Semisimple holonomic $\mathcal{D}$-modules

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1 Conjecture

There are a theory of Weil sheaves of Pierre Deligne (and BBG [1]) in characteristic p and a theory of mixed Hodge modules of Morihiko Saito ([3]) in characteristic 0. Weil sheaves or Hodge modules satisfy the following properties. In the statements, we write a pure perverse sheaf instead of a pure perverse Weil sheaf or a Hodge module.

(1) Let $f : X \to Y$ be a projective morphism and F a pure perverse sheaf on X. Then $\mathrm{R}f_*(F)$ is a direct sum of the $\mathrm{R}^k f_*(F)[-k]$'s, and $\mathrm{R}^k f_*(F)[-k]$ is pure.

(2) The graduation of the near-by cycle (or vanishing) sheaf of a pure perverse sheaf with respect to the weight monodromy filtration is again pure.

(3) The hard Lefschetz theorem holds for pure perverse sheaves.

etc., etc..

I conjecture that (1), (2) and (3) should hold even if we replace "pure sheaves" with "semisimple perverse sheaves", or more generally with "semisimple holonomic D-modules". Here the varieties are complex quasi-projective varieties.

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2 Precise statement

In this note, we mean by an algebraic manifold a separated quasi-compact smooth scheme over the complex number field $\mathbb{C}$. For an algebraic manifold X let us denote by $\mathcal{D}_X$ the sheaf of rings of differential operators on X. Let $\mathrm{D}^b(\mathcal{D}_X)$ denote the derived category of bounded complexes of left $\mathcal{D}_X$-modules and let $\mathrm{D}^b_h(\mathcal{D}_X)$ denote the full subcategory of $\mathrm{D}(\mathcal{D}_X)$ consisting of bounded complexes of $\mathcal{D}_X$-modules with holonomic cohomologies. The left derived functor of $\otimes_{\mathcal{O}_X}$ gives the bifunctor

$$\cdot \overset{\mathbf{D}}{\otimes} \cdot : \mathrm{D}^b_h(\mathcal{D}_X) \times \mathrm{D}^b_h(\mathcal{D}_X) \to \mathrm{D}^b_h(\mathcal{D}_X).$$

Let $f : X \to Y$ be a morphism of algebraic manifolds. Then $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$ has a structure of a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule. Then the left derived functor of $\mathcal{M} \mapsto \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{M}$ defines the pull-back functor

$$\mathbf{D}f^* \ : \mathrm{D}^b_h(\mathcal{D}_Y) \to \mathrm{D}^b_h(\mathcal{D}_X).$$

Set $\mathcal{D}_{Y\leftarrow X} = f^{-1}\mathcal{D}_Y \otimes_{f^{-1}\mathcal{O}_Y} \Omega_{X/Y}$, where Ω_X is the sheaf of the highest degree forms and $\Omega_{X/Y} = \Omega_X \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}$. Then it is a $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-module. Then the functor $\mathcal{M} \mapsto \mathrm{R}f_*(\mathcal{D}_{Y\leftarrow X} \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_X} \mathcal{M})$ defines the push-forward functor

$$\mathbf{D}f_* : \mathrm{D}^b_h(\mathcal{D}_X) \to \mathrm{D}^b_h(\mathcal{D}_Y).$$

Let us denote by t the coordinate of $\mathbb{C}$. We set $\tilde{X} = X \times \mathbb{C}$. Let us define

$$V^k(\mathcal{D}_{\tilde{X}}) = \{P \in \mathcal{D}_{\tilde{X}}; P(t^i\mathcal{O}_{\tilde{X}}) \subset t^{i+k}\mathcal{O}_{\tilde{X}} \quad \text{for any } i \text{ such that } i, i+k \geq 0\}.$$

Then it defines a filtration of $\mathcal{D}_{\tilde{X}}$. Let us choose a total ordering of $\mathbb{C}$ such that $a < b$ implies $a < a + n < b + n$ for any positive rational number n. Then for any holonomic $\mathcal{D}_{\tilde{X}}$-module $\mathcal{M}$, there exists a unique family of submodules $\{V^a(\mathcal{M})\}_{a\in\mathbb{C}}$ satisfying the following properties (see [2]):

(a) The filtration $V(\mathcal{M})$ is locally finitely generated. Namely, there exist locally finitely many $u_j \in V^{a_j}(\mathcal{M})$ such that

$$V^a(\mathcal{D}_{\tilde{X}}) = \sum_{n\in\mathbb{Z}, n+a_j\geq a} V^n(\mathcal{D}_{\tilde{X}})V^{a_j}(\mathcal{M})$$

for any $a \in \mathbb{C}$.

(b) Set $V^{>a}(\mathcal{M}) = \bigcup_{b>a} V^b(\mathcal{M})$ and $\mathrm{Gr}^a_V(\mathcal{M}) = V^a(\mathcal{M})/V^{>a}(\mathcal{M})$. Then the action of $t\partial/\partial t - a$ on $\mathrm{Gr}^a_V(\mathcal{M})$ is nilpotent.

The graduation $\mathrm{Gr}^a_V(\mathcal{M})$ does not depend on the choice of the total order of $\mathbb{C}$. Moreover $\mathrm{Gr}^a_V(\mathcal{M})$ is a holonomic $\mathcal{D}_X$-module. The homomorphisms $t : \mathrm{Gr}^{a-1}_V\mathcal{M} \to \mathrm{Gr}^a_V\mathcal{M}$ and $\partial/\partial t : \mathrm{Gr}^a_V\mathcal{M} \to \mathrm{Gr}^{a-1}_V\mathcal{M}$ are isomorphisms unless $a = 0$. Let $j : X \hookrightarrow \tilde{X}$ be the embedding by $t = 0$. Then $\mathbf{D}j^*\mathcal{M}$ is isomorphic to the complex $\mathrm{Gr}^{-1}_V\mathcal{M} \xrightarrow{t} \mathrm{Gr}^0_V\mathcal{M}$.

Let f be a regular function on X. Let $i : X \hookrightarrow \tilde{X}$ be the embedding $x \mapsto (x, f(x))$. For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we set

$$\Psi_f(\mathcal{M}) \quad = \quad \bigoplus_{-1<a\leq 0} \mathrm{Gr}^a_V(\mathbf{D}i_*\mathcal{M})$$

and call it the nearby-cycle of $\mathcal{M}$. This is a holonomic $\mathcal{D}_X$-module supported in $f^{-1}(0)$. Then $t\partial/\partial t - a$ gives a nilpotent endomorphism of $\Psi_f(\mathcal{M})$. We call it the nilpotent part of the monodromy.

A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is called semisimple if it is semisimple in the abelian category of coherent $\mathcal{D}_X$-module.

It is easy to see that it is a Zariski local property.

Lemma 2.1 *Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. Let $X = \bigcup_j U_j$ be an open covering. Then $\mathcal{M}$ is a semisimple holonomic $\mathcal{D}_X$-module if and only if $\mathcal{M}|_{U_j}$ is a semisimple holonomic $\mathcal{D}_{U_j}$-module for any j.*

In fact it is an etale local property.

Lemma 2.2 *Let $X \to Y$ be a smooth surjective morphism, and let $\mathcal{M}$ be a $\mathcal{D}_Y$-module. Then $\mathcal{M}$ is a semisimple holonomic $\mathcal{D}_Y$-module if and only if $\mathbf{D}f^*\mathcal{M}$ is a semisimple holonomic $\mathcal{D}_X$-module.*

Now the conjecture is as follows.

(C1) Let $f : X \to Y$ be a projective morphism and $\mathcal{M}$ a semisimple holonomic $\mathcal{D}_X$-module. Then $\mathbf{D}f_*(\mathcal{M})$ is isomorphic to $\bigoplus_k H^k(\mathbf{D}f_*(\mathcal{M}))[-k]$, and $H^k(\mathbf{D}f_*(\mathcal{M}))$ is a semisimple holonomic $\mathcal{D}_Y$-module.

(C2) Let f be a regular function on X, and let $\mathcal{M}$ be a semisimple holonomic $\mathcal{D}_X$-module. Let W be the weight filtration of the nilpotent part of the monodromy of $\Psi_f(\mathcal{M})$. Then $\mathrm{Gr}^W(\Psi_f(\mathcal{M}))$ is a semisimple holonomic $\mathcal{D}_X$-module.

(C3) The hard Lefschetz theorem holds for a semisimple holonomic $\mathcal{D}_X$-module.

The precise meaning of (C3) is as follows. Let $f : X \to Y$ be a projective morphism of algebraic manifolds, and L a relatively ample invertible $\mathcal{O}_X$-module. Its first Chern class $c_1(L)$ defines a morphism $\mathcal{O}_X \to \mathcal{O}_X[2]$ in $\mathrm{D}^b_h(\mathcal{D}_X)$. Then for any semisimple holonomic $\mathcal{D}_X$-module $\mathcal{M}$, it induces

$$\mathbf{D}f_*(\mathcal{M}) \cong \mathbf{D}f_*(\mathcal{M} \overset{\mathbf{D}}{\otimes} \mathcal{O}_X) \xrightarrow{c_1(L)} \mathbf{D}f_*(\mathcal{M} \overset{\mathbf{D}}{\otimes} \mathcal{O}_X[2]) \cong \mathbf{D}f_*(\mathcal{M})[2].$$

The conjecture is that for any positive integer n,

$$c_1(L)^n : H^{-n}(\mathbf{D}f_*(\mathcal{M})) \to H^n(\mathbf{D}f_*(\mathcal{M}))$$

is an isomorphism.

The above conjecture implies the following.

(C4) Let $f : X \to Y$ be a morphism of algebraic manifolds. Let $\mathcal{M}$ be a semisimple holonomic $\mathcal{D}_Y$-module. Assume that f is non-characteristic to $\mathcal{M}$. Then $\mathbf{D}f^*\mathcal{M}$, which is concentrated in degree 0 by the non-characteristic condition, is a semisimple holonomic $\mathcal{D}_X$-module.

(C5) Let $\mathcal{M}$ and $\mathcal{M}'$ be semisimple holonomic $\mathcal{D}_X$-modules. Assume that they are non-characteristic. Then $\mathcal{M} \overset{\mathbf{D}}{\otimes} \mathcal{M}'$, which is concentrated in degree 0 by the non-characteristic condition, is a semisimple holonomic $\mathcal{D}_X$-module.

Note that, by the Riemann-Hilbert correspondence, the conjecture implies the corresponding statements for semisimple perverse sheaves.

3 Evidences

Besides the theory of Hodge modules, we have now a considerable amount of evidences. Namely some of the consequences of Conjecture are already known. One is by the theory of Tannaka category. This theory asserts the following proposition. We call a holonomic $\mathcal{D}_X$-module is lisse if it is a locally free $\mathcal{O}_X$-module of finite rank.

Proposition 3.1 *The tensor product of two semisimple holonomic lisse $\mathcal{D}_X$-modules is again semisimple.*

The other is by works on Higgs bundle by N.J. Hitchin, C.T. Simpson, K. Corlette, and others (see [4]). For example, the hard Lefschetz theorem (C3) is already known for a semisimple local system on a smooth projective variety. Also the following proposition (a consequence of (C4)) is known

Proposition 3.2 ([4]) *Let X be a projective algebraic manifold and F a semisimple local system on X. Then the restriction of F to any closed smooth subvariety Z is again a semisimple local system on Z.*

References

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