SIMPLICITY OF HEADS AND SOCLES OF TENSOR PRODUCTS

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ABSTRACT. We prove that, for simple modules M and N over a quantum affine algebra, their tensor product $M \otimes N$ has a simple head and a simple socle if $M \otimes M$ is simple. A similar result is proved for the convolution product of simple modules over quiver Hecke algebras.

INTRODUCTION

Let $\mathbf{g}$ be a complex simple Lie algebra and $U_q(\mathbf{g})$ the associated quantum group. The multiplicative property of the upper global basis $\mathbf{B}$ of the negative half $U_q^-(\mathbf{g})$ was investigated in [3, 13]. Set $q^{\mathbb{Z}}\mathbf{B} = \{q^n b \mid b \in \mathbf{B}, n \in \mathbb{Z}\}$. In [3], Berenstein and Zelevinsky conjectured that, for b_1, $b_2 \in \mathbf{B}$, the product $b_1 b_2$ belongs to $q^{\mathbb{Z}}\mathbf{B}$ if and only if b_1 and b_2 q-commute, (i,e., $b_2 b_1 = q^n b_1 b_2$ for some $n \in \mathbb{Z}$). However, Leclerc found examples of $b \in \mathbf{B}$ such that $b^2 \notin q^{\mathbb{Z}}\mathbf{B}$ ([13]).

On the other hand, the algebra $U_q^-(\mathbf{g})$ is categorified by quiver Hecke algebras ([10, 11, 14]) and also by quantum affine algebras ([4, 5, 7, 8]). In this context, the products in $U_q^-(\mathbf{g})$ correspond to the convolution or the tensor products in quiver Hecke algebras or quantum affine algebras. The upper global basis corresponds to the set of isomorphism classes of simple modules over the quiver Hecke algebra or the quantum affine algebras ([2, 15, 16]) under suitable conditions. Then Leclerc conjectured several properties of products of upper global bases and also convolutions and tensor products of simple modules. The purpose of this paper is to give an affirmative answer to some of his conjectures.

In this introduction, we state our results in the case of modules over quantum affine algebras. The similar results hold also for quiver Hecke algebras (see § 3.1).

Let $\mathfrak{g}$ be an affine Lie algebra and $U_q'(\mathfrak{g})$ the associated quantum affine algebra. A simple $U_q'(\mathfrak{g})$-module M is called *real* if $M \otimes M$ is also simple.

Conjecture ([13, Conjecture 3]). *Let M and N be finite-dimensional simple $U_q'(\mathfrak{g})$-modules. We assume further that M is real. Then $M \otimes N$ has a simple socle S and a simple head H. Moreover, if S and H are isomorphic, then $M \otimes N$ is simple.*

In this paper, we shall give an affirmative answer to this conjecture (Theorem 3.12 and Corollary 3.16). In the course of the proof, R-matrices play an important role. Indeed, the simple socle of $M \otimes N$ coincides with the image of the renormalized R-matrix $\mathbf{r}_{N,M} \colon N \otimes M \to M \otimes N$ and the simple head of $M \otimes N$ coincides with the image of the renormalized R-matrix $\mathbf{r}_{M,N} \colon M \otimes N \to N \otimes M$.

Denoting by $M \nabla N$ the head of $M \otimes N$, we also prove that $N \mapsto M \nabla N$ is an automorphism of the set of the isomorphism classes of simple $U_q'(\mathfrak{g})$-modules (Corollary 3.14). The inverse is given by $N \mapsto N \nabla \,^*M$, where $^*M$ is the right dual of M. It is an analogue of Conjecture 2 in [13] originally stated for global bases.

1. Quiver Hecke algebras

In this section, we briefly recall the basic facts on quiver Hecke algebras and R-matrices following [7]. Since the grading of quiver Hecke algebras is not important in this paper, we ignore the grading. Throughout the paper, *modules mean left modules.*

1.1. **Convolutions.** We shall recall the definition of quiver Hecke algebras. Let $\mathbf{k}$ be a field. Let I be an index set. Let Q be the free $\mathbb{Z}$-module with a basis $\{\alpha_i\}_{i \in I}$. Set $\mathsf{Q}^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$. For $\beta = \sum_{k=1}^{n} \alpha_{i_k} \in \mathsf{Q}^+$, we set $\mathrm{ht}(\beta) = n$. For $n \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathsf{Q}^+$ such that $\mathrm{ht}(\beta) = n$, we set

$$I^\beta = \{\nu = (\nu_1, \ldots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}.$$

Let us take a family of polynomials $(Q_{ij})_{i,j \in I}$ in $\mathbf{k}[u, v]$ which satisfy

$$Q_{ij}(u, v) = Q_{ji}(v, u) \quad \text{for any } i,\, j \in I,$$
$$Q_{ii}(u, v) = 0 \quad \text{for any } i \in I.$$

For $i, j \in I$, we set

$$\overline{Q}_{ij}(u, v, w) = \frac{Q_{ij}(u, v) - Q_{ij}(w, v)}{u - w} \in \mathbf{k}[u, v, w].$$

We denote by $\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1} \rangle$ the symmetric group on n letters, where $s_i := (i, i+1)$ is the transposition of i and $i+1$. Then $\mathfrak{S}_n$ acts on I^n by place permutations.

Definition 1.1. *For $\beta \in Q^+$ with $\mathrm{ht}(\beta) = n$, the quiver Hecke algebra $R(\beta)$ at β associated with a matrix $(Q_{ij})_{i,j \in I}$ is the **k**-algebra generated by the elements $\{e(\nu)\}_{\nu \in I^\beta}$, $\{x_k\}_{1 \le k \le n}$, $\{\tau_k\}_{1 \le k \le n-1}$ satisfying the following defining relations:*

$$e(\nu)e(\nu') = \delta_{\nu,\nu'} e(\nu), \quad \sum_{\nu \in I^\beta} e(\nu) = 1,$$

$$x_k x_m = x_m x_k, \quad x_k e(\nu) = e(\nu) x_k,$$

$$\tau_m e(\nu) = e(s_m(\nu))\tau_m, \quad \tau_k \tau_m = \tau_m \tau_k \quad if\ |k - m| > 1,$$

$$\tau_k^2 e(\nu) = Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1})e(\nu),$$

$$(\tau_k x_m - x_{s_k(m)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & if\ m = k\ and\ \nu_k = \nu_{k+1}, \\ e(\nu) & if\ m = k + 1\ and\ \nu_k = \nu_{k+1}, \\ 0 & otherwise, \end{cases}$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \begin{cases} \overline{Q}_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}, x_{k+2}) & if\ \nu_k = \nu_{k+2}, \\ 0 & otherwise. \end{cases}$$

For an element w of the symmetric group $\mathfrak{S}_n$, let us choose a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$, and set

$$\tau_w = \tau_{i_1} \cdots \tau_{i_\ell}.$$

In general, it depends on the choice of reduced expressions of w. Then we have the PBW decomposition

$$(1.1) \qquad\qquad R(\beta) = \bigoplus_{\nu \in I^\beta,\ w \in \mathfrak{S}_n} \mathbf{k}[x_1, \ldots, x_n]e(\nu)\tau_w.$$

We denote by $R(\beta)$-mod the category of $R(\beta)$-modules M such that M is finite-dimensional over $\mathbf{k}$ and the action of x_k on M is nilpotent for any k.

For an $R(\beta)$-module M, the dual space

$$M^* := \mathrm{Hom}_{\mathbf{k}}(M, \mathbf{k})$$

is endowed with the $R(\beta)$-module structure given by

$$(r \cdot f)(u) := f(\psi(r)u) \quad \text{for } f \in M^*,\ r \in R(\beta) \text{ and } u \in M,$$

where ψ denotes the **k**-algebra anti-involution on $R(\beta)$ which fixes the generators $\{e(\nu)\}_{\nu \in I^\beta}$, $\{x_k\}_{1 \le k \le n}$, $\{\tau_k\}_{1 \le k \le n-1}$.

For $\beta, \gamma \in Q^+$ with $\mathrm{ht}(\beta) = m$ and $\mathrm{ht}(\gamma) = n$, set

$$e(\beta, \gamma) = \sum_{\substack{\nu \in I^{m+n}, \\ (\nu_1, \ldots, \nu_m) \in I^\beta, \\ (\nu_{m+1}, \ldots, \nu_{m+n}) \in I^\gamma}} e(\nu) \in R(\beta + \gamma).$$

Then $e(\beta, \gamma)$ is an idempotent. Let

$$R(\beta) \otimes R(\gamma) \to e(\beta, \gamma) R(\beta + \gamma) e(\beta, \gamma)$$

be the **k**-algebra homomorphism given by

$$
\begin{aligned}
e(\mu) \otimes e(\nu) &\mapsto e(\mu * \nu) & (\mu \in I^\beta, \ \nu \in I^\gamma) \\
x_k \otimes 1 &\mapsto x_k e(\beta, \gamma) & (1 \le k \le m), \\
1 \otimes x_k &\mapsto x_{m+k} e(\beta, \gamma) & (1 \le k \le n), \\
\tau_k \otimes 1 &\mapsto \tau_k e(\beta, \gamma) & (1 \le k < m), \\
1 \otimes \tau_k &\mapsto \tau_{m+k} e(\beta, \gamma) & (1 \le k < n).
\end{aligned}
$$

Here $\mu * \nu$ is the concatenation of μ and ν, i.e.,

$$\mu * \nu = (\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n).$$

For an $R(\beta)$-module M and an $R(\gamma)$-module N, we define their *convolution product* $M \circ N$ by

$$(1.2) \qquad M \circ N = R(\beta + \gamma) e(\beta, \gamma) \underset{R(\beta) \otimes R(\gamma)}{\otimes} (M \otimes N).$$

Set $m = \mathrm{ht}(\beta)$ and $n = \mathrm{ht}(\gamma)$. Set

$$\mathfrak{S}_{m,n} := \left\{ w \in \mathfrak{S}_{m+n} \mid w|_{[1,m]} \text{ and } w|_{[m+1,m+n]} \text{ are increasing} \right\}.$$

Here $[a, b] := \{ k \in \mathbb{Z} \mid a \le k \le b \}$. Then we have

$$(1.3) \qquad M \circ N = \bigoplus_{w \in \mathfrak{S}_{m,n}} \tau_w (M \otimes N).$$

We also have (see [12, Theorem 2.2 (2)])

$$(1.4) \qquad (M \circ N)^* \simeq N^* \circ M^*.$$

1.2. *R*-matrices for quiver Hecke algebras.

1.2.1. *Intertwiners.* For $\mathrm{ht}(\beta) = n$ and $1 \le a < n$, we define $\varphi_a \in R(\beta)$ by

$$
(1.5) \qquad \varphi_a e(\nu) =
\begin{cases}
\begin{aligned}
(\tau_a x_a &- x_a \tau_a) e(\nu) \\
&= (x_{a+1} \tau_a - \tau_a x_{a+1}) e(\nu) \\
&= (\tau_a (x_a - x_{a+1}) + 1) e(\nu) \\
&= ((x_{a+1} - x_a) \tau_a - 1) e(\nu) & \text{if } \nu_a = \nu_{a+1}, \\
\end{aligned} \\
\tau_a e(\nu) & \text{otherwise.}
\end{cases}
$$

They are called the *intertwiners*.

Lemma 1.2.

(i) $\varphi_a^2 e(\nu) = \left(Q_{\nu_a, \nu_{a+1}}(x_a, x_{a+1}) + \delta_{\nu_a, \nu_{a+1}} \right) e(\nu).$

(ii) $\{\varphi_k\}_{1 \leq k < n}$ *satisfies the braid relation.*

(iii) *For $w \in \mathfrak{S}_n$, let $w = s_{a_1} \cdots s_{a_\ell}$ be a reduced expression of w and set $\varphi_w = \varphi_{a_1} \cdots \varphi_{a_\ell}$. Then φ_w does not depend on the choice of reduced expressions of w.*

(iv) *For $w \in \mathfrak{S}_n$ and $1 \leq k \leq n$, we have $\varphi_w x_k = x_{w(k)} \varphi_w$.*

(v) *For $w \in \mathfrak{S}_n$ and $1 \leq k < n$, if $w(k+1) = w(k) + 1$, then $\varphi_w \tau_k = \tau_{w(k)} \varphi_w$.*

For $m, n \in \mathbb{Z}_{\geq 0}$, let us denote by $w[m, n]$ the element of $\mathfrak{S}_{m+n}$ defined by

$$(1.6) \qquad w[m, n](k) = \begin{cases} k + n & \text{if } 1 \leq k \leq m, \\ k - m & \text{if } m < k \leq m + n. \end{cases}$$

Let $\beta, \gamma \in \mathsf{Q}^+$ with $\mathrm{ht}(\beta) = m$, $\mathrm{ht}(\gamma) = n$, and let M be an $R(\beta)$-module and N an $R(\gamma)$-module. Then the map $M \otimes N \to N \circ M$ given by $u \otimes v \longmapsto \varphi_{w[n,m]}(v \otimes u)$ is $R(\beta) \otimes R(\gamma)$-linear by the above lemma, and it extends to an $R(\beta + \gamma)$-module homomorphism

$$(1.7) \qquad R_{M,N} \colon M \circ N \longrightarrow N \circ M.$$

Then we obtain the following commutative diagrams:

$$(1.8) \quad L \circ M \circ N \xrightarrow{R_{L,M}} M \circ L \circ N \quad \text{and} \quad L \circ M \circ N \xrightarrow{R_{M,N}} L \circ N \circ M$$

with maps $R_{L,M\circ N}$ and $R_{L,N}$ to $M \circ N \circ L$, and maps $R_{L\circ M,N}$ and $R_{L,N}$ to $N \circ L \circ M$.

1.2.2. *Spectral parameters.*

Definition 1.3. *For $\beta \in \mathsf{Q}^+$, the quiver Hecke algebra $R(\beta)$ is called* symmetric *if $Q_{i,j}(u, v)$ is a polynomial in $u - v$ for all $i, j \in \mathrm{supp}(\beta)$. Here, we set $\mathrm{supp}(\beta) = \{i_k \mid 1 \leq k \leq n\}$ for $\beta = \sum_{k=1}^n \alpha_{i_k}$.*

Assume that the quiver Hecke algebra $R(\beta)$ is symmetric. Let z be an indeterminate, and let ψ_z be the algebra homomorphism

$$\psi_z \colon R(\beta) \to \mathbf{k}[z] \otimes R(\beta)$$

given by

$$\psi_z(x_k) = x_k + z, \quad \psi_z(\tau_k) = \tau_k, \quad \psi_z(e(\nu)) = e(\nu).$$

For an $R(\beta)$-module M, we denote by M_z the $\big(\mathbf{k}[z] \otimes R(\beta)\big)$-module $\mathbf{k}[z] \otimes M$ with the action of $R(\beta)$ twisted by ψ_z. Namely,

$$(1.9) \qquad \begin{aligned} e(\nu)(a \otimes u) &= a \otimes e(\nu)u, \\ x_k(a \otimes u) &= (za) \otimes u + a \otimes (x_k u), \\ \tau_k(a \otimes u) &= a \otimes (\tau_k u) \end{aligned}$$

for $\nu \in I^\beta$, $a \in \mathbf{k}[z]$ and $u \in M$. For $u \in M$, we sometimes denote by u_z the corresponding element $1 \otimes u$ of the $R(\beta)$-module M_z.

For a non-zero $M \in R(\beta)$-mod and a non-zero $N \in R(\gamma)$-mod,

(1.10)   let s be the order of zeroes of $R_{M_z,N} \colon M_z \circ N \longrightarrow N \circ M_z$; i.e., the largest non-negative integer such that the image of $R_{M_z,N}$ is contained in $z^s(N \circ M_z)$.

Note that such an s exists because $R_{M_z,N}$ does not vanish ([7, Proposition 1.4.4 (iii)]).

Definition 1.4. *Assume that $R(\beta)$ is symmetric. For a non-zero $M \in R(\beta)$-mod and a non-zero $N \in R(\gamma)$-mod, let s be an integer as in* (1.10). *We define*

$$\mathbf{r}_{M,N} \colon M \circ N \to N \circ M$$

by

$$\mathbf{r}_{M,N} = \left(z^{-s} R_{M_z,N}\right)\big|_{z=0},$$

and call it the renormalized R-matrix.

By the definition, the renormalized R-matrix $\mathbf{r}_{M,N}$ never vanishes.

We define also

$$\mathbf{r}_{N,M} \colon N \circ M \to M \circ N$$

by

$$\mathbf{r}_{N,M} = \left((-z)^{-t} R_{N,M_z}\right)\big|_{z=0},$$

where t is the multiplicity of zero of R_{N,M_z}.

Note that if $R(\beta)$ and $R(\gamma)$ are symmetric, then s coincides with the multiplicity of zero of R_{M,N_z}, and $\left(z^{-s} R_{M_z,N}\right)\big|_{z=0} = \left((-z)^{-s} R_{M,N_z}\right)\big|_{z=0}$.

Indeed, we have

(1.11) $$R_{M_{z_1},N_{z_2}}\left((u)_{z_1} \otimes (v)_{z_2}\right) = \varphi_{w[n,m]}\left((v)_{z_2} \otimes (u)_{z_1}\right)$$
$$\in \sum\nolimits_{w,u',v'} \mathbf{k}[z_1 - z_2]\tau_w\left((v')_{z_2} \otimes (u')_{z_1}\right)$$

for $u \in M$ and $v \in N$. Here w ranges over

$$\mathfrak{S}_{n,m} := \left\{w \in \mathfrak{S}_{m+n} \mid w|_{[1,n]} \text{ and } w|_{[n+1,n+m]} \text{ are strictly increasing}\right\}$$

and $v' \in N$ and $u' \in M$. Hence, $\mathbf{r}_{M,N}$ is well defined whenever at least one of $R(\beta)$ and $R(\gamma)$ is symmetric.

The proof of (1.11) will be given later in Section 4.

2. Quantum affine algebras

In this section, we briefly review the representation theory of quantum affine algebras following [1, 9]. When concerned with quantum affine algebras, we take the algebraic closure of $\mathbb{C}(q)$ in $\cup_{m>0}\mathbb{C}((q^{1/m}))$ as a base field $\mathbf{k}$.

2.1. **Integrable modules.** Let I be an index set and $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix of affine type.

We choose $0 \in I$ as the leftmost vertices in the tables in [6, pages 54, 55] except $A_{2n}^{(2)}$-case in which case we take the longest simple root as α_0. Set $I_0 = I \setminus \{0\}$.

The weight lattice P is given by

$$P = \left(\bigoplus_{i \in I} \mathbb{Z} \Lambda_i \right) \oplus \mathbb{Z} \delta,$$

and the simple roots are given by

$$\alpha_i = \sum_{j \in I} a_{ji} \Lambda_j + \delta(i = 0) \delta.$$

The weight δ is called the imaginary root. There exist $d_i \in \mathbb{Z}_{>0}$ such that

$$\delta = \sum_{i \in I} d_i \alpha_i.$$

Note that $d_i = 1$ for $i = 0$. The simple coroots $h_i \in P^\vee := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ are given by

$$\langle h_i, \Lambda_j \rangle = \delta_{ij}, \quad \langle h_i, \delta \rangle = 0.$$

Hence we have $\langle h_i, \alpha_j \rangle = a_{ij}$.

Let $c = \sum_{i \in I} c_i h_i$ be a unique element such that $c_i \in \mathbb{Z}_{>0}$ and

$$\mathbb{Z} c = \left\{ h \in \bigoplus_{i \in I} \mathbb{Z} h_i \mid \langle h, \alpha_i \rangle = 0 \text{ for any } i \in I \right\}.$$

Let us take a $\mathbb{Q}$-valued symmetric bilinear form $(\bullet, \bullet)$ on P such that

$$\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \text{ and } (\delta, \lambda) = \langle c, \lambda \rangle \text{ for any } \lambda \in P.$$

Let q be an indeterminate. For each $i \in I$, set $q_i = q^{(\alpha_i, \alpha_i)/2}$.

Definition 2.1. *The* quantum group $U_q(\mathfrak{g})$ *associated with* (A, P) *is the* **k**-*algebra generated by* e_i, f_i ($i \in I$) *and* q^λ ($\lambda \in P$) *satisfying following relations:*

$$q^0 = 1, \ q^\lambda q^{\lambda'} = q^{\lambda + \lambda'} \ \text{ for } \lambda, \lambda' \in P,$$

$$q^\lambda e_i q^{-\lambda} = q^{(\lambda, \alpha_i)} e_i, \quad q^\lambda f_i q^{-\lambda} = q^{-(\lambda, \alpha_i)} f_i \ \text{ for } \lambda \in P, i \in I,$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad \text{where } K_i = q^{\alpha_i},$$

$$\sum_{r=0}^{1 - a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_i e_i^{1 - a_{ij} - r} e_j e_i^r = 0 \quad \text{if } i \neq j,$$

$$\sum_{r=0}^{1 - a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_i f_i^{1 - a_{ij} - r} f_j f_i^r = 0 \quad \text{if } i \neq j.$$

Here, we set $[n]_i = \dfrac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$, $[n]_i! = \prod_{k=1}^n [k]_i$ and $\begin{bmatrix} m \\ n \end{bmatrix}_i = \dfrac{[m]_i!}{[m-n]_i![n]_i!}$ for each $n \in \mathbb{Z}_{\geq 0}$, $i \in I$ and $m \geq n$.

We denote by $U_q'(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, $K_i^{\pm 1}(i \in I)$, and call it *quantum affine algebra*. The algebra $U_q'(\mathfrak{g})$ has a Hopf algebra structure with the coproduct:

(2.1)
$$\begin{aligned}
\Delta(K_i) &= K_i \otimes K_i, \\
\Delta(e_i) &= e_i \otimes K_i^{-1} + 1 \otimes e_i, \\
\Delta(f_i) &= f_i \otimes 1 + K_i \otimes f_i.
\end{aligned}$$

Set
$$P_{\mathrm{cl}} = P/\mathbb{Z}\delta$$

and call it the *classical weight lattice*. Let $\mathrm{cl} \colon P \to P_{\mathrm{cl}}$ be the projection. Then $P_{\mathrm{cl}} = \bigoplus_{i \in I} \mathbb{Z}\,\mathrm{cl}(\Lambda_i)$. Set $P_{\mathrm{cl}}^0 = \{\lambda \in P_{\mathrm{cl}} \mid \langle c, \lambda \rangle = 0\} \subset P_{\mathrm{cl}}$.

A $U_q'(\mathfrak{g})$-module M is called an *integrable module* if

(a) M has a weight space decomposition
$$M = \bigoplus_{\lambda \in P_{\mathrm{cl}}} M_\lambda,$$

where $M_\lambda = \left\{ u \in M \mid K_i u = q_i^{\langle h_i, \lambda \rangle} u \text{ for all } i \in I \right\}$,

(b) the actions of e_i and f_i on M are locally nilpotent for any $i \in I$.

Let us denote by $U_q'(\mathfrak{g})$-mod the abelian tensor category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules.

If M is a simple module in $U_q'(\mathfrak{g})$-mod, then there exists a non-zero vector $u \in M$ of weight $\lambda \in P_{\mathrm{cl}}^0$ such that λ is dominant (i.e., $\langle h_i, \lambda \rangle \geq 0$ for any $i \in I_0$) and all the weights of M lie in $\lambda - \sum_{i \in I_0} \mathbb{Z}_{\geq 0}\alpha_i$. We say that λ is the *dominant extremal weight* of M and u is a *dominant extremal vector* of M. Note that a dominant extremal vector of M is unique up to a constant multiple.

Let z be an indeterminate. For a $U_q'(\mathfrak{g})$-module M, let us denote by M_z the module $\mathbf{k}[z, z^{-1}] \otimes M$ with the action of $U_q'(\mathfrak{g})$ given by

$$e_i(u_z) = z^{\delta_{i,0}}(e_i u)_z, \quad f_i(u_z) = z^{-\delta_{i,0}}(f_i u)_z, \quad K_i(u_z) = (K_i u)_z.$$

Here, for $u \in M$, we denote by u_z the element $1 \otimes u \in \mathbf{k}[z, z^{-1}] \otimes M$.

2.2. R-matrices.

We recall the notion of R-matrices [9, § 8]. Let us choose the following *universal R-matrix*. Let us take a basis $\{P_\nu\}_\nu$ of $U_q^+(\mathfrak{g})$ and a basis $\{Q_\nu\}_\nu$ of $U_q^-(\mathfrak{g})$ dual to each other with respect to a suitable coupling between $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$. Then

for $U_q'(\mathfrak{g})$-modules M and N define

$$(2.2) \qquad R_{MN}^{\mathrm{univ}}(u \otimes v) = q^{(\mathrm{wt}(u),\mathrm{wt}(v))} \sum_\nu P_\nu v \otimes Q_\nu u \,,$$

so that R_{MN}^{univ} gives a $U_q'(\mathfrak{g})$-linear homomorphism from $M \otimes N$ to $N \otimes M$ provided that the infinite sum has a meaning.

Let M and N be $U_q'(\mathfrak{g})$-modules in $U_q'(\mathfrak{g})$-mod, and let z_1 and z_2 be indeterminates. Then $R_{M_{z_1},N_{z_2}}^{\mathrm{univ}}$ converges in the (z_2/z_1)-adic topology. Hence we obtain a morphism of $\mathbf{k}[[z_2/z_1]] \otimes_{\mathbf{k}[z_2/z_1]} \mathbf{k}[z_1^{\pm 1}, z_2^{\pm 1}] \otimes U_q'(\mathfrak{g})$-modules

$$R_{M_{z_1},N_{z_2}}^{\mathrm{univ}} \colon \mathbf{k}[[z_2/z_1]] \underset{\mathbf{k}[z_2/z_1]}{\otimes} (M_{z_1} \otimes N_{z_2}) \to \mathbf{k}[[z_2/z_1]] \underset{\mathbf{k}[z_2/z_1]}{\otimes} (N_{z_2} \otimes M_{z_1}).$$

If there exist $a \in \mathbf{k}((z_2/z_1))$ and a $\mathbf{k}[z_1^{\pm 1}, z_2^{\pm 1}] \otimes U_q'(\mathfrak{g})$-linear homomorphism

$$R \colon M_{z_1} \otimes N_{z_2} \to N_{z_2} \otimes M_{z_1}$$

such that $R_{M_{z_1},N_{z_2}}^{\mathrm{univ}} = aR$, then we say that $R_{M_{z_1},N_{z_2}}^{\mathrm{univ}}$ is *rationally renormalizable*.

Now assume further that M and N are non-zero. Then, we can choose R so that, for any $c_1, c_2 \in \mathbf{k}^\times$, the specialization of R at $z_1 = c_1$, $z_2 = c_2$

$$R|_{z_1=c_1,z_2=c_2} \colon M_{c_1} \otimes N_{c_2} \to N_{c_2} \otimes M_{c_1}$$

does not vanish. Such an R is unique up to a multiple of $\mathbf{k}[(z_1/z_2)^{\pm 1}]^\times = \sqcup_{n \in \mathbb{Z}} \mathbf{k}^\times z_1^n z_2^{-n}$. We write

$$\mathbf{r}_{M,N} := R|_{z_1=z_2=1} \colon M \otimes N \to N \otimes M,$$

and call it the *renormalized R-matrix*. The renormalized R-matrix $\mathbf{r}_{M,N}$ is well defined up to a constant multiple when $R_{M_{z_1},N_{z_2}}^{\mathrm{univ}}$ is rationally renormalizable. By the definition, $\mathbf{r}_{M,N}$ never vanishes.

Now assume that M_1 and M_2 are simple $U_q'(\mathfrak{g})$-modules in $U_q'(\mathfrak{g})$-mod. Then, the universal R-matrix $R_{(M_1)_{z_1},(M_2)_{z_2}}^{\mathrm{univ}}$ is rationally renormalizable. More precisely, we have the following. Let u_1 and u_2 be dominant extremal weight vectors of M_1 and M_2, respectively. Then there exists $a(z_2/z_1) \in \mathbf{k}[[z_2/z_1]]^\times$ such that

$$R_{(M_1)_{z_1},(M_2)_{z_2}}^{\mathrm{univ}} \big((u_1)_{z_1} \otimes (u_2)_{z_2}\big) = a(z_2/z_1)\big((u_2)_{z_2} \otimes (u_1)_{z_1}\big).$$

Then $R_{M_1,M_2}^{\mathrm{norm}} := a(z_2/z_1)^{-1} R_{(M_1)_{z_1},(M_2)_{z_2}}^{\mathrm{univ}}$ is a unique $\mathbf{k}(z_1, z_2) \otimes U_q'(\mathfrak{g})$-module homomorphism

$$(2.3) \qquad \begin{aligned} R_{M_1,M_2}^{\mathrm{norm}} \colon \mathbf{k}(z_1, z_2) &\otimes_{\mathbf{k}[z_1^{\pm 1}, z_2^{\pm 1}]} \big((M_1)_{z_1} \otimes (M_2)_{z_2}\big) \\ &\longrightarrow \mathbf{k}(z_1, z_2) \otimes_{\mathbf{k}[z_1^{\pm 1}, z_2^{\pm 1}]} \big((M_2)_{z_2} \otimes (M_1)_{z_1}\big) \end{aligned}$$

satisfying

$$(2.4) \qquad R_{M_1,M_2}^{\mathrm{norm}}\big((u_1)_{z_1} \otimes (u_2)_{z_2}\big) = (u_2)_{z_2} \otimes (u_1)_{z_1}.$$

Note that $\mathbf{k}(z_1, z_2) \otimes_{\mathbf{k}[z_1^{\pm 1}, z_2^{\pm 1}]} \left((M_1)_{z_1} \otimes (M_2)_{z_2} \right)$ is a simple $\mathbf{k}(z_1, z_2) \otimes U_q'(\mathfrak{g})$-module ([9, Proposition 9.5]). We call $R^{\mathrm{norm}}_{M_1, M_2}$ the *normalized R-matrix.*

Let $d_{M_1, M_2}(u) \in \mathbf{k}[u]$ be a monic polynomial of the smallest degree such that the image of $d_{M_1, M_2}(z_2/z_1) R^{\mathrm{norm}}_{M_1, M_2}$ is contained in $(M_2)_{z_2} \otimes (M_1)_{z_1}$. We call $d_{M_1, M_2}(u)$ the *denominator of* $R^{\mathrm{norm}}_{M_1, M_2}$. Then we have

$$(2.5) \qquad d_{M_1, M_2}(z_2/z_1) R^{\mathrm{norm}}_{M_1, M_2} : (M_1)_{z_1} \otimes (M_2)_{z_2} \longrightarrow (M_2)_{z_2} \otimes (M_1)_{z_1},$$

and the renormalized R-matrix

$$\mathbf{r}_{M_1, M_2} : M_1 \otimes M_2 \longrightarrow M_2 \otimes M_1$$

is equal to the specialization of $d_{M_1, M_2}(z_2/z_1) R^{\mathrm{norm}}_{M_1, M_2}$ at $z_1 = z_2 = 1$ up to a constant multiple.

Note that R^{univ} satisfies the following properties: the following diagrams commute

$$M_1 \otimes M_2 \otimes N \xrightarrow{M_1 \otimes R^{\mathrm{univ}}_{M_2, N}} M_1 \otimes N \otimes M_2 \xrightarrow{R^{\mathrm{univ}}_{M_1, N} \otimes M_2} N \otimes M_1 \otimes M_2,$$

with $R^{\mathrm{univ}}_{M_1 \otimes M_2, N}$ the composite arc,

$$M \otimes N_1 \otimes N_2 \xrightarrow{R^{\mathrm{univ}}_{M, N_1} \otimes N_2} N_1 \otimes M \otimes N_2 \xrightarrow{N_1 \otimes R^{\mathrm{univ}}_{M, N_2}} N_1 \otimes N_2 \otimes M$$

with $R^{\mathrm{univ}}_{M, N_1 \otimes N_2}$ the composite arc,

for M, M_1, M_2, N, N_1, N_2 in $U_q'(\mathfrak{g})$-mod. Hence, if $R^{\mathrm{univ}}_{(M_1)_{z_1}, N_{z_2}}$ and $R^{\mathrm{univ}}_{(M_2)_{z_1}, N_{z_2}}$ are rationally renormalizable, then $R^{\mathrm{univ}}_{(M_1 \otimes M_2)_{z_1}, N_{z_2}}$ is also rationally renormalizable. Moreover, we have

$$(2.6) \qquad \left(\mathbf{r}_{M_1, N} \otimes M_2 \right) \circ \left(M_1 \otimes \mathbf{r}_{M_2, N} \right) = c\, \mathbf{r}_{M_1 \otimes M_2, N} \quad \text{for some } c \in \mathbf{k}.$$

Note that c may vanish.

In particular, if M_1, M_2 and N are simple modules in $U_q'(\mathfrak{g})$-mod, then $R^{\mathrm{univ}}_{(M_1 \otimes M_2)_{z_1}, N_{z_2}}$ is rationally renormalizable.

3. Simple heads and socles of tensor products

In this section we give a proof of Conjecture in Introduction for the quiver Hecke algebra case and the quantum affine algebra case.

3.1. **Quiver Hecke algebra case.** We shall first discuss the quiver Hecke algebra case.

Lemma 3.1. *Let $\beta_k \in Q^+$ and $M_k \in R(\beta_k)$-mod ($k = 1, 2, 3$). Let X be an $R(\beta_1 + \beta_2)$-submodule of $M_1 \circ M_2$ and Y an $R(\beta_2 + \beta_3)$-submodule of $M_2 \circ M_3$ such that $X \circ M_3 \subset M_1 \circ Y$ as submodules of $M_1 \circ M_2 \circ M_3$. Then there exists an $R(\beta_2)$-submodule N of M_2 such that $X \subset M_1 \circ N$ and $N \circ M_3 \subset Y$.*

Proof. Set $n_k = \mathrm{ht}(\beta_k)$. Set $N = \{u \in M_2 \mid u \otimes M_3 \subset Y\}$. Then N is the largest $R(\beta_2)$-submodule of M_2 such that $N \circ M_3 \subset Y$. Let us show $X \subset M_1 \circ N$. Let us take a basis $\{v_a\}_{a \in A}$ of M_1.

By (1.3), we have
$$M_1 \circ M_2 = \bigoplus_{w \in \mathfrak{S}_{n_1, n_2}} \tau_w(M_1 \otimes M_2).$$

Hence, any $u \in X$ can be uniquely written as
$$u = \sum_{w \in \mathfrak{S}_{n_1, n_2}, a \in A} \tau_w(v_a \otimes u_{a,w})$$

with $u_{a,w} \in M_2$. Then, for any $s \in M_3$, we have
$$u \otimes s = \sum_{w \in \mathfrak{S}_{n_1, n_2}, a \in A} \tau_w(v_a \otimes u_{a,w} \otimes s) \in X \circ M_3 \subset M_1 \circ Y.$$

Since
$$M_1 \circ Y = \bigoplus_{w \in \mathfrak{S}_{n_1, n_2 + n_3}} \tau_w(M_1 \otimes Y)$$

and $\mathfrak{S}_{n_1, n_2} \subset \mathfrak{S}_{n_1, n_2 + n_3}$, we have
$$u_{a,w} \otimes s \in Y \quad \text{for any } a \in A \text{ and } w \in \mathfrak{S}_{n_1, n_2}.$$

Therefore we have $u_{a,w} \in N$. $\qquad \square$

Theorem 3.2. *Let $\beta, \gamma \in Q^+$ and $M \in R(\beta)$-mod and $N \in R(\gamma)$-mod. We assume further the following condition:*

(3.1)
      (a) *$R(\beta)$ is symmetric and $\mathbf{r}_{M,M} \in \mathbf{k}\,\mathrm{id}_{M \circ M}$,*
      (b) *M is non-zero,*
      (c) *N is a simple $R(\gamma)$-module.*

Then we have

  (i) *$M \circ N$ has a simple socle and a simple head. Similarly, $N \circ M$ has a simple socle and a simple head.*
 (ii) *Moreover, $\mathrm{Im}(\mathbf{r}_{N,M})$ is equal to the socle of $M \circ N$ and also equal to the head of $N \circ M$. Similarly, $\mathrm{Im}(\mathbf{r}_{M,N})$ is equal to the socle of $N \circ M$ and to the head of $M \circ N$.*

In particular, M is a simple module.

Proof. Let us show that $\mathrm{Im}(\mathbf{r}_{N,M})$ is a unique simple submodule of $M \circ N$. Let $S \subset M \circ N$ be an arbitrary non-zero $R(\beta+\gamma)$-submodule. Let m and m' be the multiplicity of zero of $R_{N,(M)_z} \colon N \circ (M)_z \to (M)_z \circ N$ and $R_{M,(M)_z} \colon M \circ (M)_z \to (M)_z \circ M$ at $z = 0$, respectively. Then by the definition, $\mathbf{r}_{N,M} = (z^{-m} R_{N,(M)_z})|_{z=0} \colon N \circ M \to M \circ N$ and $\mathbf{r}_{M,M} = (z^{-m'} R_{M,(M)_z})|_{z=0} \colon M \circ M \to M \circ M$. Now, we have a commutative diagram

$$
\begin{array}{ccc}
S \circ (M)_z & \xrightarrow{\;\; z^{-m-m'} R_{S,(M)_z} \;\;} & (M)_z \circ S \\
\downarrow & & \downarrow \\
M \circ N \circ (M)_z & \xrightarrow[\; M \circ z^{-m} R_{N,(M)_z} \;]{} M \circ (M)_z \circ N \xrightarrow[\; z^{-m'} R_{M,(M)_z} \circ N \;]{} & (M)_z \circ M \circ N.
\end{array}
$$

Therefore $z^{-m-m'} R_{S,(M)_z} \colon S \circ (M)_z \to (M)_z \circ S$ is well-defined, and we obtain the following commutative diagram by specializing the above diagram at $z = 0$.

$$
\begin{array}{ccc}
S \circ M & \xrightarrow{\hspace{4cm}} & M \circ S \\
\downarrow & & \downarrow \\
M \circ N \circ M & \xrightarrow[\; M \circ \mathbf{r}_{N,M} \;]{} M \circ M \circ N \xrightarrow[\; \mathrm{id}_{M \circ M \circ N} \;]{} & M \circ M \circ N.
\end{array}
$$

Here, we have used the assumption that $\mathbf{r}_{M,M}$ is equal to $\mathrm{id}_{M \circ M}$ up to a constant multiple.

Hence we obtain $\big(M \circ \mathbf{r}_{N,M}\big)(S \circ M) \subset M \circ S$, or equivalently

$$
S \circ M \subset M \circ (\mathbf{r}_{N,M})^{-1}(S).
$$

By the preceding lemma, there exists an $R(\gamma)$-submodule K of N such that $S \subset M \circ K$ and $K \circ M \subset (\mathbf{r}_{N,M})^{-1}(S)$. By the first inclusion, we have $K \neq 0$. Since N is simple, we have $K = N$ and we obtain $N \circ M \subset (\mathbf{r}_{N,M})^{-1}(S)$, or equivalently, $\mathrm{Im}(\mathbf{r}_{N,M}) \subset S$. Noting that S is an arbitrary non-zero submodule of $M \circ N$, we conclude that $\mathrm{Im}(\mathbf{r}_{N,M})$ is a unique simple submodule of $M \circ N$.

The proof of the other statements in (i) and (ii) is similar.

The simplicity of M follows from (i) and (ii) by taking the one-dimensional $R(0)$-module $\mathbf{k}$ as N. Note that $\mathbf{r}_{M,\mathbf{k}}$ and $\mathbf{r}_{\mathbf{k},M}$ coincide with the identity morphism id_M. $\quad\square$

A simple $R(\beta)$-module M is called *real* if $M \circ M$ is simple

Then the following corollary is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Assume that $R(\beta)$ is symmetric and M is a non-zero $R(\beta)$-module in $R(\beta)$-mod. Then the following conditions are equivalent:*

(a) *M is a real simple $R(\beta)$-module,*

(b) *$\mathbf{r}_{M,M} \in \mathbf{k}\,\mathrm{id}_{M \circ M}$,*

(c) $\mathrm{End}_{R(2\beta)}(M \circ M) \simeq \mathbf{k}\,\mathrm{id}_{M\circ M}$.

We have also the following corollary.

Corollary 3.4. *If $R(\beta)$ is symmetric and M is a real simple $R(\beta)$-module, then $M^{\circ n}:=$*
$$\overbrace{M \circ \cdots \circ M}^{n} \text{ is a simple } R(n\beta)\text{-module for any } n \geq 1.$$

Proof. The quiver Hecke algebra version of (2.6) implies that $\mathbf{r}_{M^{\circ m},M^{\circ n}}$ is equal to $\mathrm{id}_{M^{\circ(m+n)}}$ up to a constant multiple. $\square$

Thus we have established the first statement of Conjecture in the introduction in the quiver Hecke algebra case.

Lemma 3.5. *Let $\beta,\gamma \in Q^+$, and let $M \in R(\beta)$-mod and $L \in R(\beta+\gamma)$-mod. Then there exist $X, Y \in R(\gamma)$-mod satisfying the following universal properties:*

$$(3.2) \qquad\qquad \mathrm{Hom}_{R(\beta+\gamma)}(M \circ Z, L) \simeq \mathrm{Hom}_{R(\gamma)}(Z, X),$$

$$(3.3) \qquad\qquad \mathrm{Hom}_{R(\beta+\gamma)}(L, Z \circ M) \simeq \mathrm{Hom}_{R(\gamma)}(Y, Z)$$

functorially in $Z \in R(\gamma)$-mod.

Proof. Set $X = \mathrm{Hom}_{R(\beta+\gamma)}(M \circ R(\gamma), L)$. Then we have

$$\mathrm{Hom}_{R(\beta+\gamma)}(M \circ Z, L) \simeq \mathrm{Hom}_{R(\beta)\otimes R(\gamma)}(M \otimes Z, L)$$
$$\simeq \mathrm{Hom}_{R(\gamma)}\big(Z, \mathrm{Hom}_{R(\beta)}(M, L)\big).$$

Similarly set $Y = \Big(\mathrm{Hom}_{R(\beta+\gamma)}(M^* \circ R(\gamma), L^*)\Big)^*$. Then we have by using (1.4)

$$\mathrm{Hom}_{R(\beta+\gamma)}(L, Z \circ M) \simeq \mathrm{Hom}_{R(\beta+\gamma)}(M^* \circ Z^*, L^*)$$
$$\simeq \mathrm{Hom}_{R(\beta)\otimes R(\gamma)}(M^* \otimes Z^*, L^*)$$
$$\simeq \mathrm{Hom}_{R(\gamma)}(Z^*, Y^*) \simeq \mathrm{Hom}_{R(\gamma)}(Y, Z).$$

$\square$

Proposition 3.6. *Let $\beta,\gamma \in Q^+$. Assume that $R(\beta)$ is symmetric, and let M be a real simple module in $R(\beta)$-mod, and L a simple module in $R(\beta+\gamma)$-mod. Then the $R(\gamma)$-module $X := \mathrm{Hom}_{R(\beta+\gamma)}(M \circ R(\gamma), L)$ is either zero or has a simple socle.*

Proof. The $R(\gamma)$-module X satisfies the functorial property (3.2). Assume that $X \neq 0$. Let $p\colon M \circ X \to L$ be the canonical morphism. Since L is simple, it is an epimorphism. Let Y be as in Lemma 3.5, and let $i\colon L \to Y \circ M$ be the canonical morphism. For an arbitrary simple $R(\gamma)$-submodule S of X, since $\mathrm{Hom}_{R(\beta+\gamma)}(M\circ S, L) \simeq \mathrm{Hom}_{R(\gamma)}(S, X)$, the composition $M \circ S \to M \circ X \xrightarrow{p} L$ does not vanish. Hence, by Theorem 3.2, L is the simple head of $M \circ S$ and is the simple socle of $S \circ M$. Moreover, $L \cong \mathrm{Im}(\mathbf{r}_{M,S})$. Since

the monomorphism $L \to S \circ M$ factors through i by (3.3), the morphism $i \colon L \to Y \circ M$ is a monomorphism.

As in the proof of Theorem 3.2, we have a commutative diagram

$$
\begin{array}{ccc}
M \circ L & \longrightarrow & L \circ M \\
\downarrow{\scriptstyle M \circ i} & & \downarrow{\scriptstyle i \circ M} \\
M \circ Y \circ M & \xrightarrow{\ \mathbf{r}_{M,Y} \circ M\ } & Y \circ M \circ M\,.
\end{array}
$$

Then we obtain $M \circ i(L) \subset (\mathbf{r}_{M,Y})^{-1}(i(L)) \circ M$. Hence, by Lemma 3.1, there exists an $R(\gamma)$-submodule Z of Y such that $\mathbf{r}_{M,Y}(M \circ Z) \subset i(L)$ and $i(L) \subset Z \circ M$. The last inclusion induces a morphism $L \to Z \circ M$ and it induces a morphism $Y \to Z$ by (3.3). Since the composition $Y \to Z \to Y$ is the identity again by (3.3), we have $Z = Y$. Hence $\mathrm{Im}(\mathbf{r}_{M,Y}) \subset i(L)$, which gives the commutative diagram

$$
\begin{array}{ccc}
 & \xrightarrow{\ \ \ \mathbf{r}_{M,Y}\ \ \ } & \\
M \circ Y \longrightarrow L \rightarrowtail & \xrightarrow[\ i\ ]{} & Y \circ M\,.
\end{array}
$$

By the argument dual to the above one (also see the proof of Proposition 3.8), we have a commutative diagram

$$
\begin{array}{ccc}
 & \xrightarrow{\ \ \ \mathbf{r}_{M,X}\ \ \ } & \\
M \circ X \xtwoheadrightarrow{\ p\ } L & \xrightarrow{\ \xi\ } & X \circ M\,.
\end{array}
$$

Hence $\xi \colon L \to X \circ M$ is a monomorphism, and $\mathrm{Im}(\mathbf{r}_{M,X})$ is isomorphic to L. By (3.3), there exists a unique morphism $\varphi \colon Y \to X$ such that ξ factors as

$$
\begin{array}{ccc}
 & \xrightarrow{\ \ \ \xi\ \ \ } & \\
L \rightarrowtail \xrightarrow{\ i\ } Y \circ M & \xrightarrow{\ \varphi \circ M\ } & X \circ M.
\end{array}
$$

Let us show that $\mathrm{Im}(\varphi)$ is a unique simple submodule of X. In order to see this, let S be an arbitrary simple $R(\beta)$-submodule of X. We have seen that L is isomorphic to the head of $M \circ S$ and isomorphic to $\mathrm{Im}(\mathbf{r}_{M,S})$. Since the composition $M \circ S \to M \circ X \xrightarrow{\mathbf{r}_{M,X}} X \circ M$ does not vanish, we have a commutative diagram by [7, Lemma 1.4.8]

$$
\begin{array}{ccc}
M \circ S & \xrightarrow{\ \mathbf{r}_{M,S}\ } & S \circ M \\
\downarrow & & \downarrow \\
M \circ X & \xrightarrow{\ \mathbf{r}_{M,X}\ } & X \circ M.
\end{array}
$$

Since $\mathrm{Im}(\mathbf{r}_{M,S}) \simeq \mathrm{Im}(\mathbf{r}_{M,X}) \simeq L$, the morphism $\xi \colon L \to X \circ M$ factors as $L \to S \circ M \to X \circ M$. Hence (3.3) implies that $\varphi \colon Y \to X$ factors through $Y \to S \to X$. Thus we obtain $\mathrm{Im}(\varphi) \subset S$. Since S is an arbitrary simple submodule of X, we conclude that $\mathrm{Im}(\varphi)$ is a unique simple submodule of X. $\qquad\square$

Let $\beta, \gamma \in \mathsf{Q}^+$. For a simple $R(\beta)$-module M and a simple $R(\gamma)$-module N, let us denote by $M \nabla N$ the head of $M \circ N$.

Corollary 3.7. *Let $\beta, \gamma \in \mathsf{Q}^+$. Assume that $R(\beta)$ is symmetric, and let M be a real simple module in $R(\beta)$-mod. Then, the map $N \mapsto M \nabla N$ is injective from the set of the isomorphism classes of simple objects of $R(\gamma)$-mod to the set of the isomorphism classes of simple objects of $R(\beta + \gamma)$-mod.*

Proof. Indeed, for a simple $R(\gamma)$-module N, $M \nabla N$ is a simple $R(\beta + \gamma)$-module by Theorem 3.2, and $N \subset X := \mathrm{Hom}_{R(\beta+\gamma)}(M \circ R(\gamma), M \nabla N)$ is the socle of X by the preceding proposition. $\qquad\square$

If $L(i)$ is the one-dimensional simple $R(\alpha_i)$-module, then $L(i)$ is real and $M \nabla L(i)$ corresponds to the crystal operator $\tilde{f}_i M$ and $L(i) \nabla M$ to the dual crystal operator $\tilde{f}_i^{\vee} M$ in [12]. Hence, ∇ is a generalization of the crystal operator as suggested in [13].

Proposition 3.8. *Let $\beta, \gamma \in \mathsf{Q}^+$. Assume that $R(\beta)$ is symmetric, and let M be a real simple module in $R(\beta)$-mod, and N a simple module in $R(\gamma)$-mod. Then we have $\mathrm{End}_{R(\beta+\gamma)}(M \circ N) \simeq \mathbf{k}\,\mathrm{id}_{M \circ N}$.*

Proof. Set $L = M \circ N$. Let $X, Y \in R(\gamma)$-mod be as in Lemma 3.5. Let $p \colon M \circ X \to L$ and $i \colon L \to Y \circ M$ be the canonical morphisms. Then the isomorphism $M \circ N \to L$ induces a morphism $j \colon N \to X$ such that the composition $M \circ N \xrightarrow{M \circ j} M \circ X \xrightarrow{p} L$ is that isomorphism. Hence $p \colon M \circ X \to L$ is an epimorphism. Since N is simple and j does not vanish, the morphism $j \colon N \to X$ is a monomorphism.

We have a commutative diagram

$$
\begin{array}{ccccc}
M \circ M \circ X & \xrightarrow{\ \mathbf{r}_{M,M} \circ X\ } & M \circ M \circ X & \xrightarrow{\ M \circ \mathbf{r}_{M,X}\ } & M \circ X \circ M \\
\ \downarrow{\scriptstyle M \circ p} & & & & \ \downarrow{\scriptstyle p \circ M} \\
M \circ L & \xrightarrow{\hspace{6cm}} & & & L \circ M.
\end{array}
$$

Since $\mathbf{r}_{M,M}$ is $\mathrm{id}_{M \circ M}$ up to a constant multiple, we obtain a commutative diagram:

$$
\begin{array}{ccc}
M \circ (M \circ X) & \xrightarrow{\ M \circ \mathbf{r}_{M,X}\ } & M \circ X \circ M \\
\ \downarrow{\scriptstyle M \circ p} & & \ \downarrow{\scriptstyle p \circ M} \\
M \circ L & \xrightarrow{\hspace{3cm}} & L \circ M.
\end{array}
$$

Therefore we have

$$M \circ \big(\mathbf{r}_{M,X}(\mathrm{Ker}\, p)\big) \subset (\mathrm{Ker}\, p) \circ M.$$

Hence Lemma 3.1 implies that there exists $Z \subset X$ such that $\mathbf{r}_{M,X}(\mathrm{Ker}\, p) \subset Z \circ M$ and $M \circ Z \subset \mathrm{Ker}\, p$. The last inclusion shows that $M \circ Z \to M \circ X \to L$ vanishes. Hence by (3.2), the morphism $Z \to X$ vanishes, or equivalently, $Z = 0$. Hence we have $\mathbf{r}_{M,X}(\mathrm{Ker}\, p) = 0$. Therefore $\mathbf{r}_{M,X}$ factors through p:

$$
\begin{array}{ccc}
& \mathbf{r}_{M,X} & \\
M \circ X \xrightarrow{\quad p \quad} L & \xrightarrow{\quad \xi \quad} & X \circ M \,.
\end{array}
$$

Since $\mathbf{r}_{M,X} \neq 0$, the morphism ξ does not vanish. By (3.3), there exists $\varphi \colon Y \to X$ such that $\xi \colon L \to X \circ M$ coincides with the composition $L \xrightarrow{i} Y \circ M \xrightarrow{\varphi \circ M} X \circ M$. Then we have a commutative diagram with the solid arrows:

$$
\begin{array}{ccc}
M \circ N & \xrightarrow{\ \mathbf{r}_{M,N}\ } & N \circ M \\
\downarrow{\scriptstyle M \circ j} & \quad L \quad & \downarrow{\scriptstyle j \circ M} \\
M \circ X & \xrightarrow{\ \mathbf{r}_{M,X}\ } & X \circ M.
\end{array}
$$

Indeed, the commutativity follows from [7, Lemma 1.4.8] and the fact that the composition $M \circ N \xrightarrow{M \circ j} M \circ X \xrightarrow{\mathbf{r}_{M,X}} X \circ M$ does not vanish because it coincides with $M \circ N \xrightarrow{\sim} L \xrightarrow{\xi} X \circ M$.

Thus $\xi \colon L \to X \circ M$ coincides with the composition

$$L \simeq M \circ N \xrightarrow{\ \mathbf{r}_{M,N}\ } N \circ M \xrightarrow{\ j \circ M\ } X \circ M.$$

Hence (3.3) implies that $\varphi \colon Y \to X$ decomposes as

$$Y \xrightarrow{\ \psi\ } N \rightarrowtail\xrightarrow{\ j\ } X.$$

Since N is simple, ψ is an epimorphism, and we conclude that N is the image of $\varphi \colon Y \to X$.

Now let us prove that any $f \in \mathrm{End}_{R(\beta+\gamma)}(L)$ satisfies $f \in \mathbf{k}\,\mathrm{id}_L$. By the universal properties (3.2) and (3.3), the endomorphism f induces endomorphisms $f_X \in$

$\operatorname{End}_{R(\gamma)}(X)$ and $f_Y \in \operatorname{End}_{R(\gamma)}(Y)$ such that the diagrams with the solid arrows

(3.4)

$$
\begin{array}{ccccc}
M \circ X & \xrightarrow{\ \ p\ \ } & L & \cdots\xrightarrow{\ \xi\ } & X \circ M \\
\Big\downarrow{\scriptstyle M\circ f_X} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f_X\circ M} \\
M \circ X & \xrightarrow{\ \ p\ \ } & L & \cdots\xrightarrow{\ \xi\ } & X \circ M
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
L & \xrightarrow{\ i\ } & Y \circ M \\
\Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f_Y\circ M} \\
L & \xrightarrow{\ i\ } & Y \circ M
\end{array}
$$

with $\mathbf{r}_{M,X}$ arrows above and below.

commute. Since $\mathbf{r}_{M,X}$ commutes with f, the left diagram with dotted arrows commutes. Hence, the following diagram with the solid arrows

(3.5)

$$
\begin{array}{ccccc}
Y & \xrightarrow{\ \psi\ } & N & \xrightarrow{\ j\ } & X \\
\Big\downarrow{\scriptstyle f_Y} & & \Big\downarrow{\scriptstyle f_N} & & \Big\downarrow{\scriptstyle f_X} \\
Y & \xrightarrow{\ \psi\ } & N & \xrightarrow{\ j\ } & X
\end{array}
$$
with φ arrows above and below.

commutes. Then we can add the dotted arrow f_N so that the whole diagram (3.5) commutes. Since N is simple, we have $f_N = c\,\mathrm{id}_N$ for some $c \in \mathbf{k}$. By replacing f with $f - c\,\mathrm{id}_L$, we may assume from the beginning that $f_N = 0$. Then $f_X \circ j = 0$. Now, $f = 0$ follows from the commutativity of the diagram

$$
\begin{array}{ccccc}
M \circ N & \xrightarrow{\ M\circ j\ } & M \circ X & \xrightarrow{\ p\ } & L \\
& {\scriptstyle 0}\searrow & \Big\downarrow{\scriptstyle M\circ f_X} & & \Big\downarrow{\scriptstyle f} \\
& & M \circ X & \xrightarrow{\ p\ } & L.
\end{array}
$$
with $\sim$ arrow above.

$\square$

Corollary 3.9. *Let $\beta, \gamma \in \mathsf{Q}^+$, and assume that $R(\beta)$ is symmetric. Let M be a real simple module in $R(\beta)$-mod, and N a simple module in $R(\gamma)$-mod.*

 (i) *If the head of $M \circ N$ and the socle of $M \circ N$ are isomorphic, then $M \circ N$ is simple and $M \circ N \simeq N \circ M$.*

 (ii) *If $M \circ N \simeq N \circ M$, then $M \circ N$ is simple. Conversely, if $M \circ N$ is simple, then $M \circ N \simeq N \circ M$.*

Proof. (i) Let S be the head of $M \circ N$ and the socle of $M \circ N$. Then S is simple. Now we have the morphisms

$$M \circ N \twoheadrightarrow S \rightarrowtail M \circ N.$$

By the preceding proposition, the composition is equal to $\mathrm{id}_{M \circ N}$ up to a constant multiple. Hence $M \circ N$ and $N \circ M$ are isomorphic to S.

(ii) Assume first $M \circ N \simeq N \circ M$. Then the simplicity of $M \circ N$ immediately follows from (i) because the socle of $M \circ N$ is isomorphic to the head of $N \circ M$ by Theorem 3.2.

If $M \circ N$ is simple, then $\mathbf{r}_{M,N}$ is injective. Since $\dim(M \circ N) = \dim(N \circ M)$, $\mathbf{r}_{M,N} \colon M \circ N \to N \circ M$ is an isomorphism. $\qquad\square$

Note that, when $R(\beta)$ and $R(\gamma)$ are symmetric, for a real simple $R(\beta)$-module M and a real simple $R(\gamma)$-module N, their convolution $M \circ N$ is real simple if $M \circ N \simeq N \circ M$.

3.2. **Quantum affine algebra case.** The similar results to Theorem 3.2, Corollary 3.7 and Corollary 3.9 hold also for quantum affine algebras. Let $U_q'(\mathfrak{g})$ be the quantum affine algebra as in § 2. Recall that $U_q'(\mathfrak{g})$-mod denotes the category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules.

First note that the following lemma, an analogue of Lemma 3.1 in the quantum affine algebra case, is almost trivial. Indeed, the similar result holds for any rigid monoidal category which is abelian and the tensor functor is additive.

Lemma 3.10. *Let M_k be a module in $U_q'(\mathfrak{g})$-mod ($k = 1, 2, 3$). Let X be a $U_q'(\mathfrak{g})$-submodule of $M_1 \otimes M_2$ and Y a $U_q'(\mathfrak{g})$-submodule of $M_2 \otimes M_3$ such that $X \otimes M_3 \subset M_1 \otimes Y$ as submodules of $M_1 \otimes M_2 \otimes M_3$. Then there exists a $U_q'(\mathfrak{g})$-submodule N of M_2 such that $X \subset M_1 \otimes N$ and $N \otimes M_3 \subset Y$.*

Corollary 3.11.

(i) *Let M_k be a module in $U_q'(\mathfrak{g})$-mod ($k = 1, 2, 3$), and let $\varphi_1 \colon L \to M_1 \otimes M_2$ and $\varphi_2 \colon M_2 \otimes M_3 \to L'$ be non-zero morphisms. Assume further that M_2 is a simple module. Then the composition*

(3.6) $$L \otimes M_3 \xrightarrow{\varphi_1 \otimes M_3} M_1 \otimes M_2 \otimes M_3 \xrightarrow{M_1 \otimes \varphi_2} M_1 \otimes L'$$

does not vanish.

(ii) *Let M, N_1 and N_2 be simple modules in $U_q'(\mathfrak{g})$-mod. Then the following diagram commutes up to a constant multiple:*

$$
\begin{array}{ccccc}
& & \mathbf{r}_{M, N_1 \otimes N_2} & & \\
M \otimes N_1 \otimes N_2 & \xrightarrow{\ \mathbf{r}_{M,N_1} \otimes N_2\ } & N_1 \otimes M \otimes N_2 & \xrightarrow{\ N_1 \otimes \mathbf{r}_{M,N_2}\ } & N_1 \otimes N_2 \otimes M.
\end{array}
$$

Proof. (i) Assume that the composition (3.6) vanishes. Then we have $\operatorname{Im} \varphi_1 \otimes M_3 \subset M_1 \otimes \operatorname{Ker} \varphi_2$. Hence, by the preceding lemma, there exists $N \subset M_2$ such that $\operatorname{Im} \varphi_1 \subset M_1 \otimes N$ and $N \otimes M_3 \subset \operatorname{Ker} \varphi_2$. The first inclusion implies $N \neq 0$ and the last inclusion implies $N \neq M_2$. It contradicts the simplicity of M_2.

(ii) By (i) $(N_1 \otimes \mathbf{r}_{M,N_2}) \circ (\mathbf{r}_{M,N_1} \otimes N_2)$ does not vanish. Hence it is equal to $\mathbf{r}_{M, N_1 \otimes N_2}$ up to a constant multiple by (2.6). $\qquad\square$

Since the proof of the following theorem is similar to the quiver Hecke algebra case, we just state the result omitting its proof.

Theorem 3.12. *Let M and N be simple modules in $U_q'(\mathfrak{g})$-mod. We assume further*

$$(3.7) \qquad \mathbf{r}_{M,M} \in \mathbf{k}\,\mathrm{id}_{M \otimes M}.$$

Then we have

(i) *$M \otimes N$ has a simple socle and a simple head.*
(ii) *Moreover, $\mathrm{Im}(\mathbf{r}_{M,N})$ is equal to the head of $M \otimes N$ and is also equal to the socle of $N \otimes M$.*

Recall that a simple $U_q'(\mathfrak{g})$-module M is called *real* if $M \otimes M$ is simple. Hence M in Theorem 3.12 is real.

For a module M in $U_q'(\mathfrak{g})$-mod, let us denote by $^*M$ and M^* the right dual and the left dual of M, respectively. Hence we have isomorphisms

$$
\begin{aligned}
(3.8) \qquad
& \mathrm{Hom}_{U_q'(\mathfrak{g})}(M \otimes X, Y) \simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}(X, {}^*M \otimes Y), \\
& \mathrm{Hom}_{U_q'(\mathfrak{g})}(X \otimes {}^*M, Y) \simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}(X, Y \otimes M), \\
& \mathrm{Hom}_{U_q'(\mathfrak{g})}(M^* \otimes X, Y) \simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}(X, M \otimes Y), \\
& \mathrm{Hom}_{U_q'(\mathfrak{g})}(X \otimes M, Y) \simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}(X, Y \otimes M^*)
\end{aligned}
$$

functorial in $X, Y \in U_q'(\mathfrak{g})$-mod.

Corollary 3.13. *Under the assumption of the theorem above, the head of $\mathrm{Im}\,\mathbf{r}_{M,N} \otimes {}^*M$ is isomorphic to N.*

Proof. Set $S = \mathrm{Im}\,\mathbf{r}_{M,N}$. Since $\mathrm{Hom}_{U_q'(\mathfrak{g})}(S, N \otimes M) \simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}(S \otimes {}^*M, N)$, there exists a non-trivial morphism $S \otimes {}^*M \to N$. Since N is simple, we have an epimorphism

$$S \otimes {}^*M \twoheadrightarrow N.$$

Since $^*M \otimes {}^*M \simeq {}^*(M \otimes M)$ is a simple module, the tensor product $S \otimes {}^*M$ has a simple head by the preceding theorem. Hence, we obtain the desired result. $\qquad\square$

For simple $U_q'(\mathfrak{g})$-modules M and N, let us denote by $M \nabla N$ the head of $M \otimes N$.

Corollary 3.14. *Let M be a real simple module in $U_q'(\mathfrak{g})$-mod. Then, the map $N \mapsto M \nabla N$ is bijective on the set of the isomorphism classes of simple $U_q'(\mathfrak{g})$-modules in $U_q'(\mathfrak{g})$-mod, and its inverse is given by $N \mapsto N \nabla {}^*M$.*

Lemma 3.15. *Let M be a real simple module in $U_q'(\mathfrak{g})$-mod and N a simple module in $U_q'(\mathfrak{g})$-mod. Then we have $\mathrm{End}_{U_q'(\mathfrak{g})}(M \otimes N) \simeq \mathbf{k}\,\mathrm{id}_{M \otimes N}$.*

Proof. By Corollary 3.11, we have a commutative diagram up to a constant multiple

$$
\begin{array}{ccc}
M^* \otimes M \otimes N \xrightarrow{\ \mathbf{r}_{M^*,M} \otimes N\ } M \otimes M^* \otimes N \xrightarrow{\ M \otimes \mathbf{r}_{M^*,N}\ } M \otimes N \otimes M^*.
\end{array}
$$
(with upper curved arrow $\mathbf{r}_{M^*, M \otimes N}$)

By Theorem 3.12, $\mathrm{Im}(\mathbf{r}_{M^*,M})$ is the simple socle of $M \otimes M^*$, and hence $\mathbf{r}_{M^*,M}$ is equal to the composition

$$
M^* \otimes M \xrightarrow{\ \varepsilon\ } \mathbf{1} \longrightarrow M \otimes M^*
$$

up to a constant multiple. Here $\mathbf{1}$ denotes the trivial representation of $U_q'(\mathfrak{g})$. Hence we have a commutative diagram up to a constant multiple

$$
M^* \otimes M \otimes N \xrightarrow{\ \varepsilon \otimes N\ } N \rightarrowtail M \otimes N \otimes M^*.
$$
(with upper curved arrow $\mathbf{r}_{M^*, M \otimes N}$)

Let $f \in \mathrm{End}_{U_q'(\mathfrak{g})}(M \otimes N)$. Let us show that $f \in \mathbf{k}\,\mathrm{id}_{M \otimes N}$. Since $\mathbf{r}_{M^*,M \otimes N}$ commutes with f, the following diagram with the solid arrows

(3.9)
$$
\begin{array}{ccc}
M^* \otimes M \otimes N & \twoheadrightarrow & N \rightarrowtail M \otimes N \otimes M^* \\
\downarrow{\scriptstyle M^* \otimes f} & \quad \downarrow{\scriptstyle f_N} & \quad \downarrow{\scriptstyle f \otimes M^*} \\
M^* \otimes M \otimes N & \xrightarrow{\ \varepsilon \otimes N\ } & N \rightarrowtail M \otimes N \otimes M^*
\end{array}
$$

is commutative. Hence we can add the dotted arrow f_N so that the whole diagram (3.9) commutes. Since N is simple, we have $f_N = c\,\mathrm{id}_N$ for some $c \in \mathbf{k}$. Then by replacing f with $f - c\,\mathrm{id}_{M \otimes N}$, we may assume from the beginning that $f_N = 0$. Hence the composition

$$
M^* \otimes M \otimes N \xrightarrow{\ M^* \otimes f\ } M^* \otimes M \otimes N \xrightarrow{\ \varepsilon \otimes N\ } N
$$

vanishes. Therefore (3.8) implies that $M \otimes N \xrightarrow{\ f\ } M \otimes N$ vanishes. $\qquad\square$

Corollary 3.16. *Let M be a real simple module in $U_q'(\mathfrak{g})$-mod, and N a simple module in $U_q'(\mathfrak{g})$-mod.*

(i) *If the head of $M \otimes N$ and the socle of $M \otimes N$ are isomorphic, then $M \otimes N$ is simple and $M \otimes N \simeq N \otimes M$.*

(ii) *If $M \otimes N \simeq N \otimes M$, then $M \otimes N$ is simple.*

This corollary follows from the preceding lemma by an argument similar to the one in the proof of Corollary 3.9.

4. Proof of (1.11)

We shall show (1.11). We keep the notations in Section 1. We set

$$\tilde{x}_{a,b} = \sum_{\substack{\nu \in I^{\beta+\gamma}, \\ \nu_a, \nu_b \in \text{supp}(\beta) \cap \text{supp}(\gamma)}} (x_a - x_b) e(\nu) \quad \text{and} \quad \widetilde{\tau}_c = \sum_{\substack{\nu \in I^{\beta+\gamma}, \\ \nu_c \in \text{supp}(\gamma), \nu_{c+1} \in \text{supp}(\beta)}} \tau_c e(\nu)$$

for $1 \le a, b \le m+n$ and $1 \le c < m+n$. They are elements of $R(\beta+\gamma)$.

We denote by A the commutative subalgebra of $R(\beta+\gamma)$ generated by $\tilde{x}_{a,b}$ and $e(\nu)$ where $1 \le a < b \le m+n$ and $\nu \in I^{\beta+\gamma}$. Let us denote by $\widetilde{R}_{\gamma,\beta}$ the subalgebra of $R(\beta+\gamma)$ generated by A and $\widetilde{\tau}_c$ where $1 \le c < m+n$.

Then $\varphi_{w[n,m]} e(\gamma, \beta)$ belongs to $\widetilde{R}_{\gamma,\beta}$.

These generators satisfy the following commutation relations:

$$(4.1) \quad \begin{cases} \tilde{x}_{a,b}\widetilde{\tau}_c - \widetilde{\tau}_c \tilde{x}_{s_c(a),s_c(b)} \\ \quad = \displaystyle\sum_{\nu_c = \nu_{c+1} \in \text{supp}(\beta) \cap \text{supp}(\gamma)} \big(\delta(a = c+1) - \delta(a = c) - \delta(b = c+1) + \delta(b = c) \big) e(\nu), \\[2mm] \widetilde{\tau}_a^2 = \displaystyle\sum_{\nu_a, \nu_{a+1} \in \text{supp}(\beta) \cap \text{supp}(\gamma)} Q_{\nu_a, \nu_{a+1}}(x_a, x_{a+1}) e(\nu), \\[2mm] \widetilde{\tau}_a \widetilde{\tau}_b - \widetilde{\tau}_b \widetilde{\tau}_a = 0 \quad \text{if } |a - b| > 1, \\[2mm] \widetilde{\tau}_{a+1} \widetilde{\tau}_a \widetilde{\tau}_{a+1} - \widetilde{\tau}_a \widetilde{\tau}_{a+1} \widetilde{\tau}_a \\ \quad = \displaystyle\sum_{\nu_a, \nu_{a+1} \in \text{supp}(\beta) \cap \text{supp}(\gamma), \, \nu_a = \nu_{a+2}} \overline{Q}_{\nu_a, \nu_{a+1}}(x_a, x_{a+1}, x_{a+2}) e(\nu). \end{cases}$$

Indeed, the last equality follows from

$$\widetilde{\tau}_{a+1} \widetilde{\tau}_a \widetilde{\tau}_{a+1} = \sum_\nu \tau_{a+1} \tau_a \tau_{a+1} \, e(\nu) \quad \text{and}$$

$$\widetilde{\tau}_a \widetilde{\tau}_{a+1} \widetilde{\tau}_a = \sum_\nu \tau_a \tau_{a+1} \tau_a \, e(\nu).$$

Here the sums in the both formulas range over $\nu \in I^{\beta+\gamma}$ satisfying the conditions: $\nu_a \in \text{supp}(\gamma)$, $\nu_{a+1} \in \text{supp}(\beta) \cap \text{supp}(\gamma)$, and $\nu_{a+2} \in \text{supp}(\beta)$.

Note that the error terms (i.e., the right hand sides of the equalities in (4.1)) belong to the algebra A because we assume that $R(\beta)$ and $R(\gamma)$ are symmetric. Hence we have

$$(4.2) \quad \begin{cases} \tilde{x}_{a,b}\widetilde{\tau}_c - \widetilde{\tau}_c \tilde{x}_{s_c(a),s_c(b)} \in A \\ \widetilde{\tau}_a^2 \in A, \\ \widetilde{\tau}_a \widetilde{\tau}_b = \widetilde{\tau}_b \widetilde{\tau}_a \quad \text{if } |a - b| > 1, \\ \widetilde{\tau}_{a+1} \widetilde{\tau}_a \widetilde{\tau}_{a+1} - \widetilde{\tau}_a \widetilde{\tau}_{a+1} \widetilde{\tau}_a \in A. \end{cases}$$

Now for each element $w \in \mathfrak{S}_{m+n}$ let us choose a reduced expression $w = s_{a_1} \cdots s_{a_\ell}$. We then set

$$\widetilde{\tau}_w = \widetilde{\tau}_{a_1} \cdots \widetilde{\tau}_{a_\ell}.$$

Then, similarly to a proof of the PBW decomposition (1.1) (e.g., see [10, 14]), the commutation relations (4.2) imply

$$\widetilde{R}_{\gamma,\beta} = \sum_{w \in \mathfrak{S}_{m+n}} \widetilde{\tau}_w A.$$

In particular, we obtain

$$\widetilde{R}_{\gamma,\beta} \subset \bigoplus_{\substack{w \in \mathfrak{S}_{n,m}, \\ w_1 \in \mathfrak{S}_n, w_2 \in \mathfrak{S}_m}} \tau_w \big(\tau_{w_1} \otimes \tau_{w_2} \big) A.$$

It immediately implies (1.11), because we have, for $1 \le a < b \le m+n$, $\nu \in I^\gamma$, $\mu \in I^\beta$, $v \in e(\nu)N$ and $u \in e(\mu)M$,

$$\tilde{x}_{a,b}\big((v)_{z_2} \otimes (u)_{z_1} \big)$$

$$= \begin{cases} \big((x_a - x_b)v \big)_{z_2} \otimes (u)_{z_1} & \text{if } 1 \le a < b \le n \text{ and } \nu_a, \nu_b \in \operatorname{supp}(\beta), \\[2mm] (z_2 - z_1)\big((v)_{z_2} \otimes (u)_{z_1} \big) + (x_a v)_{z_2} \otimes (u)_{z_1} - (v)_{z_2} \otimes (x_{b-n}u)_{z_1} \\ \hspace{2cm} \text{if } 1 \le a \le n < b \le m+n \text{ and} \\ \hspace{4cm} \nu_a \in \operatorname{supp}(\beta), \mu_{b-n} \in \operatorname{supp}(\gamma), \\[2mm] (v)_{z_2} \otimes \big((x_{a-n} - x_{b-n})u \big)_{z_1} & \text{if } n < a < b \le m+n \text{ and } \mu_{a-n}, \mu_{b-n} \in \operatorname{supp}(\gamma), \\[2mm] 0 & \text{otherwise,} \end{cases}$$

and $\big(\tau_{w_1} \otimes \tau_{w_2} \big) \big((v)_{z_2} \otimes (u)_{z_1} \big) = \big(\tau_{w_1} v \big)_{z_2} \otimes \big(\tau_{w_2} u \big)_{z_1}$.

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