,k};\lambda)\to\mathcal{O}(D, V_\tau)$. We consider the Hilbert space

$$M_\lambda = \{\mathscr{F}_\lambda(\phi);\ \phi\in L^2(\mathrm{M}_{n,k};\lambda)\} \quad \text{with } \|\mathscr{F}_\lambda(\phi)\|^2 = \|\phi\|^2.$$

Then M_λ is a Hilbert space of holomorphic functions for which point evaluations is continuous (Lemma (4.3)). Let Λ be the highest weight of the representation $\tau\otimes(\det g)^{-k/2}$ of the two-sheeted covering group of $\mathrm{GL}(n,\mathbb{C})$. Then the corresponding infinitesimal module M of all K-finite vector is equivalent to $\mathrm{L}(\Lambda)$, and $\Lambda\in P$.

We recall that $\Lambda\in D$, i.e. the corresponding representation is a member of the discrete series if and only $\langle\Lambda+\rho, H_{\gamma_1}\rangle<0$. In particular we see on the description of Λ given by applying Propositions (6.9) and (6.13), the following result.

(8.2) If $k\geqq 2n+1$, then all the representations associated to λ obtained are members of the holomorphic discrete series;

If $k\leqq 2n-1$, none of the representations obtained are discrete.

If $k=2n$, the representation associated to $\lambda=(m_1,m_2,\dots,m_n)_+$ is discrete if and only if $m_n\neq 0$. In the cases we described, i.e. when $\langle\Lambda+\rho, H_{\gamma_1}\rangle<0$, we know that the Hilbert space M_λ has to be $\mathscr{H}(\Lambda)$. In the coordinates $z=x+iy$ of D, it comes that

(8.3) $M_\lambda = \{f;\ V_\tau$-valued holomorphic functions on D such that

$$\int \langle \tau(y)^{-1} f(x+iy), f(x+iy)\rangle (\det y)^{k/2-(n+1)}\, dx\, dy < \infty\}.$$

If the representation $L_k(\lambda)$ is not in the holomorphic discrete series, then it is not easy to give a description of the Hilbert space M_λ of holomorphic functions obtained. For example, let us remark that if $k<n$, the functions $\mathscr{F}_\lambda f$ being the Fourier transform of a measure supported on $b_k(C)$ will satisfy the differential equations corresponding to the equations of $b_k(C)$ (all the minors of rank $k+1$ has to be zero). The converse problem involves to know conditions for a distribution solution of differential equations to be the Fourier transform of a measure.

(8.4) Let $\lambda \in \Sigma$ and $M_\lambda = \{\mathscr{F}_\lambda \phi;\ \phi \in L^2(M_{n,k};\lambda)\}$ with $\|\mathscr{F}_\lambda \phi\| = \|\phi\|$. We will compute the reproducing kernel K_λ of the space M_λ, i.e. K_λ is a function $K_\lambda(z,w)$ on $D\times D$ holomorphic in z, antiholomorphic in w with values in $\mathrm{End}(V_\tau)$ such that for every $v\in V_\tau$ and for every $w\in D$, the function $K_\lambda(\cdot,w)\cdot v$ belongs to M_λ and for every $f\in M_\lambda$ $\langle f(w), v\rangle_{V^\tau} = \langle f, K_\lambda(\cdot,w)\cdot v\rangle_{M_\lambda}$.

(8.5) **Lemma.** *Up to a constant factor, we have*

$$K_\lambda(z,w) = \tau\left(\frac{z-\bar w}{i}\right) \det\left(\frac{z-\bar w}{i}\right)^{-k/2}.$$

Proof. We have, under the correspondence $F(z) = f\left(\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\mathbf{c}^{-1}\right)$, $F(i)=f(1)$, consequently by the Remark (I.3.8), we know that

$$K_\lambda(z,i)\cdot v_A = \tau\left(\frac{z+i}{i}\right)\det\left(\frac{z+i}{i}\right)^{-k/2}\cdot v_A.$$

Using the fact that the representation $T(\tau\otimes\det^{-k/2})$ is unitary, we get covariance relations of $K_\lambda(z,w)$ under G_2 and the lemma follows.

(8.6) **Corollary.** *Let $\lambda\in\Sigma_k$, then we have*

$$\tau\left(\frac{z-\bar w}{i}\right)\det\left(\frac{z-\bar w}{i}\right)^{-\frac{k}{2}} = \int_{M_{n,k}} e^{\frac{i\,\mathrm{Tr}\,{}^t x(z-\bar w)x}{2}} I(x)^* I(x)\, dx.$$

Proof. Let $\phi\in L^2(M_{n,k};\lambda)$ then for $v\in V_\tau$

$$\begin{aligned}\langle(\mathscr{F}_\lambda\phi)(w), v\rangle &= \int e^{\frac{i}{2}\mathrm{Tr}\,{}^t x w x}\langle I(x)^*\phi(x), v\rangle\, dx\\ &= \int \langle \phi(x), e^{-\frac{i}{2}\mathrm{Tr}\,{}^t x \bar w x} I(x)\, v\rangle\, dx.\end{aligned}$$

Let $K_\lambda(z,\bar w) = \mathscr{F}_\lambda k_\lambda(x,\bar w)$ be the reproducing kernel of M_λ. Then we have:

$$\begin{aligned}\langle(\mathscr{F}_\lambda\phi)(w), v\rangle &= \langle \mathscr{F}_\lambda\phi, \mathscr{F}_\lambda k_\lambda(\cdot,\bar w)\cdot v\rangle\\ &= \int\langle\phi(x), k_\lambda(x,\bar w)\cdot v\rangle\, dx\end{aligned}$$

It follows that $k_\lambda(x,\bar w)\cdot v = e^{-i/2\,\mathrm{Tr}\,{}^t x\bar w x} I(x)$. Hence the Corollary (8.6).

From the Corollary (8.6), the highest weight vector

$$\tau\left(\frac{z+i}{i}\right)\det\left(\frac{z+i}{i}\right)^{-\frac{k}{2}}\cdot v_A=\int e^{\frac{i}{2}\operatorname{Tr}{}^t xzx}\, e^{-\operatorname{Tr}\frac{{}^t xx}{2}} I(x)^* I(x)\, v_A\, dx$$

i.e. the highest weight vector of the representation $\mathrm{L}_{k,\lambda}$ in $L^2(\mathrm{M}_{n,k};\lambda)$ is the function $e^{-\operatorname{Tr}\frac{{}^t xx}{2}} I_\lambda(x)\cdot v_A$; this function is given explicitly by Propositions (6.6) and (6.11).

(8.7) Let S be the set of all irreducible representations of $\widetilde{\mathrm{GL}}(n,\mathbb{C})$ of the form $\tau(\lambda)\otimes(\det)^{-k/2}$ for some $\lambda\in\mathrm{O}(\hat{k})$.

(8.8) **Theorem.** *We have*

$$S=\Big\{\tau=(-m_1,-m_2,\dots,-m_n);\ m_1\in\mathbb{Z}/2$$

$$m_1\leqq\dots\leqq m_n,\ m_{i+1}-m_i \text{ integer, and}$$

$$m_1\geqq\min\Big(n-1,\frac{m_2-m_1}{2}+n-2,\frac{(m_3-m_1)+(m_2-m_1)}{2}+n-3,$$

$$\dots,\frac{(m_n-m_1)+\dots+(m_2-m_1)}{2}\Big)\Big\}.$$

Proof. We set S' the right hand side of this formula. Combining Proposition (6.9) and (6.13), we have

$$S=\{(-l,\dots,-l,-m_1,\dots,-m_l);\ 0\leqq l\leqq n,\ m_1\geqq l,\ m_1\in\mathbb{Z}\}$$

$$\cup\{(\underbrace{-l,\dots,-l}_{n-2l+j},\underbrace{-(l+1),\dots,-(l+1)}_{2(l-j)},-m_1,\dots,-m_j);\ n-2l+j\geqq 1,\ l\geqq j+1\}$$

$$\cup\{(-(l+\tfrac{1}{2}),\dots,-(l+\tfrac{1}{2}),-m_1,\dots,-m_l);$$

$$0\leqq l\leqq n,\ m_1\geqq l+\tfrac{1}{2},\ m_1\in\tfrac{1}{2}+\mathbb{Z}\}$$

$$\cup\{(\underbrace{-(l+\tfrac{1}{2}),\dots,-(l+\tfrac{1}{2})}_{n-2l+j+1},\underbrace{-(l+1+\tfrac{1}{2}),\dots,-(l+1+\tfrac{1}{2})}_{2(l-j)+1},-m_1,\dots,-m_j);$$

$$0\leqq j\leqq l,\ n>2l-j+1\}.$$

Then, it is easy to check that S is contained in S'. We shall show that S contains S'. Let $\tau=(-m_1,\dots,-m_n)$ be a member of S'. If $m_1\geqq n-1$, τ is obviously contained in S. So, assume that $m_1<n-1$. Then, there is k $(1\leqq k\leqq n-1)$ such that

$$m_1\geqq\frac{(m_2-m_1)+\dots+(m_{k+1}-m_1)}{2}+(n-k-1)$$

and

$$m_1<\frac{(m_2-m_1)+\dots+(m_k-m_1)}{2}+(n-k).$$

Therefore we have

$$0>\frac{m_{k+1}-m_1}{2}-1.$$

Since $0 \leqq m_{k+1} - m_1$, we have $m_{k+1} = m_1$ or $m_{k+1} = m_1 + 1$. Suppose that $m_{k+1} = m_1$. Then $m_1 = \cdots = m_{k+1}$, and hence $n - k - 1 \leqq m_1 < n - k$. Therefore $m_1 = l$ or $m_1 = l + \frac{1}{2}$ with $l = n - k - 1$, and hence

$$\tau = (-l, \ldots, -l, \underbrace{-m_{k+2}, \ldots, -m_n}_{l})$$

or

$$\tau = (-(l+\tfrac{1}{2}), \ldots, -(l+\tfrac{1}{2}), \underbrace{-m_{k+2}, \ldots, -m_n}_{l}).$$

Thus τ belongs to S.

Suppose that $m_{k+1} = m_1 + 1$. Then, there is v such that

$$m_1 = \cdots = m_v, m_{v+1} = \cdots = m_{k+1} = m_1 + 1. \quad \text{and} \quad 1 \leqq v \leqq k.$$

Thus, we have

$$\frac{k-v}{2} + n - k > m_1 \geqq \frac{k-v+1}{2} + n - k - 1$$

and hence

$$m_1 = \frac{k-v}{2} + n - k - \tfrac{1}{2} = n - \frac{k+v+1}{2}.$$

If m_1 is an integer, setting $l = m_1$, we have

$$\tau = (\underbrace{-l, \ldots, -l}_{v}, \underbrace{-(l+1), \ldots, -(l+1)}_{k-v+1}, \underbrace{-m_{k+2}, \ldots, -m_n}_{n-k-1}).$$

Since $v = n - 2l + (n-k-1)$ and $k - v + 1 = 2(l - (n-k-1))$, τ belongs to S.

If m_1 is a half integer, setting $m_1 = l + \frac{1}{2}$, we have

$$\tau = (\underbrace{-(l+\tfrac{1}{2}), \ldots, -(l+\tfrac{1}{2})}_{v}, \underbrace{-(l+\tfrac{3}{2}), \ldots, -(l+\tfrac{3}{2})}_{k-v+1}, \underbrace{-m_{k+2}, \ldots, -m_n}_{n-k-1})$$

and hence τ belongs to S.

III. Tensor Products of the Harmonic Representation of $U(p, q)$

1. Description of the Harmonic Representation

(1.1) We consider the complex vector space $\mathbb{C}^q \oplus \mathbb{C}^r \oplus \mathbb{C}^q$ with basis

$$e_1, e_2, \ldots, e_q, v_1, v_2, \ldots, v_r, f_1, f_2, \ldots, f_q \ (p = q + r).$$

We will write any $(q+r+q) \times (q+r+q)$ complex matrix by blocs, i.e.

$$x = \left(\begin{array}{c|c|c} x_{11} & x_{12} & x_{13} \\ \hline x_{21} & x_{22} & x_{23} \\ \hline \underset{q}{x_{31}} & \underset{r}{x_{32}} & \underset{q}{x_{33}} \end{array}\right) \begin{array}{l} \updownarrow q \\ \updownarrow r. \\ \updownarrow q \end{array}$$

Let us consider the hermitian matrix

$$h=\left(\begin{array}{c|c|c} 0 & 0 & -i \\ \hline 0 & 1 & 0 \\ \hline i & 0 & 0 \end{array}\right),$$

We denote by (x, y) the canonical complex symmetric bilinear form, and by $\langle x, y\rangle=(x, \bar{y})$ the canonical hermitian form; then $h(x, y)=\langle hx, y\rangle$ is a hermitian form on $\mathbb{C}^q\oplus\mathbb{C}^r\oplus\mathbb{C}^q$ of signature (p, q).

We consider the group $G=U(p, q)=\{g;\, g^* h g=h\}$.

(1.2) Let us describe Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g}=\left\{x=\left(\begin{array}{c|c|c} a & \alpha_1 & b \\ \hline \beta_1 & \gamma & \beta_2 \\ \hline c & \alpha_2 & d \end{array}\right);\ \begin{array}{l} a+d^*=0,\ b=b^*,\ c=c^* \\ \alpha_1=i\beta_2^*,\ \alpha_2=-i\beta_1^*,\ \gamma+\gamma^*=0 \end{array}\right\}.$$

We denote by $M(k_1, k_2; \mathbb{C})=\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{k_1}, \mathbb{C}^{k_2})$ and $H(k)=\{x\in M(k,k;\mathbb{C});\, x=x^*\}$ the real vector space of hermitian $k\times k$ matrices.

(1.3) We consider the following elements of G:

$$g(a)=\left(\begin{array}{c|c|c} a & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & (a^*)^{-1} \end{array}\right); \qquad a\in \mathrm{GL}(q, \mathbb{C}),$$

$$t(x)=\left(\begin{array}{c|c|c} 1 & 0 & x \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right); \qquad x\in H(q),$$

$$k(\alpha)=\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \alpha & 0 \\ \hline 0 & 0 & 1 \end{array}\right); \qquad \alpha\in U(r),$$

$$n(u)=\left(\begin{array}{c|c|c} 1 & iu^* & i\dfrac{u^*u}{2} \\ \hline 0 & 1 & u \\ \hline 0 & 0 & 1 \end{array}\right); \qquad u\in M(r, q; \mathbb{C})$$

and

$$\sigma=\left(\begin{array}{c|c|c} 0 & 0 & -1 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right)$$

These elements generate the group G.

(1.4) Let us consider the Heisenberg group H defined on

$$\mathbb{C}^q\oplus\mathbb{C}^r\oplus\mathbb{C}^q\oplus\mathbb{R}E$$

by the law

$$(u,t)\cdot(u',t')=(u+u',t+t'-\operatorname{Im} h(u,u')).$$

We write also $(u,t)=(w_1,v_0,w_2,t)$ with $w_1\in\mathbb{C}^q$, $v_0\in\mathbb{C}^r$, $w_2\in\mathbb{C}^q$, $t\in\mathbb{R}$.

We consider the Hilbert space;

$\mathscr{L}=\{$classes of measurable functions $\psi(w,v)=\psi(\operatorname{Re} w,\operatorname{Im} w,v)$ on $\mathbb{C}^q\times\mathbb{C}^r$ such that:

a) $\psi(w,v)$ is holomorphic in v,

b) $\int|\psi(w,v)|^2\,e^{-\langle v,v\rangle}|dw|^2\,|dv|^2<\infty\}$.

Here, $|dw|^2$ is the euclidean measure $2^q d(\operatorname{Re} w_1)\,d(\operatorname{Im} w_1)\dots d(\operatorname{Re} w_q)\,d(\operatorname{Im} w_q)$ on $\mathbb{C}^q=\mathbb{R}^{2q}$ and $|dv|^2$ is the euclidean measure on $\mathbb{C}^r=\mathbb{R}^{2r}$ defined in the same manner. Then the unique class of unitary irreducible representation T of H such that $T(0,t)=e^{it}\operatorname{Id}$ can be realized as acting on $\mathscr{L}$ by

$$(T(w_1,v_0,w_2;t)\,\psi)(w,v)$$
$$=e^{it}\,e^{\langle v,v_0\rangle}\,e^{-\frac{1}{2}\langle v_0,v_0\rangle}\,e^{-2i\operatorname{Re}\langle w,w_1\rangle}\,e^{i\operatorname{Re}\langle w_1,w_2\rangle}\,\psi(w-w_2,v-v_0).$$

Let $g\in G$; the action $g\cdot(u,t)=(g\cdot u,t)$ realizes G as a subgroup of automorphisms of H leaving the center of H stable, i.e. as a subgroup of $\operatorname{Sp}(p+q;\mathbb{R})$. We consider the restriction L of the harmonic representation of the symplectic group (considered as a projective representation) i.e. for each $g\in G$ we construct $L(g)$ the unitary operator on $\mathscr{L}$ (defined up to a scalar of modulus one) such that

$$L(g)\,T(u,t)\,L(g)^{-1}=T(g\cdot u,t).$$

We will see that actually the following formulas for $L(g)$ leads to a representation of G (not only a projective one):

$$(L(g(a))\,\psi)(w,v)=(\det\cdot a)\,\psi(a^*w,v),$$
$$(L(k(\alpha))\,\psi)(w,v)=\psi(w,\alpha^{-1}v),$$
$$(L(t(x))\,\psi)(w,v)=e^{-i\langle xw,w\rangle}\,\psi(w,v),$$
$$(L(n(u))\,\psi)(w,v)=e^{-\frac{1}{2}\langle uw,uw\rangle}\,e^{\langle v,uw\rangle}\,\psi(w,v-uw),$$
$$(L(\sigma)\,\psi)(w,v)=\left(\frac{i}{2\pi}\right)^q\int_{\mathbb{C}^q}e^{2i\operatorname{Re}\langle w,w'\rangle}\,\psi(w',v)\,|dw'|^2$$

i.e. $L(\sigma)$ is the partial Fourier transform with respect to the variable w.

Let us consider any non negative integer k and L_k the k-th tensor product of the representation L.

We consider $w\in M(q,k;\mathbb{C})$ and $v\in M(r,k;\mathbb{C})$

(1.5) The representation L_k is hence realized in the Hilbert space

$\mathscr{L}_k=\{$classes of measurable functions $\psi(w,v)=\psi(\operatorname{Re} w,\operatorname{Im} w,v)$ such that

a) $\psi(w,v)$ is holomorphic in v,

b) $\displaystyle\int_{M(q,k;\mathbb{C})\times M(r,k;\mathbb{C})} e^{-\operatorname{Tr} vv^*}\,|\psi(w,v)|^2\,|dw|^2\,|dv|^2<\infty\}$.

by the formulas:

$$
\begin{aligned}
(1.6)\quad &(L_k(g(a))\,\psi)(w,v)=(\det a)^k\,\psi(a^* w, v),\\
&(L_k(k(\alpha))\,\psi)(w,v)=\psi(w,\alpha^{-1}v),\\
&(L_k(t(x))\,\psi)(w,v)=e^{-i\operatorname{Tr} w^* x w}\,\psi(w,v),\\
&(L_k n(u)\,\psi)(w,v)=e^{-\frac{1}{2}\operatorname{Tr} u w w^* u^*}\,e^{\operatorname{Tr} v w^* u^*}\,\psi(w,v-uw),\\
&(L_k(\sigma)\,\psi)(w,v)=\left(\frac{i}{2\pi}\right)^{qk}\int_{M(q,k;\mathbb{C})} e^{2i\operatorname{Re}\operatorname{Tr} w w'^*}\,\psi(w',v)\,|dw'|^2.
\end{aligned}
$$

2. *The Action of the Unitary Group* $U(k)$

On $\mathscr{L}_k$ the action of the unitary group $\mathrm{U}(k)$ given by $(h\cdot f)(x)=f(xh)$ commutes with the representation L_k; and hence we will decompose $\mathscr{L}_k$ under $\mathrm{U}(k)$.

For any representation space M of $\mathrm{U}(k)$, and any $\lambda\in U(k)^\wedge$, we denote M_λ the isotopic component of type λ.

We have $\mathscr{L}_k=\bigoplus_{\lambda\in U(k)^\wedge}\mathscr{L}_{k,\lambda}$. Each $\mathscr{L}_{k,\lambda}$ is stable by L_k.

Let V_λ be the space of λ, we define the Hilbert space:

(2.1) $\mathscr{L}(\lambda)=\{$classes of measurable function $\psi(w,v)=\psi(\operatorname{Re} w, \operatorname{Im} w, v)$ with values in V_λ, such that

a) $\psi(wc, vc)=\lambda(c)^{-1}\,\psi(w,v)$,

b) $\psi(w,v)$ is holomorphic in v,

c) $\int \|\psi(w,v)\|_{V_\lambda}^2\, e^{-\operatorname{Tr} v v^*}\,|dw|^2\,|dv|^2<\infty\}$.

The group G acts on $\mathscr{L}(\lambda)$ by the same formulas (1.6) as L_k. We denote by $\mathrm{L}_k(\lambda)$ the corresponding representation. Let λ' be the contragredient representation of λ in the dual space V'_λ, and (x,f) the bilinear paring.

Then

$$\mathscr{L}_{k,\lambda}=\mathscr{L}(\lambda')\otimes V_\lambda=\operatorname{Hom}_{\mathbb{C}}(V_{\lambda'};\mathscr{L}(\lambda')).$$

Therefore we have

$$\mathscr{L}_k=\bigoplus_{\lambda\in U(k)^\wedge}\mathscr{L}_k(\lambda')\otimes V_\lambda$$

as a representation on space of $G\times U(k)$.

(2.2) It is easy to see that the set of functions $\{e^{-\operatorname{Tr} w w^*}P(w,v)\}$ is dense in $\mathscr{L}$, where $P(w,v)=P(\operatorname{Re} w, \operatorname{Im} w, v)$ is a real polynomial in w, and holomorphic in v.

The map $P_\lambda\colon \mathscr{L}\to\operatorname{Hom}(V_\lambda,\mathscr{L}(\lambda))=\mathscr{L}_{\lambda'}$ given by

$$((P_\lambda\,\psi)\cdot f)(w,v)=\int_{U(k)}\psi(wc,vc)\,\lambda(c)^{-1}f\,dc$$

is the projector of $\mathscr{L}$ to the component of type λ'. It follows that the set of functions $\{e^{-\operatorname{Tr} w w^*}P(w,v)$; where P is a V_λ-valued polynomial in $\operatorname{Re} w$, $\operatorname{Im} w, v$ satisfying $P(wc,vc)=\lambda(c)^{-1}P(w,v)\}$ is dense in $\mathscr{L}(\lambda)$.

(2.3) In the sequel we shall prove that the representation $\mathrm{L}_k(\lambda)$ is irreducible and we will identify $\mathscr{L}(\lambda)$ with a Hilbert space of holomorphic functions on the associated hermitian symmetric space.

3. Representations of G on Spaces of Holomorphic Functions

(3.1) We consider X the complex grassmannian of q-dimensional subspaces λ of $\mathbb{C}^p \oplus \mathbb{C}^q$. We consider G imbedded in $\mathrm{GL}(p+q, \mathbb{C})$ which acts on X homogeneously in the obvious way. Let D be the open subset of X consisting of all λ's such that the hermitian form h is negative definite on λ.

Let $\lambda \in D$, then $\lambda \cap ((\mathbb{C}\, e_1 \oplus \cdots \oplus \mathbb{C}\, e_q) \oplus (\mathbb{C}\, v_1 \oplus \cdots \oplus \mathbb{C}\, v_r)) = \{0\}$, i.e.

$$\lambda = \{w + z w + u w;\ w \in \mathbb{C}\, f_1 \oplus \cdots \oplus \mathbb{C}\, f_q\}$$
$$u \in M(r, q; \mathbb{C})\ u\colon \mathbb{C}\, f_1 \oplus \cdots \oplus \mathbb{C}\, f_q \to \mathbb{C}\, v_1 \oplus \cdots \oplus \mathbb{C}\, v_r$$
$$z \in M(q, q; \mathbb{C})\ z\colon \mathbb{C}\, f_1 \oplus \cdots \oplus \mathbb{C}\, f_q \to \mathbb{C}\, e_1 \oplus \cdots \oplus \mathbb{C}\, e_q.$$

Let us write $z = x + i\, y$, with $x = x^*$, $y = y^*$ the condition $h(w + zw + uw, w + zw + uw) < 0$ is equivalent to $y > \dfrac{u^* u}{2}$.

We will identify the complex manifold D with the open subset, still denoted by D, of $M(q, q; \mathbb{C}) \times M(r, q; \mathbb{C})$ defined by

$$D = \left\{(x + i\, y, u);\ x = x^*,\ y = y^*,\ y > \frac{u^* u}{2}\right\}.$$

We write also p for the couple $\begin{pmatrix} z \\ u \end{pmatrix}$ and we consider p as an element of $M(p, q; \mathbb{C})$.

Let us write an element g of $\mathrm{GL}(p+q, \mathbb{C})$ as

$$g = \left(\begin{array}{c|c} \overset{\overset{p}{\longleftrightarrow}}{\alpha} & \overset{\overset{q}{\longleftrightarrow}}{\beta} \\ \hline \gamma & \delta \end{array}\right).$$

Then the action of the group G on D is given by $g \cdot \mathrm{p} = (\alpha \mathrm{p} + \beta)(\gamma \mathrm{p} + \delta)^{-1}$.

We will consider the following element $e_0 = (i, 0)$ of D.

The stabiliser of e_0 is a maximal compact subgroup K of G. Let

$$\mathbf{c} = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c|c} 1 & 0 & i \\ \hline 0 & \sqrt{2} & 0 \\ \hline i & 0 & 1 \end{array}\right).$$

Let

$$h_0 = \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & -1 \end{array}\right), \qquad \text{then } h = \mathbf{c}\, h_0\, \mathbf{c}^*.$$

Let $x_0 = \mathbb{C} f_1 \oplus \cdots \oplus \mathbb{C} f_q$, then $e_0 = \mathbf{c}(x_0)$.

Let

$$G_0=\{g;\, g\in \mathrm{GL}(p+q;\mathbb{C});\ g^* h_0 g=h_0\}$$

and

$$K_0=\{g;\, g\in G_0;\ g\cdot x_0=x_0\}.$$

Obviously $K_0=\left(\begin{array}{c|c}\alpha & 0\\ \hline 0 & \delta\end{array}\right)$; $\alpha\in U(p)$, $\delta\in U(q)$.

We then have that the map $(\alpha,\delta)\mapsto \mathbf{c}\left(\begin{array}{c|c}\alpha & 0\\ \hline 0 & \delta\end{array}\right)\mathbf{c}^{-1}$ is an isomorphism of $U(p)\times U(q)$ with K.

The Lie algebra $\mathfrak{g}_0^{\mathbb{C}}$ is canonically identified with $\mathfrak{gl}(p+q;\mathbb{C})$. We choose

$$\mathfrak{p}^+=\mathrm{Ad}\,\mathbf{c}\left\{\left(\begin{array}{c|c}\overbrace{0}^{p} & \overbrace{x}^{q}\\ \hline 0 & 0\end{array}\right);\quad x\in M(p,q;\mathbb{C})\right\},$$

$$\mathfrak{p}^-=\mathrm{Ad}\,\mathbf{c}\left\{\left(\begin{array}{c|c}0 & 0\\ \hline y & 0\end{array}\right);\quad y\in M(q,p;\mathbb{C})\right\},$$

$$\mathfrak{h}^{\mathbb{C}}=\mathrm{Ad}\,\mathbf{c}\left\{\left(\begin{array}{ccc|ccc}a_1 & & & & & \\ & \ddots & & & 0 & \\ & & a_p & & & \\ \hline & & & b_1 & & \\ & 0 & & & \ddots & \\ & & & & & b_q\end{array}\right),\quad a_i, b_i\in\mathbb{C}\right\}.$$

We choose the system Δ^+ of positive roots, such that

$$\mathfrak{k}^+=\mathrm{Ad}(\mathbf{c})\left\{\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix};\quad \alpha \text{ and } \delta \text{ are upper triangular}\right\} \text{ and } \mathfrak{g}^+=\mathfrak{k}^+\oplus\mathfrak{p}^+.$$

We suppose $q\neq 0$. Let γ_1 be the highest non compact root, then

$$H_{\gamma_1}=\mathrm{Ad}\,\mathbf{c}\begin{pmatrix}1 & & & & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & -1\end{pmatrix} \quad\text{and}\quad \rho(H_{\gamma_1})=p+q-1.$$

(3.2) Let τ be a holomorphic representation of $GL(p,\mathbb{C})\times GL(q,\mathbb{C})$ on a finite vector space V_τ. The representation $\mathbf{c}\begin{pmatrix}\alpha & 0\\ 0 & \delta\end{pmatrix}\mathbf{c}^{-1}\mapsto\tau(\alpha,\delta)$ of $K_{\mathbb{C}}$ is also denoted by τ.

Let us consider $G_u=\{g\in GL(p+q;\mathbb{C});\ g\cdot e_0\in D\}$ i.e. $G_u=GK_{\mathbb{C}}P_-$.

Let $p\in D$, by definition

$$\begin{pmatrix}1 & p\\ 0 & 1\end{pmatrix}\cdot x_0=p=\begin{pmatrix}1 & p\\ 0 & 1\end{pmatrix}\mathbf{c}^{-1}\cdot e_0.$$

So any element of G_u can be written uniquely

$$g=\left(\begin{array}{c|c}1 & p\\ \hline 0 & 1\end{array}\right)\mathbf{c}^{-1}g_0, \qquad \text{where } p\in D,\ g_0\in K_{\mathbb{C}}P_-$$

i.e.

$$g=\left(\begin{array}{c|c}1 & p\\ \hline 0 & 1\end{array}\right)\left(\begin{array}{c|c}\alpha & 0\\ \hline \gamma & \delta\end{array}\right)\mathbf{c}^{-1}.$$

We recall that $\mathcal{O}(\tau)$ is the space of holomorphic section of $G\underset{K}{\times}V_\tau$; A function in $\mathcal{O}(\tau)$ can obviously be prolonged to $G_u=GK_{\mathbb{C}}P_-$ by $f(gkp)=\tau(k)^{-1}f(g)$, $g\in G$, $k\in K_{\mathbb{C}}$, $p\in P_-$.

We identify f as a V_τ-valued holomorphic function on D by

$$f(p)=f\left(\left(\begin{array}{c|c}1 & p\\ \hline 0 & 1\end{array}\right)\mathbf{c}^{-1}\right).$$

We denote by $J_1(g,p)$, $J_2(g,p)$ the elements of $\mathrm{GL}(p,\mathbb{C})$, $\mathrm{GL}(q,\mathbb{C})$ such that

$$g\cdot\left(\begin{array}{c|c}1 & p\\ \hline 0 & 1\end{array}\right)=\left(\begin{array}{c|c}1 & g\cdot p\\ \hline 0 & 1\end{array}\right)\left(\begin{array}{c|c}J_1(g,p) & 0\\ \hline \gamma & J_2(g,p)\end{array}\right).$$

The action of G on $\mathcal{O}(\tau)$ becomes the action on the space $\mathcal{O}(D,V_\tau)$ of all V_τ-valued holomorphic functions $f(p)$ on V_τ by

(3.3) $\quad (T(\tau)(g)f)(p)=\tau(J_1(g^{-1},p),J_2(g^{-1},p))^{-1}f(g^{-1}p).$

We will write here explicitly the formula (3.3) for the special elements of G we have pointed out;

$$(T(\tau)(g(a))f)(z,u)=\tau\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix},a^{*-1}\right)f(a^{-1}za^{*-1},ua^{*-1}),$$

$$(T(\tau)(k(\alpha))f)(z,u)=\tau\left(\begin{pmatrix}1 & 0\\ 0 & \alpha\end{pmatrix},\mathrm{id}\right)f(z,\alpha^{-1}u),$$

$$(T(\tau)(t(x))f)(z,u)=f(z-x,u),$$

$$(T(\tau)(n(u_0))f)(z,u)=\tau\left(\begin{pmatrix}1 & iu_0^*\\ 0 & 1\end{pmatrix},\mathrm{id}\right)f\left(z-iu_0^*u+i\frac{u_0^*u_0}{2},u-u_0\right),$$

$$(T(\tau)(\sigma)f)(z,u)=\tau\left(\begin{pmatrix}-z & 0\\ -u & 1\end{pmatrix},-z^{-1}\right)\cdot f(-z^{-1},-uz^{-1}).$$

(3.4) Let us assume that τ is irreducible, then $\tau=\tau_1\otimes\tau_2$ with τ_1 (resp. τ_2) an irreducible holomorphic representation of $\mathrm{GL}(p,\mathbb{C})$ (resp. $\mathrm{GL}(q,\mathbb{C})$).

Let v_{τ_1} (resp. v_{τ_2}) be the highest weight vector of V_{τ_1} (resp. V_{τ_2}), then the highest weight vector ψ_τ of $\mathcal{O}(\tau)$ (I.3.2) becomes the function:

$$f_\tau(z,u)=\left(\tau_1\begin{pmatrix}\frac{z+i}{i} & 0\\ \frac{u}{i} & \sqrt{2}\end{pmatrix}\otimes\tau_2\left(\frac{z+i}{i}\right)\right)^{-1}\cdot v_{\tau_1}\otimes v_{\tau_2}.$$

4. Intertwining Operators

We identify the complixification of the real vector space

$$M(q,k;\mathbb{C}) \quad \text{with} \quad M(q,k;\mathbb{C})\times M(q,k;\mathbb{C}) \quad \text{via} \quad w\mapsto(w,\overline{w}).$$

So if $P(\operatorname{Re} w, \operatorname{Im} w)$ is a polynomial on the real vector space $M(q,k;\mathbb{C})$, then there exists a complex polynomial on $M(q,k;\mathbb{C})\times M(q,k;\mathbb{C})$ still denoted by P, such that $P(w,\overline{w})=P(\operatorname{Re} w, \operatorname{Im} w)$.

Let λ be an irreducible unitary representation of $U(k)$ that we extend holomorphically to $\mathrm{GL}(k,\mathbb{C})$ and τ a finite dimensional holomorphic representation of $\mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C})$ on V_τ; we choose a scalar product V_τ such that $\tau(g_1^*, g_2^*)=\tau(g_1,g_2)^*$. We denote by $\mathfrak{H}(\lambda,\tau)$ the space of polynomials $P(x,y)$ on $M(p,k;\mathbb{C})\times M(q,k;\mathbb{C})$ with values in $\operatorname{Hom}(V_\tau, V_\lambda)$ such that

a) $P(a_1 x c, {}^t a_2^{-1} y\, {}^t c^{-1})=\lambda(c)^{-1} P(x,y)\,\tau(a_1,a_2)^{-1}$
$\quad a_1\in\mathrm{GL}(p,\mathbb{C}), \quad a_2\in\mathrm{GL}(q,\mathbb{C}), \quad c\in\mathrm{GL}(k,\mathbb{C}),$

b) the function

$$w\mapsto P\left(\begin{pmatrix} w \\ v\end{pmatrix}, \overline{w}\right)$$

is a harmonic function in w, i.e.

$$\sum_{i=1}^{q}\sum_{v=1}^{k}\frac{\partial^2}{\partial w_{iv}\,\partial\overline{w}_{iv}} P\left(\begin{pmatrix} w \\ v\end{pmatrix}, \overline{w}\right)=0.$$

Note that condition a) and b) imply

$$(\Delta_{i,j}P)(x,y)=0 \qquad \text{for } 1\leqq i\leqq p,\ 1\leqq j\leqq q$$

where Δ_{ij} is the differential operator:

$$\sum_{v=1}^{h}\frac{\partial^2}{\partial x_{iv}\,\partial y_{jv}}.$$

If $P\in\mathfrak{H}(\lambda,\tau)$, we define

$$(IP)(w,v)=P\left(\begin{pmatrix} w \\ i\,v\end{pmatrix}, \overline{w}\right)$$

then IP is a polynomial on $M(q,k;\mathbb{C})\times M(r,k;\mathbb{C})$ harmonic in w, and holomorphic in v. We have the following relations for IP:

α) $(IP)(w\,c, v\,c)=\lambda(c)^{-1}(IP)(w,v); \quad c\in U(k),$

β) $(IP)(a\,w,v)=(IP)(w,v)\,\tau\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}, (a^*)^{-1}\right)^{-1}; \quad a\in\mathrm{GL}(q,\mathbb{C}),$

γ) $(IP)(w,\alpha\,v)=(IP)(w,v)\left(\tau\left(\begin{pmatrix} 1 & 0\\ 0 & \alpha\end{pmatrix}, 1\right)\right)^{-1}; \quad \alpha\in U(k),$

δ) $(IP)(w, v+u_0 w)=(IP)(w,v)\left(\tau\left(\begin{pmatrix} 1 & 0\\ i\,u_0 & 1\end{pmatrix}, \mathrm{id}\right)\right)^{-1}; \quad u_0\in M(r,q;\mathbb{C}).$

(4.1) **Proposition.** *Let P be a function in $\mathfrak{H}(\lambda, \tau)$.*

a) *Let ψ be in $\mathscr{L}(\lambda)$* (2.1), *then the integral*

$$(\mathscr{F}(P)\psi)(z,u)=\int e^{i\operatorname{Tr} w^* z w}\, e^{\operatorname{Tr} w v^* u}\, e^{-\operatorname{Tr} v v^*}(IP(w,v))^*\psi(w,v)\,|dw|^2\,|dv|^2$$

is an absolutely convergent integral whenever $(z,u)\in D$ and defines a holomorphic function on D with values in V_τ.

b) *The map $\psi\mapsto\mathscr{F}(P)\psi$ intertwines the representation $L_k(\lambda)$ with the representation $T(\tau\otimes\delta_k)$ on $\mathcal{O}(D, V_\tau)$ given by*

$$(T(\tau\otimes\delta_k)(g)f)(p)=(\tau\otimes\delta_k)(J_1(g^{-1},p), J_2(g^{-1},p))^{-1} f(g^{-1}p).$$

Here $\delta_k(g_1, g_2)=(\det g_2)^k$ for $g_1\in \mathrm{GL}(p,\mathbb{C})$, $g_2\in\mathrm{GL}(q,\mathbb{C})$.

Proof. We shall prove first a).

Let $(z,u)\in D$ and $f\in V_\tau$; we write $z=x+i\left(t+\frac{u^*u}{2}\right)$, with a positive definite hermitian matrix t, then

$$\begin{aligned}|\langle(\mathscr{F}(P)\psi)(z,u), f\rangle|^2&\\ \leqq\Big(\int e^{-\operatorname{Tr} w w^* t}\, e^{-\operatorname{Tr}\frac{w w^* u^* u}{2}}\, e^{\operatorname{Re}\operatorname{Tr} w v^* u}\, e^{-\operatorname{Tr}\frac{v v^*}{2}}\, e^{-\operatorname{Tr}\frac{v v^*}{2}}&\\ \cdot\langle (IP)^*(w,v)\psi(w,v), f\rangle\,|dw|^2\,|dv|^2\Big)^2.&\end{aligned}$$

We have

$$|\langle IP^*(w,v)\psi(w,v), f\rangle|=|\langle\psi(w,v),(IP)(w,v)f\rangle|\leqq\|\psi(w,v)\|\,\|IP(w,v)f\|.$$

Applying Schwarz inequality, we get

$$\begin{aligned}&|\langle(\mathscr{F}(P)\psi)(z,u), f\rangle|^2\\ &\leqq\Big(\int e^{-2\operatorname{Tr} w w^* t}\, e^{-\operatorname{Tr} w w^* u^* u}\, e^{\operatorname{Tr} w v^* u}\, e^{\operatorname{Tr} u^* v w^*}\, e^{-\operatorname{Tr} v v^*}\\ &\quad \|(IP)(w,v)f\|^2\,|dw|^2\,|dv|^2\Big)\|\psi\|^2\\ &\leqq\int e^{-2\operatorname{Tr} w w^* t}\, e^{-\operatorname{Tr}(v-ww)(v-uw)^*}\,\|(IP)(w,v)f\|^2\,|dw|^2\,|dv|^2\cdot\|\psi\|^2_{\mathscr{L}(\lambda)}\\ &\leqq\Big(\int e^{-2\operatorname{Tr} w w^* t}\, e^{-\operatorname{Tr} v v^*}\,\|(IP(w, v+uw)f\|^2\,|dw|^2\,|dv|^2\Big)\|\psi\|^2_{\mathscr{L}(\lambda)}\end{aligned}$$

which is obviously convergent, because IP is a polynomial.

Let us verify b). The commutation relations with the elements of the form $g(a)$, $k(\alpha)$, $n(u)$ follow immediately from the relations β) γ) δ). The commutation relation with $t(x)$ is obvious. The only problem is to check the action of σ. We prove first:

(4.2) **Lemma.**

$$\begin{aligned}&\int e^{i\operatorname{Tr} w w'^*}\, e^{i\operatorname{Tr} w^* w'}\, e^{i\operatorname{Tr} w w^* z}\, e^{\operatorname{Tr} w v^* u}(IP)(w,v)^*\,|dw|^2\\ &=(-2\pi i)^{qk}(\tau\otimes\delta_k)\left(\begin{pmatrix}-z & 0\\ -u & 1\end{pmatrix}, -z^{-1}\right) e^{-i\operatorname{Tr} w' w'^* z^{-1}}\, e^{-\operatorname{Tr} w' v^* u z^{-1}}\, IP(w',v)^*.\end{aligned}$$

Proof. As both members are holomorphic functions of z, it is enough to verify this for $z=i\,y^2$, where y is a positive definite hermitian matrix. Using the change of variables $w\mapsto y^{-1}w$, and the relation β) for IP, it is enough to prove the formula for $z=i$. i.e.:

$$(4.3)\quad \int e^{i\operatorname{Tr} w w'^*} e^{i\operatorname{Tr} w^* w'} e^{-\operatorname{Tr} w w^*} e^{\operatorname{Tr} w v^* u}(IP(w,v))^* |dw|^2$$
$$=(-2\pi i)^{qk}\,\tau\left(\begin{pmatrix}-i & 0\\ -u & 1\end{pmatrix}, i\right)(i)^{qk} e^{-\operatorname{Tr} w' w'^*} e^{i\operatorname{Tr} w' v^* u}(IP)(w',v)^*$$

or taking the adjoints of both members:

$$(4.4)\quad \int e^{-i\operatorname{Tr} w w'^*} e^{-i\operatorname{Tr} w^* w'} e^{-\operatorname{Tr} w w^*} e^{\operatorname{Tr} u^* v w^*} P\left(\begin{pmatrix}w\\ i\,v\end{pmatrix}, \bar{w}\right)|dw|^2$$
$$=(2\pi)^{qk} e^{-\operatorname{Tr} w' w'^*} e^{-i\operatorname{Tr} u^* v w'^*} P\left(\begin{pmatrix}w'\\ i\,v\end{pmatrix}, \bar{w}'\right)\tau\left(\begin{pmatrix}i & -u^*\\ 0 & 1\end{pmatrix}, -i\right).$$

Let us consider the value $u=0$, then it follows from the mean value relation for harmonic polynomials:

$$(4.5)\quad \int_{\mathbb{R}^n} e^{-2i(x,y)} e^{-(x,x)} F(x)\,dx=\pi^{n/2} e^{-(y,y)} F(-i\,y)$$

where F is an harmonic polynomial on $\mathbb{R}^n$ (extended holomorphically to $\mathbb{C}^n$), that:

$$\int e^{-i\operatorname{Tr} w w'^*} e^{-i\operatorname{Tr} w^* w'} e^{-\operatorname{Tr} w w^*} P\left(\begin{pmatrix}w\\ i\,v\end{pmatrix}, \bar{w}\right)|dw|^2$$
$$=2^{qk}\pi^{qk} e^{-\operatorname{Tr} w' w'^*} P\left(\begin{pmatrix}-i\,w'\\ i\,v\end{pmatrix}, -i\,\bar{w}'\right)$$

which is what we want, as P verifies the condition a).

We consider this equality, as an equation between real analytic function of w'; by analytic continuation it follows that for any $(w_1, w_2)\in M(q,k;\mathbb{C})\times M(q,k;\mathbb{C})$ that

$$\int e^{-i\operatorname{Tr} w^* w_1} e^{-i\operatorname{Tr} w\, w_2^t} e^{-\operatorname{Tr} w w^*} P\left(\begin{pmatrix}w\\ i\,v\end{pmatrix}, \bar{w}\right)|dw|^2$$
$$=(2\pi)^{qk} e^{-\operatorname{Tr} w_1{}^t w_2} P\left(\begin{pmatrix}-i\,w_1\\ i\,v\end{pmatrix}, -i\,w_2\right)$$

as these two members are equal for the value $(w_1, \bar{w}_1)$.

Let us now consider the values

$$w_1=w'+i\,u^*\,v$$
$$w_2=\bar{w}'$$

we obtain that the left hand side of (4.4) is equal to

$$(2\pi)^{qk}\, e^{-\operatorname{Tr} w' w'^*}\, e^{-i\operatorname{Tr} u^* v w'^*}\, P\left(\begin{pmatrix} -i w' + u^* v \\ i v \end{pmatrix}, -i\bar{w}'\right).$$

Since P is in $\mathfrak{H}(\lambda, \tau)$ and

$$\begin{pmatrix} -i & -i u^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w' \\ i v \end{pmatrix} = \begin{pmatrix} -i w' + u^* v \\ i v \end{pmatrix}.$$

We obtain the equality (4.4).

Now we have

$$(\mathscr{F}(P)\, L_k(\lambda)(\sigma)\, \psi)(z, u) = \int e^{-i\operatorname{Tr} w^* z w}\, e^{\operatorname{Tr} w v^* u}\, e^{-\operatorname{Tr} v v^*}((IP)(w, v))^*$$
$$\cdot \left(\frac{i}{2\pi}\right)^{qk} \left(\int e^{2i\operatorname{Re}\operatorname{Tr} w w'^*}\, \psi(w', v)\, |dw'|^2\right) |dw|^2\, |dv|^2.$$

By Plancherel this is equal to

$$\left(\frac{1}{-2\pi i}\right)^{qk} \int\left(\int e^{2i\operatorname{Re}\operatorname{Tr} w w'^*}\, e^{i\operatorname{Tr} w^* z w}\, e^{\operatorname{Tr} w v^* u}\, IP(w, v)^*\, |dw|^2\right)$$
$$\cdot\, e^{-\operatorname{Tr} v v^*}\, \psi(w', v)\, |dw'|^2\, |dv|^2$$

applying the Lemma (4.2), we obtain that this is $(T(\tau \otimes \delta_k)(\sigma)\, \mathscr{F}(P)\, \psi)(z, u)$.

5. *Pluriharmonic Polynomials*

As seen in the preceding discussions, we will follow the same arguments as for $\mathrm{Sp}(n, \mathbb{R})$.

We consider the system of differential equations

(5.1) $\quad (\Delta_{ij} f)(x, y) = 0 \qquad \text{for } 1 \leqq i \leqq p,\ 1 \leqq j \leqq q$

where

$$\Delta_{i,j} = \sum_{\nu=1}^{k} \frac{\partial^2}{\partial x_{i\nu}\, \partial y_{j\nu}}; \qquad x \in M(p, k; \mathbb{C}), \qquad y \in M(q, k; \mathbb{C}).$$

We shall call a solution of this system pluriharmonic. We denote by $\mathfrak{H}$ the space of all pluriharmonic polynomials on $M(p, k; \mathbb{C}) \times M(q, k; \mathbb{C})$.

The group $\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C})$ acts on $M(p, k; \mathbb{C}) \times M(q, k; \mathbb{C})$ by $(g_1, g_2, c) \cdot (x, y) \to (g_1 x c^{-1}, {}^t g_2^{-1} y\, {}^t c)$.

The system (5.1) is invariant by this action; so $\mathfrak{H}$ is a representation space of $\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C})$.

(5.2) Let us denote by X the variety $M(p, k; \mathbb{C}) \times M(q, k; \mathbb{C})$ and by $\mathbb{C}[X]$ the space of all polynomials on X. We note by $\mathbb{C}[x\,{}^t y]$ the subspace of $\mathbb{C}[X]$ generated by $(x\,{}^t y)_{i,j}$; $1 \leqq i \leqq p$, $1 \leqq j \leqq q$. We introduce, for λ an irreducible holomorphic representation of $\mathrm{GL}(k;\mathbb{C})$, $\mathbb{C}[X](\lambda)$ (resp. $\mathfrak{H}(\lambda)$) the space of all V_λ-valued polynomials (resp. pluriharmonic polynomials) such that $P(xc, y^t c^{-1}) = \lambda(c)^{-1} P(x,y)$.

We have

$$\mathfrak{H} = \bigoplus_{\lambda \in \mathrm{GL}(k,\mathbb{C})^\wedge} \mathfrak{H}(\lambda) \otimes V_{\lambda'}.$$

On $\mathbb{C}[X]$, we consider the inner product $\langle P; Q\rangle$ deduced from the inner product $\mathrm{Tr}\, x x^* + \mathrm{Tr}\, y y^*$ on $X = M(p,k;\mathbb{C}) \times M(q,k;\mathbb{C})$.

In the same way as Lemma (II.5.3), we know:

(5.3) **Lemma.** $\mathbb{C}[X] = \mathbb{C}[x^t y]\, \mathfrak{H}$.

(5.4) **Corollary.** $\mathbb{C}[X](\lambda) = \mathbb{C}[x^t y]\, \mathfrak{H}(\lambda)$.

(5.5) We shall denote by Σ the subset of $\lambda \in U(k)^\wedge$ such that $\mathfrak{H}(\lambda) \neq 0$. This condition is equivalent to $\mathbb{C}[X](\lambda) \neq 0$ (5.4) and hence to $\mathscr{L}(\lambda) \neq 0$, as the subspace

$$\left\{ e^{-\mathrm{Tr}\, w^* w} P\left(\begin{pmatrix} w \\ v \end{pmatrix}, \bar{w} \right);\ p \in \mathbb{C}[X](\lambda) \right\}$$

is dense in $\mathscr{L}(\lambda)$.

For $\lambda \in \Sigma$ we denote by $\tau(\lambda)$ the representation of $\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})$ on $\mathfrak{H}(\lambda)$ given by $((g_1, g_2) \cdot P)(x,y) = P(g_1^{-1} x, {}^t g_2 y)$.

(5.6) **Proposition.** *$\tau = \tau(\lambda)$ is an irreducible representation of* $\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})$.

α) Let us suppose first that $p = q$. Let $P \in \mathbb{C}[X]$, we consider the real polynomial P' on $M(p,k;\mathbb{C})$ given by

$$P'(x + iy) = P\left(\frac{x + iy}{\sqrt{2}}, \frac{x - iy}{\sqrt{2}} \right), \qquad \text{for } x, y \in M(p,k;\mathbb{R}).$$

Then

$$\langle P', Q'\rangle = \left(P'\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \cdot \bar{Q}' \right)(0) = \langle P, Q\rangle.$$

If P, Q are pluriharmonic polynomials, it follows from (II.5.5) that

$$\langle P, Q\rangle = \text{const.} \int e^{-\mathrm{Tr}\, w w^*} P'(w)\, \overline{Q'(w)}\, |dw|^2.$$

It follows then by the same line of argument as in the proof of the Proposition (II.5.7) that if $k \leq p$ the representation τ is irreducible.

β) Let p, q, k be arbitrary; let $f(x,y)$ on $M(p,k;\mathbb{C}) \times M(q,k{:}\,\mathbb{C})$ be a highest weight vector of $\mathfrak{H}(\lambda)$ with respect to the Borel subalgebra $\begin{pmatrix} & 0 \\ \blacktriangle & \end{pmatrix} \times \begin{pmatrix} & \blacktriangledown \\ 0 & \end{pmatrix}$ of $\mathfrak{gl}(p,\mathbb{C}) \times \mathfrak{gl}(q;\mathbb{C})$; Let us choose $N \geq k, p, q$; then the function $\tilde{f}(\tilde{x}, \tilde{y})$ on $M(N,k;\mathbb{C}) \times M(N,k;\mathbb{C})$ defined by $\tilde{f}\left(\begin{pmatrix} x \\ * \end{pmatrix}, \begin{pmatrix} y \\ * \end{pmatrix} \right) = f(x,y)$ is a highest weight vector of the corresponding representation of $\mathrm{GL}(N,\mathbb{C}) \times \mathrm{GL}(N,\mathbb{C})$, hence $\tilde{f}$ is unique up to a scalar multiple by α) and so is f.

(5.7) For $\lambda \in \Sigma$, we define a $\mathrm{Hom}(\mathfrak{H}(\lambda), V_\lambda)$-valued polynomial $P_\lambda(x, y)\cdot$ by $P_\lambda(x, y)\cdot f = f(x, y)$.

It is immediate that $P_\lambda \in \mathfrak{H}(\lambda, \tau)$. Hence we get a map $\mathscr{F}_\lambda = \mathscr{F}(P_\lambda)$ intertwining the representation $L_k(\lambda)$ and $T(\tau \otimes \delta_k)$.

(5.8) **Proposition.** *The map $\mathscr{F}_\lambda$ is injective.*

Proof. Let $\psi \in \mathscr{L}(\lambda)$ such that $\mathscr{F}_\lambda \psi = 0$, i.e. for any $f \in V_\tau = \mathfrak{H}(\lambda)$ and $(z, u) \in D$, we have

$$\begin{aligned} 0 &= \langle (\mathscr{F}_\lambda \psi)(z, u), f \rangle_{V_\tau} \\ &= \int e^{i\mathrm{Tr}\, w^* z w}\, e^{\mathrm{Tr}\, w v^* u}\, e^{-\mathrm{Tr}\, v v^*} \langle \psi(w, v), IP(w, v) f \rangle |dw|^2 |dv|^2 \\ &= \int e^{-\mathrm{Tr}\, v v^*} \left\langle \psi(w, v), e^{-i\mathrm{Tr}\, \bar{z} w w^*} e^{\mathrm{Tr}\, u^* v w^*} f\left(\begin{pmatrix} w \\ i v \end{pmatrix}, \bar{w} \right) \right\rangle |dw|^2 |dv|^2. \end{aligned}$$

It is then sufficient to prove that the closure $\mathscr{L}$ of the space generated by the functions:

$$\psi(z, u, f)(w, v) = e^{-i\mathrm{Tr}\, \bar{z} w w^*} e^{\mathrm{Tr}\, u^* v w^*} f\left(\begin{pmatrix} w \\ i v \end{pmatrix}, \bar{w} \right)$$

for $(z, u) \in D$, $f \in \mathfrak{H}(\lambda)$ is $\mathscr{L}(\lambda)$.

It is easy to see that for any polynomial P the partial derivatives

$$P\left(\frac{\partial}{\partial \bar{z}_{ij}}, \frac{\partial}{\partial \bar{u}_{kl}} \right) \psi(z, u, f)$$

of the function ψ are in L. Hence $\mathscr{L}$ contains all functions of the form

$$e^{-i\mathrm{Tr}\, \bar{z} w w^*} e^{\mathrm{Tr}\, u^* v w^*} P((w w^*)_{ij}, (v w^*)_{kl}) f\left(\begin{pmatrix} w \\ i v \end{pmatrix}, \bar{w} \right).$$

We have (5.4) $\mathbb{C}[X](\lambda) = \mathbb{C}[x\, {}^t y]\, \mathfrak{H}(\lambda)$. It follows that any polynomial $Q\left(\begin{pmatrix} w \\ i v \end{pmatrix}, \bar{w} \right)$ in $\mathbb{C}[X](\lambda)$ is a linear combination of polynomials of the form $P((w w^*)_{ij}, (v w^*)_{kl}) f\left(\begin{matrix} w \\ iv \end{matrix}, \bar{w} \right)$, $f \in \mathfrak{H}(\lambda)$. So $\mathscr{L}$ contains in particular the functions

$$\left\{ e^{-\mathrm{Tr}\, w w^*} P\left(\begin{pmatrix} w \\ i v \end{pmatrix}, \bar{w} \right);\ P \in \mathbb{C}[X](\lambda) \right\}$$

which are dense in $\mathscr{L}(\lambda)$.

6. *Description of $\mathfrak{H}(\lambda)$*

Now we shall describe the set Σ and the correspondence $\lambda \mapsto \tau(\lambda)$ in terms of highest weight.

We consider the Borel subalgebra $\mathfrak{b}_k^+$ (resp. $\mathfrak{b}_p^-, \mathfrak{b}_q^+$) of $\mathfrak{gl}(k; \mathbb{C})$ (resp. $\mathfrak{gl}(p, \mathbb{C})$, $\mathfrak{gl}(q, \mathbb{C})$) as the upper (resp. lower, upper) triangular subalgebra; $GL(p, \mathbb{C}) \times GL(q, \mathbb{C}) \times GL(k; \mathbb{C})$ acts on $\mathfrak{H}$ by $((g_1, g_2, c)\cdot f)(x, y) = f(g_1^{-1} x c, {}^t g_2\, y\, {}^t c^{-1})$.

We set

$$\Delta_j(x)=\det\begin{pmatrix}x_{11},\dots,x_{1j}\\ x_{j1},\dots,x_{jj}\end{pmatrix}\qquad 0\leqq j\leqq k,p,$$

$$\tilde{\Delta}_j(y)=\det\begin{pmatrix}y_{1,k-j+1},\dots,y_{1k}\\ y_{j,k-j+1},\dots,y_{jk}\end{pmatrix}.\qquad 0\leqq j\leqq k,q.$$

(6.1) **Proposition.** *Let $f(x,y)\in\mathfrak{H}$ be a highest weight vector. Then f is of the form:*

(6.2) $\Delta_1^{\alpha_1}(x)\dots\Delta_i^{\alpha_i}(x)\cdot\tilde{\Delta}_1(y)^{\beta_1}\dots\tilde{\Delta}_j^{\beta_j}(y)$

with $0\leqq i\leqq p$, $0\leqq j\leqq q$, $i+j\leqq k$, α_i, β_i non negative integers.

Proof. It is easy to see that these functions are pluriharmonic and highest weight vectors.

To prove the proposition, we follow the same argument as in Chapter II, Proposition (6.6); i.e. We suppose first that $p=q$ and that $k\leqq p$.

Let λ be any representation of $GL(k;\mathbb{C})$, we parametrise λ by its highest weight vector with respect to $\mathfrak{b}_k$, i.e.

$$\lambda=(m_1,m_2,\dots,m_i,\ 0,\dots,0,\ -n_j,\dots,\ -n_1)$$
$$m_1\geqq m_2\geqq\cdots\geqq m_i\geqq 0;\qquad n_1\geqq\cdots\geqq n_j g\qquad i+j\leqq k\leqq p$$

then the function $\Delta_1(x)^{m_1-m_2}\dots\Delta_i^{m_i}(x)\cdot\tilde{\Delta}_1(y)^{n_1-n_2}\dots\tilde{\Delta}_j(y)^{n_j}$ is a highest weight vector under $GL(k;\mathbb{C})$ of type λ; As the representation $\tau(\lambda')$ is irreducible, it follows that this f is the unique element of $\mathfrak{H}$ highest weight vector for $\mathfrak{b}_k\times\mathfrak{b}_p\times\mathfrak{b}_q$ of type λ, with respect to $GL(k;\mathbb{C})$.

If p,q,k are arbitrary, we take $N\geqq p,q,k$, the function $\tilde{f}(\tilde{x},\tilde{y})=f(x,y)$ with $\tilde{x}=\begin{pmatrix}x\\ *\end{pmatrix}$, $\tilde{y}=\begin{pmatrix}y\\ *\end{pmatrix}$, is a pluriharmonic function and is a highest weight vector for the group $GL(N,\mathbb{C})\times GL(N,\mathbb{C})\times GL(k;\mathbb{C})$. Hence $\tilde{f}$ is of the form (6.2), which implies that so is f.

We parametrise now an irreducible representation $\tau=\tau_1\otimes\tau_2$ of $GL(p,\mathbb{C})\times GL(q,\mathbb{C})$ by its highest weight vector with respect to the product of upper triangular Borel subalgebras $\mathfrak{b}_p^+\times\mathfrak{b}_q^+$.

We have then

(6.3) **Theorem.**

a) $\Sigma=\{\lambda=(n_1,\dots,n_i,0,\dots,0,-m_1,-m_2,\dots,-m_j)\ n_1\geqq\cdots\geqq n_i>0,$
$0<m_1\leqq m_2\leqq\cdots\leqq m_j,\ 0\leqq i\leqq q,\ 0\leqq j\leqq p,\ i+j\leqq k.$

b) *If $\lambda\in\Sigma$, the representation $\tau(\lambda)$ is*
$\tau_1(0,0,\dots,0,-m_1,-m_2,\dots,-m_j)\otimes\tau_2(n_1,\dots,n_i,0,\dots,0).$

c) *For any k, the map $\lambda\mapsto\tau(\lambda)$ is injective.*

7. Decomposition of the Representation $\mathscr{L}_k$

(7.1) We have proved that the unitary representation $\mathscr{L}_k(\lambda)$ is imbedded as a subrepresentation of the representation $T(\tau(\lambda)\otimes\delta_k)$. Since $\tau(\lambda)\otimes\delta_k$ is an irreducible representation, we see that $L_k(\lambda)$ is an irreducible unitary representation of G of highest weight $\tau(\lambda)\otimes\delta_k$.

On the other hand we have $\mathscr{L}_k=\bigoplus_{\lambda\in\Sigma}\mathscr{L}_k(\lambda)\otimes V_\lambda$, and hence $\mathscr{L}_k=\bigoplus_{\lambda\in\Sigma}(\dim V_\lambda)\,\mathscr{L}_k(\lambda)$. Since $\lambda\mapsto\tau(\lambda)$ is injective (6.3) we have

(7.2) **Theorem.** 1) *For $\lambda\in\Sigma$, the representation* $\mathrm{L}_k(\lambda)$ *appears in* $\mathrm{L}_k\dim V_\lambda$*-times, and we have*

$$\mathrm{L}_k=\bigoplus_{\lambda\in\Sigma}(\dim V_\lambda)\,\mathrm{L}_k(\lambda).$$

2) *For $\lambda\in\Sigma$, the representation* $\mathrm{L}_k(\lambda)$ *is an irreducible unitary representation of G of highest weight $\tau(\lambda)\otimes\delta_k$.*

3) *The description of Σ and $\tau(\lambda)$ is given in* (6.3).

8. Unitary Representation of $U(p,q)$ with Highest Weight

(8.1) Let k be an integer, let $\lambda\in\Sigma_k$ and $\tau=\tau(\lambda)$ the irreducible representation of $\mathrm{GL}(n,\mathbb{C})$ associated to λ.

We can then construct the map $\mathscr{F}_\lambda\colon\mathscr{L}_k(\lambda)\to\mathscr{O}(D,V_\tau)$.

We consider the Hilbert space

$$\mathscr{M}_\lambda=\{\mathscr{F}_\lambda(\psi);\ \psi\in\mathscr{L}_k(\lambda)\}\qquad\text{with } \|\mathscr{F}_\lambda(\psi)\|^2=\|\psi\|^2.$$

Then $\mathscr{M}_\lambda$ is a Hilbert space of holomorphic functions for which point evaluation is continuous. Let Λ be the highest weight of the representation $\tau(\lambda)\otimes\delta_k$; the corresponding infinitesimal module M of all K-finite vectors of the representation $T(\tau(\lambda)\otimes\delta_k)$ in $\mathscr{M}_\lambda$ is equivalent to $\mathscr{L}(\Lambda)$, hence $\Lambda\in P$.

We recall that $\Lambda\in D$, i.e. the corresponding representation is a member of the discrete series if and only if $\langle\Lambda+\rho,H_{\gamma_1}\rangle<0$ (when $q\neq0$).

In particular we see on the description of Λ given by applying Proposition (6.3):

– if $k\geqq(p+q)$ all the representations occurring in L_k are members of the holomorphic discrete series

– if $k<q$, none of the representations obtained are discrete.

– if $q\leqq k<p+q$, some of the representations obtained are discrete, some are not.

(8.2) In the cases where $\Lambda\in D$, i.e., where $\langle\Lambda+\rho,H_{\gamma_1}\rangle<0$, we know that the Hilbert space $\mathscr{M}_\lambda$ has to be $\mathscr{H}(\Lambda)$.

In the coordinates $(z=x+iy,u)$ of D, it comes for $\tau=\tau_1\otimes\tau_2=\tau(\lambda)$, $\mathscr{M}_\lambda=\Big\{f$; V_τ-valued holomorphic functions on D such that:

$$\int\left\langle\left(\tau_1\begin{pmatrix} y & iu^* \\ \dfrac{-iu}{2} & \dfrac{1}{2}\end{pmatrix}^{-1}\otimes\tau_2(y)\right)\cdot f(x+iy,u),f(x+iy,u)\right\rangle$$
$$\cdot(\det y)^{k-(p+q)}\,dx\,dy\,|du|^2<\infty\Big\}.$$

(8.3) If λ is not such that $\Lambda\in D$, it is not easy to give a description of the Hilbert space $\mathscr{M}_\lambda$ obtained. For $k=1$ and $p=q=2$, for example the Hilbert space obtained are solutions of the Dirac and Maxwell equations (see [6]).

We compute now the reproducing kernel K_λ of the space $\mathscr{M}_\lambda$

(8.4) $K_\lambda((z_1, u_1), (z_2, u_2))$

$$=\det\left(\frac{z_1-z_2^*-i\,u_2^*\,u_1}{2i}\right)^{-k}\tau_1\begin{pmatrix}\dfrac{z_1-z_2^*+i\,u_2^*\,u_1}{2i} & i\,u_2^*\\ i\,u_1 & 1\end{pmatrix}$$

$$\otimes\tau_2\left(\frac{z_1-z_2^*-i\,u_2^*\,u_1}{2i}\right)^{-1}.$$

Proof. We have under the correspondence

$$F(z,u)=f\left(\begin{pmatrix}1&0&z\\0&1&u\\0&0&1\end{pmatrix}\mathbf{c}^{-1}\right),$$

$$F(i,0)=\text{const.}\,\tau\left(\begin{pmatrix}\sqrt{2}&0\\0&1\end{pmatrix},1\right)^{-1}f(1).$$

This gives the value $K_\lambda((z,u),(i,0))$ of the reproducing kernel the formula (8.4) follows then from the covariance properties of K_λ.

(8.5) **Lemma.** *Let $\lambda\in\Sigma$, then*

$K_\lambda((z_1, u_1), (z_2, u_2))$

$$=\int e^{i\operatorname{Tr}w^*(z_1-z_2^*)w}\,e^{\operatorname{Tr}wv^*u_1}\,e^{\operatorname{Tr}u_2^*vw^*}\,e^{-\operatorname{Tr}vv^*}\,IP(w,v)^*\,IP(w,v)\,|dw|^2\,|dv|^2.$$

Proof. See (II.8.5).

Let S be the space of all finite dimensional irreducible representation of $\mathrm{GL}(p,\mathbb{C})\times\mathrm{GL}(q,\mathbb{C})$ of the form $\tau(\lambda)\otimes\delta_k$ with some k and some $\lambda\in\Sigma$. Then, if $\tau\in S$. $\mathcal{O}(\tau)$ contains a non-trivial unitary representation.

(8.6) **Proposition.**

$$S=\{\tau=(n_1,\dots,n_p)\otimes(m_1,\dots,m_q);\ 0\geqq n_1\geqq\dots\geqq n_p,\ 0\leqq m_1\leqq\dots\leqq m_q,$$
$$m_q\geqq\min_{\substack{0\leqq i\leqq p\\1\leqq j\leqq q}}(-n_1-\dots-n_i+m_j+\dots+m_{q-1}-(q-j)\,m_q+p-i+j-1)\}.$$

Proof. Let S' be the right-hand side. It is immediate that S' contains S. If $m_q\geqq p+q$, then τ is contained in S. Suppose that $m_q<p+q$, and τ is in S'. Then there is $(i,j)\neq(p,q)$ such that

$$m_q\geqq -n_1-\dots-n_i+m_j+\dots+m_{q-1}-(q-j)\,m_q+p-i+j-1,$$
$$m_q< -n_1-\dots-n_{i-1}+m_j+\dots+m_{q-1}-(q-j)\,m_q+p-i+j\qquad(\text{if } i\neq 0),$$
$$m_q< -n_1-\dots-n_i+m_{j+1}+\dots+m_{q-1}-(q-j-1)\,m_q+p-i+j\qquad(\text{if } j\neq q).$$

Hence, $1>-n_i$ if $i\neq 0$ and $m_j-m_q<1$ if $j\neq q$. Therefore $n_1=\dots=n_i=0$ (for $i\neq 0$), and $m_j=\dots=m_q$ (for $j\neq q$). Hence, $p-i+j-1\leqq m_q<p-i+j$, which implies

that $m_q = p - i + j - 1$. Letting $k = p - i + j - 1$, we have

$$\tau = (0, \ldots, 0, \overset{p-i}{\overleftrightarrow{* \cdots *}}) \otimes (\overset{j-1}{\overleftrightarrow{* \cdots *}}, k, \ldots, k).$$

and $k = (p - i) + (j - 1)$. Thus, τ is in S. Q.E.D.

According to our results, it is natural to form the following conjecture.

Conjecture. S equals the set P of the finite-dimensional irreducible representations τ of $\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C})$ such that $\mathcal{O}(\tau)$ contains non-zero unitary subrepresentations.

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