ON LEVEL ZERO REPRESENTATIONS OF QUANTIZED AFFINE ALGEBRAS

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Abstract. We study the properties of level zero modules over quantized affine algebras. The proof of the conjecture on the cyclicity of tensor products by Akasaka and the present author is given. Several properties of modules generated by extremal vectors are proved. The weights of a module generated by an extremal vector are contained in the convex hull of the Weyl group orbit of the extremal weight. The universal extremal weight module with level zero fundamental weight as an extremal weight is irreducible, and isomorphic to the affinization of an irreducible finite-dimensional module.

Contents

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1. Introduction

In this paper, we study the level zero representations of quantum affine algebras. This paper is divided into three parts, on extremal weight modules, on the conjecture in [1] on the cyclicity of the tensor products of fundamental representations, and on the global basis of the Fock space.

In [12], as a generalization of highest weight vectors, the notion of extremal weight vectors is introduced, and it is shown that the universal module generated by an extremal weight vector has favorable properties: this has a crystal base, a global basis, etc. The main purpose of the first part (§ 2—§ 5) is to study such modules in the *affine case* and to prove the following two properties.

(a) If a module is generated by an extremal vector with weight λ, then all the weights of this module are contained in the convex hull of the Weyl group orbit of λ.
(b) Any module generated by an extremal vector with a level zero fundamental weight ϖ_i is irreducible, and isomorphic to the affinization of an irreducible finite-dimensional module $W(\varpi_i)$ (see Theorem 5.17 and Proposition 5.16 for an exact statement).

In the second part, we shall prove the following theorem [1], which is conjectured in [1] and proved in the case of $A_n^{(1)}$ and $C_n^{(1)}$.

Theorem. *If $a_\nu/a_{\nu+1}$ has no pole at $q = 0$ ($\nu = 1, \dots, m-1$), then $W(\varpi_{i_1})_{a_1} \otimes \cdots \otimes W(\varpi_{i_m})_{a_m}$ is generated by the tensor product of the extremal vectors.*

In the course of the proof, one uses the global basis on the tensor products of the affinizations of $W(\varpi_{i_\nu})$, especially the fact that the transformation matrix between the global basis of the tensor products and the tensor products of global bases is triangular.

Among the consequences of this theorem (see § 9), we mention here the following one. Under the conditions of the theorem above, there is a unique homomorphism up to a constant multiple

$$W(\varpi_{i_1})_{a_1} \otimes \cdots \otimes W(\varpi_{i_m})_{a_m} \longrightarrow W(\varpi_{i_m})_{a_m} \otimes \cdots \otimes W(\varpi_{i_1})_{a_1},$$

and its image is an irreducible $U'_q(\mathfrak{g})$-module. This phenomenon is analogous to the morphism from the Verma module to the dual Verma module. Conversely, combining with a result of Drinfeld ([4]), any irreducible integrable $U'_q(\mathfrak{g})$-module is isomorphic to the image for some $\{(i_1, a_1), \dots, (i_m, a_m)\}$. Moreover, $\{(i_1, a_1), \dots, (i_m, a_m)\}$ is unique up to a permutation.

In the third part (§ 12), we prove the existence of the global basis on the Fock space.

The plan of the paper is as follows. In § 2 –§ 4, we review some of the known results of crystal bases. Then, in § 5, we give a proof of (a) and (b).

In § 6, we prove a sufficient condition for a module to admit a global basis: very roughly speaking, it is enough to have a global basis in the extremal weight spaces. In § 7, we review the universal R-matrix and the universal conjugation operator. After introducing the notion of good modules (rudely speaking, a module with a global basis), we shall prove in § 9 the above theorem in the framework of good modules

After preparations in § 10–§ 11 on the combinatorial R-matrix and the energy function, we shall prove in § 12 the properties of good modules which are postulated for the existence of the wedge products and the Fock space in [13]. Finally, we shall show that the Fock space admits a global basis. In the case of the vector representation of $\mathfrak{g} = A_n^{(1)}$, the global basis of the corresponding Fock space is already constructed by B. Leclerc and J.-Y. Thibon [14] (see also [15, 21]).

[1] M. Varagnolo–E. Vasserot (Standard modules of quantum affine algebras, math.QA/0006084) prove the same conjecture in the simply-laced case by a different method.

In the last section, we present conjectures on the structure of $V(\lambda)$.

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2. Review on crystal bases

In this section, we shall review very briefly the quantized universal enveloping algebra and crystal bases. We refer the reader to [8, 9, 12].

2.1. **Quantized universal enveloping algebras.** We shall define the quantized universal enveloping algebra $U_q(\mathfrak{g})$. Assume that we are given the following data.

$$\begin{aligned}
&P : \text{a free } \mathbb{Z}\text{-module (called a weight lattice)}\\
&I : \text{an index set (for simple roots)}\\
&\alpha_i \in P \text{ for } i \in I \text{ (called a simple root)}\\
&h_i \in P^* = \operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \text{ (called a simple coroot)}\\
&(\,\cdot\,,\,\cdot\,)\colon P \times P \to \mathbb{Q} \text{ a bilinear symmetric form.}
\end{aligned}$$

We shall denote by $\langle\,\cdot\,,\,\cdot\,\rangle\colon P^* \times P \to \mathbb{Z}$ the canonical pairing.

The data above are assumed to satisfy the following axioms.

$$(\alpha_i, \alpha_i) > 0 \quad \text{for any } i \in I, \tag{2.1}$$

$$\langle h_i, \lambda\rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{for any } i \in I \text{ and } \lambda \in P, \tag{2.2}$$

$$(\alpha_i, \alpha_j) \le 0 \quad \text{for any } i,\, j \in I \text{ with } i \ne j. \tag{2.3}$$

Let us choose a positive integer d such that $(\alpha_i, \alpha_i)/2 \in \mathbb{Z}\, d^{-1}$ for any $i \in I$. Now let q be an indeterminate and set

$$K = \mathbb{Q}(q_s) \text{ where } q_s = q^{1/d}. \tag{2.4}$$

Definition 2.1. The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is the algebra over K generated by the symbols e_i, f_i ($i \in I$) and $q(h)$ ($h \in d^{-1}P^*$) with the following defining relations.

(1) $q(h) = 1$ for $h = 0$.
(2) $q(h_1)q(h_2) = q(h_1 + h_2)$ for $h_1, h_2 \in d^{-1}P^*$.
(3) $q(h)e_i\, q(h)^{-1} = q^{\langle h,\alpha_i\rangle}\, e_i$ and $q(h)f_i\, q(h)^{-1} = q^{-\langle h,\alpha_i\rangle} f_i$ for any $i \in I$ and $h \in d^{-1}P^*$.
(4) $[e_i, f_j] = \delta_{ij}\dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ for $i,\, j \in I$. Here $q_i = q^{(\alpha_i,\alpha_i)/2}$ and $t_i = q(\frac{(\alpha_i,\alpha_i)}{2} h_i)$.

(5) (Serre relation) For $i \neq j$,

$$\sum_{k=0}^{b}(-1)^k e_i^{(k)} e_j e_i^{(b-k)} = \sum_{k=0}^{b}(-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0.$$

Here $b = 1 - \langle h_i, \alpha_j \rangle$ and

$$e_i^{(k)} = e_i^k/[k]_i! \ , \qquad f_i^{(k)} = f_i^k/[k]_i! \ ,$$
$$[k]_i = (q_i^k - q_i^{-k})/(q_i - q_i^{-1}) \ , \qquad [k]_i! = [1]_i \cdots [k]_i \, .$$

Sometimes we need an algebraically closed field containing K, for example

$$\widehat{K} = \bigcup_n \mathbb{C}((q^{1/n})), \tag{2.5}$$

and to consider $U_q(\mathfrak{g})$ as an algebra over $\widehat{K}$.

We denote by $U_q(\mathfrak{g})_{\mathbb{Q}}$ the subalgebra of $U_q(\mathfrak{g})$ over $\mathbb{Q}[{q_s}^{\pm 1}]$ generated by the $e_i^{(n)}$'s, the $f_i^{(n)}$'s $(i \in I)$ and q^h $(h \in d^{-1}P^*)$.

Let us denote by W the Weyl group, the subgroup of $GL(P)$ generated by the simple reflections s_i: $s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$.

Let $\Delta \subset Q = \sum_i \mathbb{Z}\alpha_i$ be the set of roots. Let $\Delta^{\pm} = \Delta \cap Q_{\pm}$ be the set of positive and negative roots, respectively. Here $Q_{\pm} = \pm \sum_i \mathbb{Z}_{\geq 0}\alpha_i$. Let $\Delta^{\rm re}$ be the set of real roots. $\Delta^{\rm re}_{\pm} = \Delta_{\pm} \cap \Delta^{\rm re}$.

2.2. **Crystals.** We shall not review the notion of crystals, but refer the reader to [8, 9, 12]. We say that a crystal B over $U_q(\mathfrak{g})$ is *a regular crystal* if, for any $J \subset I$ such that $\{\alpha_i \, ; i \in J\}$ is of finite-dimensional type, B is, as a crystal over $U_q(\mathfrak{g}_J)$, isomorphic to the crystal bases associated with an integrable $U_q(\mathfrak{g}_J)$-module. Here $U_q(\mathfrak{g}_J)$ is the subalgebra of $U_q(\mathfrak{g})$ generated by e_j, f_j $(j \in J)$ and q^h $(h \in d^{-1}P^*)$. By [12], the Weyl group W acts on any regular crystal. This action S is given by

$$S_{s_i} b = \begin{cases} \tilde{f}_i^{\langle h_i, {\rm wt}(b) \rangle} b & \text{if } \langle h_i, {\rm wt}(b) \rangle \geq 0, \\ \tilde{e}_i^{-\langle h_i, {\rm wt}(b) \rangle} b & \text{if } \langle h_i, {\rm wt}(b) \rangle \leq 0. \end{cases}$$

Let us denote by $U_q^-(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by the f_i's (resp. by the e_i's). Then $U_q^-(\mathfrak{g})$ has a crystal base denoted by $B(\infty)$ ([9]). The unique weight vector of $B(\infty)$ with weight 0 is denoted by u_∞. Similarly $U_q^+(\mathfrak{g})$ has a crystal base denoted by $B(-\infty)$, and the unique weight vector of $B(-\infty)$ with weight 0 is denoted by $u_{-\infty}$.

Let ψ be the ring automorphism of $U_q(\mathfrak{g})$ that sends q_s, e_i, f_i and $q(h)$ to q_s, f_i, e_i and $q(-h)$. It gives a bijection $B(\infty) \simeq B(-\infty)$ by which u_∞, $\tilde{e}_i$, $\tilde{f}_i$, ε_i, φ_i, wt corresponds to $u_{-\infty}$, $\tilde{f}_i$, $\tilde{e}_i$, φ_i, ε_i, $-$wt.

Let us denote by $\tilde{U}_q(\mathfrak{g})$ the modified quantized universal enveloping algebra $\oplus_{\lambda\in P} U_q(\mathfrak{g})a_\lambda$ (see [12]). Then $\tilde{U}_q(\mathfrak{g})$ has a crystal base $B(\tilde{U}_q(\mathfrak{g}))$. As a crystal, $B(\tilde{U}_q(\mathfrak{g}))$ is regular and isomorphic to

$$\bigsqcup_{\lambda\in P} B(\infty)\otimes T_\lambda \otimes B(-\infty).$$

Here, T_λ is the crystal consisting of a single element t_λ with $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$ and $\mathrm{wt}(t_\lambda) = \lambda$.

Let $*$ be the anti-involution of $U_q(\mathfrak{g})$ that sends $q(h)$ to $q(-h)$, and q_s, e_i, f_i to themselves. The involution $*$ of $U_q(\mathfrak{g})$ induces an involution $*$ on $B(\infty)$, $B(-\infty)$, $B(\tilde{U}_q(\mathfrak{g}))$. Then $\tilde{e}_i^* = * \circ \tilde{e}_i \circ *$, etc. give another crystal structure on $B(\infty)$, $B(-\infty)$, $B(\tilde{U}_q(\mathfrak{g}))$. We call it *the star crystal structure*. In the case of $B(\tilde{U}_q(\mathfrak{g}))$, these two crystal structures are compatible, and $B(\tilde{U}_q(\mathfrak{g}))$ may be considered as a crystal over $\mathfrak{g}\oplus\mathfrak{g}$. Hence, for example, S_w^*, the Weyl group action on $B(\tilde{U}_q(\mathfrak{g}))$ with respect to the star crystal structure is a crystal automorphism of $B(\tilde{U}_q(\mathfrak{g}))$ with respect to the original crystal structure. In particular, the two Weyl group actions S_w and $S_{w'}^*$ commute with each other.

The formulas concerning with $B(\tilde{U}_q(\mathfrak{g}))$ are given in Appendix B.

Note that we have always

$$\varepsilon_i(b) + \varphi_i^*(b) = \varepsilon_i^*(b) + \varphi_i(b) \geq 0 \text{ for any } b \in B(\infty). \tag{2.6}$$

2.3. **Schubert decomposition of crystal bases.** For $w \in W$ with a reduced expression $s_{i_1}\cdots s_{i_\ell}$, we define the subset $B_w(\infty)$ of $B(\infty)$ by

$$B_w(\infty) = \{\tilde{f}_{i_1}^{a_1}\cdots\tilde{f}_{i_\ell}^{a_\ell}u_\infty\,;a_1,\ldots,a_\ell \in \mathbb{Z}_{\geq 0}\}. \tag{2.7}$$

Then $B_w(\infty)$ does not depend on the choice of a reduced expression. We refer the reader to [11] on the details of $B_w(\infty)$ and its relationship with the Demazure module.

We have ([11])

(i) $B_w(\infty)^* = B_{w^{-1}}(\infty)$.
(ii) If $w' \leq w$, then $B_{w'}(\infty) \subset B_w(\infty)$.
(iii) If $s_iw < w$, then $\tilde{f}_iB_w(\infty) \subset B_w(\infty)$.
(iv) $\tilde{e}_iB_w(\infty) \subset B_w(\infty)\sqcup\{0\}$.
(v) If both b and $\tilde{f}_ib$ belong to $B_w(\infty)$, then all $\tilde{f}_i^kb$ $(k\geq 0)$ belong to $B_w(\infty)$.

Here $\leq$ is the Bruhat order. Set

$$\overline{B}_w(\infty) = B_w(\infty)\backslash\big(\bigcup_{w'<w} B_{w'}(\infty)\big).$$

P. Littelmann ([16]) showed

$$B(\infty) = \bigsqcup_{w\in W} \overline{B}_w(\infty).$$

We have

$$\overline{B}_w(\infty)^* = \overline{B}_{w^{-1}}(\infty). \tag{2.8}$$

(2.9) If $s_iw < w$, then $\tilde{e}_i^{\max}\overline{B}_w(\infty) \subset \overline{B}_{s_iw}(\infty)$,
$\tilde{f}_i\overline{B}_w(\infty) \subset \overline{B}_w(\infty)$.
In particular, $\varepsilon_i(b) > 0$ for any $b \in \overline{B}_w(\infty)$.

Here, we use the notation $\tilde{e}_i^{\max}b = \tilde{e}_i^{\varepsilon_i(b)}b$.

2.4. **Global bases.** Let $A \subset K$ be the subring of K consisting of rational functions in q_s without pole at $q_s = 0$. Let $-$ be the automorphism of K sending q_s to ${q_s}^{-1}$. Set $K_\mathbb{Q} := \mathbb{Q}[q_s, {q_s}^{-1}]$. Let V be a vector space over K, L_0 an A-submodule of V, L_∞ an $\overline{A}$- submodule, and $V_\mathbb{Q}$ a $K_\mathbb{Q}$-submodule. Set $E := L_0 \cap L_\infty \cap V_\mathbb{Q}$.

Definition 2.2 ([9]). We say that $(L_0, L_\infty, V_\mathbb{Q})$ is *balanced* if each of L_0, L_∞ and $V_\mathbb{Q}$ generates V as a K vector space, and if the following equivalent conditions are satisfied.

(i) $E \to L_0/q_sL_0$ is an isomorphism.
(ii) $E \to L_\infty/{q_s}^{-1}L_\infty$ is an isomorphism.
(iii) $(L_0 \cap V_\mathbb{Q}) \oplus ({q_s}^{-1}L_\infty \cap V_\mathbb{Q}) \to V_\mathbb{Q}$ is an isomorphism.
(iv) $A\otimes_\mathbb{Q} E \to L_0$, $\overline{A}\otimes_\mathbb{Q} E \to L_\infty$, $K_\mathbb{Q}\otimes_\mathbb{Q} E \to V_\mathbb{Q}$ and $K\otimes_\mathbb{Q} E \to V$ are isomorphisms.

Let $-$ be the ring automorphism of $U_q(\mathfrak{g})$ sending q_s, q^h, e_i, f_i to ${q_s}^{-1}$, q^{-h}, e_i, f_i.

Let $U_q(\mathfrak{g})_\mathbb{Q}$ be the $K_\mathbb{Q}$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$, $f_i^{(n)}$ and $\left\{ {q^h \atop n} \right\}$ $(h \in P^*)$.

Let M be a $U_q(\mathfrak{g})$-module. Let $-$ be an involution of M satisfying $(au)^- = \bar{a}\bar{u}$ for any $a \in U_q(\mathfrak{g})$ and $u \in M$. We call in this paper such an involution a *bar involution*. Let (L, B) be a crystal base of an integrable $U_q(\mathfrak{g})$-module M.

Let $M_\mathbb{Q}$ be a $U_q(\mathfrak{g})_\mathbb{Q}$-submodule of M such that

$$(M_\mathbb{Q})^- = M_\mathbb{Q}, \text{ and } (u - \overline{u}) \in (q_s - 1)M_\mathbb{Q} \text{ for every } u \in M_\mathbb{Q}. \tag{2.10}$$

Definition 2.3. If $(L, \overline{L}, M_{\mathbb{Q}})$ is balanced, we say that M has a global basis.

In such a case, let $G\colon L/q_sL \xrightarrow{\sim} E := L \cap \overline{L} \cap M_{\mathbb{Q}}$ be the inverse of $E \xrightarrow{\sim} L/q_sL$. Then $\{G(b); b \in B\}$ forms a basis of M. We call this basis a (lower) *global basis*. The global basis enjoys the following properties ([9, 10]):

(i) $\overline{G(b)} = G(b)$ for any $b \in B$.
(ii) For any $n \in \mathbb{Z}_{\geq 0}$, $\{G(b); \varepsilon_i(b) \geq n\}$ is a basis of the $K_{\mathbb{Q}}$-submodule $\sum_{m \geq n} f_i^{(m)} M_{\mathbb{Q}}$.
(iii) for any $i \in I$ and $b \in B$, we have

$$f_i G(b) = [1 + \varepsilon_i(b)]_i G(\tilde{f}_i b) + \sum_{b'} F^i_{b,b'} G(b').$$

Here the sum ranges over $b' \in B$ such that $\varepsilon_i(b') > 1 + \varepsilon_i(b)$. The coefficient $F^i_{b,b'}$ belongs to $q_s q_i^{1-\varepsilon_i(b')}\mathbb{Q}[q_s]$.

Similarly for $e_i G(b)$.

3. Extremal weight modules

3.1. **Extremal vectors.** Let M be an integrable $U_q(\mathfrak{g})$-module. A vector $u \in M$ of weight $\lambda \in P$ is called *extremal* (see [1, 12]), if we can find vectors $\{u_w\}_{w \in W}$ satisfying the following properties:

$$u_w = u \text{ for } w = e, \tag{3.1}$$

$$\text{if } \langle h_i, w\lambda \rangle \geq 0, \text{ then } e_i u_w = 0 \text{ and } f_i^{(\langle h_i, w\lambda \rangle)} u_w = u_{s_i w}, \tag{3.2}$$

$$\text{if } \langle h_i, w\lambda \rangle \leq 0, \text{ then } f_i u_w = 0 \text{ and } e_i^{(-\langle h_i, w\lambda \rangle)} u_w = u_{s_i w}. \tag{3.3}$$

Hence if such $\{u_w\}$ exists, then it is unique and u_w has weight $w\lambda$. We denote u_w by $S_w u$.

Similarly, for a vector b of a regular crystal B with weight λ, we say that b is an extremal vector if it satisfies the following similar conditions: we can find vectors $\{b_w\}_{w \in W}$ such that

$$b_w = b \quad \text{for } w = e, \tag{3.4}$$

$$\text{if } \langle h_i, w\lambda \rangle \geq 0 \text{ then } \tilde{e}_i b_w = 0 \text{ and } \tilde{f}_i^{\langle h_i, w\lambda \rangle} b_w = b_{s_i w}, \tag{3.5}$$

$$\text{if } \langle h_i, w\lambda \rangle \leq 0 \text{ then } \tilde{f}_i v_w = 0 \text{ and } \tilde{e}_i^{-\langle h_i, w\lambda \rangle} b_w = b_{s_i w}. \tag{3.6}$$

Then b_w must be $S_w b$.

For $\lambda \in P$, let us denote by $V(\lambda)$ the $U_q(\mathfrak{g})$-module generated by u_λ with the defining relation that u_λ is an extremal vector of weight λ. This is in fact infinitely many linear relations on u_λ. We proved in

[12] [2] that $V(\lambda)$ has a global crystal base $(L(\lambda), B(\lambda))$. Moreover the crystal $B(\lambda)$ is isomorphic to the subcrystal of $B(\infty) \otimes t_\lambda \otimes B(-\infty)$ consisting of vectors b such that b^* is an extremal vector of weight $-\lambda$. We denote by the same letter u_λ the element of $B(\lambda)$ corresponding to $u_\lambda \in V(\lambda)$. Then $u_\lambda \in B(\lambda)$ corresponds to $u_\infty \otimes t_\lambda \otimes u_{-\infty}$.

Note that, for $b_1 \otimes t_\lambda \otimes b_2 \in B(\infty) \otimes t_\lambda \otimes B(-\infty)$ belonging to $B(\lambda)$, one has

$$\varepsilon_i^*(b_1) \le \max(\langle h_i, \lambda\rangle, 0) \text{ and } \varphi_i^*(b_2) \le \max(-\langle h_i, \lambda\rangle, 0) \text{ for any } i \in I. \tag{3.7}$$

For any $w \in W$, $u_\lambda \mapsto S_{w^{-1}} u_{w\lambda}$ gives an isomorphism of $U_q(\mathfrak{g})$-modules:

$$V(\lambda) \xrightarrow{\sim} V(w\lambda).$$

Similarly, letting S_w^* be the Weyl group action on $B(\tilde{U}_q(\mathfrak{g}))$ with respect to the star crystal structure and regarding $B(\lambda)$ as a subcrystal of $B(\tilde{U}_q(\mathfrak{g}))$, $S_w^* \colon B(\tilde{U}_q(\mathfrak{g})) \xrightarrow{\sim} B(\tilde{U}_q(\mathfrak{g}))$ induces an isomorphism of crystals

$$S_w^* \colon B(\lambda) \xrightarrow{\sim} B(w\lambda).$$

For a dominant weight λ, $V(\lambda)$ is an irreducible highest weight module of highest weight λ, and $V(-\lambda)$ is an irreducible lowest weight module of lowest weight $-\lambda$.

3.2. **Dominant weights.**

Definition 3.1. For a weight $\lambda \in P$ and $w \in W$, we say that λ is w-dominant (resp. w-regular) if $\langle \beta, \lambda\rangle \ge 0$ (resp. $\langle \beta, \lambda\rangle \ne 0$) for any $\beta \in \Delta_+^{\mathrm{re}} \cap w^{-1}\Delta_-^{\mathrm{re}}$. If λ is w-dominant and w-regular, we say that λ is regularly w-dominant.

If $w = s_{i_\ell} \cdots s_{i_1}$ is a reduced expression, then we have

$$\Delta_+^{\mathrm{re}} \cap w^{-1}\Delta_-^{\mathrm{re}} = \{s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}\,;\, 1 \le k \le \ell\}.$$

Hence λ is w-dominant (resp. w-regular) if and only if

$$\begin{aligned} &\langle h_{i_k}, s_{i_{k-1}} \cdots s_{i_1}\lambda\rangle \ge 0 \\ &(\text{resp. } \langle h_k, s_{i_{k-1}} \cdots s_{i_1}\lambda\rangle \ne 0). \end{aligned} \tag{3.8}$$

Conversely one has the following lemma.

Lemma 3.2. *For $i_1, \ldots, i_l \in I$, and a weight λ, assume that*

$$\langle h_{i_k}, s_{i_{k-1}} s_{i_{k-1}} \cdots s_{i_1}\lambda\rangle > 0 \quad \textit{for } k = 1, \ldots, l.$$

Then $w = s_{i_l} \cdots s_{i_1}$ is a reduced expression.

[2]In [12], it is denoted by $V^{\max}(\lambda)$, because I thought there would be a natural $U_q(\mathfrak{g})$-module whose crystal base is the connected component of $B(\lambda)$.

Proof. By the induction on l, we may assume that $s_{i_{l-1}}\cdots s_{i_1}$ is a reduced expression. If $l(w)<l$, then there exists k with $1\le k\le l-1$ such that $s_{i_{l-1}}\cdots s_{i_{k+1}}(h_{i_k})=-h_{i_l}$. Hence

$$\langle h_{i_l}, s_{i_{l-1}}\cdots s_{i_1}\lambda\rangle = -\langle h_{i_k}, s_{i_{k-1}}s_{i_{k-1}}\cdots s_{i_1}\lambda\rangle < 0,$$

which is a contradiction. Q.E.D.

This lemma implies the following lemma.

Lemma 3.3. *Let w_1, $w_2\in W$ and let λ be an integral weight. If λ is regularly w_2-dominant and $w_2\lambda$ is regularly w_1-dominant, then $\ell(w_1w_2)=\ell(w_1)+\ell(w_2)$ and λ is regularly w_1w_2-dominant. Here $\ell\colon W\to\mathbb{Z}$ is the length function.*

Proposition 3.4. *Let $\lambda\in P$ and $b_1\in\overline{B}_{w_1}(\infty)$, $b_2\in\overline{B}_{w_2}(-\infty)$. If $b:=b_1\otimes t_\lambda\otimes b_2$ belongs to $B(\lambda)$, then one has:*

(i) *λ is regularly w_1-dominant and $-\lambda$ is regularly w_2-dominant,*
(ii) *$\ell(w_1w_2^{-1})=\ell(w_1)+\ell(w_2)$,*
(iii) *One has*

$$\begin{aligned} S^*_{w_2}(b_1\otimes t_\lambda\otimes b_2) &\in B_{w_1w_2^{-1}}(\infty)\otimes t_{w_2\lambda}\otimes u_{-\infty},\\ S^*_{w_1}(b_1\otimes t_\lambda\otimes b_2) &\in u_\infty\otimes t_{w_1\lambda}\otimes B_{w_2w_1^{-1}}(-\infty).\end{aligned}$$

More generally if $w_1=w'w''$ with $\ell(w_1)=\ell(w')+\ell(w'')$, then

$$S^*_{w''}(b_1\otimes t_\lambda\otimes b_2)\in\overline{B}_{w'}(\infty)\otimes t_{w''\lambda}\otimes B_{w_2w''^{-1}}(-\infty).$$

Proof. Assume $w_1s_i<w_1$. Then $c:=\varepsilon_i^*(b_1)>0$ by (2.9). Hence $\langle h_i,\lambda\rangle\ge c>0$ by (3.7). We have $\tilde e_i^{*\max}b_1\in\bar B_{w_1s_i}(\infty)$.

$$b'=S_i^*(b_1\otimes t_\lambda\otimes b_2) = (\tilde e_i^{*\max}b_1)\otimes t_{s_i\lambda}\otimes(\tilde e_i^{*\langle h_i,\lambda\rangle-c}b_2). \tag{3.9}$$

Hence, λ is regularly w_1-dominant by the induction on the length of w_1. The other statement in (i) is similarly proved.

(ii) follows from (i) and the preceding lemma.

In (3.9), $\tilde e_i^{*\langle h_i,\lambda\rangle-c}b_2$ belongs to $B_{w_2s_i}(-\infty)$, since (ii) implies $w_2s_i>w_2$. Repeating this, we obtain (iii). Q.E.D.

4. Affine quantum algebras

In the sequel we assume that $\mathfrak{g}$ is affine.

4.1. **Affine root systems.** Althogh the materials in this subsection are more or less classical, we shall review the affine algebras in order to fix the notations.

Let $\mathfrak{g}$ be an affine Lie algebra, and let $\mathfrak{t}$ be its Cartan subalgebra (assuming that they are defined over $\mathbb{Q}$). Let I be the index set of simple roots and let $\alpha_i \in \mathfrak{t}^*$ be the simple roots and $h_i \in \mathfrak{t}$ the simple coroots ($i \in I$). We choose a Cartan subalgebra $\mathfrak{t}$ such that $\{\alpha_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ are linearly independent and $\dim \mathfrak{t} = \operatorname{rank}\mathfrak{g} + 1$. Let us set the root lattice and coroot lattice by

$$Q = \oplus_i \mathbb{Z}\alpha_i \subset \mathfrak{t}^* \quad \text{and} \quad Q^\vee = \oplus_i \mathbb{Z}h_i \subset \mathfrak{t}.$$

Set $Q_\pm = \pm\sum_i \mathbb{Z}_{\geq 0}\alpha_i$ and $Q^\vee_\pm = \pm\sum_i \mathbb{Z}_{\geq 0}h_i$. Let $\delta \in Q_+$ be a unique element satisfying $\{\lambda \in Q\,;\langle h_i, \lambda\rangle = 0 \text{ for every } i\} = \mathbb{Z}\delta$. Similarly we define $c \in Q^\vee_+$ by $\{h \in Q^\vee\,;\langle h, \alpha_i\rangle = 0 \text{ for every } i\} = \mathbb{Z}c$. We write

$$\delta = \sum_i a_i\alpha_i \quad \text{and} \quad c = \sum_i a_i^\vee h_i. \tag{4.1}$$

We take a W-invariant non-degenerate symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{t}^*$ normalized by

$$(\delta, \lambda) = \langle c, \lambda\rangle \quad \text{for any } \lambda \in \mathfrak{t}^*. \tag{4.2}$$

Then this symmetric form has the signature $(\dim \mathfrak{t} - 1, 1)$. We sometimes identify $\mathfrak{t}$ and $\mathfrak{t}^*$ by this symmetric form. By this identification, δ and c correspond to each other.

We have

$$a_i^\vee = \frac{(\alpha_i, \alpha_i)}{2} a_i. \tag{4.3}$$

Note that $(\alpha_i, \alpha_i)/2$ takes the values 1, 2, 3, 1/2, 1/3. Hence we have for each i

$$\frac{(\alpha_i, \alpha_i)}{2} \in \mathbb{Z} \quad \text{or} \quad \frac{2}{(\alpha_i, \alpha_i)} \in \mathbb{Z}. \tag{4.4}$$

If $\mathfrak{g}$ is untwisted, then $2/(\alpha_i, \alpha_i)$ is an integer.

Let us set $\mathfrak{t}^*_{\mathrm{cl}} = \mathfrak{t}^*/\mathbb{Q}\,\delta$ and let $\mathrm{cl}\colon \mathfrak{t}^* \to \mathfrak{t}^*_{\mathrm{cl}}$ be the canonical projection. We have

$$\mathfrak{t}^*_{\mathrm{cl}} \simeq \bigoplus_{i\in I} (\mathbb{Q}\,h_i)^*.$$

Set $\mathfrak{t}^{*0} = \{\lambda \in \mathfrak{t}^*\,;\langle c, \lambda\rangle = 0\}$ and $\mathfrak{t}^{*0}_{\mathrm{cl}} = \mathrm{cl}(\mathfrak{t}^{*0}) \subset \mathfrak{t}^*_{\mathrm{cl}}$. Then $\mathfrak{t}^{*0}_{\mathrm{cl}}$ has a positive-definite symmetric form induced by the one of $\mathfrak{t}^*$.

Lemma 4.1. *For any $a \in \mathbb{Q}$,*

$$\mathrm{cl}\colon \{\lambda \in \mathfrak{t}^*\,;(\lambda, \lambda) = a \text{ and } (\lambda, \delta) \neq 0\} \to \mathfrak{t}^*_{\mathrm{cl}} \setminus \mathfrak{t}^{*0}_{\mathrm{cl}}$$

is bijective.

Proof. Let $\lambda \in \mathfrak{t}^*$ such that $(\lambda, \delta) \neq 0$.

Setting $\mu = \lambda + x\delta$ for $x \in \mathbb{Q}$, we have $(\mu, \mu) = (\lambda + x\delta, \lambda + x\delta) = (\lambda, \lambda) + 2x(\lambda, \delta)$. Hence $\lambda + x\delta$ has square length a if and only if $x = (a - (\lambda, \lambda))/2(\lambda, \delta)$. Q.E.D.

As a corollary we have

Proposition 4.2. $\mathfrak{t}^*$ *endowed with an invariant symmetric form as above, simple roots and coroots, is unique up to a canonical isomorphism.*

Proof. For example, take $\rho \in \mathfrak{t}^*$ such that $\langle h_i, \rho \rangle = 1$ for any i and $(\rho, \rho) = 0$. The preceding lemma guarantees its existence and its uniqueness. The α_i's and ρ form a basis of $\mathfrak{t}^*$. Q.E.D.

In particular, for any Dynkin diagram isomorphism ι (i.e. a bijection $\iota\colon I \to I$ such that $\langle h_{\iota(i)}, \alpha_{\iota(j)} \rangle = \langle h_i, \alpha_j \rangle$), there exists a unique isomorphism of $\mathfrak{t}^*$ that sends α_i to $\alpha_{\iota(i)}$ and leaves the symmetric form invariant.

Let $\Delta \subset \mathfrak{t}^*$ be the root system of $\mathfrak{g}$, and Δ^{re} the set of real roots: $\Delta^{\mathrm{re}} = \Delta \setminus \mathbb{Z}\,\delta$. For $\beta \in \mathfrak{t}^*$ with $(\beta, \beta) \neq 0$, we set $\beta^\vee = 2\beta/(\beta, \beta)$. Then $\Delta^\vee := \{\beta^\vee\,; \beta \in \Delta^{\mathrm{re}}\} \cup (\mathbb{Z}\,c \setminus \{0\}) \subset \mathfrak{t}$ is the root system for the dual Lie algebra of $\mathfrak{g}$. We set $\Delta^\pm = \Delta \cap Q_\pm$.

Let us denote by Δ_{cl} the image of Δ^{re} by cl. Then Δ_{cl} is a finite subset of $\mathfrak{t}^{*0}_{\mathrm{cl}}$, and $(\Delta_{\mathrm{cl}}, \mathfrak{t}^{*0}_{\mathrm{cl}})$ is a (not necessarily reduced) root system. We call an element of Δ_{cl} a *classical* root.

Let $\mathrm{O}(\mathfrak{t}^*)$ be the orthogonal group of $\mathfrak{t}^*$ with respect to the invariant symmetric form. Let $\mathrm{O}(\mathfrak{t}^*)_\delta$ be the isotropy subgroup of δ, i.e. $\mathrm{O}(\mathfrak{t}^*)_\delta = \{g \in \mathrm{O}(\mathfrak{t}^*)\,; g\delta = \delta\}$. Then there are canonical group homomorphisms

$$\mathrm{cl}\colon \mathrm{O}(\mathfrak{t}^*)_\delta \to \mathrm{GL}(\mathfrak{t}^*_{\mathrm{cl}}) \quad \text{and} \quad \mathrm{cl}_0\colon \mathrm{O}(\mathfrak{t}^*)_\delta \to \mathrm{O}(\mathfrak{t}^{*0}_{\mathrm{cl}}).$$

The homomorphism $\mathrm{cl}\colon \mathrm{O}(\mathfrak{t}^*)_\delta \to \mathrm{GL}(\mathfrak{t}^*_{\mathrm{cl}})$ is injective.

For $\beta \in \Delta^{\mathrm{re}}$, let s_β be the corresponding reflection $\lambda \mapsto \lambda - \langle \beta^\vee, \lambda \rangle \beta$. Let W be the Weyl group, i.e. the subgroup of $\mathrm{GL}(\mathfrak{t}^*)$ generated by the s_β's. Since $W \subset \mathrm{O}(\mathfrak{t}^*)_\delta$, there are group homomorphisms $W \to \mathrm{GL}(\mathfrak{t}^*_{\mathrm{cl}})$ and $W \to \mathrm{O}(\mathfrak{t}^{*0}_{\mathrm{cl}})$.

Let us denote by W_{cl} the image of $W \to \mathrm{O}(\mathfrak{t}^{*0}_{\mathrm{cl}})$. Then W_{cl} is the Weyl group of the root system $(\Delta_{\mathrm{cl}}, \mathfrak{t}^{*0}_{\mathrm{cl}})$.

For $\xi \in \mathfrak{t}^{*0}$, we set

$$T(\lambda) = \lambda + (\delta, \lambda)\xi - (\xi, \lambda)\delta - \frac{(\xi, \xi)}{2}(\delta, \lambda)\delta. \tag{4.5}$$

Then T belongs to $\mathrm{O}(\mathfrak{t}^*)_\delta$, and T depends only on $\mathrm{cl}(\xi)$. For $\xi_0 \in \mathfrak{t}^{*0}_{\mathrm{cl}}$, let us define $t(\xi_0) \in \mathrm{O}(\mathfrak{t}^*)_\delta$ as the right-hand side of (4.5) with $\xi \in \mathrm{cl}^{-1}(\xi_0)$.

Then,

(4.6) $t\colon \mathfrak{t}^{*0}_{\mathrm{cl}} \to \operatorname{Ker}\Big(\mathrm{cl}_0\colon \mathrm{O}(\mathfrak{t}^*)_\delta \to \mathrm{GL}(\mathfrak{t}^{*0}_{\mathrm{cl}})\Big)$ is a group isomorphism.

We have

$$g \circ t(\xi) \circ g^{-1} = t\Big(\mathrm{cl}_0(g)(\xi)\Big) \quad \text{for } g \in \mathrm{O}(\mathfrak{t}^*)_\delta \text{ and } \xi \in \mathfrak{t}^{*0}_{\mathrm{cl}}. \tag{4.7}$$

For $\beta \in \mathfrak{t}^*$ such that $(\beta,\beta) \neq 0$, let us denote by s_β the reflection

$$s_\beta(\lambda) = \lambda - (\beta^\vee, \lambda)\beta\,.$$

Then we have for $\beta \in \mathfrak{t}^{*0}$ such that $(\beta,\beta) \neq 0$,

$$s_{\beta - a\delta} s_\beta = t(a\beta^\vee)\,. \tag{4.8}$$

There exists i_0 such that

$$W_{\mathrm{cl}} \text{ is generated by } \{s_i\,;\, i \neq i_0\}. \tag{4.9}$$

If $\mathfrak{g}$ is not isomorphic to $A^{(2)}_{2n}$, such an i_0 is unique up to a Dynkin diagram automorphism and $(\alpha_{i_0}, \alpha_{i_0}) = 2$, $a_{i_0} = a^\vee_{i_0} = 1$. In the case of $A^{(2)}_{2n}$, there are two choices of i_0, two extremal nodes, and $(\alpha_{i_0}, \alpha_{i_0}) = 1$ or 4, and accordingly $a_{i_0} = 2$ or 1, $a^\vee_{i_0} = 1$ or 2.

For $\alpha \in \Delta^{\mathrm{re}}$ or $\alpha \in \Delta_{\mathrm{cl}}$, we set

$$c_\alpha = \max(1, \frac{(\alpha,\alpha)}{2}),$$

and $c_i = c_{\alpha_i}$. Then we have, for any $\alpha \in \Delta^{\mathrm{re}}$

$$\{n \in \mathbb{Z}\,;\, \alpha + n\delta \in \Delta\} = \mathbb{Z}\, c_\alpha. \tag{4.10}$$

We set

$$Q_{\mathrm{cl}} = \mathrm{cl}(Q),\ Q^\vee_{\mathrm{cl}} = \mathrm{cl}(Q^\vee),\ \widetilde{Q} = Q_{\mathrm{cl}} \cap Q^\vee_{\mathrm{cl}}. \tag{4.11}$$

Here $Q^\vee = \sum_{\alpha \in \Delta^{\mathrm{re}}} \mathbb{Z}\alpha^\vee$.

We have an exact sequence

$$1 \longrightarrow \widetilde{Q} \xrightarrow{\ t\ } W \xrightarrow{\ \mathrm{cl}_0\ } W_{\mathrm{cl}} \longrightarrow 1\,. \tag{4.12}$$

For any $\alpha \in \Delta^{\mathrm{re}}$, let $\tilde{\alpha}$ be the element in $\widetilde{Q} \cap \mathbb{Q}_{>0}\mathrm{cl}(\alpha)$ with the smallest length. We set

$$\tilde{\Delta} = \{\tilde{\alpha}\,;\, \alpha \in \Delta^{\mathrm{re}}\}.$$

Then $\tilde{\Delta}$ is a reduced root system, and $\widetilde{Q}$ is the root lattice of $\tilde{\Delta}$.

Remark 4.3. Any affine Lie algebra is either untwisted or the dual of an untwisted affine algebra or $A^{(2)}_{2n}$.

(i) If $\mathfrak{g}$ is untwisted, then $\widetilde{Q} = Q_{\mathrm{cl}}^{\vee} \subset Q_{\mathrm{cl}}$, $\tilde{\Delta} = \mathrm{cl}(\Delta^{\vee\,\mathrm{re}})$, $\tilde{\alpha} = \alpha^{\vee}$.

(ii) If $\mathfrak{g}$ is the dual of an untwisted algebra, then $\widetilde{Q} = Q_{\mathrm{cl}} \subset Q_{\mathrm{cl}}^{\vee}$, $\tilde{\Delta} = \mathrm{cl}(\Delta^{\mathrm{re}})$, $\tilde{\alpha} = \alpha$.

(iii) If $\mathfrak{g} = A_{2n}^{(2)}$, then $\widetilde{Q} = Q_{\mathrm{cl}} = Q_{\mathrm{cl}}^{\vee}$, $\tilde{\Delta} = \mathrm{cl}(\Delta^{\mathrm{re}}) = \mathrm{cl}(\Delta^{\vee\,\mathrm{re}})$. For any $\alpha \in \Delta^{\mathrm{re}}$, one has

$$\tilde{\alpha} = \begin{cases} \mathrm{cl}(\alpha) & \text{if } (\alpha,\alpha) \neq 4, \\ \mathrm{cl}(\alpha)/2 & \text{if } (\alpha,\alpha) = 4. \end{cases}$$

Note that $(\alpha - \delta)/2 \in \Delta^{\mathrm{re}}$ if $(\alpha,\alpha) = 4$.

If $\mathfrak{g} \neq A_{2n}^{(2)}$, then $\tilde{\alpha} = c_\alpha \alpha^{\vee}$.

Proposition 4.4. *For $\xi \in \widetilde{Q}$,*

$$l(t(\xi)) = \sum_{\beta \in \Delta_{\mathrm{cl}}} (\beta,\xi)_+/c_\beta = \frac{1}{2} \sum_{\beta \in \Delta_{\mathrm{cl}}} |(\beta,\xi)|/c_\beta = \sum_{\beta \in \tilde{\Delta}} (\beta^{\vee},\xi)_+.$$

Here $a_+ = \max(a, 0)$.

Proof. For $\beta \in \Delta_{\mathrm{cl}}$, let us denote by β' the unique element of Δ^+ such that $\mathrm{cl}(\beta') = \beta$ and $\beta' - n\delta \notin \Delta^+$ for any $n > 0$. Note that $(\beta,\xi) \in c_\beta \mathbb{Z}$. We have

$$t(\xi)^{-1}\Delta^- \bigcap \Delta^+ = \{\gamma \in \Delta^+\,;\, \gamma - (\gamma,\xi)\delta \in \Delta^-\},$$

and $l(t(\xi))$ is the number of elements in this set. By setting $\gamma = \beta' + nc_\beta\delta$, it is isomorphic to

$$\begin{aligned} &\{(\beta,n) \in \Delta_{\mathrm{cl}} \times \mathbb{Z}\,;\, n \geq 0 \text{ and } \beta' + \Bigl(nc_\beta - (\beta,\xi)\Bigr)\delta \in \Delta^-\} \\ &\quad = \{(\beta,n) \in \Delta_{\mathrm{cl}} \times \mathbb{Z}\,;\, 0 \leq n < (\beta,\xi)/c_\beta\}. \end{aligned}$$

Since $(\beta,\xi)/c_\beta$ is an integer, we have

$$l(t(\xi)) = \sum_{\beta \in \Delta_{\mathrm{cl}}} ((\beta,\xi)/c_\beta)_+.$$

The other equalities easily follow. Q.E.D.

Corollary 4.5. *For $\xi \in \widetilde{Q}$ and $w \in W_{\mathrm{cl}}$,*

$$l(t(w\xi)) = l(t(\xi)).$$

We choose a weight lattice $P \subset \mathfrak{t}^*$ satisfying

$$(4.13)\quad \begin{cases} \alpha_i \in P \text{ and } h_i \in P^* \text{ for any } i \in I. \\ \text{For every } i \in I, \text{ there exists } \Lambda_i \in P \text{ such that } \langle h_j, \Lambda_i\rangle = \delta_{ji}. \end{cases}$$

We set

(4.14) $P^0 = \{\lambda \in P\,;\langle c,\lambda\rangle = 0\}$, $P_{\rm cl} = {\rm cl}(P) \subset \mathfrak{t}^*_{\rm cl}$, and $P^0_{\rm cl} = {\rm cl}(P^0)$.

We have

$$P_{\rm cl} = \oplus_{i\in I}(\mathbb{Z}h_i)^*.$$

Lemma 4.6. *For $\lambda \in P^0$ and $\mu \in \widetilde{Q}$, the following two conditions are equivalent.*

(i) *λ and μ are in the same Weyl chamber (i.e. for any $\alpha \in \Delta^{\rm re}$, $({\rm cl}(\alpha),\mu) > 0$ implies $(\alpha,\lambda) \geq 0$).*
(ii) *λ is $t(\mu)$-dominant.*

Proof. For $\alpha \in \Delta^{\rm re}$, let us take $\alpha' \in (\alpha + \mathbb{Z}\delta)\cap\Delta^+$ such that ${\rm cl}(\alpha') = {\rm cl}(\alpha)$ and $\alpha' - n\delta \notin \Delta^+$ for any $n \in \mathbb{Z}_{>0}$. Then for $\alpha = \alpha' + n\delta \in \Delta^+$,

$$\begin{aligned}\alpha \in \Delta^+ \cap t(\mu)^{-1}\Delta^- \quad &\Leftrightarrow t(\mu)\alpha = \alpha - (\alpha,\mu)\delta \\ &\qquad\qquad\;\; = \alpha' + (n - (\alpha,\mu))\delta \in \Delta^- \\ &\Leftrightarrow 0 \leq n < (\alpha,\mu).\end{aligned}$$

(i)⇒(ii) Now assume $\alpha = \alpha' + n\delta \in \Delta^+ \cap t(\mu)^{-1}\Delta^-$. Then $0 \leq n < (\alpha,\mu)$, and (i) implies $(\alpha,\lambda) \geq 0$

(ii)⇒(i) Assume $(\alpha,\mu) > 0$. Then taking $n = 0$, $\alpha' \in \Delta^+ \cap t(\mu)^{-1}\Delta^-$, and hence $(\alpha,\lambda) = (\alpha',\lambda) \geq 0$. Q.E.D.

The following lemma is similarly proved.

Lemma 4.7. *For $\lambda \in P^0$ and $\mu \in \widetilde{Q}$, the following two conditions are equivalent.*

(i) *For any $\alpha \in \Delta_{\rm cl}$, $(\alpha,\mu) > 0$ implies $(\alpha,\lambda) > 0$,*
(ii) *λ is regularly $t(\mu)$-dominant.*

Let us choose $i_0 \in I$ as in (4.9), and let W_0 be the subgroup of W generated by $\{s_i\,; i \in I \setminus \{i_0\}\}$. Then W is a semidirect product of W_0 and $\widetilde{Q}$.

Lemma 4.8. *Let $\xi \in \widetilde{Q}$ and $w \in W_0$. If ξ is regularly w-dominant then*

$$l(t(\xi)) = l(t(\xi)w^{-1}) + l(w).$$

Proof. We shall prove the assertion by the induction on $l(w)$. Write $w = s_iw'$ with $w > w'$ and $i \neq i_0$. Then $l(t(\xi)) = l(t(\xi)w'^{-1}) + l(w')$. Hence it is enough to show $t(\xi)w'^{-1} > t(\xi)w'^{-1}s_i$, or equivalently $t(\xi)w'^{-1}\alpha_i \in \Delta^-$. We have

$$t(\xi)w'^{-1}\alpha_i = {w'}^{-1}\alpha_i - (w'\xi,\alpha_i)\delta.$$

Since $(w'\xi, \alpha_i) > 0$, the coefficient of α_{i_0} in $t(\xi)w'^{-1}\alpha_i$ is negative, and hence $t(\xi)w'^{-1}\alpha_i$ is a negative root. Q.E.D.

4.2. **Affinization.** Let P and P_{cl} be as in (4.13). We denote by $U_q(\mathfrak{g})$ the quantized universal enveloping algebra with P as a weight lattice. We denote by $U_q'(\mathfrak{g})$ the quantized universal enveloping algebra with P_{cl} as a weight lattice. Hence $U_q'(\mathfrak{g})$ is a subalgebra of $U_q(\mathfrak{g})$ generated by the e_i's, the f_i's and q^h ($h \in d^{-1}(P_{\mathrm{cl}})^*$). When we talk about an integrable $U_q(\mathfrak{g})$-module (resp. $U_q'(\mathfrak{g})$-module), the weight of its element belongs to P (resp. P_{cl}).

Let M be a $U_q'(\mathfrak{g})$-module with the weight decomposition $M = \oplus_{\lambda \in P_{\mathrm{cl}}} M_\lambda$. We define a $U_q(\mathfrak{g})$-module M_{aff} with a weight decomposition $M_{\mathrm{aff}} = \oplus_{\lambda \in P} (M_{\mathrm{aff}})_\lambda$ by

$$(M_{\mathrm{aff}})_\lambda = M_{\mathrm{cl}(\lambda)}.$$

The action of e_i and f_i are defined in an obvious way, so that the canonical homomorphism $\mathrm{cl}\colon M_{\mathrm{aff}} \to M$ is $U_q'(\mathfrak{g})$-linear. We define the $U_q'(\mathfrak{g})$-linear automorphism z of M_{aff} with weight δ by $(M_{\mathrm{aff}})_\lambda \xrightarrow{\sim} M_{\mathrm{cl}(\lambda)} = M_{\mathrm{cl}(\lambda+\delta)} \xrightarrow{\sim} (M_{\mathrm{aff}})_{\lambda+\delta}$.

Let us choose $0 \in I$ satisfying

$$W_{\mathrm{cl}} \text{ is generated by } \{s_i; i \neq 0\}, \text{ and and } a_0 = 1. \tag{4.15}$$

Recall that $\delta = \sum_i a_i \alpha_i$. When $\mathfrak{g} = A_{2n}^{(2)}$, 0 is the longest simple root.

Choose a section $s\colon P_{\mathrm{cl}} \to P$ of $\mathrm{cl}\colon P \to P_{\mathrm{cl}}$ such that $s(\mathrm{cl}(\alpha_i)) = \alpha_i$ for any $i \in I \setminus \{0\}$. Then M is embedded into M_{aff} by s as a vector space. We have an isomorphism of $U_q'(\mathfrak{g})$-modules

$$M_{\mathrm{aff}} \simeq K[z, z^{-1}] \otimes M. \tag{4.16}$$

Here, $e_i \in U_q'(\mathfrak{g})$ and $f_i \in U_q'(\mathfrak{g})$ act on the right hand side by $z^{\delta_{i0}} \otimes e_i$ and $z^{-\delta_{i0}} \otimes f_i$.

Similarly, for a crystal with weights in P_{cl}, we can define its affinization B_{aff} by

$$B_{\mathrm{aff}} = \bigsqcup_{\lambda \in P} B_{\mathrm{cl}(\lambda)}. \tag{4.17}$$

If an integrable $U_q'(\mathfrak{g})$-module M has a crystal base (L, B), then its affinization M_{aff} has a crystal base $(L_{\mathrm{aff}}, B_{\mathrm{aff}})$.

For $a \in K$, we define the $U_q'(\mathfrak{g})$-module M_a by

$$M_a = M_{\mathrm{aff}}/(z-a)M_{\mathrm{aff}}. \tag{4.18}$$

4.3. **Simple crystals.** In [1], we defined the notion of simple crystals and studied their properties.

Definition 4.9. We say that a finite regular crystal B (with weights in P^0_{cl}) is a *simple crystal* if B satisfies

(1) There exists $\lambda \in P^0_{\mathrm{cl}}$ such that the weight of any extremal vector of B is contained in $W_{\mathrm{cl}}\lambda$.
(2) $\sharp(B_\lambda) = 1$.

Simple crystals have the following properties (loc. cit.).

Lemma 4.10. *A simple crystal B is connected.*

Lemma 4.11. *The tensor product of simple crystals is also simple.*

Proposition 4.12. *A finite-dimensional integrable $U'_q(\mathfrak{g})$-module with a simple crystal base is irreducible.*

5. Affine extremal weight modules

5.1. **Extremal vectors—affine case.** We prove now one of the main results of this paper. In the sequel we employ the notations

$\tilde{e}_i^{\max} b = \tilde{e}_i^{\varepsilon_i(b)} b$, $\tilde{f}_i^{\max} b = \tilde{f}_i^{\varphi_i(b)} b$, and similarly for $\tilde{e}_i^{*\max}$ and $\tilde{f}_i^{*\max}$.

Theorem 5.1. *For any $\lambda \in P^0$, the weight of any extremal vector of $B(\lambda)$ is contained in* $\mathrm{cl}^{-1}\mathrm{cl}(W\lambda)$.

Proof. We regard $B(\lambda)$ as a subcrystal of $B(\infty) \otimes t_\lambda \otimes B(-\infty) \subset B(\tilde{U}_q(\mathfrak{g}))$.

We shall show that $\mathrm{cl}(\mathrm{wt}(b))$ and $-\mathrm{cl}(\mathrm{wt}(b^*))$ are in the same W_{cl}-orbit whenever b and b^* are extremal vectors.

For any $b_1 \otimes t_\lambda \otimes b_2$, we have

$$\tilde{f}_i^{\max}(b_1 \otimes t_\lambda \otimes b_2) \;=\; b'_1 \otimes t_\lambda \otimes \tilde{f}_i^{\max} b_2 \quad \text{for some } b'_1.$$

(For the action of $\tilde{f}_i^{\max}$, etc. on $B(\tilde{U}_q(\mathfrak{g}))$, see Appendix B.) Hence, any extremal vector $b \in B(\lambda)$ has the form $b_1 \otimes t_\lambda \otimes u_{-\infty}$ after applying the $\tilde{f}_i^{\max}$'s.

Hence, we may further assume the following conditions on b:

(5.1) b has the form $b_1 \otimes t_\lambda \otimes u_{-\infty}$,

(5.2) for any vector of the form $b'_1 \otimes t_\mu \otimes u_{-\infty}$ in $\{S_w S^*_{w'} b \,;\, w, w' \in W\}$, the length of $\mathrm{wt}(b'_1)$ is greater than or equal to the length of $\mathrm{wt}(b_1)$.

Here, the length of $\sum_i m_i\alpha_i$ is by the definition $\sum_i |m_i|$.

Take $i \in I$. We write $\lambda_i = \langle h_i, \lambda\rangle$ and $\mathrm{wt}_i(b_1) = \langle h_i, \mathrm{wt}(b_1)\rangle$ for brevity.

Note that we have $\varepsilon_i^*(b_1) \le \max(\lambda_i, 0)$.

We shall show $\mathrm{wt}_i(b_1) \ge 0$ for every i in several steps.

(1) The case $\lambda_i \le 0$ and $\lambda_i + \mathrm{wt}_i(b_1) \le 0$.
Since $b_1 \otimes t_\lambda \otimes u_{-\infty}$ is a lowest weight vector in the i-string, one has $\varphi_i(b) = \max(\varphi_i(b_1) + \lambda_i, 0) = 0$, and hence $\varphi_i(b_1) + \lambda_i \le 0$. Similarly, $\varepsilon_i^*(b) = 0$ because b^* is a highest weight vector in the i-string. Therefore, one has

$$\begin{aligned} S_i^* S_i(b_1 \otimes t_\lambda \otimes u_{-\infty}) &= \tilde{f}_i^{*-\lambda_i}(\tilde{e}_i^{\max} b_1 \otimes t_\lambda \otimes \tilde{e}_i^{-\varphi_i(b_1)-\lambda_i} u_{-\infty}) \\ &= (\tilde{f}_i^{*\varphi_i(b_1)} \tilde{e}_i^{\max} b_1) \otimes t_{s_i\lambda} \otimes u_{-\infty}. \end{aligned}$$

The last equality follows from $S_i^* S_i(b) = (\tilde{f}_i^{*k} \tilde{e}_i^{\max} b_1) \otimes t_{s_i\lambda} \otimes u_{-\infty}$ for some k.

Hence, the minimality of b_1 gives

$$0 \le \varphi_i(b_1) - \varepsilon_i(b_1) = \mathrm{wt}_i(b_1).$$

(2) The case $\lambda_i > 0$ and $\lambda_i + \mathrm{wt}_i(b_1) \le 0$.
We shall show that this case cannot occur. In this case, as in (i),

$$\varphi_i(b_1) + \lambda_i \le 0.$$

On the other hand, $\varphi_i^*(b_1 \otimes t_\lambda \otimes u_{-\infty}) = \max(\varepsilon_i^*(b_1) - \lambda_i, 0) = 0$ implies

$$\varepsilon_i^*(b_1) \le \lambda_i.$$

Hence we obtain (the first inequality by (2.6))

$$0 \le \varepsilon_i^*(b_1) + \varphi_i(b_1) = (\varepsilon_i^*(b_1) - \lambda_i) + (\varphi_i(b_1) + \lambda_i) \le 0,$$

which implies $\varepsilon_i^*(b_1) = \lambda_i$ and $\varphi_i(b_1) = -\lambda_i$. Then we have

$$\tilde{e}_i^{*\max}(b_1 \otimes t_\lambda \otimes u_{-\infty}) = (\tilde{e}_i^{*\max} b_1) \otimes t_{s_i\lambda} \otimes u_{-\infty}.$$

Hence, the minimality of $\mathrm{wt}(b_1)$ implies $\varepsilon_i^*(b_1) = 0$, and this contradicts $\varepsilon_i^*(b_1) = \lambda_i > 0$.

(3) The case $\lambda_i \ge 0$ and $\lambda_i + \mathrm{wt}_i(b_1) \ge 0$.
In this case, one has $\varepsilon_i(b) = \varphi_i^*(b) = 0$, and hence $\varphi_i(b) = \lambda_i + \mathrm{wt}_i(b_1)$, which implies $\varphi_i(b) - (\lambda_i - \varepsilon_i^*(b_1)) = \varphi_i^*(b_1) \ge 0$. Hence we have

$$\begin{aligned} S_i S_i^*(b_1 \otimes t_\lambda \otimes u_{-\infty}) &= \tilde{f}_i^{\varphi_i(b)}(\tilde{e}_i^{*\max} b_1 \otimes t_{s_i\lambda} \otimes \tilde{e}_i^{\lambda_i - \varepsilon_i^*(b_1)} u_{-\infty}) \\ &= (\tilde{f}_i^{\varphi_i^*(b)} \tilde{e}_i^{*\max} b_1) \otimes t_{s_i\lambda} \otimes u_{-\infty}. \end{aligned}$$

Hence we have $\varphi_i^*(b_1) \ge \varepsilon_i^*(b_1)$, or equivalently $\mathrm{wt}_i(b_1) \ge 0$.

(4) The case $\lambda_i \le 0$ and $\lambda_i + \mathrm{wt}_i(b_1) \ge 0$.
We have immediately $\mathrm{wt}_i(b_1) \ge 0$.

In all the cases we have $\mathrm{wt}_i(b_1) \geq 0$. Since $\mathrm{wt}(b_1)$ is of level 0, one has $0 = \langle c, \mathrm{wt}(b_1)\rangle = \sum_i a_i^\vee \mathrm{wt}_i(b_1)$, which implies that $\mathrm{wt}_i(b_1) = 0$ for every i, or equivalently $\mathrm{cl}(\mathrm{wt}(b_1)) = 0$. Q.E.D.

Corollary 5.2. *For any $\lambda \in P$, the weight of any vector in $B(\lambda)$ is contained in the convex hull of $W\lambda$.*

Proof. In the positive level case (i.e. $\langle c, \lambda\rangle > 0$), λ being conjugate to a dominant weight and $B(\lambda)$ is isomorphic to the crystal base of an irreducible highest weight module. In this case, the assertion is well-known. Similarly for negative level case.

Assume that the level of λ is zero. Note that all vector in $B(\lambda)$ can be reached at an extremal vector after applying $\tilde{e}_i^{\max}$ and $\tilde{f}_i^{\max}$ by [12]. Hence the assertion follows from the preceding theorem. Note that $\mathrm{cl}^{-1}\mathrm{cl}(W\lambda)$ is contained in the convex hull of $W\lambda$ provided that $\mathrm{cl}(\lambda) \neq 0$. Q.E.D.

The following theorem is an immediate consequence of the preceding corollary.

Theorem 5.3. *Let M be an integrable $U_q'(\mathfrak{g})$-module and u a vector in M of weight $\lambda \in P_{\mathrm{cl}}$. Then the following conditions are equivalent.*

(i) *u is an extremal vector.*
(ii) *The weights of $U_q'(\mathfrak{g})u$ are contained in the convex hull of $W_{\mathrm{cl}}\lambda$.*
(iii) *$U_q'(\mathfrak{g})_\beta u = 0$ for any $\beta \in \Delta_{\mathrm{cl}}$ such that $(\beta, \lambda) \geq 0$.*

In particular, for any $\lambda \in P$, $V(\lambda)$ is isomorphic to the $U_q(\mathfrak{g})$-module generated by a weight vector u of weight λ with (iii) in the above corollary and the following integrability condition as defining relations:

$$f_i^{1+\langle h_i,\lambda\rangle} u = 0 \text{ if } \langle h_i, \lambda\rangle \geq 0 \text{ and } e_i^{1-\langle h_i,\lambda\rangle} u = 0 \text{ if } \langle h_i, \lambda\rangle \leq 0.$$

5.2. **Fundamental representations.** Let us take $0^\vee \in I$ such that

$$W_{\mathrm{cl}} \text{ is generated by } \{s_i; i \neq 0^\vee\}, \text{ and and } a_{0^\vee}^\vee = 1. \tag{5.3}$$

Recall that $c = \sum_i a_i^\vee h_i$. When $\mathfrak{g} = A_{2n}^{(2)}$, $0^\vee$ is the shortest simple root. We set $I_{0^\vee} = I \setminus \{0^\vee\}$. For $i \in I_{0^\vee}$, we set

$$\varpi_i = \Lambda_i - a_i^\vee \Lambda_{0^\vee} \in P^0.$$

Hence we have $P_{\mathrm{cl}}^0 = \oplus_{i\in I_{0^\vee}} \mathbb{Z}\mathrm{cl}(\varpi_i)$. We say that $\lambda \in P$ is a *basic weight* if $\mathrm{cl}(\lambda)$ is W_{cl}-conjugate to some $\mathrm{cl}(\varpi_i)$ ($i \in I_{0^\vee}$). Note that this notion does not depend on the choice of $0^\vee$.

Proposition 5.4. *Assume that $\lambda = \sum_{i\in J} \varpi_i$ for some subset J of $I_{0^\vee}$. Then one has:*

(i) *any extremal vector of* $B(\lambda)$ *is in the* W*-orbit of* u_λ,

(ii) $B(\lambda)$ *is connected.*

Proof. (ii) follows from (i) because any vector is connected with extremal vector.

Let us prove (i). We use arguments similar to the proof of Theorem 5.1. Let us take an extremal vector $b \in B(\lambda)$. Among the vectors in $S_w S^*_{w'} b$ with the form $b_1 \otimes t_\mu \otimes u_{-\infty}$, we take one such that $\mathrm{wt}(b_1)$ has the smallest length. Then the proof in Theorem 5.1 shows that $\mathrm{cl}(\mathrm{wt}(b_1)) = 0$. Hence, one has

$$S_i S_i^*(b_1 \otimes t_\mu \otimes u_{-\infty}) \quad = \quad \begin{cases} \tilde{f}_i^{\varepsilon_i^*(b_1)} \tilde{e}_i^{*\max}(b_1) \otimes t_{s_i\mu} \otimes u_{-\infty} & \text{if } \mu_i \geq 0, \\ \tilde{f}_i^{*\varepsilon_i(b_1)} \tilde{e}_i^{\max}(b_1) \otimes t_{s_i\mu} \otimes u_{-\infty} & \text{if } \mu_i \leq 0. \end{cases}$$

In the both cases, the length of b_1 remains unchanged after applying $S_i S_i^*$. Therefore, applying $S_{w'^{-1}} S^*_{w'^{-1}}$, we can assume $w' = 1$ and $\mu = \lambda$.

For $i \in I \setminus J$, we have $\lambda_i \leq 0$, which implies $\varepsilon_i^*(b_1) = 0$. If $i \in J$, then $\lambda_i = 1$ and hence $\varepsilon_i^*(b_1)$ $(\leq \lambda_i)$ must be 0 or 1. On the other hand, we have

$$S_i^*(b_1 \otimes t_\lambda \otimes u_{-\infty}) = \tilde{e}_i^{*\max} b_1 \otimes t_\lambda \otimes \tilde{e}_i^{\lambda_i - \varepsilon_i^*(b_1)} u_{-\infty}.$$

If $\varepsilon_i^*(b_1) = 1$, then this contradicts the minimality of $\mathrm{wt}(b_1)$. Hence $\varepsilon_i^*(b_1) = 0$ for every $i \in J$.

Thus we have $\varepsilon_i^*(b_1) = 0$ for every $i \in I$ and hence $b_1 = u_\infty$. Thus we obtain $u_\lambda = S_w b$. Q.E.D.

The following theorem is a particular case of the preceding proposition.

Theorem 5.5. *If* $\lambda \in P$ *is a basic weight, then any extremal vector of* $B(\lambda)$ *is in the* W*-orbit of* u_λ.

We shall now study further properties of $B(\lambda)$ for a basic weight λ.

Lemma 5.6. *Let* λ *be a basic weight. Then* $\{w \in W; w\lambda = \lambda\}$ *is generated by* $\{s_\beta\,; \beta \in \Delta_+^{\mathrm{re}}, (\beta, \lambda) = 0\}$.

Proof. We may assume $\lambda = \Lambda_j - a_j^\vee \Lambda_{0^\vee}$ for some $j \in I_{0^\vee}$. Since the similar statement holds for $(W_{\mathrm{cl}}, \mathfrak{t}_{\mathrm{cl}}^{*0})$, it is enough to show that $t(\xi)$ is contained in the subgroup G generated by $\{s_\beta\,; \beta \in \Delta_+^{\mathrm{re}}, (\beta, \lambda) = 0\}$, provided that $\xi \in \widetilde{Q}$ and $(\xi, \lambda) = 0$. We have $s_{a\delta - \beta} s_\beta = t(a\beta^\vee)$ by (4.8). In particular, one has $t(c_\beta \beta^\vee) \in G$ whenever $\beta \in \Delta^{\mathrm{re}}$ satisfies $(\beta, \lambda) = 0$.

(1) The case where $\mathfrak{g} \neq A_{2n}^{(2)}$ It is enough to show that $\{\xi \in \widetilde{Q}; (\xi, \lambda) = 0\}$ is generated by $\{c_\beta \beta^\vee; \beta \in \Delta_{\mathrm{cl}}, (\beta, \lambda) = 0\}$. In this case, $\widetilde{Q}$ has a

basis $\{c_i\alpha_i^\vee\,;i\in I_{0^\vee}\}$. Hence $\{\xi\in\widetilde{Q}\,;(\xi,\lambda)=0\}$ is generated by $\{c_i\alpha_i^\vee\,;i\in I_{0^\vee}\setminus\{j\}\}$.

(2) The case where $\mathfrak{g}=A_{2n}^{(2)}$ In this case, $\widetilde{Q}=Q=\oplus_{i\in I_{0^\vee}}\mathbb{Z}\tilde{\alpha}_i$. Hence $\{\xi\in\widetilde{Q}\,;(\xi,\lambda)=0\}$ has a basis $\{\tilde{\alpha}_i\,;i\in I_{0^\vee}\setminus\{j\}\}$. Hence, the result follows from

$$t(\tilde{\alpha}_i)=\begin{cases}s_{\delta-\alpha_i}s_{\alpha_i} & \text{if } (\alpha_i,\alpha_i)=2,\\ s_{(\delta-\alpha_i)/2}s_{\alpha_i} & \text{if } (\alpha_i,\alpha_i)=4.\end{cases}$$

Note that $(\delta-\alpha_i)/2$ is a real root in the last case. Q.E.D.

Lemma 5.7. *For any $\beta\in\Delta^{\mathrm{re}}$ and any $\lambda\in P$ such that $s_\beta\lambda=\lambda$, we have $S_{s_\beta}(u_\infty\otimes t_\lambda\otimes u_{-\infty})=u_\infty\otimes t_\lambda\otimes u_{-\infty}$.*

Proof. Set $a_\lambda=u_\infty\otimes t_\lambda\otimes u_{-\infty}$. We assume $\beta\in\Delta_+^{\mathrm{re}}$. We shall prove the assertion by the induction on the length of β. If β is a simple root, it is obvious. Otherwise, we can write $\beta=s_i\gamma$ for a positive real root γ whose length is less than that of β. We have $S_{s_\beta}=S_iS_{s_\gamma}S_i$. Set $\mu=s_i\lambda$. Then $s_\gamma\mu=\mu$ and hence we have $S_{s_\gamma}a_\mu=a_\mu$ by the induction hypothesis. Since $S_iS_i^*a_\lambda=a_\mu$ or equivalently $S_ia_\lambda=S_i^*a_\mu$, we have

$$S_{s_\beta}a_\lambda=S_iS_{s_\gamma}S_ia_\lambda=S_iS_{s_\gamma}S_i^*a_\mu=S_iS_i^*a_\mu=a_\lambda.$$

Q.E.D.

Lemma 5.6 and Lemma 5.7 imply the following proposition.

Proposition 5.8. *Let λ be a basic weight.*

(i) *If $w\in W$ satisfies $w\lambda=\lambda$, then $S_wu_\lambda=u_\lambda$ and $S_w^*u_\lambda=u_\lambda$.*
(ii) *For $\mu\in W\lambda$, the isomorphism $S_w^*\colon B(\lambda)\xrightarrow{\sim}B(\mu)$ does not depend on $w\in W$ such that $\mu=w\lambda$.*

Here we regard $B(\lambda)$ and $B(\mu)$ as subcrystals of $B(\tilde{U}_q(\mathfrak{g}))$.

Remark 5.9. For a general $\lambda\in P^0$, it is not true that the extremal weights of $B(\lambda)$ belong to $W\lambda$. For example in $\lambda=2(\Lambda_1-\Lambda_0)$ in the $A_1^{(1)}$-case $f_0f_1u_\lambda$ is an extremal vector with weight $\lambda-\delta$.

Remark 5.10. It is not true in general $w\lambda=\lambda$ implies $S_wu_\lambda=u_\lambda$. For example in the case of $\mathfrak{g}=A_2^{(1)}$, and $\lambda=\Lambda_1+\Lambda_2-2\Lambda_0$, set $w_1=t(\alpha_1)=s_1s_0s_2s_1$ and $w_2=t(\alpha_2)=s_1s_0s_1s_2$. Then $w_1\lambda=w_2\lambda=\lambda-\delta$, but $S_{w_1}u_\lambda\neq S_{w_2}u_\lambda$.

Conjecture 5.11. For any $\lambda\in P$, $S_wu_\lambda=u_\lambda$ if and only if $w\in W$ is in the subgroup generated by $\{s_\beta;\beta$ is a real root such that $(\beta,\lambda)=0\}$.

Theorem 5.5 and Proposition 5.8 immediately imply the following result.

Proposition 5.12. *Assume that λ is a basic weight.*

(i) $B(\lambda)_\lambda = \{u_\lambda\}$.

(ii) $B(\lambda)$ *is connected.*

Proof. Let $b \in B(\lambda)_\lambda$. Then Theorem 5.5 implies $b = S_w u_\lambda$ for some $w \in W$ with $w\lambda = \lambda$, and Proposition 5.8 implies $S_w u_\lambda = u_\lambda$. Q.E.D.

In order to show the finite multiplicity theorem for $B(\varpi_i)$, we shall need the following result.

Lemma 5.13. *Assume $\lambda = \mathrm{cl}(\Lambda_{i_1} - a^\vee_{i_1}\Lambda_0)$ for some $i_1 \in I_{0^\vee}$ and $\mu \in W_{\mathrm{cl}}\lambda$. If $w \in W$ satisfies $l(w) \geq \sharp W_{\mathrm{cl}}$ and μ is regularly w-dominant, then there exist w', $w'' \in W$ such that $w = w'w''$, $l(w) = l(w') + l(w'')$ and $\lambda = w''\mu$.*

Proof. Let $w = s_{i_1}\cdots s_{i_l}$ be a reduced expression of w. Since $l \geq \sharp W_{\mathrm{cl}}$, there exists $0 \leq j < k \leq l$ such that $\mathrm{cl}(s_{i_1}\cdots s_{i_j}) = \mathrm{cl}(s_{i_1}\cdots s_{i_k})$. Hence $s_{i_{j+1}}\cdots s_{i_k} = t(\xi)$ for some $\xi \in \widetilde{Q}\setminus\{0\}$. Replacing μ with $s_{i_{k+1}}\cdots s_{i_\ell}\mu$, we reduce the lemma to the following sublemma. Q.E.D.

Sublemma 5.14. *If $\xi \in \widetilde{Q}\setminus\{0\}$ and $\mu \in W_{\mathrm{cl}}\lambda$ is regularly $t(\xi)$-dominant, then there exists $w_1 \in W$ such that $\lambda = w_1\mu$ and $l(t(\xi)) = l(t(\xi)w_1^{-1}) + l(w_1)$.*

Proof. Let us take $w \in W_{0^\vee} := \langle s_i; i \in I_{0^\vee}\rangle$ such that $\mu = w\lambda$ and λ is regularly w-dominant. By Lemma 4.7, for $\beta \in \Delta_{\mathrm{cl}}$, $(\beta,\xi) > 0$ implies $(\beta,\mu) > 0$. Hence $(\beta, w^{-1}\xi) > 0$ implies $(\beta,\lambda) > 0$. In particular, $(\beta,\lambda) = 0$ (resp. $(\beta,\lambda) > 0$) implies $(\beta, w^{-1}\xi) = 0$ (resp. $(\beta, w^{-1}\xi) \geq 0$). For $i \in I_{0^\vee}\setminus\{i_1\}$, $(\alpha_i, w^{-1}\xi) = 0$ because $(\alpha_i,\lambda) = 0$. Moreover $(\alpha_{i_1}, w^{-1}\xi) \geq 0$ because $(\alpha_{i_1},\lambda) > 0$. Hence we have $w^{-1}\xi = c\lambda$ for $c > 0$. Hence $w^{-1}\xi$ is regularly w-dominant. Corollary 4.5 and Lemma 4.8 imply that

$$l(t(\xi)) \;=\; l(t(w^{-1}\xi)) = l(t(w^{-1}\xi)w^{-1}) + l(w) = l(w) + l(w^{-1}t(\xi)).$$

Then the sublemma follows by setting $w_1 = w^{-1}t(\xi)$. Q.E.D.

Proposition 5.15. *Let $\lambda \in P$ be a basic weight. Then for every $\xi \in P$, $B(\lambda)_\xi$ is a finite set.*

Proof. For $w \in W$ and $\mu \in W\lambda$, we define a subset $A_w(\mu)$ of $B(\tilde{U}_q(\mathfrak{g}))$ by

$$A_w(\mu) = \{b \otimes t_\mu \otimes u_{-\infty} \in B(\mu); b \in \overline{B}_w(\infty)\},$$

and then set $A_w = \bigsqcup_{\mu\in W\lambda} A_w(\mu)$. Note that $A_w(\mu)$ is a finite set. One has

$$B(\lambda) \subset \bigcup_{w,w_1\in W} S^*_{w_1}(A_w(w_1^{-1}\lambda)).$$

We shall first show

$$B(\lambda) \subset \bigcup_{\substack{w_1 \in W, \\ w \in W \ \text{with} \ \ell(w) \le N}} S^*_{w_1}(A_w). \tag{5.4}$$

Here $N := \sharp W_{\mathrm{cl}}$.

For $b := b_1 \otimes t_\mu \otimes u_{-\infty}$ in A_w, we shall show

$$b \in \bigcup_{\substack{w_1 \in W, \\ w' \in W \ \text{with} \ \ell(w') \le N}} S^*_{w_1}(A_{w'})$$

by the induction on $\ell(w)$.

Proposition 3.4 implies that μ is regularly w-dominant. We may assume $\ell(w) > N$. By Lemma 5.13, there exists $w_1 = w'w''$ such that $l(w) = l(w') + l(w'')$, $w' \ne 1$ and $\lambda' := w''\mu$ satisfies $\mathrm{cl}(\lambda') = \mathrm{cl}(\lambda)$.

By Proposition 3.4, one has

$$S^*_{w''}(b_1 \otimes t_\mu \otimes u_{-\infty}) = b'_1 \otimes t_{\lambda'} \otimes b'_2$$

with $b'_1 \in \overline{B}_{w'}(\infty)$ and $b'_2 \in B_{w''^{-1}}(-\infty)$. Take $i \in I$ such that $w's_i < w'$. Then $\lambda'_i > 0$ implies $i = i_1$. Hence $c := \varepsilon^*_i(b'_1) \le \lambda'_i = 1$. One has

$$S^*_i(b'_1 \otimes t_{\lambda'} \otimes b'_2) = (\tilde{e}^{*\max}_i b'_1) \otimes t_{s_i\lambda'} \otimes \tilde{e}^{*\lambda'_i - c}_i b'_2. \tag{5.5}$$

If $c = 1$, then $\lambda'_i - c = 0$. Take $x \in W$ such that $b'_2 \in \overline{B}_x(-\infty)$. Then $x \le w''^{-1}$, since $b'_2 \in B_{w''^{-1}}(-\infty)$. Since $\tilde{e}^{*\max}_i b'_1 \in \overline{B}_{w's_i}(\infty)$, Proposition 3.4 implies

$$S^*_x((\tilde{e}^{*\max}_i b'_1) \otimes t_{s_i\lambda} \otimes b'_2) \in B_{w's_ix^{-1}}(\infty) \otimes t_{xs_i\lambda} \otimes u_{-\infty}.$$

Since $\ell(w's_ix^{-1}) < \ell(w)$, the induction proceeds.

Next assume $c = 0$. Then $\lambda_j \le 0$ for $j \in I \setminus \{i_1\}$ implies $\varepsilon^*_j(b'_1) = 0$ for every $j \in I$. Hence $b'_1 = u_\infty$. This contradicts $w' \ne 1$ and $b'_1 \in \overline{B}_{w'}(\infty)$. Thus we have proved (5.4).

For $\mu \in W\lambda$, set

$$C(\mu) = \bigcup_{w \in W \ \text{with} \ \ell(w) \le N} A_w(\mu).$$

Taking $w \in W$ such that $\mu = w\lambda$, we set

$$\tilde{C}(\mu) := S^*_{w^{-1}} C(\mu) \subset B(\lambda),$$

By Proposition 5.8, $\tilde{C}(\mu)$ does not depend on the choice of w. We have

(i) $\tilde{C}(\mu)$ is a finite set,

(ii) there is a finite subset F of Q independent of μ such that $\mathrm{Wt}(\tilde{C}(\mu)) \subset \mu + F$.

Hence, for any $\xi \in P$,

$$B(\lambda)_\xi \subset \bigcup_{\mu \in W\lambda} \tilde{C}(\mu)_\xi = \bigcup_{\mu \in W\lambda \cap (\xi - F)} \tilde{C}(\mu)_\xi$$

is a finite set. Q.E.D.

We have thus obtained the following properties of $V(\lambda)$.

Proposition 5.16. *Let $\lambda \in P^0$ be a basic weight.*

(i) $\mathrm{Wt}(V(\lambda))$ *is contained in the intersection of $\lambda + Q$ and the convex hull of $W\lambda$.*
(ii) $\dim V(\lambda)_\mu = 1$ *for any $\mu \in W\lambda$.*
(iii) $\dim V(\lambda)_\mu < \infty$ *for any $\mu \in P$.*
(iv) $\mathrm{Wt}(V(\lambda)) \cap (\lambda + \mathbb{Z}\,\delta) \subset W\lambda$.
(v) *$V(\lambda)$ is an irreducible $U_q(\mathfrak{g})$-module.*
(vi) *Any non-zero integrable $U_q(\mathfrak{g})$-module generated by an extremal weight vector of weight λ is isomorphic to $V(\lambda)$.*

Moreover $V(\lambda)$ has a global base.

For any $\mu \in W\lambda$, let us denote by u_μ the unique global basis in $V(\lambda)_\mu$. Since u_μ is an extremal vector with weight μ, we have the $U_q(\mathfrak{g})$-linear homomorphism $V(\mu) \to V(\lambda)$ that sends $u_\mu \in V(\mu)$ to $u_\mu \in V(\lambda)$. This homomorphism is in fact an isomorphism.

Set $\lambda = \varpi_i$. One has

$$\{n \in \mathbb{Z}; \varpi_i + n\delta \in W\varpi_i\} = \mathbb{Z}d_i, \tag{5.6}$$

where $d_i = (\varpi_i, \tilde{\alpha}_i)$. Note that $d_i = \max(1, (\alpha_i, \alpha_i)/2) \in \mathbb{Z}$ except the case $d_i = 1$ when $\mathfrak{g} = A_{2n}^{(2)}$ and α_i is the longest root. Hence one has

$$\bigoplus_{\mu \in \mathrm{cl}^{-1}\mathrm{cl}(\varpi_i)} V(\varpi_i)_\mu = \bigoplus_{n \in \mathbb{Z}} V(\varpi_i)_{\lambda + nd_i}.$$

We have a $U_q(\mathfrak{g})$-linear isomorphism $V(\varpi_i + d_i\delta) \xrightarrow{\sim} V(\varpi_i)$. Since there is a $U_q'(\mathfrak{g})$-linear isomorphism $V(\varpi_i) \xrightarrow{\sim} V(\varpi_i + d_i\delta)$ that sends u_{ϖ_i} to $u_{\varpi_i + d_i\delta}$, we obtain a $U_q'(\mathfrak{g})$-linear automorphism z_i of $V(\varpi_i)$ of weight $d_i\delta$, which sends u_{ϖ_i} to $u_{\varpi_i + d_i\delta}$.

Let us define the $U_q'(\mathfrak{g})$-module $W(\varpi_i)$ by

$$W(\varpi_i) = V(\varpi_i)/(z_i - 1)V(\varpi_i). \tag{5.7}$$

The following result is now obvious.

Theorem 5.17. (i) *$W(\varpi_i)$ is a finite-dimensional irreducible integrable $U_q'(\mathfrak{g})$-module.*
(ii) *$W(\varpi_i)$ has a global basis with a simple crystal.*

(iii) *For any* $\mu \in \mathrm{Wt}(V(\varpi_i))$,

$$W(\varpi_i)_{\mathrm{cl}(\mu)} \simeq V(\varpi_i)_\mu.$$

(iv) $\dim W(\varpi_i)_{\mathrm{cl}(\varpi_i)} = 1$.

(v) *The weight of any extremal vector of* $W(\varpi_i)$ *belongs to* $W\mathrm{cl}(\varpi_i)$.

(vi) $\mathrm{Wt}(W(\varpi_i))$ *is the intersection of* $\mathrm{cl}(\varpi_i) + Q_{\mathrm{cl}}$ *and the convex hull of* $W\mathrm{cl}(\varpi_i)$.

(vii) $K[z_i^{1/d_i}] \otimes_{K[z_i]} V(\varpi_i) \simeq W(\varpi_i)_{\mathrm{aff}}$. *Here the action of* z_i^{1/d_i} *on the left hand side corresponds to the action of* z *on the right hand side defined in* § *4.2*.

(viii) $V(\varpi_i)$ *is isomorphic to the submodule* $K[z^{d_i}, z^{-d_i}] \otimes W(\varpi_i)$ *of* $W(\varpi_i)_{\mathrm{aff}}$ *as a* $U_q(\mathfrak{g})$*-module. Here we identify* $W(\varpi_i)_{\mathrm{aff}}$ *with* $K[z, z^{-1}] \otimes W(\varpi_i)$ *as in* (4.16).

(ix) *Any irreducible finite-dimensional integrable* $U'_q(\mathfrak{g})$*-module with* $\mathrm{cl}(\varpi_i)$ *as an extremal weight is isomorphic to* $W(\varpi_i)_a$ *for some* $a \in K \setminus \{0\}$.

Proof. The irreducibility of $W(\varpi_i)$ follows for example by Proposition 4.12, and the other assertions are now obvious. Q.E.D.

We call $W(\varpi_i)$ a *fundamental representation* (of level 0).

6. Existence of Global bases

6.1. **Regularized modified operators.** For $n \in \mathbb{Z}$ and $i \in I$, let us define the operator $\widetilde{F}_i^{(n)}$

$$\widetilde{F}_i^{(n)} = \sum_{k \geq 0, -n} f_i^{(n+k)} e_i^{(k)} a_k(t_i). \tag{6.1}$$

Here

$$a_k(t_i) = (-1)^k q_i^{k(1-n)} t_i^k \prod_{\nu=0}^{k-1} (1 - q_i^{n+2\nu}).$$

Then it acts on any integrable $U_q(\mathfrak{g})$-module M. Moreover it acts also on any $U_q(\mathfrak{g})_{\mathbb{Q}}$-submodule $M_{\mathbb{Q}}$. In this sense, $\widetilde{F}_i^{(n)}$ has no pole except $q = 0$, ∞. Let (L, B) be a crystal base of M. Then we have the following result, which says that $\widetilde{F}_i^{(n)}$ has no pole at $q = 0$ and coincides with $\tilde{f}_i^n$ at $q = 0$.

Proposition 6.1. *We have* $\widetilde{F}_i^{(n)} L \subset L$, *and the action of* $\widetilde{F}_i^{(n)}$ *on* $L/q_s L$ *coincides with* $\tilde{f}_i^n$.

Proof. In order to prove this, it is sufficient to prove the following statement. For any weight vector $u \in M$ with $e_i u = 0$ and $m \in \mathbb{Z}_{\geq 0}$, we have

$$\widetilde{F}_i^{(n)} f_i^{(m)} u = c f_i^{(m+n)} u$$

for some $c \in K := \mathbb{Q}(q_s)$ regular at $q_s = 0$ and $c(0) = 1$. Set $t_i u = q_i^l u$. Then we can assume

$$l \geq n + m.$$

We have

$$a_k(t_i) f_i^{(m)} u = a_k(q_i^{l-2m}) f_i^{(m)} u.$$

Hence

$$\begin{aligned} f_i^{(m)} u &= \sum_{k \geq 0} a_k(q_i^{l-2m}) f_i^{(n+k)} e_i^{(k)} f_i^{(m)} u \\ &= \sum_{k=0}^{m} a_k(q_i^{l-2m}) f_i^{(n+k)} \begin{bmatrix} l-m+k \\ k \end{bmatrix}_i f_i^{(m-k)} u \\ &= \sum_{k=0}^{m} a_k(q_i^{l-2m}) \begin{bmatrix} n+m \\ m-k \end{bmatrix}_i \begin{bmatrix} l-m+k \\ k \end{bmatrix}_i f_i^{(m+n)} u. \end{aligned}$$

Here,

$$\begin{bmatrix} n \\ m \end{bmatrix}_i = \frac{[n]_i!}{[m]_i![n-m]_i!}$$

is the q-binomial coefficient. Hence it is enough to show that

$$A := \sum_{k=0}^{m} a_k(q_i^{l-2m}) \begin{bmatrix} n+m \\ m-k \end{bmatrix}_i \begin{bmatrix} l-m+k \\ k \end{bmatrix}_i \in 1 + q_i \mathbb{Z}[q_i].$$

This follows immediately from the following formula, whose proof due to Anne Schilling is given in Appendix A.

$$A = \sum_{k=0}^{m} q_i^{k(2l-2m-n+2)} \prod_{j=1}^{k} \frac{1 - q_i^{n+2(j-1)}}{1 - q_i^{2j}} \prod_{j=1}^{m-k} \frac{1 - q_i^{n+2j}}{1 - q_i^{2j}}. \tag{6.2}$$

Q.E.D.

6.2. **Existence theorem.** We shall use the notations and terminologies in § 2.4. Let M be an integrable $U_q(\mathfrak{g})$-module, $-$ a bar involution of M, and (L, B) a crystal base of M. Let $M_{\mathbb{Q}}$ be a $U_q(\mathfrak{g})_{\mathbb{Q}}$-submodule of M such that $(M_{\mathbb{Q}})^- = M_{\mathbb{Q}}$. Set $E := L \cap \overline{L} \cap M_{\mathbb{Q}}$.

Theorem 6.2. *Let S be a subset of P. We assume the following conditions:*

(i) $\{(\xi, \xi); \xi \in \mathrm{Wt}(M)\}$ *is bounded from above.*
(ii) $u - \bar{u} \in (q_s - 1) M_{\mathbb{Q}}$ *for any $u \in M_{\mathbb{Q}}$.*

(iii) *$M_{\mathbb{Q}}$ generates M as a vector space over K.*
(iv) *For any $\xi \in P \setminus S$, $(L_\xi, \overline{L}_\xi, (M_{\mathbb{Q}})_\xi)$ is balanced.*
(v) *Any extremal weight (i.e. the weight of an extremal vector) of B is in $P \setminus S$.*
(vi) $q_s L \cap \overline{L} \cap M_{\mathbb{Q}} = 0$.

Then we have

(a) *$(L, \overline{L}, M_{\mathbb{Q}})$ is balanced.*
(b) *For any n, we have*

$$f_i^n M = \bigoplus_{\varepsilon_i(b)\geq n} \mathbb{Q}(q_s)G(b) \quad \textit{and} \quad e_i^n M = \bigoplus_{\varphi_i(b)\geq n} \mathbb{Q}(q_s)G(b).$$

(c) *$M_{\mathbb{Q}} = \sum_{\xi\in P\setminus S} U_q(\mathfrak{g})_{\mathbb{Q}}(M_{\mathbb{Q}})_\xi$ and $M = \sum_{\xi\in P\setminus S} U_q(\mathfrak{g})M_\xi$.*

The rest of this section is devoted to the proof of this theorem.

Lemma 6.3. *The action of $-$ on E is the identity.*

Proof. For $u \in E$, we have $(u - \bar{u})/(1 - {q_s}^{-1}) \in q_s L \cap \overline{L} \cap M_{\mathbb{Q}} = 0$. Q.E.D.

By (vi), the homomorphism $E \to L/q_s L$ is injective. Let us denote by B' the intersection of B and the image of this homomorphism. To see (a), it is enough to show that $B = B'$. For $b \in B'$, let us denote by $G(b)$ the element E such that $b \equiv G(b) \bmod q_s L$. Note that $G(b)^- = G(b)$ by Lemma 6.3. We shall prove the following statements by the descending induction on (ξ, ξ):

(6.3) $B_\xi = B'_\xi$, or equivalently, $(L_\xi,\ \overline{L}_\xi,\ (M_{\mathbb{Q}})_\xi)$ is balanced,

(6.4) $G(b) - f_i^{(\varepsilon_i(b))} G(\tilde{e}_i^{\max} b) \in \sum_{\varepsilon_i(b')>\varepsilon_i(b)} \mathbb{Q}[q_s, {q_s}^{-1}]G(b')$ for any $b \in B_\xi$,

(6.5) $$\sum_{b\in B_\xi,\, \varepsilon_i(b)\geq n} \mathbb{Q}[q_s, {q_s}^{-1}]G(b) = \sum_{m\geq n} f_i^{(m)}(M_{\mathbb{Q}})_{\xi+m\alpha_i} \quad \text{for any } n \geq \max(0, -\langle h_i, \xi\rangle).$$

as well as the similar statements replacing f_i with e_i.

If (ξ, ξ) is big enough, those statements are trivially satisfied by (i). Now assuming (6.3)–(6.5) for ξ such that $(\xi, \xi) > a$, let us prove them for ξ with $(\xi, \xi) = a$.

Lemma 6.4. *Let $i \in I$. Set $k = \max(0, -\langle h_i, \xi\rangle)$.*

(a) *If* $\tilde{e}_i^{\max}b \in B'$*, then* $b \in B'$ *and*

$$G(b) - f_i^{(\varepsilon_i(b))}G(\tilde{e}_i^{\max}b) \in \sum_{\substack{b' \in B'_\xi \\ \varepsilon_i(b') > \varepsilon_i(b)}} \mathbb{Q}[q_s, {q_s}^{-1}]G(b').$$

In particular, any $b \in B_\xi$ *with* $\varepsilon_i(b) > k$ *is contained in* B'*.*

(b) $\sum\limits_{\substack{b \in B'_\xi \\ \varepsilon_i(b) \geq n}} \mathbb{Q}[q_s, {q_s}^{-1}]G(b) = \sum_{m \geq n} f_i^{(m)}(M_{\mathbb{Q}})_{\xi + m\alpha_i}$ *for any* $n > k$*.*

The similar statements hold after exchanging e_i *and* f_i*.*

Proof. Let us prove the lemma by the descending induction on n (in the case (a), n means $\varepsilon_i(b)$, and hence $n \geq k$). If n is big enough, they are true by the hypothesis (i) in Theorem 6.2. Let us prove (a). Set $b_1 = \tilde{e}_i^{\max}b$. Then $u = \widetilde{F}_i^{(n)}G(b_1)$ satisfies $b \equiv u \bmod q_s L$ and

$$u - f_i^{(n)}G(b_1) \in \sum_{m>0} \mathbb{Z}[q_s, {q_s}^{-1}]f_i^{(m+n)}e_i^{(m)}G(b_1) \subset \sum_{m>n} f_i^{(m)}(M_{\mathbb{Q}})_{\xi + m\alpha_i}.$$

The induction hypothesis (b) implies that the last space is contained in

$$\sum_{\substack{b' \in B'_\xi \\ \varepsilon_i(b') > n}} \mathbb{Q}[q_s, {q_s}^{-1}]G(b').$$

Hence we can write $u - f_i^{(n)}G(b_1) = \sum_{b'} c_{b'}G(b')$ where b' ranges over $b' \in B'$ with $\varepsilon_i(b') > n$ and $c_{b'} \in \mathbb{Q}[q_s, {q_s}^{-1}]$. Hence we can write $c_{b'} - \overline{c_{b'}} = c'_{b'} - \overline{c'_{b'}}$ with $c'_{b'} \in q_s\mathbb{Q}[q_s]$. Then $v := u - \sum_{b'} c'_{b'}G(b') = f_i^{(n)}G(b_1) + \sum_{b'}(c_{b'} - c'_{b'})G(b')$ satisfies $\overline{v} = v$ and hence it belongs to E. Moreover one has $b \equiv v \bmod q_s L$. Hence b belongs to B', and $G(b) = v$.

To complete the proof of (a), it is enough to remark $\tilde{e}_i^{\max}b \in B'$ when $\varepsilon_i(b) > k$, because $(\mathrm{wt}(\tilde{e}_i^{\max}(b)), \mathrm{wt}(\tilde{e}_i^{\max}(b))) > (\mathrm{wt}((b), \mathrm{wt}(b))$.

Let us prove (b). The left hand side is contained in the right hand side by (a) and the induction hypothesis on n. Let us show the opposite inclusion. Set $\eta = \xi + n\alpha_i$ with $n > k$. Then we have $(\eta, \eta) > (\xi, \xi)$, and (6.5) holds for η. Hence we have

$$(M_{\mathbb{Q}})_\eta \subset \sum_{\varepsilon_i(b)=0,\, b \in B'_\eta} \mathbb{Q}[q_s, {q_s}^{-1}]G(b) + \sum_{m>0} f_i^{(m)}(M_{\mathbb{Q}})_{\eta + m\alpha_i},$$

which implies

$$
\begin{aligned}
f_i^{(n)}(M_{\mathbb{Q}})_\eta &\subset \sum_{\varepsilon_i(b)=0,\, b\in B'_\eta} \mathbb{Q}[q_s, {q_s}^{-1}] f_i^{(n)} G(b) + \sum_{m>n} f_i^{(m)} M_{\mathbb{Q}} \\
&\subset \sum_{\substack{\varepsilon_i(b)=0\\ b\in B'_\eta}} \mathbb{Q}[q_s, {q_s}^{-1}] f_i^{(n)} G(b) + \sum_{\substack{\varepsilon_i(b)>n\\ b\in B'_\xi}} \mathbb{Q}[q_s, {q_s}^{-1}] G(b).
\end{aligned}
$$

The desired inclusion follows from (a).

Q.E.D.

Lemma 6.5. *$B_\xi \subset B'$.*

Proof. Let $b \in B_\xi$. By the hypothesis (v), there exists $X_l \cdots X_1 b$ whose weight is outside S, where X_ν is $\tilde{e}_i^{\max}$ or $\tilde{f}_i^{\max}$. Hence by the induction on l we may assume that $\tilde{e}_i^{\max} b$ or $\tilde{f}_i^{\max} b$ is contained in B'. Then the preceding lemma implies $b \in B'$. Q.E.D.

The properties (6.3) and (6.4) are now obvious, and (6.5) easily follows from Lemma 6.4 and Lemma6.5.

Thus the induction proceeds, and we complete the proof of (a), (b) in Theorem 6.2.

Finally let us prove (c). Set $M' = \sum_{\xi\in P\setminus S} U_q(\mathfrak{g}) M_\xi$ and $M'_{\mathbb{Q}} = \sum_{\xi\in P\setminus S} U_q(\mathfrak{g})_{\mathbb{Q}} (M_{\mathbb{Q}})_\xi$. Set $L' = L \cap M'$. Then L' is invariant by $\tilde{e}_i$ and $\tilde{f}_i$. By the hypothesis (v), any vector in B is connected with a vector whose weight is outside S. Hence B is contained in $L'/q_s L' \subset L/q_s L$. This shows that (L', B) is a crystal base of M', and $L'/q_s L' = L/q_s L$. Thus we can apply Theorem 6.2 to M'. Hence we obtain $L' \cap \overline{L'} \cap M'_{\mathbb{Q}} = L \cap \overline{L} \cap M_{\mathbb{Q}}$, and $M'_{\mathbb{Q}} = K_{\mathbb{Q}} \otimes (L' \cap \overline{L'} \cap M'_{\mathbb{Q}}) = M_{\mathbb{Q}}$. This completes the proof of Theorem 6.2.

7. Universal R-matrix

In this section, we shall review the universal R-matrix introduced by Drinfeld and the universal bar involution introduced by Lusztig.

Although we mainly use the following coproduct Δ in this article

$$
\begin{aligned}
\Delta(q^h) &= q^h \otimes q^h \\
\Delta(e_i) &= e_i \otimes t_i^{-1} + 1 \otimes e_i \\
\Delta(f_i) &= f_i \otimes 1 + t_i \otimes f_i,
\end{aligned}
\tag{7.1}
$$

we shall introduce another coproduct $\overline{\Delta} = (- \otimes -) \circ \Delta \circ -$

$$
\begin{aligned}
\overline{\Delta}(q^h) &= q^h \otimes q^h \\
\overline{\Delta}(e_i) &= e_i \otimes t_i + 1 \otimes e_i \\
\overline{\Delta}(f_i) &= f_i \otimes 1 + t_i^{-1} \otimes f_i.
\end{aligned}
\tag{7.2}
$$

Let M_ν ($\nu = 1, 2$) be a $U_q(\mathfrak{g})$-module with weight decomposition. Let us denote by $M_1 \otimes M_2$ the tensor product of M_1 and M_2 with the $U_q(\mathfrak{g})$-module structure induced by Δ, and $M_1 \overline{\otimes} M_2$ the $U_q(\mathfrak{g})$-module induced by $\overline{\Delta}$.

Then there is an isomorphism

$$q^{-(\cdot,\cdot)}\colon M_1 \overline{\otimes} M_2 \to M_2 \otimes M_1$$

given by

$$q^{-(\cdot,\cdot)}\,(x \overline{\otimes} y) = q^{-(\mathrm{wt}(x), \mathrm{wt}(y))}\, y \otimes x.$$

Let us define the ring $U_q^+(\mathfrak{g}) \widehat{\otimes} U_q^-(\mathfrak{g})$ by

$$U_q^+(\mathfrak{g}) \widehat{\otimes} U_q^-(\mathfrak{g}) = \bigoplus_{\xi \in Q} \prod_{\xi = \lambda + \mu} \big(U_q^+(\mathfrak{g})_\lambda \otimes U_q^-(\mathfrak{g})_\mu \big). \tag{7.3}$$

The counits $U_q^+(\mathfrak{g}) \to K$ and $U_q^-(\mathfrak{g}) \to K$ induces $\varepsilon\colon U_q^+(\mathfrak{g}) \widehat{\otimes} U_q^-(\mathfrak{g}) \to K$. Modifying Drinfeld's construction ([3]) of a universal R-matrix, Lusztig has shown that there exists a unique intertwiner $\Xi \in U_q^+(\mathfrak{g}) \widehat{\otimes} U_q^-(\mathfrak{g})$ satisfying the following properties:

$$\Xi \circ \Delta(a) = \overline{\Delta}(a) \circ \Xi \text{ for any } a \in U_q(\mathfrak{g}),$$

normalized by $\varepsilon(\Xi) = 1$. Then it satisfies

$$\overline{\Xi} \circ \Xi = \Xi \circ \overline{\Xi} = 1. \tag{7.4}$$

We introduce the completion of the tensor products as follows. We set

$$F_{(\lambda,\mu)}(M_1 \widehat{\otimes} M_2) = \prod_{\gamma \in Q_+} (M_1)_{\lambda+\gamma} \otimes (M_2)_{\mu-\gamma}$$

$$F_{>(\lambda,\mu)}(M_1 \widehat{\otimes} M_2) = \prod_{\gamma \in Q_+ \setminus \{0\}} (M_1)_{\lambda+\gamma} \otimes (M_2)_{\mu-\gamma},$$

and then

$$M_1 \widehat{\otimes} M_2 = \sum_{\lambda,\mu \in P} F_{(\lambda,\mu)}(M_1 \widehat{\otimes} M_2) \subset \prod_{\lambda,\mu \in P} (M_1)_\lambda \otimes (M_2)_\mu.$$

Sometimes we use another completion $M_1 \widetilde{\otimes} M_2$ in the opposite direction:

$$F_{(\lambda,\mu)}(M_1 \widetilde{\otimes} M_2) = \prod_{\gamma \in Q_+} (M_1)_{\lambda-\gamma} \otimes (M_2)_{\mu+\gamma}$$

and then

$$M_1 \widetilde{\otimes} M_2 = \sum_{\lambda,\mu \in P} F_{(\lambda,\mu)}(M_1 \widetilde{\otimes} M_2) \subset \prod_{\lambda,\mu \in P} (M_1)_\lambda \otimes (M_2)_\mu.$$

They have a structure of a $U_q(\mathfrak{g})$-module by Δ and containing $M_1 \otimes M_2$ as a $U_q(\mathfrak{g})$-submodule.

We denote by $M_1\widehat{\widehat{\otimes}}M_2$ the same vector space $M_1\widehat{\otimes}M_2$ with the action of $U_q(\mathfrak{g})$ induced by $\overline{\Delta}$. Then $M_1\widehat{\widehat{\otimes}}M_2$ contains $M_1\overline{\otimes}M_2$ as a $U_q(\mathfrak{g})$-submodule.

We have an isomorphism

$$q^{-(\,\cdot\,,\,\cdot\,)}\colon M_1\widehat{\widehat{\otimes}}M_2 \xrightarrow{\sim} M_2\widetilde{\otimes}M_1.$$

The operator Ξ induces an isomorphism

$$M_1\widehat{\otimes}M_2 \xrightarrow{\sim} M_1\widehat{\widehat{\otimes}}M_2.$$

Then Ξ sends $F_{(\lambda,\mu)}(M_1\widehat{\otimes}M_2)$ to $F_{(\lambda,\mu)}(M_1\widehat{\widehat{\otimes}}M_2)$, and

The homomorphism induced by Ξ

$$\begin{aligned} &M_{1\lambda}\otimes M_{2\mu} \simeq F_{(\lambda,\mu)}(M_1\widehat{\otimes}M_2)/F_{>(\lambda,\mu)}(M_1\widehat{\otimes}M_2) \\ &\qquad \longrightarrow F_{(\lambda,\mu)}(M_1\widehat{\widehat{\otimes}}M_2)/F_{>(\lambda,\mu)}(M_1\widehat{\widehat{\otimes}}M_2) \simeq M_{1\lambda}\otimes M_{2\mu} \end{aligned} \tag{7.5}$$

is equal to the identity.

The intertwiner $R^{\mathrm{univ}}\colon M_1\widehat{\otimes}M_2 \to M_2\widetilde{\otimes}M_1$, called the *universal R-matrix*, is given by by

$$R^{\mathrm{univ}}\colon M_1\widehat{\otimes}M_2 \xrightarrow{\Xi} M_1\widehat{\widehat{\otimes}}M_2 \xrightarrow{q^{-(\,\cdot\,,\,\cdot\,)}} M_2\widetilde{\otimes}M_1. \tag{7.6}$$

It is an isomorphism.

Assume that M_1 and M_2 have a bar involution. Then (7.4) implies that

$$\mathrm{c}^{\mathrm{univ}}\colon M_1\widehat{\otimes}M_2 \xrightarrow{\Xi} M_1\widehat{\widehat{\otimes}}M_2 \xrightarrow{-\otimes-} M_1\widehat{\otimes}M_2$$

is a bar involution on $M_1\widehat{\otimes}M_2$ as observed by G. Lusztig ([17]). We call it the *universal bar involution*.

8. Good modules

Let us take a finite-dimensional integrable $U_q'(\mathfrak{g})$-module M. We consider the following conditions on M:

(8.1) M has a bar involution,

(8.2) M has a crystal base $(L(M), B(M))$,

(8.3) M has a global base,

(8.4) $B(M)$ is a simple crystal.

In this paper, we say that a $U_q'(\mathfrak{g})$-module M is a *good* $U_q'(\mathfrak{g})$-module if M satisfies the above conditions. The level zero fundamental representations $W(\varpi_i)$ is a good $U_q'(\mathfrak{g})$-module. A good $U_q'(\mathfrak{g})$-module is always irreducible (Proposition 4.12).

Let M_1 and M_2 be good $U_q'(\mathfrak{g})$-modules. Then we have

$$(M_1)_{\mathrm{aff}}\widehat{\otimes}(M_2)_{\mathrm{aff}} = K[[z_1/z_2]] \bigotimes_{K[z_1/z_2]} \Big((M_1)_{\mathrm{aff}} \otimes (M_2)_{\mathrm{aff}}\Big),$$
$$(M_2)_{\mathrm{aff}}\widetilde{\otimes}(M_1)_{\mathrm{aff}} = K[[z_1/z_2]] \bigotimes_{K[z_1/z_2]} \Big((M_2)_{\mathrm{aff}} \otimes (M_1)_{\mathrm{aff}}\Big).$$

Here z_ν is the $U_q'(\mathfrak{g})$-linear automorphism of weight δ on $(M_\nu)_{\mathrm{aff}}$ introduced in § 4.2.

Lemma 8.1. $K(z_1/z_2) \otimes_{K[z_1/z_2]} \big((M_1)_{\mathrm{aff}} \otimes (M_2)_{\mathrm{aff}}\big)$ *is an irreducible module over* $K(z_1/z_2) \otimes_{K[z_1/z_2]} U_q(\mathfrak{g})[z_1^{\pm1}, z_2^{\pm1}]$.

Proof. Since $M_1 \otimes M_2$ has a simple crystal base by Lemma 4.11, it is irreducible by Proposition 4.12. Then the lemma follows from the fact that the specialization of $(M_1)_{\mathrm{aff}} \otimes (M_2)_{\mathrm{aff}}$ at the special point $z_1/z_2 = 1$ is irreducible. Q.E.D.

By the result of the previous section, we have the bar involution

$$\mathrm{c}^{\mathrm{univ}} \colon (M_1)_{\mathrm{aff}}\widehat{\otimes}(M_2)_{\mathrm{aff}} \to (M_1)_{\mathrm{aff}}\widehat{\otimes}(M_2)_{\mathrm{aff}}.$$

It commutes with z_1 and z_2. Let u_ν be the extremal vector with dominant weight λ_ν of M_ν $(\nu = 1, 2)$, and set $u = u_1 \otimes u_2$. Then we have $\big((M_1)_{\mathrm{aff}}\widehat{\otimes}(M_2)_{\mathrm{aff}}\big)_{\lambda_1+\lambda_2} = K((z_1/z_2))u$. Hence, by (7.5), we have

$$\mathrm{c}^{\mathrm{univ}}(u) = \overline{\varphi(z_1/z_2)}u \quad \text{or equivalently} \quad \Xi(u) = \varphi(z_1/z_2)u \tag{8.5}$$

for some $\varphi(z_1/z_2) \in K[[z_1/z_2]]$ with $\varphi(0) = 1$. We define

$$\mathrm{c}^{\mathrm{norm}} \colon (M_1)_{\mathrm{aff}}\widehat{\otimes}(M_2)_{\mathrm{aff}} \to (M_1)_{\mathrm{aff}}\widehat{\otimes}(M_2)_{\mathrm{aff}}$$

by $\mathrm{c}^{\mathrm{norm}} = \mathrm{c}^{\mathrm{univ}} \circ \varphi(z_1/z_2)^{-1}$. Then it satisfies

$$\mathrm{c}^{\mathrm{norm}}(u) = u.$$

Lemma 8.2. $\mathrm{c}^{\mathrm{norm}}$ *is a unique endomorphism of* $(M_1)_{\mathrm{aff}}\widehat{\otimes}(M_2)_{\mathrm{aff}}$ *satisfying* $\mathrm{c}^{\mathrm{norm}}(u_1 \otimes u_2) = u_1 \otimes u_2$ *and* $\mathrm{c}^{\mathrm{norm}}(av) = \overline{a}\mathrm{c}^{\mathrm{norm}}(v)$ *for any* $a \in U_q(\mathfrak{g})((z_1/z_2))[z_2^{\pm1}]$, $v \in (M_1)_{\mathrm{aff}}\widehat{\otimes}(M_2)_{\mathrm{aff}}$.

Proof. It is enough to show that a $U_q(\mathfrak{g})[z_1^{\pm1}, z_2^{\pm1}]$-linear homomorphism

$$f \colon (M_1)_{\mathrm{aff}} \otimes (M_2)_{\mathrm{aff}} \to (M_1)_{\mathrm{aff}}\widehat{\otimes}(M_2)_{\mathrm{aff}}$$

vanishes if $f(u_1 \otimes u_2) = 0$. By Lemma 8.1, $K(z_1/z_2) \otimes_{K[z_1/z_2]} (M_1)_{\mathrm{aff}} \otimes (M_2)_{\mathrm{aff}}$ is an irreducible module over $K(z_1/z_2)[z_2^{\pm1}] \otimes U_q(\mathfrak{g})$. Hence the assertion follows. Q.E.D.

Hence, c^{norm} defines a bar involution on $(M_1)_{\mathrm{aff}}\widehat{\otimes}(M_2)_{\mathrm{aff}}$, which we call the *normalized bar involution*. In particular we have

$$\varphi(z)\overline{\varphi(z)} = 1.$$

In the sequel, we use the normalized bar involution to define a global basis.

The universal R-matrix:

$$R^{\mathrm{univ}}\colon (M_1)_{\mathrm{aff}}\widehat{\otimes}(M_2)_{\mathrm{aff}} \to (M_2)_{\mathrm{aff}}\widetilde{\otimes}(M_1)_{\mathrm{aff}}$$

sends $u_1 \otimes u_2$ to $q^{-(\lambda_1,\lambda_2)}\varphi(z_1/z_2)u_2 \otimes u_1$ with the same function φ given in (8.5). Hence setting $R^{\mathrm{norm}} = q^{(\lambda_1,\lambda_2)}\varphi(z_1/z_2)^{-1}R^{\mathrm{univ}}$, we have an intertwiner

$$R^{\mathrm{norm}}\colon (M_1)_{\mathrm{aff}}\widehat{\otimes}(M_2)_{\mathrm{aff}} \to (M_2)_{\mathrm{aff}}\widetilde{\otimes}(M_1)_{\mathrm{aff}}$$

that sends $u_1 \otimes u_2$ to $u_2 \otimes u_1$. We call R^{norm} the *normalized R-matrix*. Both R-matrices commute with z_1 and z_2.

By (7.6) and (7.5), we have, for any $v_\nu \in (M_\nu)_{\mathrm{aff}}$,

$$(8.6)\quad R^{\mathrm{norm}}(v_1 \otimes v_2) \equiv q^{\langle\lambda_1,\lambda_2\rangle - \langle \mathrm{wt}(v_1),\mathrm{wt}(v_2)\rangle} v_2 \otimes v_1 \\ \mod \prod_{\xi \in Q_+\setminus\{0\}} ((M_2)_{\mathrm{aff}})_{\mathrm{wt}(v_2)-\xi} \otimes ((M_1)_{\mathrm{aff}})_{\mathrm{wt}(v_1)+\xi}.$$

We have also

$$R^{\mathrm{norm}}\colon (M_1)_{\mathrm{aff}} \otimes (M_2)_{\mathrm{aff}} \to K(z_1/z_2) \otimes_{K[z_1/z_2]} \big((M_2)_{\mathrm{aff}} \otimes (M_1)_{\mathrm{aff}}\big) \\ \hookrightarrow (M_2)_{\mathrm{aff}}\widetilde{\otimes}(M_1)_{\mathrm{aff}}.$$

We shall generalize these observations to the case of tensor products of several modules. Let M_ν ($\nu = 1, \dots, m$) be a good $U'_q(\mathfrak{g})$-modules with a crystal base (L_ν, B_ν). Let $(M_\nu)_{\mathrm{aff}}$ be its affinization. Then $(M_\nu)_{\mathrm{aff}}$ has a crystal base $\big((L_\nu)_{\mathrm{aff}}, (B_\nu)_{\mathrm{aff}}\big)$. Let $\lambda_\nu \in P$ be a dominant extremal weight of $(M_\nu)_{\mathrm{aff}}$, and u_ν the extremal global basis with weight λ_ν. We denote the canonical automorphism $(M_\nu)_{\mathrm{aff}}$ of weight δ by z_ν.

Then

$$M := \otimes_{\nu=1}^m (M_\nu)_{\mathrm{aff}} = (M_1)_{\mathrm{aff}} \otimes \cdots \otimes (M_m)_{\mathrm{aff}}$$

has a structure of $K[z_1^{\pm 1}, \dots, z_m^{\pm 1}]$-module. Set

$$(8.7)\qquad M = (M_1)_{\mathrm{aff}} \otimes \cdots \otimes (M_m)_{\mathrm{aff}},$$

$$(8.8)\qquad M_{\mathbb{Q}} = (M_{1\mathbb{Q}})_{\mathrm{aff}} \otimes \cdots \otimes (M_{m\mathbb{Q}})_{\mathrm{aff}},$$

and let $\big(L(M), B(M)\big)$ be the tensor product of the crystal bases of the $(M_\nu)_{\mathrm{aff}}$'s. We set

$$\widehat{M} = K[[z_1/z_2, \dots, z_{m-1}/z_m]] \mathop{\otimes}_{K[z_1/z_2,\dots,z_{m-1}/z_m]} \otimes_{\nu=1}^m (M_\nu)_{\mathrm{aff}}.$$

We set also

$$L(\widehat{M}) = A[[z_1/z_2, \dots, z_{m-1}/z_m]] \underset{A[z_1/z_2,\dots,z_{m-1}/z_m]}{\otimes} \otimes_{\nu=1}^m (L_\nu)_{\mathrm{aff}},$$
$$\widehat{M}_{\mathbb{Q}} = \mathbb{Q}[[z_1/z_2, \dots, z_{m-1}/z_m]] \underset{\mathbb{Q}[z_1/z_2,\dots,z_{m-1}/z_m]}{\otimes} \otimes_{\nu=1}^m ((M_\nu)_{\mathrm{aff}\,\mathbb{Q}}).$$

Similarly to the case of the tensor product of two modules, we can define the universal bar involution of $\widehat{M}$ by

$$\mathrm{c}^{\mathrm{univ}} = (- \otimes \cdots \otimes -) \circ \prod_{1 \le i < j \le m} \Xi_{ij},$$

where Ξ_{ij} is the operator Ξ acting on the i-th and j-th components of the tensor product. Normalizing $\mathrm{c}^{\mathrm{univ}}$, we obtain the normalized bar involution $\mathrm{c}^{\mathrm{norm}}$ on $\widehat{M}$. It satisfies, by setting $u = u_1 \otimes \cdots \otimes u_m$,

$$\mathrm{c}^{\mathrm{norm}}(u) = u.$$

Moreover it satisfies for $v_\nu \in (M_\nu)_{\mathrm{aff}}$

(8.9) $$\mathrm{c}^{\mathrm{norm}}(v_1 \otimes \cdots \otimes v_m) \equiv \overline{v_1} \otimes \cdots \otimes \overline{v_m} \mod \prod_{\xi_1,\dots,\xi_m} ((M_1)_{\mathrm{aff}})_{\mathrm{wt}(v_1)+\xi_1} \otimes \cdots \otimes ((M_m)_{\mathrm{aff}})_{\mathrm{wt}(v_m)+\xi_m}.$$

Here the product ranges over $\xi_1, \dots, \xi_m \in Q$ with $\sum_{\nu=1}^m \xi_\nu = 0$ and $\sum_{\nu=1}^\mu \xi_\nu \in Q^+ \setminus \{0\}$ $(\mu = 1, \dots, m-1)$.

Since $\mathrm{c}^{\mathrm{norm}}$ is expressed by a triangular matrix, the well-known argument of triangular matrices implies the following result.

Lemma 8.3. (i) $q_s L(\widehat{M}) \bigcap \big(\mathrm{c}^{\mathrm{norm}} L(\widehat{M})\big) \bigcap \widehat{M}_{\mathbb{Q}} = 0$.

(ii) *For any $b = b_1 \otimes \cdots b_m \in B(M)$, there exists a unique $G(b) \in L(\widehat{M})$ such that $\mathrm{c}^{\mathrm{norm}}(G(b)) = G(b)$ and $b \equiv G(b) \mod q_s L(\widehat{M})$.*

(iii) *Moreover $G(b)$ has the form*

$$G(b) = G(b_1) \otimes \cdots \otimes G(b_m) + \sum c_{b'_1,\cdots,b'_m} G(b'_1) \otimes \cdots \otimes G(b'_m).$$

Here the infinite sum ranges over $b'_1 \otimes \cdots \otimes b'_m \in B(M)$ such that $\sum_{\nu=1}^m \mathrm{wt}(b'_\nu) = \sum_{\nu=1}^m \mathrm{wt}(b_\nu)$ and $\sum_{\nu=1}^\mu (\mathrm{wt}(b'_\nu) - \mathrm{wt}(b_\nu)) \in Q_+ \setminus \{0\}$ $(\mu = 1, \dots, m-1)$. Moreover $c_{b'_1,\cdots,b'_m} \in q_s \mathbb{Q}[q_s]$.

Later we shall see that this infinite sum is in fact a finite sum.

Set

$$N = U_q(\mathfrak{g})[z_1^{\pm 1}, \dots, z_m^{\pm 1}]u.$$

Then N is a submodule of $\widehat{M}$ stable by the bar involution $\mathrm{c}^{\mathrm{norm}}$. Set $\lambda = \sum_{\nu=1}^{m} \lambda_\nu$. Then we have

$$\begin{aligned} N_{\lambda+\mathbb{Z}\delta} &:= \oplus_{n\in\mathbb{Z}} N_{\lambda+n\delta} = \big(\otimes_{\nu=1}^{m} (M_\nu)_{\mathrm{aff}}\big)_{\lambda+\mathbb{Z}\delta} \\ &= \otimes_{\nu=1}^{m} \big((M_\nu)_{\mathrm{aff}}\big)_{\lambda_\nu+\mathbb{Z}\delta} = K[z_1^{\pm1}, \dots, z_m^{\pm1}](u_1 \otimes \cdots \otimes u_m). \end{aligned}$$

Hence one has

$$N_\mu = M_\mu \text{ for any } \mu \in W\lambda + \mathbb{Z}\delta. \tag{8.10}$$

Define

$$\begin{aligned} L(N) &= L(M) \cap N, \\ N_{\mathbb{Q}} &= M_{\mathbb{Q}} \cap N, \\ B(N) &= B(M). \end{aligned}$$

Then $L(N)/q_s L(N) \subset L(M)/q_s L(M)$.

Lemma 8.4. *$B(N)$ is a basis of $L(N)/q_s L(N)$, and $(L(N), B(N))$ is a crystal base of N.*

Proof. Since $B(N)$ is a basis of $L(M)/q_s L(M)$, it is enough to show that $B(N)$ is contained in $L(N)/q_s L(N)$. Since every vector in $B(N)$ is connected with an extremal vector with weight in $\lambda+\mathbb{Z}\delta$, and extremal vectors with such a weight is u up to the action of $z_1^{\pm1}, \dots, z_m^{\pm1}$, we obtain the desired result. Q.E.D.

Setting $S = \mathrm{Wt}(M) \setminus (W\lambda + \mathbb{Z}\delta)$, we can apply Theorem 6.2 to N. The hypotheses in the theorem are satisfied by Lemma 8.3 and Lemma 8.4, and we obtain the following theorem.

Theorem 8.5. (i) *$(L(N), \mathrm{c}^{\mathrm{norm}} L(N), N_{\mathbb{Q}})$ is balanced. Hence N has a global base.*
(ii) *$N_{\mathbb{Q}} = U_q(\mathfrak{g})_{\mathbb{Q}}[z_1, \dots z_m]u$.*

Furthermore, Lemma 8.3 implies the following proposition.

Proposition 8.6. *For any $b_\nu \in B((M_\nu)_{\mathrm{aff}})$ $(\nu = 1, \dots m)$, we have*

$$G(b_1 \otimes \cdots \otimes b_m) = G(b_1) \otimes \cdots \otimes G(b_m) + \sum c_{b'_1,\cdots,b'_m} G(b'_1) \otimes \cdots \otimes G(b'_m).$$

Here the sum ranges over $(b'_1, \dots, b'_m) \in \prod_{\nu=1}^{m} B((M_\nu)_{\mathrm{aff}})$ such that $\sum_{\nu=1}^{m} \mathrm{wt}(b'_\nu) = \sum_{\nu=1}^{m} \mathrm{wt}(b_\nu)$ and $\sum_{\nu=1}^{\mu} (\mathrm{wt}(b'_\nu) - \mathrm{wt}(b_\nu)) \in Q_+ \setminus \{0\}$ $(\mu = 1, \dots, m-1)$. Moreover $c_{b'_1,\cdots,b'_m} \in q_s\mathbb{Q}[q_s]$, and $c_{b'_1,\cdots,b'_m}$ vanishes except for finitely many $(b'_1, \cdots, b'_m)$.

By specializing at $z_\nu = 1$, we obtain the following proposition.

Proposition 8.7. *The tensor product of good $U'_q(\mathfrak{g})$-modules is also a good $U'_q(\mathfrak{g})$-module.*

9. Main theorem

The following theorem is conjectured in [1] in the special case when all the M_ν are fundamental representations. Note that, as seen by the proof, the theorem holds even if we consider $U_q(\mathfrak{g})$ as an algebra over the algebraically closed field $\widehat{K} := \sum_{n>0} \mathbb{C}((q^{1/n}))$, and a_ν as elements of $\widehat{K}$, and replace A with the subring $\widehat{A} := \sum_{n>0} \mathbb{C}[[q^{1/n}]]$ of $\widehat{K}$.

Theorem 9.1. (i) *Let M_ν $(\nu = 1, \dots, m)$ be good $U'_q(\mathfrak{g})$-modules. Let $a_\nu \in K$. Assume that $a_\nu/a_{\nu+1} \in A$ for $\nu = 1, \dots, m-1$. Then $(M_1)_{a_1} \otimes (M_2)_{a_2} \otimes \cdots \otimes (M_m)_{a_m}$ is generated by $u_1 \otimes \cdots \otimes u_m$.*

(ii) *Assume that $(M_\nu)^*$ $(\nu = 1, \dots, m)$ is a good $U'_q(\mathfrak{g})$-module, and $a_{\nu+1}/a_\nu \in A$ for $\nu = 1, \dots, m-1$. Then any non-zero submodule of $(M_1)_{a_1} \otimes (M_2)_{a_2} \otimes \cdots \otimes (M_m)_{a_m}$ contains $u_1 \otimes \cdots \otimes u_m$.*

Proof. Since (ii) is the dual statement of (i), it is enough to prove (i).

Let us embed the crystal B_ν of M_ν into $(B_\nu)_{\mathrm{aff}}$ as in (4.16). Let $\psi\colon (M_1)_{\mathrm{aff}} \otimes \cdots \otimes (M_m)_{\mathrm{aff}} \to (M_1)_{a_1} \otimes \cdots \otimes (M_m)_{a_m}$ be the canonical projection. Then $\psi(G(b_1) \otimes \cdots \otimes G(b_m))$ $(b_\nu \in B_\nu)$ forms a basis of $(M_1)_{a_1} \otimes (M_2)_{a_2} \otimes \cdots \otimes (M_m)_{a_m}$. Since $\psi(G(b_1 \otimes \cdots \otimes b_m))$ are in $U'_q(\mathfrak{g})(u_1 \otimes \cdots \otimes u_m)$, it is enough to show that they also generate $(M_1)_{a_1} \otimes (M_2)_{a_2} \otimes \cdots \otimes (M_m)_{a_m}$ as a vector space.

By Proposition 8.6, we can write

$$G(b_1 \otimes \cdots \otimes b_m) = G(b_1) \otimes \cdots \otimes G(b_m) + \sum c^{k_1,\dots,k_m}_{b'_1,\cdots,b'_m} G(z^{k_1}b'_1) \otimes \cdots \otimes G(z^{k_m}b'_m).$$

Here, the summation ranges over the set of $(b'_1, \dots, b'_m) \in \prod_{\nu=1}^m B_\nu$ and $(k_1, \dots, k_m) \in \mathbb{Z}^m$ such that $\sum_{\nu=1}^m k_\nu = 0$ and $k_1 + \cdots + k_\nu \geq 0$ $(\nu = 1, \dots, m)$. Moreover we have $c^{k_1,\dots,k_m}_{b'_1,\cdots,b'_m} \in q_s\mathbb{Q}[q_s]$.

On the other hand, we have

$$\begin{aligned}
&\psi(G(z^{k_1}b'_1) \otimes \cdots \otimes G(z^{k_m}b'_m)) \\
&\quad = (a_1^{k_1} \cdots a_m^{k_m})\psi\big(G(b'_1) \otimes \cdots \otimes G(b'_m)\big) \\
&\qquad = (a_1/a_2)^{k_1}(a_2/a_3)^{k_1+k_2} \cdots \psi\big(G(b'_1) \otimes \cdots \otimes G(b'_m)\big) \in L,
\end{aligned}$$

where $L = \bigoplus_{b_\nu \in B(M_\nu)} A\big(G(b_1) \otimes \cdots \otimes G(b_m)\big)$.

Hence we have

$$\psi\Big(G\big(b_1 \otimes \cdots \otimes b_m\big)\Big) \equiv \psi\big(G(b_1) \otimes \cdots \otimes G(b_m)\big) \bmod q_s L.$$

Then Nakayama's lemma implies that $\{\psi\big(G(b_1 \otimes \cdots \otimes b_m)\big)\,; b_\nu \in B(M_\nu)\}$ generates $(M_1)_{a_1} \otimes \cdots \otimes (M_m)_{a_m}$. Q.E.D.

We can apply the theorem above to the fundamental representations.

Theorem 9.2. *Let $a_\nu \in K$, and $i_\nu \in I_{0^\vee}$ $(\nu = 1, \ldots, m)$.*

(i) *Assume $a_\nu/a_{\nu+1} \in A$ for $\nu = 1, \ldots, m-1$. Then $W(\varpi_{i_1})_{a_1} \otimes W(\varpi_{i_2})_{a_2} \otimes \cdots \otimes W(\varpi_{i_m})_{a_m}$ is generated by $u_{\varpi_{i_1}} \otimes \cdots \otimes u_{\varpi_{i_m}}$.*

(ii) *Assume $a_{\nu+1}/a_\nu \in A$ for $\nu = 1, \ldots, m-1$. Then any non-zero submodule of $W(\varpi_{i_1})_{a_1} \otimes W(\varpi_{i_2})_{a_2} \otimes \cdots \otimes W(\varpi_{i_m})_{a_m}$ contains $u_{\varpi_{i_1}} \otimes \cdots \otimes u_{\varpi_{i_m}}$.*

Since these theorem hold even if we replace K and A with $\widehat{K}$ and $\widehat{A}$, they imply the following consequences as shown in [1].

Proposition 9.3. *Assume that M_j is a good $U_q'(\mathfrak{g})$-module with dominant extremal vector u_j. The normalized R-matrix*

$$R^{\mathrm{norm}}_{i,j}(x, y)\colon (M_i)_x \otimes (M_j)_y \to (M_j)_y \otimes (M_i)_x$$

does not have a pole at $x/y = a \in \widehat{A}$.

Here $R^{\mathrm{norm}}_{i,j}(x, y)$ is the intertwiner $(M_i)_x \otimes (M_j)_y \to (M_j)_y \otimes (M_i)_x$ so normalized that it sends $u_i \otimes u_j$ to $u_j \otimes u_i$.

Let $\psi_{ij}(x, y)$ be the denominator of $R^{\mathrm{norm}}_{i,j}(x, y)$. Then one has

$$\psi_{ij}(x, y) \in 1 + A[x/y]q_s x/y. \tag{9.1}$$

For the sake of simplicity, we assume that M_j as well as its dual M_j^* is a good $U_q'(\mathfrak{g})$-module, and let u_j be a dominant extremal vector of M_j.

Proposition 9.4. (i) *The extremal vector $u_1 \otimes \cdots \otimes u_m$ generates $(M_1)_{a_1} \otimes \cdots \otimes (M_m)_{a_m}$ if and only if $R^{\mathrm{norm}}_{i,j}(x, y)$ has no pole at $x/y = a_i/a_j$ for any $1 \le j < i \le m$.*

(ii) *Any non-zero submodule $(M_1)_{a_1} \otimes \cdots \otimes (M_m)_{a_m}$ contains $u_1 \otimes \cdots \otimes u_m$, if and only if $R^{\mathrm{norm}}_{i,j}(x, y)$ has no pole at $x/y = a_i/a_j$ for any $1 \le i < j \le m$.*

(iii) *$(M_1)_{a_1} \otimes \cdots \otimes (M_m)_{a_m}$ is irreducible if and only if $R^{\mathrm{norm}}_{i,j}(x, y)$ does not have a pole at $x/y = a_i/a_j$ for any $1 \le i, j \le m$ $(i \ne j)$.*

Proposition 9.5. *If M and M' are irreducible finite-dimensional integrable $U_q'(\mathfrak{g})$-modules, then $M \otimes M'_z$ is an irreducible $U_q'(\mathfrak{g})$-module except for finitely many z.*

10. Combinatorial R-matrices

Let M_1 and M_2 be two good $U'_q(\mathfrak{g})$-modules. Let u_ν be the extremal vector of M_ν with dominant weight $(\nu = 1, 2)$.

Let $\psi(z_1/z_2)$ be the denominator of the normalized R-matrix, normalized by $\psi \in K[z_1/z_2]$ with $\psi(0) = 1$. Then, by (9.1), we have

$$\psi(z) \in 1 + q_s z A[z]. \tag{10.1}$$

We have an intertwiner

$$\psi(z_1/z_2) R^{\mathrm{norm}} \colon (M_1)_{\mathrm{aff}} \otimes (M_2)_{\mathrm{aff}} \to (M_2)_{\mathrm{aff}} \otimes (M_1)_{\mathrm{aff}}.$$

We shall first prove the follwoing proposition.

Proposition 10.1.

$$\psi(z_1/z_2) R^{\mathrm{norm}} \big(L(M_1)_{\mathrm{aff}} \otimes L(M_2)_{\mathrm{aff}} \big) \subset L(M_2)_{\mathrm{aff}} \otimes L(M_1)_{\mathrm{aff}}.$$

Proof. Set $M = (M_1)_{\mathrm{aff}} \otimes (M_2)_{\mathrm{aff}}$, and let L be the smallest crystal lattice of M containing $A[z_1^{\pm 1}, z_2^{\pm 1}](u_1 \otimes u_2)$. Then L is contained in $L(M)$. Since every vector in $B(M)$ is connected with some $z_1^m u_1 \otimes z_2^n u_2$, $L/q_s L \to L(M)/q_s L(M)$ is surjective. Hence by the following well-known lemma, there exists g such that

$$g \in 1 + q_s A[z_1^{\pm 1}, z_2^{\pm 1}] \text{ and } g L(M) \subset L.$$

Lemma 10.2. *Let R be a commutative ring, $a \in R$ and F a finitely generated R-module. If $F = aF$, then there exists $b \in 1 + aR$ such that $bF = 0$.*

Let us define M' and L' in the similar way as M and L by exchanging M_1 and M_2. The operator $T = \psi(z_1/z_2) R^{\mathrm{norm}} \colon M \to M'$ commutes with $\tilde{e}_i$, $\tilde{f}_i$, z_1, z_2, and it satisfies $T(u_1 \otimes u_2) \in L(M')$ by (10.1). Hence we have

$$TL \subset L(M').$$

Taking g as above, we obtain

$$gT(L(M)) \subset TL \subset L(M').$$

Since $L(M')$ is a free $A[z_1^{\pm 1}, z_2^{\pm 1}]$-module of finite rank, the proposition follows from the following lemma. Q.E.D.

Lemma 10.3. *Let F be a free $A[z_1^{\pm 1}, z_2^{\pm 1}]$-module, and g an element in $1 + q_s A[z_1^{\pm 1}, z_2^{\pm 1}]$. If $u \in K \underset{A}{\otimes} F$ satisfies $gu \in F$, then u belongs to F.*

Since the proof is elementary, we do not give its proof.

As a corollary of Proposition 10.1 and (10.1), we obtain

Corollary 10.4.

$$R^{\mathrm{norm}}\Big(L\big((M_1)_{\mathrm{aff}}\widehat{\otimes}(M_2)_{\mathrm{aff}}\big)\Big) \subset L\big((M_2)_{\mathrm{aff}}\widetilde{\otimes}L(M_1)_{\mathrm{aff}}\big).$$

Conjecture 10.5.

$$\psi(z_1/z_2)\big((M_1)_{\mathrm{aff}}\otimes(M_2)_{\mathrm{aff}}\big) \subset U_q(\mathfrak{g})[z_1^{\pm 1}, z_2^{\pm 1}](u_1\otimes u_2).$$

Set

$$\begin{aligned} N &= U_q(\mathfrak{g})[z_1^{\pm 1}, z_2^{\pm 1}](u_1\otimes u_2) \subset (M_1)_{\mathrm{aff}}\otimes(M_2)_{\mathrm{aff}}, \\ N' &= U_q(\mathfrak{g})[z_1^{\pm 1}, z_2^{\pm 1}](u_2\otimes u_1) \subset (M_2)_{\mathrm{aff}}\otimes(M_1)_{\mathrm{aff}}. \end{aligned}$$

Then R^{norm} gives an isomorphism

$$R^{\mathrm{norm}} \colon N \xrightarrow{\sim} N'.$$

In § 8, we saw that N (resp. N') has a crystal base $(L(N), B(M_1)_{\mathrm{aff}}\otimes B(M_2)_{\mathrm{aff}})$ (resp. $(L(N'), B(M_2)_{\mathrm{aff}}\otimes B(M_1)_{\mathrm{aff}})$). Hence R^{norm} induces an isomorphism:

$$R^{\mathrm{comb}} \colon B(M_1)_{\mathrm{aff}}\otimes B(M_2)_{\mathrm{aff}} \xrightarrow{\sim} B(M_2)_{\mathrm{aff}}\otimes B(M_1)_{\mathrm{aff}}.$$

We have

$$\begin{aligned} R^{\mathrm{comb}}(zb_1\otimes b_2) &= (1\otimes z)R^{\mathrm{comb}}(b_1\otimes b_2), \\ R^{\mathrm{comb}}(b_1\otimes zb_2) &= (z\otimes 1)R^{\mathrm{comb}}(b_1\otimes b_2). \end{aligned}$$

Hence we have a commutative diagram:

$$(10.2)\qquad \begin{array}{ccc} B(M_1)_{\mathrm{aff}}\otimes B(M_2)_{\mathrm{aff}} & \xrightarrow{R^{\mathrm{comb}}} & B(M_2)_{\mathrm{aff}}\otimes B(M_1)_{\mathrm{aff}} \\ \downarrow & & \downarrow \\ B(M_1)\otimes B(M_2) & \xrightarrow{\sim} & B(M_2)\otimes B(M_1). \end{array}$$

Hence one obtains the following proposition.

Proposition 10.6. *If B_1 and B_2 are a crystal base of a good $U_q'(\mathfrak{g})$-module, then $B_1\otimes B_2 \simeq B_2\otimes B_1$.*

By Corollary 10.4, we have

$$(10.3)\qquad R^{\mathrm{norm}}(G(b_1\otimes b_2)) = G(R^{\mathrm{comb}}(b_1\otimes b_2)).$$

Setting $R^{\mathrm{comb}}(b_1\otimes b_2) = b_2'\otimes b_1'$ with b_ν, $b_\nu' \in B(M_\nu)_{\mathrm{aff}}$, we define

$$S(b_1\otimes b_2) = \mathrm{wt}(b_1') - \mathrm{wt}(b_1) = \mathrm{wt}(b_2) - \mathrm{wt}(b_2') \in Q.$$

By (8.6), we have $S(b_1\otimes b_2) \in Q_+ := \sum_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$. On the other hand, we have $S(z_1b_1\otimes b_2) = S(b_1\otimes z_2b_2) = S(b_1\otimes b_2)$, and hence it induces a map:

$$(10.4)\qquad S\colon B(M_1)\otimes B(M_2) \to Q_+.$$

This map S is characterized by the following properties (note that $B(M_1)\otimes B(M_2)$ is connected):

$$S(u_1\otimes u_2)=0, \tag{10.5}$$

$$S(\tilde f_i(b_1\otimes b_2)) = \begin{cases} S(b_1\otimes b_2)+\alpha_i & \text{if } \tilde f_i(b_1\otimes b_2)=(\tilde f_i b_1)\otimes b_2 \text{ and} \\ & \qquad \tilde f_i(b'_2\otimes b'_1)=(\tilde f_i b'_2)\otimes b'_1, \\ S(b_1\otimes b_2)-\alpha_i & \text{if } \tilde f_i(b_1\otimes b_2)=b_1\otimes(\tilde f_i b_2) \\ & \qquad \tilde f_i(b'_2\otimes b'_1)=b'_2\otimes(\tilde f_i b'_1), \\ S(b_1\otimes b_2) & \text{otherwise.}\end{cases} \tag{10.6}$$

11. Energy function

In this section, we assume that M is good, and we investigate the properties of ${M_{\mathrm{aff}}}^{\otimes 2}$. In this case, we have

$$R^{\mathrm{norm}}=\bar\iota\circ \mathrm{c}^{\mathrm{norm}}\colon M_{\mathrm{aff}}\otimes M_{\mathrm{aff}}\to M_{\mathrm{aff}}\widetilde{\otimes}M_{\mathrm{aff}}. \tag{11.1}$$

Here $\bar\iota\colon M_{\mathrm{aff}}\widehat{\otimes}M_{\mathrm{aff}}\to M_{\mathrm{aff}}\widetilde{\otimes}M_{\mathrm{aff}}$ is given by

$$\bar\iota(v\otimes v')=q^{(\lambda,\lambda)-(\mathrm{wt}(v),\mathrm{wt}(v'))}\overline{v'}\otimes\overline{v}.$$

Indeed, R^{norm} and $\bar\iota\circ\mathrm{c}^{\mathrm{norm}}$ are $U_q(\mathfrak g)$-linear homomorphisms sending $u\otimes u$ to itself, $z\otimes 1$ to $1\otimes z$ and $1\otimes z$ to $z\otimes 1$. Such a homomorphism is unique.

Similarly the identity being a unique automorphism of $B(M)^{\otimes 2}$, (10.2) implies that there exists a unique map $H\colon {B(M)_{\mathrm{aff}}}^{\otimes 2}\to\mathbb Z$ such that

$$R^{\mathrm{comb}}(b_1\otimes b_2)=(z^{-H(b_1\otimes b_2)}b_1)\otimes(z^{H(b_1\otimes b_2)}b_2).$$

Hence, one has

$$S(b_1\otimes b_2)=\mathrm{wt}(b_2)-\mathrm{wt}(b_1)+H(b_1\otimes b_2)\delta.$$

We call H the *energy function*. We have $H((zb_1)\otimes b_2)=H(b_1\otimes(z^{-1}b_2))=H(b_1\otimes b_2)+1$. It is easy to see that

$${B(M)_{\mathrm{aff}}}^{\otimes 2}=\sqcup_{n\in\mathbb Z}H^{-1}(n)$$

is the decomposition of $B(M)_{\mathrm{aff}}^{\otimes 2}$ into the minimal regular subcrystals invariant by $z\otimes z$ (cf. [6]).

Embedding $B(M)$ into $B(M)_{\mathrm{aff}}$ as in (4.16), the energy function restricted on $B(M)^{\otimes 2}$ is also characterized by the following two properties:

$$H(v \otimes v) = 0 \quad \text{for any extremal vector } v \text{ of } B(M),$$

$$H(\tilde{f}_i(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2) & \text{if } i \neq 0 \text{ and } \tilde{f}_i(b_1 \otimes b_2) \neq 0, \\ H(b_1 \otimes b_2) + 1 & \text{if } i = 0 \text{ and } \\ & \qquad \tilde{f}_i(b_1 \otimes b_2) = (\tilde{f}_i b_1) \otimes b_2 \neq 0, \\ H(b_1 \otimes b_2) - 1 & \text{if } i = 0 \text{ and } \\ & \qquad \tilde{f}_i(b_1 \otimes b_2) = b_1 \otimes (\tilde{f}_i b_2) \neq 0. \end{cases}$$

Set

$$\begin{aligned} N &:= U_q(\mathfrak{g})[(z \otimes z)^{\pm 1}, z \otimes 1 + 1 \otimes z](u \otimes u) \subset M_{\mathrm{aff}}^{\otimes 2}, \\ N' &:= \operatorname{Ker}\bigl(R^{\mathrm{norm}} - 1 \colon M_{\mathrm{aff}}^{\otimes 2} \to K(z \otimes z^{-1}) \otimes_{K[z \otimes z^{-1}]} M_{\mathrm{aff}}^{\otimes 2}\bigr). \end{aligned}$$

Then we have $N \subset N' \subset M_{\mathrm{aff}}^{\otimes 2}$. Define $L(N) = L(M_{\mathrm{aff}})^{\otimes 2} \cap N$ and similarly for $L(N')$. Then one has $L(N)/q_sL(N) \subset L(N')/q_sL(N') \subset L(M_{\mathrm{aff}}^{\otimes 2})/q_sL(M_{\mathrm{aff}}^{\otimes 2})$. Set

$$B_0(M_{\mathrm{aff}}^{\otimes 2}) := \{b_1 \otimes b_2 \in B(M_{\mathrm{aff}})^{\otimes 2}; H(b_1 \otimes b_2) = 0\}.$$

Then

$$\{(z^n \otimes 1)b; n \in \mathbb{Z}_{\geq 0},\, b \in B_0(M_{\mathrm{aff}}^{\otimes 2})\} \sqcup \{(1 \otimes z^n)b; n \in \mathbb{Z}_{>0},\, b \in B_0(M_{\mathrm{aff}}^{\otimes 2})\}$$

is a basis of $L(M_{\mathrm{aff}}^{\otimes 2})/q_sL(M_{\mathrm{aff}}^{\otimes 2})$.

Let us define the subset B' of $L(M_{\mathrm{aff}}^{\otimes 2})/q_sL(M_{\mathrm{aff}}^{\otimes 2})$ by

$$B' := \{(z^n \otimes 1 + \delta(n \neq 0)(1 \otimes z^n))b; n \in \mathbb{Z}_{\geq 0},\, b \in B_0(M_{\mathrm{aff}}^{\otimes 2})\}.$$

Here, for a statement P, we define $\delta(P)$ by

$$\delta(P) = \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{if } P \text{ is false.} \end{cases}$$

Then B' is linearly independent.

Lemma 11.1. *We have $B' \subset L(N)/q_sL(N)$. Moreover, $(L(N), B')$ and $(L(N'), B')$ are a crystal base of N and N', respectively.*

Proof. It is enough to show that $B' \subset L(N)/q_sL(N)$ and B' is a basis of $L(N')/q_sL(N')$. Since $B_0(M_{\mathrm{aff}}^{\otimes 2})$ is a minimal subcrystal invariant by $z \otimes z$, we have $B_0(M_{\mathrm{aff}}^{\otimes 2}) \subset L(N)/q_sL(N)$. Since $L(N)/q_sL(N)$ is invariant by $z^n \otimes 1 + 1 \otimes z^n$, we have $B' \subset L(N)/q_sL(N)$. It remains to prove that B' generates $L(N')/q_sL(N')$.

Since $R^{\mathrm{norm}} = 1$ on N', we have $R^{\mathrm{norm}} = 1$ on $L(N')/q_sL(N')$, and hence $L(N')/q_sL(N') \subset F := \{v \in (L(M_{\mathrm{aff}})/q_sL(M_{\mathrm{aff}}))^{\otimes 2}; R^{\mathrm{norm}}(v) = v\}$.

Since the action of R^{norm} on $(L(M_{\mathrm{aff}})/q_sL(M_{\mathrm{aff}}))^{\otimes 2} = \mathbb{Q}^{\oplus B(M_{\mathrm{aff}})^{\otimes 2}}$ is given by R^{comb}, we can see easily that B' is a basis of F. Q.E.D.

Proposition 11.2. *$N = N'$ and it has a global basis $\{G(b); b \in B'\}$.*

Proof. We shall apply Theorem 6.2 for N and N'. Set

$$N_{\mathbb{Q}} := U_q(\mathfrak{g})_{\mathbb{Q}}[(z\otimes z)^{\pm 1}, z\otimes 1 + 1\otimes z](u\otimes u) \subset M_{\mathrm{aff}}^{\otimes 2},$$

and similarly for $N'_{\mathbb{Q}}$. Set $S = \mathrm{Wt}(M_{\mathrm{aff}}^{\otimes 2}) \setminus (W(2\lambda) + \mathbb{Z}\delta)$. For $\xi = 2w\lambda + n\delta$ ($w \in W$, $n \in \mathbb{Z}$), setting $H = \oplus_{\nu+\mu=n} K\,(z^\nu \otimes z^\mu + z^\mu \otimes z^\nu)u_{w\lambda}^{\otimes 2}$, we have $H \subset N_\xi \subset N'_\xi \subset H$. Hence $N_\xi = N'_\xi = H$, and the condition (iv) in Theorem 6.2 is satisfied for N and N'. The condition (v) follows from the fact that the weight of any extremal vector of $B(M_{\mathrm{aff}})^{\otimes 2}$ is in $W(2\lambda) + \mathbb{Z}\delta$. Hence all the conditions in Theorem 6.2 are satisfied for N and N', and both N and N' have a global basis. These two global bases coincide, and hence $N = N'$. Q.E.D.

Corollary 11.3. *If b_1, $b_2 \in B(M)_{\mathrm{aff}}$ satisfy $H(b_1 \otimes b_2) = 0$, then*

$$G(b_1 \otimes b_2) \in U_q(\mathfrak{g})_{\mathbb{Q}}[(z\otimes z)^{\pm 1}, z\otimes 1 + 1\otimes z](u\otimes u).$$

Moreover, denoting by N_0 the vector subspace generated by $\{G(b_1 \otimes b_2)\,;\, H(b_1 \otimes b_2) = 0\}$, one has

$$U_q(\mathfrak{g})_{\mathbb{Q}}[(z\otimes z)^{\pm 1}, z\otimes 1 + 1\otimes z](u\otimes u) = \mathbb{Q}[z\otimes 1 + 1\otimes z] \otimes_{\mathbb{Q}} N_0,$$
$$M_{\mathrm{aff}}{}^{\otimes 2} = \mathbb{Q}[z^{\pm 1}\otimes 1, 1\otimes z^{\pm 1}] \otimes_{\mathbb{Q}[(z\otimes z)^{\pm 1}]} N_0 = \mathbb{Q}[z^{\pm 1}\otimes 1] \otimes_{\mathbb{Q}} N_0.$$

12. Fock space

12.1. **Some properties of good modules.** In [13], we defined the wedge spaces and the Fock spaces for a finite-dimensional integrable $U'_q(\mathfrak{g})$-module V. In that paper, we assumed several conditions on V. In this section, we shall show that all those conditions are satisfied whenever V is a good module with a perfect crystal base. In [13], we employed the reversed coproduct. Adapting the notations to ours, those conditions read as follows. We set $N := U_q(\mathfrak{g})[(z\otimes z)^{\pm 1}, z\otimes 1 + 1\otimes z](u\otimes u) \subset (V_{\mathrm{aff}})^{\otimes 2}$ with an extremal vector u of V of weight λ.

(G) V is good. Let (L, B) be the crystal base of V.

(P) B is a perfect crystal.

(L) Let $s\colon Q \to \mathbb{Z}$ be the additive function such that $s(\alpha_i) = 1$, and $\ell\colon B_{\mathrm{aff}} \to \mathbb{Z}$ be the function defined by $\ell(b) = s(\mathrm{wt}(b) - \mathrm{wt}(u))$. Then one has

$$H(b_1 \otimes b_2) \le 0 \Rightarrow \ell(b_1) \le \ell(b_2).$$

(D) $\psi \in 1 + q_s zA[z]$. Here $\psi(x/y)$ is the denominator of the normalized R-matrix $R^{\mathrm{norm}}\colon V_x \otimes V_y \to V_y \otimes V_x$.

(R) For every pair (b_1, b_2) in B_{aff} with $H(b_1 \otimes b_2) = 0$, there exists $C_{b_1,b_2} \in N$ of the form

$$C_{b_1,b_2} = G(b_1) \otimes G(b_2) - \sum_{b_1',b_2'} a_{b_1',b_2'} G(b_1') \otimes G(b_2').$$

Here the sum ranges over $(b_1', b_2') \in B_{\mathrm{aff}}^2$ such that

$$H(b_1' \otimes b_2') > 0,$$
$$\ell(b_1) < \ell(b_1') \le \ell(b_2),$$
$$\ell(b_1) \le \ell(b_2') < \ell(b_2),$$

and the coefficients $a_{b_1',b_2'}$ belong to $\mathbb{Q}[q_s, {q_s}^{-1}]$.

Theorem 12.1 ([13]). *We assume* (G), (L), (D) *and* (R). *Then the wedge space* $\bigwedge^m V_{\mathrm{aff}}$ *has a basis* $\{G(b_1) \wedge \cdots \wedge G(b_m)\}$, *where* $(b_1, \dots, b_m)$ *ranges over* $(B_{\mathrm{aff}})^m$ *with* $H(b_j \otimes b_{j+1}) > 0$ $(j = 1, \dots, m-1)$.

For the other consequences and the Fock space, see § 12.2, § 12.3 and [13].

In this section we shall prove the following theorem.

Theorem 12.2. *Assume that* V *is a good* $U_q'(\mathfrak{g})$*-module. Then all the properties above except* (P) *are satisfied.*

In fact, we shall prove here a little bit stronger results. In the sequel, we assume that V is a good $U_q'(\mathfrak{g})$-module. The property (D) has already been proved in (10.1). The following lemma immediately implies (L).

Lemma 12.3. *If* $H(b_1 \otimes b_2) \le 0$, *then* $\mathrm{wt}(b_2) - \mathrm{wt}(b_1) \in Q_+$.

Proof. By (10.4), we have $S(b_1 \otimes b_2) = \mathrm{wt}(b_2) - \mathrm{wt}(b_1) + H(b_1 \otimes b_2)\delta \in Q_+$. Hence if $H(b_1 \otimes b_2) \le 0$, then $\mathrm{wt}(b_2) - \mathrm{wt}(b_1) \in Q_+$. Q.E.D.

In order to prove the remaining property (R), we shall prove the following result on global bases.

Proposition 12.4. *Assume* $H(b_1 \otimes b_2) = 0$. *Write*

$$G(b_1 \otimes b_2) = \sum_{b_1',\, b_2' \in B(M)_{\mathrm{aff}}} a_{b_1',b_2'} G(b_1') \otimes G(b_2'). \tag{12.1}$$

Then we have

$$a_{b_1,b_2} = 1,$$
$$a_{b_2',b_1'} = q^{(\lambda,\lambda)-(\mathrm{wt} b_1', \mathrm{wt} b_2')} \overline{a_{b_1',b_2'}}.$$

If $a_{b'_1,b'_2}\neq 0$*, then*

$$\mathrm{wt}(b'_1)\in\Big(\mathrm{wt}(b_1)+Q_+\Big)\cap\Big(\mathrm{wt}(b_2)-Q_+\Big),$$
$$\mathrm{wt}(b'_2)\in\Big(\mathrm{wt}(b_1)+Q_+\Big)\cap\Big(\mathrm{wt}(b_2)-Q_+\Big).$$

Moreover $\mathrm{wt}(b'_1)=\mathrm{wt}(b_1)$ *implies* $(b'_1,b'_2)=(b_1,b_2)$*, and* $\mathrm{wt}(b'_1)=\mathrm{wt}(b_2)$ *implies* $(b'_1,b'_2)=(b_2,b_1)$

Proof. We have seen $\mathrm{wt}(b'_1)\in\mathrm{wt}(b_1)+Q_+$, $\mathrm{wt}(b'_2)\in\mathrm{wt}(b_2)-Q_+$. and $\mathrm{wt}(b'_1)=\mathrm{wt}(b_1)$ implies $(b'_1,b'_2)=(b_1,b_2)$.

Since $R^{\mathrm{norm}}G(b_1\otimes b_2)=G(b_1\otimes b_2)$, and $\mathrm{c}^{\mathrm{norm}}G(b_1\otimes b_2)=G(b_1\otimes b_2)$, we have $\bar\iota G(b_1\otimes b_2)=G(b_1\otimes b_2)$ by (11.1). Hence we have

$$G(b_1\otimes b_2)=\sum_{b'_1,b'_2\in B(M)_{\mathrm{aff}}}q^{(\lambda,\lambda)-(\mathrm{wt}b'_1,\mathrm{wt}b'_2)}\overline{a_{b'_1,b'_2}}G(b'_2)\otimes G(b'_1),$$

which gives $a_{b'_2,b'_1}=q^{(\lambda,\lambda)-(\mathrm{wt}b'_1,\mathrm{wt}b'_2)}\overline{a_{b'_1,b'_2}}$. Hence we obtain the remaining assertions. Q.E.D.

Conjecture 12.5. Conjecturally, we have $H(b'_1\otimes b'_2)\geq 0$ if $a_{b'_1,b'_2}\neq 0$.

Let us set

$$\begin{aligned}I_+(b)&=\{b'\in B_{\mathrm{aff}};\mathrm{wt}(b')-\mathrm{wt}(b)\in Q_+\setminus\{0\}\}\sqcup\{b\},\\ I_-(b)&=\{b'\in B_{\mathrm{aff}};\mathrm{wt}(b)-\mathrm{wt}(b')\in Q_+\setminus\{0\}\}\sqcup\{b\}.\end{aligned}$$

The following lemma immediately implies (R).

Lemma 12.6. *For every pair* (b_1,b_2) *in* B_{aff}*, there exists* $C_{b_1,b_2}\in N$ *of the form*

$$C_{b_1,b_2}=G(b_1)\otimes G(b_2)-\sum_{b'_1,b'_2}a_{b'_1,b'_2}G(b'_1)\otimes G(b'_2).$$

Here the sum ranges over $(b'_1,b'_2)\in B_{\mathrm{aff}}^2$ *such that* $H(b'_1\otimes b'_2)>0$ *and* $b'_1,b'_2\in I_+(b_1)\cap I_-(b_2)$*, and the coefficients* $a_{b'_1,b'_2}$ *belong to* $\mathbb{Q}[q_s,{q_s}^{-1}]$*.*

Proof. We shall prove this by the induction on $\ell(b_2)-\ell(b_1)$. Note that the assertion is trivial when $H(b_1\otimes b_2)>0$. We may assume $H(b_1\otimes b_2)\leq 0$. Then (L) implies $\ell(b_2)-\ell(b_1)\geq 0$.

Set $n:=-H(b_1\otimes b_2)$. Then $H(z^nb_1\otimes b_2)=0$. Hence $G(z^nb_1\otimes b_2)\in N$. By Proposition 12.4, we can write

$$G(z^nb_1\otimes b_2)=z^nG(b_1)\otimes G(b_2)+\sum_{b'_1,b'_2}a_{b'_1,b'_2}G(b'_1)\otimes G(b'_2)$$

where the sum ranges over (b_1', b_2') with $b_1', b_2' \in I_+(z^n b_1) \cap I_-(b_2)$ and $b_1' \neq z^n b_1$. In particular, one has $\ell(z^{-n}b_1') > \ell(b_1)$. Then,

$$\begin{aligned}
&(z^{-n} \otimes 1 + \delta(n>0) 1 \otimes z^{-n}) G(z^n b_1 \otimes b_2)\\
&\quad = G(b_1) \otimes G(b_2) + \delta(n>0) G(z^n b_1) \otimes G(z^{-n} b_2)\\
&\quad + \sum_{(b_1',b_2') \in I} a_{b_1',b_2'} \bigl(G(z^{-n}b_1') \otimes G(b_2') + \delta(n>0) G(b_1') \otimes G(z^{-n}b_2')\bigr)
\end{aligned}$$

belongs to $N_{\mathbb{Q}} = N \cap (M_{\mathrm{aff}}^{\otimes 2})_{\mathbb{Q}}$. Hence, modulo $N_{\mathbb{Q}}$, $G(b_1) \otimes G(b_2)$ is a linear combination of $G(z^n b_1) \otimes G(z^{-n} b_2)$ $(n > 0)$, $G(z^{-n}b_1') \otimes G(b_2')$ and $G(b_1') \otimes G(z^{-n}b_2')$.

When $n > 0$, we have $\ell(z^{-n}b_2) - \ell(z^n b_1) < \ell(b_2) - \ell(b_1)$, and the induction hypothesis implies that $G(z^n b_1) \otimes G(z^{-n}b_2)$ is, modulo $N_{\mathbb{Q}}$, a linear combination of $G(b_1'') \otimes G(b_2'')$ with $H(b_1'' \otimes b_2'') > 0$ and $b_1'', b_2'' \in I_+(z^n b_1) \cap I_-(z^{-n}b_2) \subset I_+(b_1) \cap I_-(b_2)$.

Similarly, we have $\ell(b_2') - \ell(z^{-n}b_1') < \ell(b_2) - \ell(b_1)$. Hence, modulo $N_{\mathbb{Q}}$, $G(z^{-n}b_1') \otimes G(b_2')$ is a linear combination of $G(b_1'') \otimes G(b_2'')$ with $H(b_1'' \otimes b_2'') > 0$ and $b_1'', b_2'' \in I_+(z^{-n}b_1') \cap I_-(b_2') \subset I_+(b_1) \cap I_-(b_2)$.

Finally, since $\ell(z^{-n}b_2') - \ell(b_1') \leq \ell(b_2') - \ell(z^{-n}b_1') < \ell(b_2) - \ell(b_1)$, the induction hypothesis implies that $G(b_1') \otimes G(z^{-n}b_2')$ modulo $N_{\mathbb{Q}}$ is a linear combination of $G(b_1'') \otimes G(b_2'')$ with $H(b_1'' \otimes b_2'') > 0$ and $b_1'', b_2'' \in I_+(b_1') \cap I_-(z^{-n}b_2') \subset I_+(b_1) \cap I_-(b_2)$. Q.E.D.

12.2. **Wedge spaces.** Let us recall the construction of the wedge space in [13]. Let V be a good $U_q'(\mathfrak{g})$-module with an extremal global basis u. Let us set

$$N = U_q(\mathfrak{g})[(z \otimes z)^{\pm 1}, z \otimes 1 + 1 \otimes z](u \otimes u) \subset V_{\mathrm{aff}}^{\otimes 2}, \tag{12.2}$$

$$N_m = \sum_{j=0}^{m-2} V_{\mathrm{aff}}{}^{\otimes j} \otimes N \otimes V^{\otimes(m-2-j)} \subset V_{\mathrm{aff}}^{\otimes m}. \tag{12.3}$$

The wedge space $\bigwedge^m V_{\mathrm{aff}}$ is defined by

$$\bigwedge^m V_{\mathrm{aff}} = V_{\mathrm{aff}}^{\otimes m} / N_m.$$

For $v_1, \ldots, v_m \in V_{\mathrm{aff}}$, let $v_1 \wedge \cdots \wedge v_m$ denote the image of $v_1 \otimes \cdots \otimes v_m$ by the projection $V_{\mathrm{aff}}^{\otimes m} \to \bigwedge^m V_{\mathrm{aff}}$. Let $L(\bigwedge^m V_{\mathrm{aff}}) \subset \bigwedge^m V_{\mathrm{aff}}$ be the image of $L(V_{\mathrm{aff}}^{\otimes m})$. For $b = b_1 \otimes \cdots \otimes b_m \in B(V_{\mathrm{aff}}^{\otimes m})$, we set

$$G^{\mathrm{pure}}(b) = G(b_1) \wedge \cdots \wedge G(b_m).$$

For $b \in B(V_{\mathrm{aff}}^{\otimes m})$, let $G(b)$ be the global basis of $V_{\mathrm{aff}}^{\otimes m}$ and $G^{\wedge}(b)$ be its image in $\bigwedge^m V_{\mathrm{aff}}$. We set

$$B(\bigwedge^m V_{\mathrm{aff}}) = \{b_1 \otimes \cdots \otimes b_m \in B(V_{\mathrm{aff}}^{\otimes m}); \\ H(b_\nu \otimes b_{\nu+1}) > 0 \text{ for } \nu = 1, \ldots, m-1\}.$$

Then in [13], the following properties are proved

(i) $\{G^{\mathrm{pure}}(b)\,; b \in B(\bigwedge^m V_{\mathrm{aff}})\}$ is a basis of $L(\bigwedge^m V_{\mathrm{aff}})$.
(ii) Identifying $B(\bigwedge^m V_{\mathrm{aff}})$ with a subset of $L(\bigwedge^m V_{\mathrm{aff}})/q_s L(\bigwedge^m V_{\mathrm{aff}})$ by G^{pure}, $(L(\bigwedge^m V_{\mathrm{aff}}), B(\bigwedge^m V_{\mathrm{aff}}))$ is a crystal base of $\bigwedge^m V_{\mathrm{aff}}$.

On the other hand, the following proposition follows from Proposition 8.6.

Proposition 12.7. *For $b_1 \in B(V_{\mathrm{aff}}^{\otimes m_1})$ and $b_2 \in B(V_{\mathrm{aff}}^{\otimes m_2})$, one has the equality in $V_{\mathrm{aff}}^{\otimes (m_1+m_2)}$*

$$G(b_1 \otimes b_2) = G(b_1) \otimes G(b_2) + \sum_{b'_1, b'_2} c_{b'_1, b'_2} G(b'_1) \otimes G(b'_2).$$

Here the sum ranges over $(b'_1, b'_2) \in B(V_{\mathrm{aff}}^{\otimes m_1}) \times B(V_{\mathrm{aff}}^{\otimes m_2})$ such that $\mathrm{wt}(b'_1) - \mathrm{wt}(b_1) = \mathrm{wt}(b_2) - \mathrm{wt}(b'_2) \in Q_+ \setminus \{0\}$, and the coefficients satisfy $c_{b'_1, b'_2} \in q_s \mathbb{Q}[q_s]$.

Set

$$B_0(V_{\mathrm{aff}}^{\otimes m}) = \{b_1 \otimes \cdots \otimes b_m \in B(V_{\mathrm{aff}}^{\otimes m}); H(b_\nu \otimes b_{\nu+1}) = 0 \\ \text{for } \nu = 1, \ldots, m-1\},$$

$$N_m^0 = \bigoplus_{b \in B_0(V_{\mathrm{aff}}^{\otimes m})} K G(b).$$

The similar arguments as in Proposition 11.2 and Corollary 11.3 show the following proposition.

Proposition 12.8.

$$\big(U_q(\mathfrak{g}) \otimes \mathbb{Q}[z_1^{\pm 1}, \ldots, z_m^{\pm 1}]^{\mathrm{sym}}\big) u^{\otimes m} \\ = \mathbb{Q}[z_1^{\pm 1}, \ldots, z_m^{\pm 1}]^{\mathrm{sym}} \underset{\mathbb{Q}[(z_1 \cdots z_m)^{\pm 1}]}{\otimes} N_m^0.$$

Here z_ν is the automorphism of $V_{\mathrm{aff}}^{\otimes m}$ induced by the action of z on the ν-th factor, and $\mathbb{Q}[z_1^{\pm 1}, \ldots, z_m^{\pm 1}]^{\mathrm{sym}}$ is the ring of symmetric Laurent polynomials.

In particular, for any Laurent polynomial $f(z_1, \ldots, z_m)$ symmetric in $(z_\nu, z_{\nu+1})$ for some ν,

$$f(z_1, \ldots, z_m) N_m^0 \subset N_m.$$

Since N_m is a $\mathbb{Q}[z_1^{\pm 1}, \dots, z_m^{\pm 1}]^{\mathrm{sym}}$-module, $\bigwedge^m V_{\mathrm{aff}}$ has a structure of a $\mathbb{Q}[z_1^{\pm 1}, \dots, z_m^{\pm 1}]^{\mathrm{sym}}$-module. We denote by B_n the operator on $\bigwedge^m V_{\mathrm{aff}}$ given by $\sum_{\nu=1}^m z_\nu^n$. Then the B_n's commute with one another.

Lemma 12.9. *For any $b \in B(V_{\mathrm{aff}}^{\otimes m})$, one has either $G^\wedge(b) = 0$ or $G^\wedge(b) = \pm G^\wedge(b')$ for some $b' \in B(\bigwedge^m V_{\mathrm{aff}})$.*

Proof. Set $b = z^{a_1} b_1 \otimes \cdots \otimes z^{a_m} b_m$ with $H(b_\nu \otimes b_{\nu+1}) = 0$. Then we have by Proposition 12.8

$$G(b) \equiv \pm G(z^{a_{\sigma(1)}} b_1 \otimes \cdots \otimes z^{a_{\sigma(m)}} b_m)$$

for any permutation σ. Hence we may assume that $(a_1, \dots, , a_m)$ is a decreasing sequence. If there is ν such that $a_\nu = a_{\nu+1}$ then $G(b) \in N_m$ by the preceding proposition. Otherwise b belongs to $B(\bigwedge^m V_{\mathrm{aff}})$. Q.E.D.

12.3. **Global basis of the Fock space.** The purpose of this subsection is to define the global basis of the Fock space.

Let us now assume that V is a good $U_q'(\mathfrak{g})$-module with perfect crystal base (L, B) of level ℓ. Let us recall that a simple crystal B is called *perfect* of level ℓ if it satisfies the following conditions.

(P1) Any $b \in B$ satisfies $\langle c, \varepsilon(b) \rangle = \langle c, \varphi(b) \rangle \geq \ell$. Here $\varepsilon(b) = \sum_i \varepsilon_i(b) \mathrm{cl}(\Lambda_i) \in P_{\mathrm{cl}}$ and $\varphi(b) = \sum_i \varphi_i(b) \mathrm{cl}(\Lambda_i) \in P_{\mathrm{cl}}$.

(P2) Set $P_{\mathrm{cl}}^{(\ell)} = \{\lambda \in P_{\mathrm{cl}}\, ; \langle c, \lambda \rangle = \ell$ and $\langle h_i, \lambda \rangle \geq 0$ for every $i\}$, the set of dominant weights of level ℓ, and $B_{\min} = \{b \in B; \langle c, \varepsilon(b) \rangle = \ell\}$. Then the two maps

$$\varepsilon \colon B_{\min} \longrightarrow P_{\mathrm{cl}}^{(\ell)} \quad \text{and} \quad \varphi \colon B_{\min} \longrightarrow P_{\mathrm{cl}}^{(\ell)}$$

are bijective.

For example, the vector representation of $A_n^{(1)}$ is a good $U_q'(\mathfrak{g})$-module with a perfect crystal base of level 1. Let $(B_{\mathrm{aff}})_{\min}$ be the inverse image of $B_{\min}$ by the map $B_{\mathrm{aff}} \to B$. Let us take a sequence $\{b_n\}_{n \in \mathbb{Z}}$ in $(B_{\mathrm{aff}})_{\min}$ such that

$$\varphi(b_n^\circ) = \varepsilon(b_{n-1}^\circ) \text{ and } H(b_n^\circ \otimes b_{n-1}^\circ) = 1.$$

Such a sequence is called a *ground state*. Take a sequence $\{\lambda_n\}_{n \in \mathbb{Z}}$ in P such that

$$\lambda_n = \lambda_{n-1} + \mathrm{wt}(b_n^\circ) \text{ and } \mathrm{cl}(\lambda_n) = \varphi(b_n^\circ) = \varepsilon(b_{n-1}^\circ).$$

In [13], the Fock spaces $\mathcal{F}_r$ ($r \in \mathbb{Z}$) are constructed, and they satisfy the following properties.

(F1) $\mathcal{F}_r$ is an integrable $U_q(\mathfrak{g})$-module.

(F2) $\mathrm{Wt}(\mathcal{F}_r) \subset \lambda_r + Q_-$.

(F3) There exist $U_q'(\mathfrak{g})$-linear endomorphisms B_n $(n \in \mathbb{Z} \setminus \{0\})$ of $\mathcal{F}_r$ with weight $n\delta$ satisfying the boson commutation relations $[B_n, B_m] = \delta_{-n,m} a_n$ for some $a_n \in K \setminus \{0\}$.

(F4) There exists a $U_q(\mathfrak{g})$-linear map $\cdot \wedge \cdot : \mathcal{F}_r \otimes \bigwedge^m V_{\mathrm{aff}} \to \mathcal{F}_{r-m}$ such that $(u \wedge v) \wedge v' = u \wedge (v \wedge v')$ for $u \in \mathcal{F}_r$, $v \in \bigwedge^m V_{\mathrm{aff}}$ and $v' \in \bigwedge^{m'} V_{\mathrm{aff}}$.

(F5) $B_n(u \wedge v) = (B_n u) \wedge v + u \wedge (z^n v)$ for $n \in \mathbb{Z} \setminus \{0\}$, $u \in \mathcal{F}_r$ and $v \in V_{\mathrm{aff}}$.

(F6) There is a non-zero vector $\mathrm{vac}_r \in \mathcal{F}_r$ of weight λ_r, $(\mathcal{F}_r)_{\lambda_r} = K\mathrm{vac}_r$. Moreover one has $\mathrm{vac}_{r+1} \wedge G(b_r^\circ) = \mathrm{vac}_r$.

(F7) $\{u \in \mathcal{F}_r\,; B_n u = 0$ for any $n > 0$ and $e_i u = 0$ for any $i\}$
$= K\,\mathrm{var}_r$.

(F8) Let $K[B_{-1}, B_{-2}, \ldots] = K[B_n; n \neq 0]/\big(\sum_{m>0} K[B_n; n \neq 0]B_m\big)$ be the Fock space of the boson algebra. Then $K[B_{-1}, B_{-2}, \ldots] \otimes V(\lambda_r) \xrightarrow{\sim} \mathcal{F}_r$ as a $K[B_n; n \neq 0] \otimes U_q(\mathfrak{g})$-module. Here $1 \otimes u_{\lambda_r}$ corresponds to vac_r.

(F9) Let $B(\mathcal{F}_r)$ be the set of sequences $\{b_n\}_{n \geq r}$ satisfying

$$H(b_{n+1} \otimes b_n) > 0 \text{ for any } n \geq r,$$
$$b_n = b_n^\circ \text{ for } n \gg r.$$

For $b = \{b_n\}_{n \geq r} \in B(\mathcal{F}_r)$, set $G^{\mathrm{pure}}(b) = \mathrm{vac}_n \wedge G(b_{n-1}) \wedge \cdots \wedge G(b_r)$ for $n \gg r$. Then $\{G^{\mathrm{pure}}(b)\,; b \in B(\mathcal{F}_r)\}$ is a basis of $\mathcal{F}_r$.

(F10) Set $L(\mathcal{F}_r) = \bigcup_{n \geq r} \mathrm{vac}_n \wedge L(\bigwedge^{n-r} V_{\mathrm{aff}})$. Then $(L(\mathcal{F}_r), B(\mathcal{F}_r))$ is a crystal base of $\mathcal{F}_r$. Here $B(\mathcal{F}_r)$ is identified with a subset of $L(\mathcal{F}_r)/q_s L(\mathcal{F}_r)$ by G^{pure}.

(F11) $f_i^{(k)} \mathrm{vac}_r = \mathrm{vac}_{r+1} \wedge G(\tilde{f}_i^k b_r^\circ)$.

Now we shall show that the Fock space $\mathcal{F}_r$ has a global basis.

First let us define a bar involution c on $\mathcal{F}_r$ such that

$$\mathrm{c}(\mathrm{vac}_r) = \mathrm{vac}_r, \tag{12.4}$$

$$[B_n, \mathrm{c}] = 0 \text{ for any } n > 0. \tag{12.5}$$

By (F8), there exists a unique bar involution on $\mathcal{F}_r$ satisfying the conditions above. Note that $\mathrm{c} \circ B_{-n} \circ \mathrm{c} = \overline{a_n} {a_n}^{-1} B_{-n}$ for $n > 0$, since $[B_n, {a_n}^{-1} B_{-n}] = 1$ implies ${a_n}^{-1} B_{-n}$ is c-invariant.

We set

$$(\mathcal{F}_r)_{\mathbb{Q}} = \sum_{m \geq r} \mathrm{vac}_m \wedge \bigwedge^{m-r} (V_{\mathrm{aff}})_{\mathbb{Q}}.$$

Lemma 12.10. *Let $b := b_1 \otimes \cdots \otimes b_m$ be an element of $B_{\mathrm{aff}}^{\otimes m}$.*

(a) *If* $H(b_r^\circ \otimes b_1) \le 0$, *then* $\mathrm{vac}_r \wedge G^\wedge(b) = \mathrm{vac}_r \wedge G^{\mathrm{pure}}(b) = 0$ *hold in* $\mathcal{F}_{r-m}$.

(b) $\mathrm{vac}_{r+1} \wedge G^\wedge(b_r^\circ \otimes b) = \mathrm{vac}_r \wedge G^\wedge(b)$.

Proof. (a) We have

$$G(b) = \sum c_{b'_1, b'} G(b'_1) \otimes G(b'),$$

where the sum ranges over $b'_1 \in B_{\mathrm{aff}}$ and $b' \in B_{\mathrm{aff}}{}^{\otimes(m-1)}$ such that $\mathrm{wt}(b'_1) - \mathrm{wt}(b_1) \in Q_+$. Since $H(b_r^\circ \otimes b_1) \le 0$, we have $\ell(b_{r-1}^\circ) < \ell(b_1) \le \ell(b'_1)$ by Lemma 4.2.2 in [13]. Since $\mathrm{Wt}(\mathcal{F}_{r-1}) \subset \lambda_{r-1} + Q_-$ by (F2), one has $\mathrm{vac}_r \wedge G(b'_1) = 0$. Hence we obtain $\mathrm{vac}_r \wedge G^\wedge(b) = 0$. The proof of $\mathrm{vac}_r \wedge G^{\mathrm{pure}}(b) = 0$ is similar.

(b) The proof is similar. One has

$$G(b_r^\circ \otimes b) = G(b_r^\circ) \otimes G(b) + \sum c_{b'_0, b'} G(b'_0) \otimes G(b'),$$

where the sum ranges over $b'_0 \in B_{\mathrm{aff}}$ and $b' \in B_{\mathrm{aff}}{}^{\otimes m}$ such that $\mathrm{wt}(b'_0) - \mathrm{wt}(b_r^\circ) \in Q_+ \setminus \{0\}$. Then by the same reasoning on the weight of $\mathrm{Wt}(\mathcal{F}_r)$, we have $\mathrm{vac}_{r+1} \wedge G(b'_0) = 0$. Q.E.D.

By the lemma above, for $b = \{b_n\}_{n \ge r} \in B(\mathcal{F}_r)$,

$$G(b) := \mathrm{vac}_m \wedge G^\wedge(b_{m-1} \otimes \cdots \otimes b_r)$$

does not depend on m such that $b_j = b_j^\circ$ for $j \ge m$.

Lemma 12.11. $\{G(b)\,; b \in B(\mathcal{F}_r)\}$ *is a basis of the* A*-module* $L(\mathcal{F}_r)$.

Proof. Since $b \equiv G(b) \bmod q_s L(\mathcal{F}_r)$, $\{G(b)\,; b \in B(\mathcal{F}_r)\}$ is linearly independent. Hence it is enough to show that it generates $L(\mathcal{F}_r)$.

Let $b = (b_1, \dots, b_m) \in B(\bigwedge^m V_{\mathrm{aff}})$. For any integer N, we can write

$$G^{\mathrm{pure}}(b) = \sum_{b'} a_{b'} G^\wedge(b') + \sum_{b''} c_{b''} G^{\mathrm{pure}}(b'').$$

Here b' ranges over $B(V_{\mathrm{aff}}^{\otimes m})$ and $b'' = b''_1 \otimes \cdots \otimes b''_m$ ranges over $B(V_{\mathrm{aff}}^{\otimes m})$ with $\ell(b''_1) > N$. Taking $\ell(b_{m+r-1}^\circ)$ as N, one has $\mathrm{vac}_{m+r} \wedge G^{\mathrm{pure}}(b'') = 0$. Hence one has

$$\mathrm{vac}_{m+r} \wedge G^{\mathrm{pure}}(b) = \sum_{b' \in B(V_{\mathrm{aff}}^{\otimes m})} a_{b'} \mathrm{vac}_{m+r} \wedge G^\wedge(b').$$

Now it is enough to apply Lemma 12.9 and Lemma 12.10. Q.E.D.

Theorem 12.12. $\{G(b)\,; b \in B(\mathcal{F}_r)\}$ *is a global basis of* $\mathcal{F}_r$.

Proof. It remains to prove that the $G(b)$'s are invariant by the bar involution c. Let E be the vector space over $\mathbb{Q}$ generated by $\{G(b)\,;b\in B(\mathcal{F}_r)\}$. Then $\mathrm{vac}_{r+m}\wedge G^{\wedge}(b)$ is contained in E for any $b\in B(V_{\mathrm{aff}}^{\otimes m})$ by Lemma 12.9 and Lemma 12.10. We define the involution c' of $\mathcal{F}_r$ by

$$\mathrm{c}'(v)=v \text{ for any } v\in E \text{ and}$$
$$\mathrm{c}'(av)=\overline{a}\,\mathrm{c}'(v) \text{ for any } v\in\mathcal{F}_r \text{ and } a\in K.$$

We shall show that $\mathrm{c}'=\mathrm{c}$. In order to see this, it is enough to show the following properties:

(12.6) $\quad \mathrm{c}'(\mathrm{vac}_r)=\mathrm{vac}_r,$

(12.7) $\quad \mathrm{c}'$ commutes with B_n if $n>0$,

(12.8) $\quad \mathrm{c}'(av)=\overline{a}\mathrm{c}'(v)$ for any $v\in\mathcal{F}_r$ and $a\in U_q(\mathfrak{g})$.

The property (12.6) is obvious.

Let us first show that c' commutes with B_n $(n>0)$. This follows from the fact that $B_n(\mathrm{vac}_{r+m}\wedge G^{\wedge}(b))=\mathrm{vac}_{r+m}\wedge B_nG^{\wedge}(b)$ holds for $b\in B(\bigwedge^m V_{\mathrm{aff}})$, and the fact that $B_nG^{\wedge}(b)$ belongs to E.

Let us show (12.8). We have evidently $q^h\circ\mathrm{c}'=\mathrm{c}'\circ q^{-h}$ for every $h\in P^*$.

The conjugation c' commutes with e_i, because, for $b\in B(\mathcal{F}_r)$, $e_iG(b)$ belongs to $\mathbb{Q}[q_s+{q_s}^{-1}]\otimes E$.

Finally, let us show that c' commutes with f_i. To see this, we shall prove $f_i\mathrm{c}'(v)=\mathrm{c}'(f_iv)$ for any weight vector $v\in\mathcal{F}_r$ by the induction on $\mathrm{wt}(v)$. For any $j\in I$, one has, by using the commutativity of c' and e_i

$$\begin{aligned}&e_j(f_i\mathrm{c}'(v)-\mathrm{c}'(f_iv))\\&\quad=(f_ie_j+\delta_{ij}\frac{t_i-t_i^{-1}}{q_i-q_i^{-1}})\mathrm{c}'(v)-\mathrm{c}'\big((f_ie_j+\delta_{ij}\frac{t_i-t_i^{-1}}{q_i-q_i^{-1}})v\big)\\&\quad=f_i\mathrm{c}'(e_jv)-\mathrm{c}'(f_ie_jv).\end{aligned}$$

Since this vanishes by the induction hypothesis, $f_i\mathrm{c}'(v)-\mathrm{c}'(f_iv)$ is a highest weight vector. Similarly it is annihilated by all the B_n's $(n>0)$. Since the weight of $f_i\mathrm{c}'(v)-\mathrm{c}'(f_iv)$ is not λ_r, it must vanish by (F7). Thus we obtain (12.8). Q.E.D.

Remark 12.13. In the case when $\mathfrak{g}=A_n^{(1)}$ and V is the vector representation, the global basis of the Fock space was introduced by B. Leclerc and J.-Y. Thibon ([14, 15]). D. Uglov ([20]) generalized this to the case when $\mathfrak{g}=A_n^{(1)}\oplus A_m^{(1)}$ and V is the tensor product of the vector representations. The connection of global bases of Fock space and Kazhdan-Lusztig polynomials are also studied by M. Varagnolo–E. Vasserot ([21]) and O. Schiffmann ([19]).

13. Conjectural structure of $V(\lambda)$

In this section, we shall present conjectures that clarify the structure of $V(\lambda)$ and its crystal base $B(\lambda)$ for $\lambda \in P^0$. The paper by Beck, Chari and Pressley ([2]) should help to solve them. These conjectures are closely related with those of G. Lusztig ([18]).

Let λ be a dominant integral weight of level 0. We write $\lambda = \sum_{i\in I_{0^\vee}} m_i\varpi_i$. Then the module $\otimes_{i\in I_{0^\vee}} V(m_i\varpi_i)$ contains the extremal vector $\underset{i\in I_{0^\vee}}{\otimes} u_{m_i\varpi_i}$ whose weight is λ. Here we can take any ordering of $I_{0^\vee}$ to define the tensor product. Hence we have a $U_q(\mathfrak{g})$-linear morphism

$$\Phi_\lambda\colon V(\lambda) \to \otimes_{i\in I_{0^\vee}} V(m_i\varpi_i)$$

sending u_λ to $\underset{i\in I_{0^\vee}}{\otimes} u_{m_i\varpi_i}$.

Conjecture 13.1. (i) Φ_λ is a monomorphism.
(ii) $\Phi_\lambda^{-1}\big(\otimes_{i\in I_{0^\vee}} L(m_i\varpi_i)\big) = L(\lambda)$.
(iii) By Φ_λ, we have an isomorphism of crystals

$$B(\lambda) \xrightarrow{\sim} \bigotimes_{i\in I_{0^\vee}} B(m_i\varpi_i).$$

Next we shall consider the case when λ is a multiple of a fundamental weight. There is a morphism of $U_q(\mathfrak{g})$-modules

$$\Psi_{m,i}\colon V(m\varpi_i) \to V(\varpi_i)^{\otimes m}$$

sending $u_{m\varpi_i}$ to $u_{\varpi_i}^{\otimes m}$. Let z_i be the $U_q'(\mathfrak{g})$-linear automorphism of $V(\varpi_i)$ of weight $d_i\delta$ introduced in § 5.2, and let z_ν ($\nu = 1, \dots, m$) be the operator of $V(\varpi_i)^{\otimes m}$ obtained by the action of z_i on the ν-th factor. It is again a $U_q'(\mathfrak{g})$-linear automorphism of $V(\varpi_i)^{\otimes m}$ of weight $d_i\delta$. Let $B_0(m\varpi_i)$ be the connected component of $B(m\varpi_i)$ containing $u_{m\varpi_i}$, and let $B_0(V(\varpi_i)^{\otimes m})$ be the connected component of $B(\varpi_i)^{\otimes m}$ containing $u_{\varpi_i}^{\otimes m}$.

Conjecture 13.2. (i) $\Psi_{m,i}$ is a monomorphism.
(ii) $\Psi_{m,i}^{-1} L(V(\varpi_i)^{\otimes m}) = L(m\varpi_i)$.
(iii) $B_0(m\varpi_i) \xrightarrow{\sim} B_0(V(\varpi_i)^{\otimes m})$ by $\Psi_{m,i}$. Moreover the global basis $G(b)$ with $b \in B_0(m\varpi_i)$ is sent to the corresponding global basis of $U_q(\mathfrak{g})u_{\varpi_i}^{\otimes m} \subset W(\varpi_i)_{\mathrm{aff}}^{\otimes m}$ constructed in Theorem 8.5.

(iv) Let S be the set of Schur Laurent polynomials in $z_1, \dots, z_m$, i.e. the set of characters of $GL(m)$ ($(z_1, \dots, z_m)$ being the components of the diagonal matrices). Then $\{G(b)\,; b \in B(m\varpi_i)\}$ is by $\Psi_{m,i}$ sent to $\{aG(b); b \in B_0(V(\varpi_i)^{\otimes m}), a \in S\}$.

Note that, for a, $a' \in S$ and b, $b' \in B_0(V(\varpi_i)^{\otimes m})$, $a\,G(b) = a'\,G(b')$ holds if and only if $a' = (z_1 \cdots z_m)^r a$ and $b = (z_1 \cdots z_m)^r b'$ for some $r \in \mathbb{Z}$.

These conjectures imply the following conjecture on $\tilde{U}_q(\mathfrak{g})$ analogous to Peter-Weyl theorem. For $\lambda \in P$, let $B_0(\lambda)$ be the connected component of $B(\lambda)$ containing u_λ. Note that if $\langle c, \lambda \rangle \neq 0$, then $B_0(\lambda) = B(\lambda)$. We consider $\bigsqcup_{\lambda \in P} B_0(\lambda) \times B(-\lambda)$ as a crystal over $\mathfrak{g} \oplus \mathfrak{g}$. The Weyl group W acts on $\bigsqcup_{\lambda \in P} B_0(\lambda) \times B(-\lambda)$ by $S_w^* \times S_w^* \colon B_0(\lambda) \times B(-\lambda) \to B_0(w\lambda) \times B(-w\lambda)$.

Conjecture 13.3. $\Big(\bigsqcup_{\lambda \in P} B_0(\lambda) \times B(-\lambda)\Big)/W \xrightarrow{\sim} B(\tilde{U}_q(\mathfrak{g}))$ as a crystal over $\mathfrak{g} \times \mathfrak{g}$.

Here the usual crystal structure on $B(\tilde{U}_q(\mathfrak{g}))$ corresponds to the one of $B_0(\lambda)$ and the star crystal structure on $B(\tilde{U}_q(\mathfrak{g}))$ corresponds to the one of $B(-\lambda)$. The isomorphism sends $u_\lambda \otimes b \in B_0(\lambda) \times B(-\lambda)$ to $b^* \in B(\tilde{U}_q(\mathfrak{g}))$.

Appendix A.

In this appendix, we shall give a proof of (6.2) due to Anne Schilling. Let us define

$$(a)_n = (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i).$$

Then in terms of $(a; q)_n$, the q-binomial in this paper is given as

$$\begin{bmatrix} m \\ n \end{bmatrix}_i = q^{n(n-m)} \frac{(q^2; q^2)_m}{(q^2; q^2)_n (q^2; q^2)_{m-n}}.$$

Hence replacing $n \to 2n$ and $q \to q^{1/2}$ in (6.2), it reads as follows:

Lemma A.1.

$$\sum_{k=0}^{m} (-1)^k q^{\frac{1}{2}k(k+1-2m)-nm} \frac{(q^n)_k (q)_{2n+m} (q)_{\ell-m+k}}{(q)_{m-k} (q)_{2n+k} (q)_k (q)_{\ell-m}} = \sum_{k=0}^{m} q^{k(\ell-m-n+1)} \frac{(q^n)_k (q^{n+1})_{m-k}}{(q)_k (q)_{m-k}}. \tag{A.1}$$

Proof. Using [5, I.10]

$$(a)_{m-k} = \frac{(a)_m}{(q^{1-m}/a)_k}(-\frac{q}{a})^k q^{\binom{k}{2}-mk},$$

the equation (A.1) may be rewritten in hypergeometric notation as

$$q^{-nm}\frac{(q^{2n+1})_m}{(q)_m}{}_3\Phi_2\left[\begin{matrix}q^{-m}, q^n, q^{\ell-m+1}\\ q^{2n+1}, 0\end{matrix}; q\right] = \frac{(q^{n+1})_m}{(q)_m}{}_2\Phi_1\left[\begin{matrix}q^{-m}, q^n\\ q^{-m-n}\end{matrix}; q^{\ell-m-2n+1}\right].$$

However, this formula readily follows from [5, III.7] with the replacements

$$n \to m,\ b \to q^n,\ c \to q^{-n-m},\ z \to q^{\ell-m-2n+1}.$$

Q.E.D.

Appendix B. Formulas for the crystal $B(\tilde{U}_q(\mathfrak{g}))$

In this table, $b_1 \in B(\infty)$, $b_2 \in B(-\infty)$, $\lambda \in P$, $b = b_1 \otimes t_\lambda \otimes b_2$, $\lambda_i = \langle h_i, \lambda\rangle$ and $\mathrm{wt}_i(b_1) = \langle h_i, \mathrm{wt}(b_1)\rangle$.

$$\begin{aligned}
b^* &= b_1^* \otimes t_{-\lambda-\mathrm{wt}(b_1)-\mathrm{wt}(b_2)} \otimes b_2^*\,,\\
\varepsilon_i(b) &= \max(\varepsilon_i(b_1),\, \varepsilon_i(b_2) - \lambda_i - \mathrm{wt}_i(b_1))\,,\\
\varphi_i(b) &= \max(\varphi_i(b_1) + \lambda_i + \mathrm{wt}_i(b_2),\, \varphi_i(b_2))\,,\\
\mathrm{wt}^*(b) &= \mathrm{wt}(b^*) = -\lambda_i\,,\\
\varepsilon_i^*(b) &= \max(\varepsilon_i^*(b_1),\, \varphi_i^*(b_2) + \lambda_i)\,,\\
\varphi_i^*(b) &= \max(\varepsilon_i^*(b_1) - \lambda_i,\, \varphi_i^*(b_2))\,,
\end{aligned}$$

$$\tilde{e}_i b = \begin{cases} \tilde{e}_i b_1 \otimes t_\lambda \otimes b_2 & \text{if } \varphi_i(b_1) \ge \varepsilon_i(b_2) - \lambda_i\,,\\ b_1 \otimes t_\lambda \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) - \lambda_i\,,\end{cases}$$

$$\tilde{f}_i b = \begin{cases} \tilde{f}_i b_1 \otimes t_\lambda \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) - \lambda_i\,,\\ b_1 \otimes t_\lambda \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \le \varepsilon_i(b_2) - \lambda_i\,,\end{cases}$$

$$\tilde{e}_i^* b = \begin{cases} \tilde{e}_i^* b_1 \otimes t_{\lambda-\alpha_i} \otimes b_2 & \text{if } \varepsilon_i^*(b_1) \ge \varphi_i^*(b_2) + \lambda_i\,,\\ b_1 \otimes t_{\lambda-\alpha_i} \otimes \tilde{e}_i^* b_2 & \text{if } \varepsilon_i^*(b_1) < \varphi_i^*(b_2) + \lambda_i\,,\end{cases}$$

$$\tilde{f}_i^* b = \begin{cases} \tilde{f}_i^* b_1 \otimes t_{\lambda+\alpha_i} \otimes b_2 & \text{if } \varepsilon_i^*(b_1) > \varphi_i^*(b_2) + \lambda_i\,,\\ b_1 \otimes t_{\lambda+\alpha_i} \otimes \tilde{f}_i^* b_2 & \text{if } \varepsilon_i^*(b_1) \le \varphi_i^*(b_2) + \lambda_i\,,\end{cases}$$

$$\begin{aligned}
\tilde e_i^{\max} b &= \tilde e_i^{\max} b_1 \otimes t_\lambda \otimes \tilde e_i{}^c b_2 \\
&\qquad\qquad\text{where } c = \max(\varepsilon_i(b_2) - \varphi_i(b_1) - \lambda_i, 0), \\
\tilde f_i^{\max} b &= \tilde f_i{}^c b_1 \otimes t_\lambda \otimes \tilde f_i^{\max} b_2 \\
&\qquad\qquad\text{where } c = \max(\varphi_i(b_1) - \varepsilon_i(b_2) + \lambda_i, 0)\,,
\end{aligned}$$

$$\tilde e_i^{*\max} b = \begin{cases} \tilde e_i^{*\max} b_1 \otimes t_{\lambda - (\varphi_i^*(b_2)+\lambda_i)\alpha_i} \otimes \tilde e_i^{*\varphi_i^*(b_2)-\varepsilon_i^*(b_1)+\lambda_i} b_2 \\ \qquad\qquad \text{if } \varepsilon_i^*(b_1) - \varphi_i^*(b_2) - \lambda_i \le 0\,, \\ \tilde e_i^{*\max} b_1 \otimes t_{\lambda - \varepsilon_i^*(b_1)\alpha_i} \otimes b_2 \\ \qquad\qquad \text{if } \varepsilon_i^*(b_1) - \varphi_i^*(b_2) - \lambda_i \ge 0\,, \end{cases}$$

$$\tilde f_i^{*\max} b = \begin{cases} \tilde f_i^{*\varepsilon_i^*(b_1)-\varphi_i^*(b_2)-\lambda_i} b_1 \otimes t_{\lambda + (\varepsilon_i^*(b_1)-\lambda_i)\alpha_i} \otimes \tilde f_i^{*\max} b_2 \\ \qquad\qquad \text{if } \varepsilon_i^*(b_1) - \varphi_i^*(b_2) - \lambda_i \ge 0\,, \\ b_1 \otimes t_{\lambda + \varphi_i^*(b_2)\alpha_i} \otimes \tilde f_i^{*\max} b_2 \\ \qquad\qquad \text{if } \varepsilon_i^*(b_1) - \varphi_i^*(b_2) - \lambda_i \le 0\,. \end{cases}$$

Assume now $b = b_1 \otimes t_\lambda \otimes u_{-\infty}$. If b is extremal,

$$S_i b = \begin{cases} \tilde f_i^{\,\mathrm{wt}_i(b_1)+\lambda_i} b_1 \otimes t_\lambda \otimes u_{-\infty} & \text{if } \varepsilon_i(b) = 0, \\ \tilde e_i^{\max} b_1 \otimes t_\lambda \otimes \tilde e_i{}^{-\varphi_i(b_1)-\lambda_i} u_{-\infty} & \text{if } \varphi_i(b) = 0. \end{cases}$$

If b^* is extremal,

$$S_i^* b = \begin{cases} \tilde f_i^{*-\lambda_i} b_1 \otimes t_{s_i\lambda} \otimes u_{-\infty} & \text{if } \varepsilon_i^*(b) = 0, \\ \tilde e_i^{*\max} b_1 \otimes t_{s_i\lambda} \otimes \tilde e_i^{*\lambda_i - \varepsilon_i^*(b_1)} u_{-\infty} & \text{if } \varphi_i^*(b) = 0. \end{cases}$$

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