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Similarity of Crystal Bases

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ABSTRACT. We show that the crystal $B(\lambda)$ associated with the irreducible highest weight module with highest weight λ is embedded into the crystal $B(m\lambda)$ for any positive integer m. As an application, we prove that Littelmann's path crystal coincides with $B(\lambda)$.

1. Introduction

In [**6**], Littelmann introduced a crystal structure on the space of paths. This has a following similarity property. For a positive integer m, let us denote by S_m the dilatation by m, i.e. $S_m(\pi)(t) = m\pi(t)$ for a path π. Then it satisfies

$$S_m(\widetilde{e}_i\pi) = \widetilde{e}_i^m S_m(\pi) \text{ and } S_m(\widetilde{f}_i\pi) = \widetilde{f}_i^m S_m(\pi) \quad \text{for any path } \pi.$$

In this note, we show that a similar property holds for the crystals associated with irreducible highest weight modules. As an application, we prove Littelmann's conjecture : the path crystal of L-S paths is isomorphic to the crystal associated with irreducible highest weight modules.

2. Review on Crystals

Let us recall briefly the notion of crystals (see [**3**], [**5**]).
We are given following data:

$$P : \text{a free } \mathbb{Z}\text{-module (called a weight lattice)},$$
$$I : \text{an index set for simple roots},$$
$$\alpha_i \in P : \text{called a simple root } (i \in I),$$
$$h_i \in P^* = \mathrm{Hom}(P, \mathbb{Z}) : \text{called a simple coroot } (i \in I).$$

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We assume that $(\langle h_i, \alpha_j\rangle)_{i,j\in I}$ is a symmetrizable generalized Cartan matrix. By definition, a crystal B is a set equipped with a map

$$\begin{aligned} &wt : B \to P\,, \\ &\tilde{e}_i : B \to B \sqcup \{0\}\,, \\ &\tilde{f}_i : B \to B \sqcup \{0\} \quad \text{for } i \in I\,, \\ &\varepsilon_i, \varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\}. \end{aligned}$$

Here 0 is a ghost element. We assume the following conditions.

(C1): $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b)\rangle$ for any $b \in B$ and any $i \in I$.

(C2): If $b \in B$ satisfies $\tilde{e}_i b \neq 0$, then we have

$$\begin{aligned} &wt(\tilde{e}_i b) = wt(b) + \alpha_i \quad \text{and} \\ &\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \\ &\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1. \end{aligned}$$

(C3): If $b \in B$ satisfies $\tilde{f}_i b \neq 0$, then we have

$$\begin{aligned} &wt(\tilde{f}_i b) = wt(b) - \alpha_i, \\ &\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \\ &\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1. \end{aligned}$$

(C4): For $b_1, b_2 \in B$ and $i \in I$, $b_1 = \tilde{f}_i b_2$ is equivalent to $b_2 = \tilde{e}_i b_1$.

(C5): If $b \in B$ satisfies $\varphi_i(b) = \varepsilon_i(b) = -\infty$, then

$$\tilde{e}_i b = \tilde{f}_i b = 0.$$

Then the crystals form a tensor category (see [**3**], [**5**]).

The crystal T_λ is a crystal $\{t_\lambda\}$ with

$$wt(t_\lambda) = \lambda \ \text{ and } \ \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty\,.$$

For $i \in I$, B_i is a crystal $\{b_i(n); n \in \mathbb{Z}\}$ with

$$\begin{aligned} &\varphi_j(b_i(n)) = \varepsilon_j(b_i(n)) = -\infty \text{ for } \ j \neq i\,, \\ &\varphi_i(b_i(n)) = n,\ \varepsilon_i(b_i(n)) = -n \ \text{ and} \\ &\tilde{e}_i b_i(n) = b_i(n+1),\ \tilde{f}_i b_i(n) = b_i(n-1). \end{aligned}$$

The element $b_i(0)$ is also denoted by b_i.

For $\lambda \in P_+ = \{\lambda \in P; \langle h_i, \lambda\rangle \geq 0 \text{ for any } i \in I\}$, let us denote by $B(\lambda)$ the crystal associated with the irreducible highest weight module with highest weight λ. The unique vector of $B(\lambda)$ of weight λ is denoted by u_λ. Similarly let us denote $B(\infty)$ the crystal associated with $U_q^-(\mathfrak{g})$ (cf. [**3**], [**5**]). Then there is an embedding $B(\lambda) \hookrightarrow B(\infty) \otimes T_\lambda$. There is also an embedding $B(\infty) \hookrightarrow B(\infty) \otimes B_i$ for any $i \in I$.

3. Similarity

Let us fix a positive integer m. Take $\lambda \in P_+$.

The purpose of this section is to show the following theorem.

THEOREM 3.1. *There exists a unique injective map*

$$S_\lambda : B(\lambda) \to B(m\lambda)$$

satisfying the following conditions:

(3.1) *For any* $b \in B(\lambda)$, *we have*

$$wt\big(S_\lambda(b)\big) = m\ wt(b),$$
$$\varepsilon_i\big(S_\lambda(b)\big) = m\ \varepsilon_i(b),$$
$$\varphi_i\big(S_\lambda(b)\big) = m\ \varphi_i(b).$$

(3.2) *For* $b \in B(\lambda)$ *and* $i \in I$, *we have*

$$S_\lambda(\tilde{e}_i b) = \tilde{e}_i^{\,m} S_\lambda(b),\ \ S_\lambda(\tilde{f}_i b) = \tilde{f}_i^{\,m} S_\lambda(b).$$

Here $S_\lambda(0)$ *is understood to be* 0.

In particular we have

$$S_\lambda(u_\lambda) = u_{m\lambda}. \tag{3.3}$$

Similarly, we have

THEOREM 3.2. *There is a unique injective map*

$$S_\infty : B(\infty) \to B(\infty)$$

satisfying the two properties similar to (3.1) and (3.2).

PROOF OF THEOREMS 3.1, 3.2. The uniqueness is obvious. Since $B(\lambda)$ is embedded into $B(\infty) \otimes T_\lambda$, Theorem 3.1 is an immediate consequence of Theorem 3.2.

Let us prove Theorem 3.2. Let us take a sequence $\{i_1, i_2, \cdots\}$ in I such that $\{n; i_n = i\}$ is an infinity set for every $i \in I$. Then by [**3**], $B(\infty)$ is embedded into the crystal $B = \{\cdots \otimes f_{i_2}^{a_2} b_{i_2} \otimes f_{i_1}^{a_1} b_{i_1} \in \cdots \otimes B_{i_2} \otimes B_{i_1}; a_k \geq 0$ for every k and $a_k = 0$ for $k \gg 0\}$. Let $\Psi : B(\infty) \to B$ be the embedding. Let us define the map $S : B \to B$ by

$$S(\cdots \otimes f_{i_2}^{a_2} b_{i_2} \otimes f_{i_1}^{a_1} b_{i_1}) = \cdots \otimes f_{i_2}^{ma_2} b_{i_2} \otimes f_{i_1}^{ma_1} b_{i_1}.$$

Then we can easily verify that S satisfies the conditions (3.1) and (3.2). Hence the composition $B(\infty) \xrightarrow{\Psi} B \xrightarrow{S} B$ decomposes into $B(\infty) \xrightarrow{S_\infty} B(\infty) \xrightarrow{\Psi} B$ and S_∞ satisfies the desired property. □

4. Littelmann's Path Crystal

Littelmann defined a crystal structure on the path space $\mathcal{P}$ on $P_{\mathbb{R}} = P \otimes_{\mathbb{Z}} \mathbb{R}$. He also conjectured that for any $\lambda \in P_+$ the crystal $\mathcal{P}_\lambda$ generated by the straight path π_λ connecting 0 and λ is isomorphic to $B(\lambda)$. In this section, let us give a proof of his conjecture (another proof is given by A. Joseph).

A path is, by definition, a continuous piecewise linear map $\pi : [0,1] \to P_{\mathbb{R}}$ such that $\pi(0) = 0$ and $\pi(1) \in P$. We say that two paths π_1 and π_2 are equivalent if there exist surjective continuous (not necessarily strictly) increasing maps $\psi_1, \psi_2 : [0,1] \to [0,1]$ such that $\pi_1 \circ \psi_1 = \pi_2 \circ \psi_2$.

Let $\mathcal{P}$ be the set of equivalence classes of paths. Littelmann defined two crystal structures on $\mathcal{P}$ (see [**6**], [**7**]); they are almost similar but one behaves well under tensor product and the other under similarity. To fix idea, we shall use the last definition (in [**7**]). We shall not recall the definition but we only recall their properties.

(4.1) For $\pi \in \mathcal{P}$, $wt(\pi) = \pi(1)$.

(4.2) For $\pi \in \mathcal{P}$, $\varepsilon_i(\pi) = \max\left(\mathbb{Z} \cap \{-\langle h_i, \pi(t)\rangle; 0 \le t \le 1\}\right)$.

For a positive integer m, let us define $S_m : \mathcal{P} \to \mathcal{P}$ by $S_m(\pi)(t) = m\pi(t)$.

(4.3) S_m satisfies the properties (3.1) and (3.2).

The crystal $\mathcal{P}$ behaves well under tensor product with a small reservation. For $\pi_1, \pi_2 \in \mathcal{P}$ and $i \in I$, let $\pi_1 * \pi_2$ denote the concatenation of π_1 and π_2, namely:

$$(\pi_1 * \pi_2)(t) = \begin{cases} \pi_1(2t) & 0 \le t \le 1/2\,, \\ \pi_1(1) + \pi_2(2t-1) & 1/2 \le t \le 1\,. \end{cases}$$

Let us denote by $\mathcal{P}_{int}$ the largest full subcrystal of $\mathcal{P}$ such that

$$\varepsilon_i(\pi) = \max\{-\langle h_i, \pi(t)\rangle; 0 \le t \le 1\} \text{ for any } i \in I \text{ and } \pi \in \mathcal{P}_{int}.$$

(4.4) The concatenation induces a morphism of crystals

$$\begin{array}{ccc} \mathcal{P}_{int} \otimes \mathcal{P}_{int} & \longrightarrow & \mathcal{P}_{int} \\ \uplus & & \uplus \\ \pi_1 \otimes \pi_2 & \longmapsto & \pi_1 * \pi_2. \end{array}$$

For $\lambda \in P_+$, let $\mathcal{P}_\lambda$ be the smallest full subcrystal containing π_λ, where $\pi_\lambda(t) = t\lambda$ $(0 \le t \le 1)$. Littelmann([**6**]) proved $\mathcal{P}_\lambda \subset \mathcal{P}_{int}$.

THEOREM 4.1. *There is a unique isomorphism of crystals $B(\lambda) \to \mathcal{P}_\lambda$ sending the highest weight vector u_λ to π_λ.*

PROOF. The uniqueness is obvious. Let us prove that there is a morphism

$$B(\lambda) \to \mathcal{P}_\lambda$$

sending u_λ to π_λ. In order to see this, it is enough to show

(4.5) For $i_1, \cdots, i_n$ and $j_1, \cdots, j_n \in I$,
$\widetilde{f}_{i_1} \cdots \widetilde{f}_{i_n} u_\lambda = \widetilde{f}_{j_1} \cdots \widetilde{f}_{j_n} u_\lambda \Leftrightarrow \widetilde{f}_{i_1} \cdots \widetilde{f}_{i_n} \pi_\lambda = \widetilde{f}_{j_1} \cdots \widetilde{f}_{j_n} \pi_\lambda$.

(4.6) For $i_1, \cdots, i_n \in I$,
$\widetilde{f}_{i_1} \cdots \widetilde{f}_{i_n} u_\lambda = 0 \Leftrightarrow \widetilde{f}_{i_1} \cdots \widetilde{f}_{i_n} \pi_\lambda = 0$.

The proof being similar, let us only prove (4.5).

Let W be the Weyl group. Then, for $w \in W$, $B(\lambda)_{w\lambda}$ consists of a single element, which we shall denote by $u_{w\lambda}$. We have, for $i \in I$ and $w \in W$

(4.7) If $\langle h_i, w\lambda \rangle \geq 0$, then we have
$\varepsilon_i(u_{w\lambda}) = 0,\ \varphi_i(u_{w\lambda}) = \langle h_i, w\lambda \rangle$ and $\widetilde{f}_i^{\langle h_i, w\lambda \rangle} = u_{s_i w\lambda}$.

(4.8) If $\langle h_i, w\lambda \rangle \leq 0$, then we have
$\varepsilon_i(u_{w\lambda}) = -\langle h_i, w\lambda \rangle,\ \varphi_i(u_{w\lambda}) = 0$ and$\widetilde{e}_i^{-\langle h_i, w\lambda \rangle} = u_{s_i w\lambda}$.

For a reduced expression $w = s_{i_1} \cdots s_{i_n}$, we set

$$F_w = \widetilde{f}_{i_1}^{\langle h_{i_1}, s_{i_2} \cdots s_n \lambda \rangle} \cdots \widetilde{f}_{i_n}^{\langle h_{i_n}, \lambda \rangle}.$$

Then we have

$$F_w u_\lambda = u_{w\lambda} .$$

The similar properties hold with $\pi_{w\lambda}$ instead of $u_{w\lambda}$.

For a positive integer m, let $G_m : B(m\lambda) \to B(m)^{\otimes m}$ be the morphism that sends $u_{m\lambda}$ to $u_\lambda^{\otimes m}$. Then $G_m \circ S_m : B(\lambda) \to B(m)^{\otimes m}$ is a map that satisfies (3.1) and (3.2).

Now take $i_1, \cdots, i_n \in I$ and $j_1, \cdots, j_n \in I$. Then for an integer m that contains sufficiently many divisors, we have

$$G_m \circ S_m(\widetilde{f}_{i_1} \cdots \widetilde{f}_{i_n} u_\lambda) = F_{w_1} u_\lambda \otimes \cdots \otimes F_{w_m} u_\lambda \quad \text{and}$$
$$G_m \circ S_m(\widetilde{f}_{j_1} \cdots \widetilde{f}_{j_n} u_\lambda) = F_{w'_1} u_\lambda \otimes \cdots \otimes F_{w'_m} u_\lambda .$$

for some $w_1, \cdots, w_m, w'_1, \cdots w'_m \in W$.

Then we have

$$G_m \circ S_m(\widetilde{f}_{i_1} \cdots \widetilde{f}_{i_n} \pi_\lambda) = F_{w_1} \pi_\lambda * \cdots * F_{w_m} \pi_\lambda \quad \text{and}$$
$$G_m \circ S_m(\widetilde{f}_{j_1} \cdots \widetilde{f}_{j_n} \pi_\lambda) = F_{w'_1} \pi_\lambda * \cdots * F_{w'_m} \pi_\lambda .$$

Finally, we conclude

$$\begin{aligned}
& \tilde{f}_{i_1} \cdots \tilde{f}_{i_n} u_\lambda = \tilde{f}_{j_1} \cdots \tilde{f}_{j_n} u_\lambda \\
\Longleftrightarrow \quad & F_{w_1} u_\lambda \otimes \cdots \otimes F_{w_m} u_\lambda = F_{w_1'} u_\lambda \otimes \cdots \otimes F_{w_m'} u_\lambda \\
\Longleftrightarrow \quad & w_1 \lambda = w_1' \lambda, \cdots, w_m \lambda = w_m' \lambda \\
\Longleftrightarrow \quad & F_{w_1} \pi_\lambda * \cdots * F_{w_m} \pi_\lambda = F_{w_1'} \pi_\lambda * \cdots * F_{w_m'} \pi_\lambda \\
\Longleftrightarrow \quad & \tilde{f}_{i_1} \cdots \tilde{f}_{i_n} \pi_\lambda = \tilde{f}_{j_1} \cdots \tilde{f}_{j_n} \pi_\lambda .
\end{aligned}$$

□

5. Variants

Let (I, P) be data as in §1. Let J be another finite set and let $\xi : I \to J$ be a surjective map. To each $i \in I$ we associate a positive integer m_i. We set $\tilde{\alpha}_j = \sum_{i \in \xi^{-1}(j)} m_i \alpha_i \in P$. Let us denote by $\widetilde{P}$ the subset of P consisting of $\lambda \in P$ such that, for any $j \in J$, $\frac{1}{m_i} \langle h_i, \lambda \rangle$ is an integer and does not depend on the choice of $i \in \xi^{-1}(j)$. Then for $j \in J$, $\tilde{h}_j \in \widetilde{P}^*$ is well defined by $\langle \tilde{h}_j, \lambda \rangle = \frac{1}{m_i} \langle h_i, \lambda \rangle$ for $i \in \xi^{-1}(j)$ and $\lambda \in \widetilde{P}$.

We assume the following properties.

(5.1) $\langle h_i, \alpha_{i'} \rangle = 0 \quad$ for $i, i' \in I$ such that $\xi(i) = \xi(i')$ and $i \neq i'$,

(5.2) $\tilde{\alpha}_j$ belongs to $\widetilde{P}$ for any $j \in J$.

Then $(J, \widetilde{P})$ defines another data.

Let $\widetilde{P}_+ = \widetilde{P} \cap P_+$. For $\lambda \in \widetilde{P}_+$, let $B(\lambda)$ be the crystal with highest weight λ over (I, P) and $B_J(\lambda)$ the crystal with highest weight λ over $(J, \widetilde{P})$,

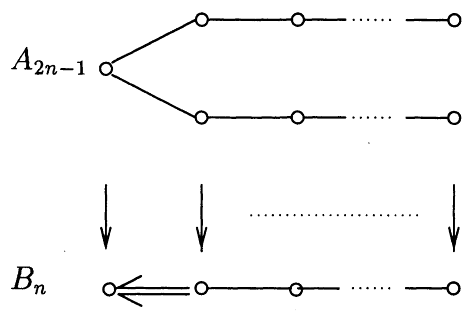
THEOREM 5.1. *There exists a unique map $S : B_J(\lambda) \to B(\lambda)$ such that*

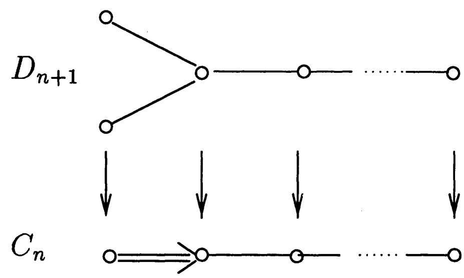
(5.3) $$wt\big(S(b)\big) = wt(b)\,,$$

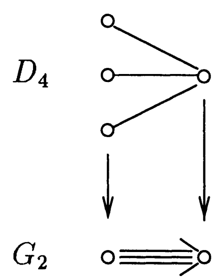
(5.4) $$S(\tilde{e}_j b) = \prod_{i \in \xi^{-1}(j)} \tilde{e}_i^{m_i} S(b)\,,$$

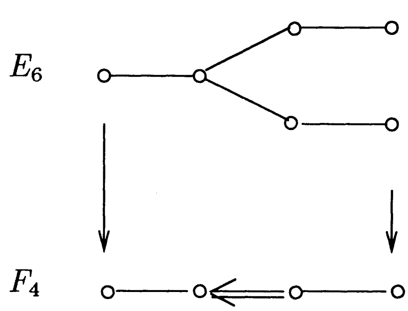
(5.5) $$S(\tilde{f}_j b) = \prod_{i \in \xi^{-1}(j)} \tilde{f}_i^{m_i} S(b).$$

Note that for $i, i' \in \xi^{-1}(j)$, $\tilde{e}_i$ and $\tilde{e}_i'$ (resp. $\tilde{f}_i$ and $\tilde{f}_i{}'$) commute by (5.1), Theorem 3.1 is a special case of this theorem where we take the identity as ξ. As in Lusztig([**8**]), an automorphism of a Dynkin diagram gives such examples (by taking $m_i = 1$).


A_{2n-1}
B_n

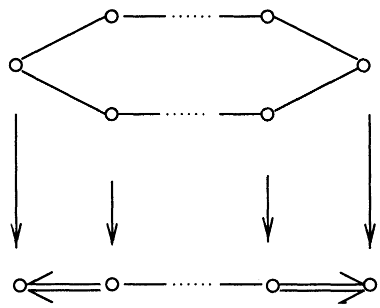

D_{n+1}
C_n


D_4
G_2


E_6
F_4

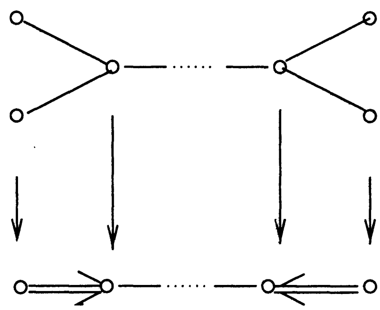
$A^{(1)}_{2n-1}$

$D^{(2)}_n$



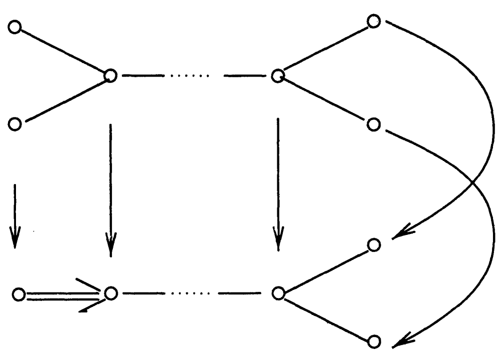
$D^{(1)}_{n+2}$

$C^{(1)}_n$



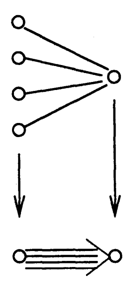
$D^{(1)}_{n+1}$

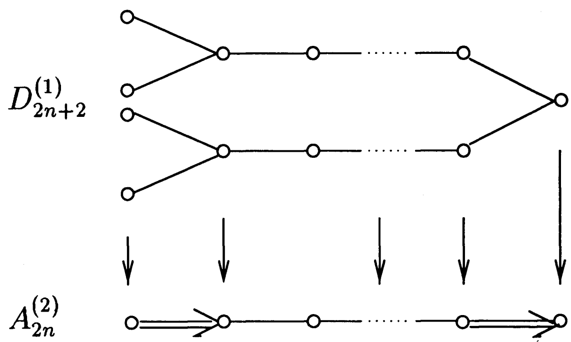
$A^{(2)}_{2n-1}$



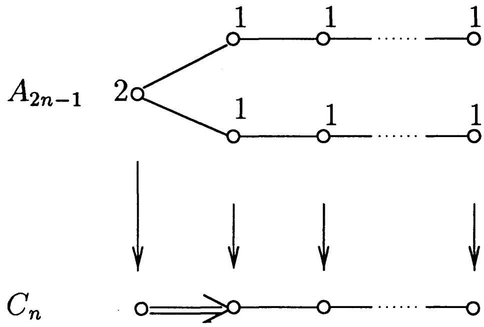
$D^{(1)}_4$

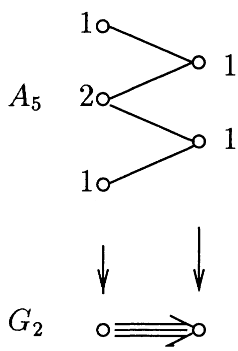
$A^{(2)}_2$





Here are other examples. The numbers indicate m_i.





PROOF OF THEOREM 5.1. The proof is similar to the proof of Theorems 3.1 and 3.2. For $j \in J$, let $\tilde{B}_j$ be the (I, P)-crystal $\otimes_{i \in \xi^{-1}(j)} B_i$. Let B^J_j be the $(J, \tilde{P})$-crystal corresponding to $j \in J$. Let $S_j : B^J_j \to \tilde{B}_j$ be the map given by $S_j(b_j(n)) = \otimes_{i \in \xi^{-1}(j)} b_i(nm_i)$. Take a sequence $j_1, j_2, \cdots$ in J in which every element of J appears infinitely many times. Set $\tilde{B}_J = \cdots \otimes B^J_{j_2} \otimes B^J_{j_1}$ and $\tilde{B} = \cdots \otimes \left(\otimes_{i \in \xi^{-1}(j_2)} B_i \right) \otimes \left(\otimes_{i \in \xi^{-1}(j_1)} B_i \right)$. Then we consider the embedings $\Psi_J : B(\infty) \to \tilde{B}_J$ and $\Psi : B(\infty) \to \tilde{B}$. Now $\tilde{S} = \otimes_n S_{j_n}$ defines a map $\tilde{B}_J \to \tilde{B}$. We can see easily that S satisfied the conditions (5.3), (5.4) and (5.5). Hence there exists $S : B_J(\infty) \to B(\infty)$ such that $\Psi \circ S = \tilde{S} \circ \Psi_J$. □

REMARK 5.2. The corresponding relation between the quantized universal enveloping algebras $U_q(\mathfrak{g})$ and $U_q(\mathfrak{g}_J)$ are not known.

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