THE FLAG MANIFOLD OF KAC-MOODY LIE ALGEBRA

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0. Introduction. In this paper, we shall construct the flag variety of a Kac-Moody Lie algebra as an infinite-dimensional scheme. There are several constructions by Kac-Peterson ([K-P]), Kazhdan-Lusztig ([K-L]), S. Kumar ([Ku]), O. Mathieu ([M]), P. Slodowy ([S]), J. Tits ([T]), but there the flag variety is understood as a union of finite-dimensional varieties.

We give here two methods of construction of the flag variety. For a Kac-Moody Lie algebra g, let $\hat{g}$ be the completion of g. The first construction is to realize the flag variety as a subscheme of $\mathrm{Grass}(\hat{g})$, the Grassmann variety of $\hat{g}$. More precisely, taking the Borel subalgebra $b_- \subset \hat{g}$ and regarding this as a point of $\mathrm{Grass}(\hat{g})$, we define the flag variety as its orbit by the infinitesimal action of $\hat{g}$ in $\mathrm{Grass}(\hat{g})$.

The other construction is to realize the flag variety as G/B_-. Of course, in the Kac-Moody Lie algebra case, we cannot expect that there is a group scheme whose Lie algebra is g. But we can construct a scheme G on which g acts infinitesimally from the left and the right. Then we define the flag variety G/B_-, where B_- is the Borel subgroup. More precisely, we consider the ring of regular functions as in [K-P]. Then its spectrum admits an infinitesimal action of g. But its action is not locally free. Roughly speaking, G is the open subscheme where g acts locally freely (Proposition 6.3.1).

The flag variety of a Kac-Moody algebra shares the similar properties to the finite-dimensional ones, such as Bruhat decompositions.

I would like to acknowledge mathematicians I saw at Tata Institute, especially S. Kumar, Moody, Verma. I also thank for hospitalities of the staffs in The Johns Hopkins University during my preparing this article.

Manuscript received 5 December 1988.

1. Scheme of countable type.

1.1. In this paper, we treat infinite-dimensional schemes such as $\mathbf{A}^\infty$, $\mathbf{P}^\infty$, etc.. We shall discuss their local properties briefly.

Let k be a commutative ring.

Definition 1.1.1. *A k-algebra A is called of countable type over k, if A is generated by k and countable numbers of elements.*

The following is easily proven just as in EGA.

LEMMA 1.1.2. *Let X be a scheme over k. Assume that there is an open affine covering $X = \bigcup U_j$ of X such that $\Gamma(U_j; \mathcal{O}_X)$ is of countable type. Then, for any open affine subset U of X, $\Gamma(U; \mathcal{O}_X)$ is of countable type.*

Definition 1.1.3. *A scheme X over k is called of countable type if for any open affine subset U of X, $\Gamma(U; \mathcal{O}_X)$ is a k-algebra of countable type.*

LEMMA 1.1.4. *Let k be a noetherian ring. Then any ideal of a k-algebra A of countable type is generated by countable elements.*

Proof. Assume A is generated by x_i $(i = 1, 2, \ldots)$. Then for any ideal I of A, $I \cap k[x_1, \ldots, x_n]$ is generated by finitely many elements.

LEMMA 1.1.5. *Let k be an algebraically closed field such that k is not a countable set, and let X be a k-scheme of countable type. If X has no k-valued point, then X is empty.*

Proof. We may assume $X = \operatorname{Spec}(A)$ and $A \cong k[T_n;\, n \in \mathbf{N}]/I$, where T_n are indeterminates. Then I is generated by countably many elements f_j. Let k' be the subring of k generated by the coefficients of the f_j. Set $A' = k'[T_n;\, n \in \mathbf{Z}]/I'$ where I' is the ideal generated by f_j. Then $A \cong k \otimes_{k'} A'$. If $A \neq 0$, there is a homomorphism $A' \to K'$ from A' to a field K'. We may assume K' is generated by the image of A' as a field. Then K' has at most countable transcendental dimension over the prime field. Hence $k' \to k$ splits $k' \to K' \xrightarrow{\varphi} k$ for some φ. Therefore X has a k-valued point.

PROPOSITION 1.1.6. *Let k be a noetherian ring, and $A \cong \varinjlim_n A_n$, where $\{A_n\}_{n\in\mathbf{N}}$ is an inductive system of k-algebra of finite type and $A_n \to A_{n+1}$ is flat. Then $\mathcal{O}_{\operatorname{Spec}(A)}$ is a coherent ring.*

Proof. Any homomorphism $\varphi : A^{\oplus m} \to A$ comes from some φ' :

$A^{\oplus m} \to A_n$. Then Ker φ' is finitely generated over A_n and hence Ker $\varphi \cong A \otimes_{A_n}$ Ker φ' is also finitely generated over A.

Let us give an example.

Example 1.1.7. *Infinite-dimensional affine space*: A^∞ = Spec $k[X_i; i \in \mathbf{N}]$. The set of k-valued points of A^∞ is $\{(x_i)_{i \in \mathbf{N}}; x_i \in k\}$. The structure ring is coherent by Proposition 1.1.6, since $k[X_i; i \in \mathbf{N}] = \bigcup_m k[X_1, \ldots, X_m]$.

2. Grassmann variety.

2.1. Let k be a base field.

Definition 2.1.1. *An l.c. k-vector space V is a k-vector space with a topology satisfying*

(i) *The addition map* $V \times V \to V$ *is continuous.*
(ii) *V is Hausdorff and complete.*
(iii) *The open k-vector subspaces form a neighborhood system of* 0.

Let V_1 and V_2 be two *l.c.* vector spaces. We set

$$V_1 \hat{\otimes} V_2 = \varprojlim_{U_1, U_2} (V_1/U_1) \otimes (V_2/U_2) \tag{2.1.1}$$

where U_j rangs over open linear subspaces of 0 in V_j ($j = 1, 2$). We endow $V_1 \hat{\otimes} V_2$ with the structure of *l.c.* vector space such that $\mathrm{Ker}(V_1 \hat{\otimes} V_2 \to (V_1/U_1) \otimes (V_2/U_2))$ form a neighborhood system of 0.

Definition 2.1.2. *An l.c. k-vector space V is called a c.l.c. k-vector space if V is an l.c. k-vector space and it satisfies furthermore*

(iv) *There is a decreasing sequence* $\{W_n\}_{n \in \mathbf{Z}}$ *of open vector subspaces forming a neighborhood system of* 0 *such that* $V = \bigcup_{n \in \mathbf{Z}} W_n$ *and* dim $W_n/W_m < \infty$ for $n \leq m$.

Remark that in this case the family $\mathfrak{F}(V)$ of open vector subspace W of V which is contained by some W_n is independent from the choice of $\{W_n\}$. In fact, $\mathfrak{F}(V)$ is the family of open vector subspaces W of V such that $\dim(W/W') < \infty$ for any open subspace $W' \subset W$.

2.2. For a *c.l.c.* vector space V, define the Grassmann variety as follows.

For a k-scheme S, set $\mathcal{O}_S \hat{\otimes} V = \varinjlim_{W \in \mathfrak{F}(V)} \mathcal{O}_S \otimes (V/W)$ and consider the functor

(2.2.1) $\mathrm{Grass}(V) : S \mapsto \{\mathfrak{F};\ \mathfrak{F}$ is a sub-$\mathcal{O}_S$-module of $\mathcal{O}_S \hat{\otimes} V$ such that locally in the Zariski topology there exists a $W \in \mathfrak{F}(V)$ such that $\mathfrak{F} \to \mathcal{O}_S \otimes (V/W)$ is an isomorphism$\}$.

For $W \in \mathfrak{F}(V)$, we set

(2.2.2) $\mathrm{Grass}_W(V) : S \mapsto \{\mathfrak{F};\ \mathfrak{F}$ is a sub-$\mathcal{O}_S$-module of $\mathcal{O}_S \hat{\otimes} V$ such that $\mathfrak{F} \to \mathcal{O}_S \otimes (V/W)$ is an isomorphism$\}$.

Hence $\mathrm{Grass}(V) = \bigcup_W \mathrm{Grass}_W(V)$ in the Zariski topology.

PROPOSITION 2.2.1. Grass(V) *is represented by a separated scheme.*

Proof. This proposition follows from the following two statements

(2.2.3) $\mathrm{Grass}_W(V)$ is represented by an affine scheme of countable type.

(2.2.4) For W, $W' \in \mathfrak{F}(V)$, there exists $f \in \Gamma(\mathrm{Grass}_W(V); \mathcal{O})$ and $f' \in \Gamma(\mathrm{Grass}_{W'}(V); \mathcal{O})$ such that $\mathrm{Grass}_W(V) \cap \mathrm{Grass}_{W'}(V)$ is represented by the open subscheme defined by $f \neq 0$ of $\mathrm{Grass}_W(V)$ and that we have $ff' = 1$ on $\mathrm{Grass}_W(V) \cap \mathrm{Grass}_{W'}(V)$.

We shall prove first (2.2.3). Let us take $\{e_i\}_{i \in I}$ in V such that $\{e_i\}$ forms a base of V/W. Take $\{u_j\}_{j \in J}$ in W such that u_j tends to 0 and any element of W is uniquely written as $\Sigma\, a_j u_j$ ($a_j \in k$). Then for a scheme S and $\mathfrak{F} \in \mathrm{Grass}_W(V)(S)$, there exist $a_{ij} \in \mathcal{O}(S)$ such that $\mathfrak{F}$ is generated by $e_i + \Sigma_j\, a_{ij} u_j$. Hence $\mathrm{Grass}_W(V)$ is represented by $\mathrm{Spec}(k[T_{ij};\, i \in I, j \in J])$.

Now, we shall prove (2.2.4).

For $\mathfrak{F} \in \mathrm{Grass}(V)(S)$, let $\mathcal{G}$ be the cokernel of $\mathfrak{F} \to \mathcal{O}_S \otimes V/(W \cap W')$, and consider the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \mathcal{O}_S \otimes W/(W \cap W') & \xrightarrow{\sim} & \mathcal{O}_S \otimes W/(W \cap W') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}_S \otimes V/(W \cap W') & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{F} & \longrightarrow & \mathcal{O}_S \otimes V/W & & & &
\end{array}
$$

Hence if $\mathcal{F} \in \mathrm{Grass}_W(V)(S)$, $\mathcal{G}$ is isomorphic to $\mathcal{O}_S \otimes W/(W \cap W')$. The similar diagram obtained by exchanging W and W' shows that $\mathcal{O}_S \otimes W'/(W' \cap W') \to \mathcal{G}$ and $\mathcal{F} \to \mathcal{O}_S \otimes V/W'$ has the same kernel and the cokernel. Hence if we denote by f the determinant of $\psi : \mathcal{O}_S \otimes W'/(W \cap W') \to \mathcal{G} \leftarrow \mathcal{O}_S \otimes W/(W \cap W')$, then $\mathrm{Grass}_W(V) \cap \mathrm{Grass}_{W'}(V)$ is defined by $f \neq 0$. On $\mathrm{Grass}_{W'}(V)$, we define f' as the determinant of ψ' : $\mathcal{O}_S \otimes W/(W \cap W') \to \mathcal{G} \leftarrow \mathcal{O}_S \otimes W'/(W \cap W')$. Then since ψ and ψ' are inverse to each other on $\mathrm{Grass}_W(V) \cap \mathrm{Grass}_{W'}(V)$, we have $ff' = 1$ there.

COROLLARY 2.2.2. $\mathrm{Grass}_W(V)$ *is open in* $\mathrm{Grass}(V)$ *and isomorphic to* $\mathbf{A}^\infty$ (*if* $\dim V = \infty$).

COROLLARY 2.2.3. (i) *For* $W, W' \in \mathcal{F}(V)$, $\mathrm{Grass}_W(V) \cap \mathrm{Grass}_{W'}(V) = \emptyset$ *if* $\dim W/(W \cap W') \neq \dim W'/(W \cap W')$.

(ii) *Fix* $W \in \mathcal{F}(V)$. *Then*

$$\mathrm{Grass}(V) = \bigcup_{d \in \mathbf{Z}} \mathrm{Grass}^d(V) \quad \text{and} \quad \mathrm{Grass}^d(V) = \bigcup_{W'} \mathrm{Grass}_{W'}(V)$$

where W' *ranges over* $\mathcal{F}(V)$ *with* $\dim W/(W \cap W') - \dim W'/(W \cap W') = d$.

2.3. Let G be an affine group scheme over a field k. We say that G acts on a k-vector space (or V is a G-module) if V is an $\mathcal{O}(G)$-comodule; i.e. there is a comultiplication $\mu : V \to \mathcal{O}(G) \otimes V$ such that

$$
(2.3.1) \qquad
\begin{array}{ccc}
V & \longrightarrow & \mathcal{O}(G) \otimes V \\
 & \searrow & \downarrow \\
 & & k \otimes V
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
V & \longrightarrow & \mathcal{O}(G) \otimes V \\
 & & {\scriptstyle \mathcal{O}(G) \otimes \mu} \downarrow \downarrow {\scriptstyle \mu_G \otimes V} \\
 & & \mathcal{O}(G) \otimes \mathcal{O}(G) \otimes V
\end{array}
$$

commutes, where $\mathcal{O}(G) \to k$ is the evaluation map at the identity and μ_G :

$\mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ is the comultiplication. As well-known, in this case, V is a union of finite-dimensional sub-G-modules.

Now, let V be an *l.c.* k-vector space. We endow $\mathcal{O}(G)$ with the discrete topology. We say that V is a (*l.c.*) G-module if there is given a continuous comultiplication $V \to \mathcal{O}(G) \hat{\otimes} V$ such that

$$\text{(2.3.2)} \qquad \begin{array}{ccc} V & \longrightarrow & \mathcal{O}(G) \hat{\otimes} V \\ & \searrow & \downarrow \\ & & k \otimes V \end{array} \quad \text{and} \quad \begin{array}{ccc} V & \longrightarrow & \mathcal{O}(G) \hat{\otimes} V \\ & & {\scriptstyle \mathcal{O}(G) \hat{\otimes} \mu} \downarrow \downarrow {\scriptstyle \mu_G \otimes V} \\ & & \mathcal{O}(G) \hat{\otimes} \mathcal{O}(G) \hat{\otimes} V \end{array}$$

commute. In this case, there exists a neighborhood system of 0 by linear subspaces U_i ($i \in I$) such that V/U_i is a G-module and $V/U_i \to V/U_i'$ is a morphism of G-modules if $U_i \subset U_i'$.

PROPOSITION 2.3.1. *If V is a c.l.c. G-module, then G acts on* Grass(V).

Proof. It is enough to construct

$$G(S) \times \mathrm{Grass}(V)(S) \to \mathrm{Grass}(V)(S)$$

functorially in S. An S-valued point of G gives $\mathcal{O}(G) \xrightarrow{a} \mathcal{O}(S)$.

Then we obtain

$$g : \mathcal{O}_S \hat{\otimes} V \xrightarrow{\mathcal{O}_S \hat{\otimes} \mu} \mathcal{O}_S \hat{\otimes} \mathcal{O}(G) \hat{\otimes} V \xrightarrow{a} \mathcal{O}_S \hat{\otimes} V.$$

This is an isomorphism. Hence for $F \subset \mathcal{O}_S \otimes V$, $\varphi(F) \subset \mathcal{O}_S \hat{\otimes} V$ and it gives the map $\mathrm{Grass}(V)(S) \to \mathrm{Grass}(V)(S)$.

3. Kac-Moody Lie algebra.

3.1. Following Kac, Moody, Mathieu, we start by the following data: a free $\mathbf{Z}$ module P, at most countably generated, and $\alpha_i \in P$ and $h_i \in \mathrm{Hom}_{\mathbf{Z}}(P, \mathbf{Z})$ indexed by an index set I.

We set $t^0 = \mathbf{C} \otimes_{\mathbf{Z}} P$, $t = \mathrm{Hom}_{\mathbf{C}}(t^0, \mathbf{C}) \cong \mathrm{Hom}_{\mathbf{Z}}(P, \mathbf{C})$ with the structure of *l.c.* vector space induced from the discrete topology of t^0. We assume the following conditions:

(3.1.1) $\{\langle\alpha_i, h_j\rangle\}_{i,j}$ is a generalized Cartan matrix, i.e. $\langle\alpha_i, h_j\rangle \in \mathbf{Z}$, $\langle\alpha_i, h_i\rangle = 2$, $\langle\alpha_i, h_j\rangle \leq 0$ for $i \neq j$ and $\langle\alpha_i, h_j\rangle = 0$ iff $\langle\alpha_j, h_i\rangle = 0$.

(3.1.2) For any i, there is $\lambda \in P$ such that $\langle\lambda, h_i\rangle > 0$ and $\langle\lambda, h_j\rangle = 0$ for any $j \neq i$.

(3.1.3) $\{\alpha_i\}_{i\in I}$ is linearly independent.

(3.1.4) For any $\lambda \in P$, $\langle h_i, \lambda\rangle = 0$ except finitely many i.

Let $\mathcal{G}$ be the Lie algebra generated by t and symbols e_i, f_i $(i \in I)$ with the following recover relations:

(3.1.5) $[h, e_i] = \alpha_i(h)e_i$ and $[h, f_i] = -\alpha_i(h)f_i$ for $h \in t$.

(3.1.6) $[e_i, f_j] = \delta_{ij}h_i$.

(3.1.7) $(ade_i)^{1-\alpha_j(h_i)}e_j = 0$ and $(adf_i)^{1-\alpha_j(h_i)}f_j = 0$ for $i \neq j$.

Let n (resp. n_-) be the Lie subalgebra generated by e_i (resp. f_i), $i \in I$. Then we have (e.g. [K])

(3.1.8) $$\mathcal{G} = n \oplus t \oplus n_-.$$

Set

(3.1.9) $$b = t \oplus n, \qquad b_- = t \oplus n_-$$

(3.1.10) $\mathcal{G}_i = t \oplus \mathbf{C}e_i \oplus \mathbf{C}f_i, \qquad p_i = \mathcal{G}_i + n, \qquad p_i^- = \mathcal{G}_i + n^-.$

Let Δ be the set of roots of $\mathcal{G}$ and Δ_+ and Δ_- the set of roots of n and n_-, respectively, and let $\mathcal{G}_\alpha$ be the root space with root $\alpha \in \Delta$. We set

(3.1.11) $$n_i = \bigoplus_{\substack{\alpha\in\Delta_+ \\ \alpha\neq\alpha_i}} \mathcal{G}_\alpha, \qquad n_i^- = \bigoplus_{\substack{\alpha\in\Delta_- \\ \alpha\neq-\alpha_i}} \mathcal{G}_\alpha.$$

Let W be the Weyl group, i.e. the subgroup of $GL(t^0)$ generated by the simple reflections s_i $(i \in I)$, where

$$s_i(\lambda) = \lambda - \langle h_i, \lambda\rangle\alpha_i. \tag{3.1.12}$$

We also denote by W' the braid group generated by s'_i $(i \in I)$ with the fundamental relation

$$\begin{aligned}
s'_i s'_j &= s'_j s'_i && \text{if} \quad \langle h_i, \alpha_j\rangle = 0\\
s'_i s'_j s'_i &= s'_j s'_i s'_j && \text{if} \quad \langle h_i, \alpha_j\rangle = \langle h_j, \alpha_i\rangle = -1\\
(s'_i s'_j)^2 &= (s'_j s'_i)^2 && \text{if} \quad \langle h_i, \alpha_j\rangle\langle h_j, \alpha_i\rangle = 2\\
(s'_i s'_j)^3 &= (s'_j s'_i)^3 && \text{if} \quad \langle h_i, \alpha_j\rangle\langle h_j, \alpha_i\rangle = 3
\end{aligned} \tag{3.1.13}$$

Then as is well-known, W is isomorphic to the quotient of W' by the subgroup generated by $ws'^2_i w^{-1}$ $(i \in I)$.

For $w \in W$, we denote by $l(w)$ the length of w, i.e. the smallest number l such that w is the product of a sequence of length l in $\{s_i\}$. Recall that

$$l(w) = \#(\Delta_+ \cap w\Delta_-). \tag{3.1.14}$$

Also recall that $l(s_i w) < l(w)$ if and only if $w^{-1}\alpha_i \in \Delta_-$. Note also there exists a unique injection $\iota : W \to W'$ such that

$$\iota(1) = 1, \qquad \iota(s_i) = s'_i \quad \text{and} \quad \iota(ww') = \iota(w)\iota(w') \quad \text{if} \quad l(ww') = l(w) + l(w'). \tag{3.1.15}$$

By this, we sometimes embed W into W'.

An element h of t is called regular if $\langle h, \alpha\rangle \neq 0$ for any $\alpha \in \Delta$. Such an element always exists. We set

$$P_+ = \{\lambda \in P;\ \langle\lambda, h_i\rangle \geq 0 \text{ for any } i\}. \tag{3.1.16}$$

For any finite set J of I, we set

$$P_{J^+} = \{\lambda \in P_+;\ \langle\lambda, h_i\rangle = 0 \text{ for } i \in I\backslash J\}. \tag{3.1.17}$$

If we set $P_0 = \{\lambda \in P;\ \langle\lambda, h_i\rangle = 0 \text{ for } i \in I\}$ then P_0 is a free **Z**-module and P_{J^+}/P_0 is a finitely generated semigroup.

3.2. Now, we shall define a completion of $\mathcal{G}$. For a subset S of Δ_+, we set

$$n_S = \bigoplus_{\alpha \in S} \mathcal{G}_\alpha. \tag{3.2.1}$$

We set

$$\hat{\mathcal{G}} = \varprojlim_S \mathcal{G}/n_S = b_- \oplus \prod_{\alpha \in \Delta_+} \mathcal{G}_\alpha \tag{3.2.2}$$

where S ranges over the subsets of Δ_+ such that $\Delta_+ \backslash S$ is finite. We define the subalgebras $\hat{P}_i$, $\hat{n}_i$, $\hat{b}$, $\hat{n}$ of $\hat{\mathcal{G}}$, similarly. We set also

$$\begin{aligned} \hat{U}_l(\mathcal{G}) &= \varprojlim_S U_l(\mathcal{G})/U_{l-1}(\mathcal{G})n_S \\ \hat{U}(\mathcal{G}) &= \bigcup_l \hat{U}_l(\mathcal{G}) \end{aligned} \tag{3.2.3}$$

Then $\hat{U}(\mathcal{G})$ is an algebra containing $U(\mathcal{G})$ as a subalgebra.

3.3. In general, let $\mathcal{G}$ be a Lie algebra. A vector v of a $\mathcal{G}$-module V is called $\mathcal{G}$-finite if v is contained in a finite-dimensional sub-$\mathcal{G}$-module of V. We call a $\mathcal{G}$-module V is locally finite if any element of V is $\mathcal{G}$-finite.

Let us define a ring homomorphism

$$\delta : U(\mathcal{G}) \to U(\mathcal{G}) \otimes U(\mathcal{G}) \tag{3.3.1}$$

by $\delta(A) = A \otimes 1 + 1 \otimes A$ for $A \in \mathcal{G}$, and an anti-ring automorphism

$$a : U(\mathcal{G}) \to U(\mathcal{G}) \tag{3.3.2}$$

by $A^a = -A$ for $A \in G$. Then δ defines $U(\mathcal{G})^* \otimes U(\mathcal{G})^* \to (U(\mathcal{G}) \otimes U(\mathcal{G}))^* \to U(\mathcal{G})^*$ and this gives a commutative ring structure on $U(\mathcal{G})^*$.

The right and left multiplication of $\mathcal{G}$ on $U(\mathcal{G})$ induces the two $\mathcal{G}$-module structures on $U(\mathcal{G})^*$:

$$(R(A)f)(P) = f(PA), \qquad (L(A)f)(P) = f(a(A)P) \tag{3.3.3}$$

for $A \in U(\mathfrak{g})$, $f \in U(\mathfrak{g})^*$ and $P \in U(\mathfrak{g})$. Then $R(A)$ and $L(A)$ are derivations of the ring $U(\mathfrak{g})^*$ for any $A \in \mathfrak{g}$.

Now let $\mathfrak{a}$ be an abelian Lie algebra acting on the Lie algebra $\mathfrak{g}$ semi-simply, t an abelian subalgebra of $\mathfrak{g}$ stable by $\mathfrak{a}$, and $P \subset t^*$ a sub-$\mathbf{Z}$-module stable by $\mathfrak{a}$. We assume that t acts semi-simply on $\mathfrak{g}$ by the adjoint action and its weights belong to P.

Then, we set

(3.3.5) $A(\mathfrak{g}, t, P, \mathfrak{a}) = \bigoplus_{\lambda \in P} \{f \in U(\mathfrak{g})^*; f$ satisfies the following conditions (3.3.6), (3.3.7) and (3.3.8)$\}$.

(3.3.6) f is $\mathfrak{g}$-finite with respect to L and R.

(3.3.7) f is a weight vector with weight λ with respect to the left action of t.

(3.3.8) f is $\mathfrak{a}$-finite.

Then $f \in U(\mathfrak{g})^*$ belongs to $A(\mathfrak{g}, t, P, \mathfrak{a})$ if and only if there exists a two-sided ideal I of $U(\mathfrak{g})$ such that

(3.3.9) $f(U(\mathfrak{g})/I) = 0$,

(3.3.10) $\dim U(\mathfrak{g})/I < \infty$,

(3.3.11) I is $\mathfrak{a}$-invariant,

(3.3.12) t acts semi-simply on $U(\mathfrak{g})/I$ by the left multiplication and its weights belong to P.

Then one can see easily that $A(\mathfrak{g}, t, P, \mathfrak{a})$ is a subring of $U(\mathfrak{g})^*$ and the multiplication map $\mu : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$ induces the homomorphism

(3.3.13)
$$\begin{array}{ccc} A(\mathfrak{g}, t, P, \mathfrak{a}) & \longrightarrow & A(\mathfrak{g}, t, P, \mathfrak{a}) \otimes A(\mathfrak{g}, t, P, \mathfrak{a}) \\ \cap & & \cap \\ U(\mathfrak{g})^* & \longrightarrow & (U(\mathfrak{g}) \otimes U(\mathfrak{g}))^* \end{array}$$

With this, $\operatorname{Spec}(A(\mathfrak{g}, t, P, \mathfrak{a}))$ becomes an affine group scheme (see [M]).

We write

$$G(\mathcal{G}, t, P, \mathfrak{a}) = \mathrm{Spec}(A(\mathcal{G}, t, P, \mathfrak{a})). \tag{3.3.14}$$

Remark that $g \mapsto g^{-1}$ is given by $a : U(\mathcal{G}) \to U(\mathcal{G})$.

When $\mathfrak{a} = 0$, we write $G(\mathcal{G}, t, P)$ for $G(\mathcal{G}, t, P, \mathfrak{a})$ for short.

3.4. Coming back to the situation in Section 3.1, we define the affine group schemes B, B_-, T, U, U_-, G_i, U_i, U_i^-, P_i, P_i^- as follows. This construction is due to Mathieu [M].

$$B = G(b, t, P),$$

$$B_- = G(b_-, t, P),$$

$$T = G(t, t, P),$$

$$U = G(n, 0, 0, t),$$

$$U_- = G(n_-, 0, 0, t),$$

$$G_i = G(\mathcal{G}_i, t, P),$$

$$U_i = G(n_i, 0, 0, t),$$

$$U_i^- = G(n_i^-, 0, 0, t),$$

$$P_i = G(p_i, t, P),$$

$$P_i^- = G(p_i^-, t, P),$$

$$G_i^+ = G(t \oplus \mathbf{C}e_i, t, P),$$

$$G_i^- = G(t \oplus \mathbf{C}f_i, t, P).$$

Then we have ([M])

$$B = T \ltimes U = G_i^+ \times U_i,$$

$$B_- = T \ltimes U_- = G_i^- \times U_i^-,$$

$$P_i = G_i \ltimes U_i \supset B \supset T,$$

$$P_i^- = G_i \ltimes U_i^- \supset B_- \supset T,$$

$$T = \operatorname{Spec} \mathbf{C}[P],$$

$$U \cong \operatorname{Spec} S\Big(\bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha^*\Big),$$

$$U_- \cong \operatorname{Spec} S\Big(\bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha^*\Big),$$

$$G_i^+ = G_i \cap B, \qquad G_i^- = G_i \cap G_-.$$

More generally, for a subset S of Δ_+ such that $(S + S) \cap \Delta_+ \subset S$, we set $n_S = \oplus\, \mathfrak{g}_\alpha$ and $U_S = G(n_s, 0, 0, t)$.

Then for $S \supset S'$ such that $S \setminus S'$ is a finite set and that $(S + S') \cap \Delta_+ \subset S'$, $n_S/n_{S'}$ is a finite-dimensional nilpotent Lie algebra and if we denote by $\exp(n_S/n_{S'})$ the associated unipotent group, we have

$$U_S \cong \varprojlim_{S'} \exp(n_S/n_{S'}).$$

3.5. The group P_i acts on the *c.l.c.* space $\hat{\mathfrak{g}}$ by the adjoint action. In fact, $ad : p_i \to \operatorname{End}(\mathfrak{g})$ extends to $ad : U(p_i) \to \operatorname{End}(\mathfrak{g})$. Moreover, for any ideal $\mathfrak{a}$ of p_i with $\operatorname{codim} p_i/\mathfrak{a} < \infty$, $\mathfrak{g}/\mathfrak{a}$ is locally p_i-finite. Hence, for any $A \in \mathfrak{g}$, there is a two-sided ideal I of $U(p_i)$ with $\dim U(p_i)/I < \infty$ and $ad(I)A \subset \mathfrak{a}$. Hence the morphism $P \mapsto ad(P)A$ from $U(p_i)$ to $\mathfrak{g}/\mathfrak{a}$ splits as $U(p_i)/I \to \mathfrak{g}/\mathfrak{a}$. Hence this gives an element of $(U(p_i)/I)^* \otimes \mathfrak{g}/\mathfrak{a} \subset U(p_i)^* \otimes (\mathfrak{g}/\mathfrak{a})$. This element clearly belongs to $A(p_i, t, P) \otimes (\mathfrak{g}/\mathfrak{a})$. Thus we obtained $\mathfrak{g}/\mathfrak{a} \to A(p_i, t, P) \otimes (\mathfrak{g}/\mathfrak{a})^*$. Since $\varprojlim \mathfrak{g}/\mathfrak{a} = \hat{\mathfrak{g}}$, we obtain $\hat{\mathfrak{g}} \to \mathcal{O}(P_i) \hat{\otimes} \hat{\mathfrak{g}}$. This gives an action of P_i on $\hat{\mathfrak{g}}$.

Clearly the action of B on $\hat{\mathfrak{g}}$ obtained from the action of P_i does not depend on $i \in I$.

Especially, P_i acts on the Grassmann variety $\operatorname{Grass}(\hat{\mathfrak{g}})$ by Proposition 2.3.1.

4. The first construction of the flag variety.

4.1. In this section, for a Kac-Moody Lie algebra $\mathfrak{g}$, we construct its flag variety as a subscheme of $\mathrm{Grass}(\hat{\mathfrak{g}})$. We keep the notations in Section 3.

4.2. Since $\hat{\mathfrak{g}}$ is a *c.l.c.* vector space, $\mathrm{Grass}(\hat{\mathfrak{g}})$ is a separated scheme. Since $\hat{\mathfrak{g}} = b_- \oplus \hat{n}$, b_- gives a **C**-valued point of $\mathrm{Grass}(\hat{\mathfrak{g}})$. We denote this point by x_0. By Section 3.5, P_i and B act on $\mathrm{Grass}(\hat{\mathfrak{g}})$.

4.3. Set $s_i' = \exp(-e_i)\exp(f_i)\exp(-e_i) \in G_i \subset P_i$. Then $s_i'^4 = 1$ and s_i' acts on $\hat{\mathfrak{g}}$. This extends to the group homomorphism:

(4.3.1) $$W' \to \mathrm{Aut}(\hat{\mathfrak{g}}).$$

In order to see this, it is enough to prove the braid relation (3.1.13) when the Lie algebra generated by e_i, e_j, f_i, f_j is finite-dimensional. Then the braid condition holds in the corresponding simply connected semi-simple group.

The morphism (4.3.1) induces

(4.3.2) $$W' \to \mathrm{Aut}(\mathrm{Grass}(\hat{\mathfrak{g}})).$$

We have also

(4.3.3) The image of $\mathrm{Ker}(W' \to W)$ in $\mathrm{Aut}(\hat{\mathfrak{g}})$ belongs to the image of T in $\mathrm{Aut}(\hat{\mathfrak{g}})$.

In fact, $\mathrm{Ker}(W' \to W)$ is generated by the $ws_i'^2w^{-1}$, which belongs to T.

Since $[t, b_-] \subset b_-$, we have

(4.3.4) $$Tx_0 = x_0.$$

Hence for $w \in W$, $w'x_0$ does not depend on the choice of a representative w' of w in W'. We denote it by wx_0.

4.4. As in (2.2.2), we set

(4.4.1) $$\mathrm{Grass}_{\hat{n}}(\hat{\mathfrak{g}}) = \{ W \in \mathrm{Grass}(\hat{\mathfrak{g}});\ W \oplus \hat{n} \xrightarrow{\sim} \hat{\mathfrak{g}} \}.$$

This is an affine open subscheme of $\mathrm{Grass}(\hat{\mathfrak{g}})$.

LEMMA 4.4.1. *The morphism* $U \to \mathrm{Grass}(\hat{\mathfrak{g}})$ *given by* $U \ni g \mapsto gx_0$ *is an embedding.*

Proof. First we shall show $Ux_0 \subset \mathrm{Grass}_{\hat{n}}(\hat{\mathfrak{g}})$. For this, it is enough to show, for any $g \in U$,

$$gb_- \oplus \hat{n} = \hat{\mathfrak{g}}. \tag{4.4.2}$$

But this is obvious because $\hat{n}$ is stable by U. Hence it is enough to show that $U \to Y = \mathrm{Grass}_{\hat{n}}(\hat{\mathfrak{g}})$ is a closed embedding. In order to see this, let us take a regular element h of t (i.e. $\langle h, \alpha\rangle \neq 0$ for any $\alpha \in \Delta$). Then for any $F \in \mathrm{Grass}_{\hat{n}}(\hat{\mathfrak{g}})$, $F \oplus \hat{n} = \hat{\mathfrak{g}}$, and hence there exists $\psi(F) \in \hat{n}$ with $h - \psi(F) \in F$. This defines a morphism

$$\psi : Y \to \hat{n}.$$

If we combine $U \to Y \overset{\psi}{\to} \hat{n}$, this is given by

$$U \ni g \mapsto h - g^{-1}h \in \hat{n}.$$

Hence it is enough to show the following lemma.

LEMMA 4.4.2. *Let* h *be a regular element of* t. *Then, the morphism* $U \to h + \hat{n}$ *given by* $g \mapsto gh$ *is an isomorphism.*

Proof. Let S be a subset of Δ_+ such that $(S + \Delta_+) \cap \Delta_+ \subset S$ and $\Delta_+ \backslash S$ is finite. Then $U \to h + \hat{n}$ induces $U/U_S \to (h + n)/n_S$, and it is enough to show that this is an isomorphism. Now U/U_S acts on b/n_S. For $A \in n/n_S$, the isotropy group at $h + A$ is the identity. In fact this follows from

$$\{E \in n;\, [h + A, E] \in n_S\} = n_S. \tag{4.4.3}$$

Since $\dim(h + n)/n_S = \dim U/U_S$, $(U/U_S)(h + A)$ is open in $(h + n)/n_S$. Thus $(U/U_S)(h + A)$ and $(U/U_S)h$ intersect. This shows $U/U_S \xrightarrow{\sim} (U/U_S)h = (h + n)/n_S$.

4.5. We have

$$Bx_0 = Ux_0 \tag{4.5.1}$$

because $Tx_0 = x_0$ and $B = UT$. For $w \in W$, let us denote

$$B \cap {}^wB = A(t \oplus \bigoplus_{\alpha \in \Delta_+ \cap {}^w\Delta_+} \mathfrak{g}_\alpha, t, P) \tag{4.5.2}$$

$$B \cap {}^wB_- = A(t \oplus \bigcap_{\alpha \in \Delta_+ \cap {}^w\Delta_-} \mathfrak{g}_\alpha, t, P).$$

They are subgroups of B. Similarly, we define $U \cap {}^wU$ and $U \cap {}^wU_-$. Then we have

$$U \simeq (U \cap {}^wU) \times (U \cap {}^wU_-) \simeq (U \cap {}^wU_-) \times (U \cap {}^wU). \tag{4.5.3}$$

We have also

$$(B \cap {}^wB_-)wx_0 = x_0. \tag{4.5.4}$$

LEMMA 4.5.1. *For $w \in W$, $Bs_i'Bwx_0 \subset Bwx_0 \cup Bs_iwx_0$.*

Proof. We have $Bs_i'Bwx_0 \subset P_iwx_0$. Since $P_i = BG_i \subset B(G_i \cap {}^wB_-) \cup Bs_i'(G_i \cap {}^wB_-)$, $P_iwx_0 \subset B(G_i \cap {}^wB_-)wx_0 \cup Bs_i'(G_i \cap {}^wB_-)wx_0 \subset Bwx_0 \cup Bs_iwx_0$.

Note that for $w_1, w_2 \in W$, $w_1Bw_2x_0$ does not depend on the representatives in W' of $w_1, w_2 \in W$. Hence we denote $w_1Bw_2x_0$ for it.

LEMMA 4.5.2. *Let $w \in W$.*

(i) *If $l(w) > l(s_iw)$, $Bs_iBwx_0 = Bs_iwx_0$.*
(ii) *If $l(w) < l(s_iw)$, $Bs_iBwx_0 = P_iwx_0 = Bwx_0 \cup Bs_iwx_0$.*

Proof. If $l(s_iw) < l(w)$, then $w^{-1}\alpha_i \in \Delta_-$. Hence $G_i^+ = G_i \cap B \subset {}^wB_-$ and $s_iB \subset s_iU_iG_i^+ \subset Bs_iG_i^+$. Hence we have $Bs_iBwx_0 = Bs_iG_i^+wx_0 = Bs_iwx_0$.

If $l(s_iw) > l(w)$, then we have $Bs_iBs_iwx_0 = Bwx_0$ since $l(s_is_iw) < l(s_iw)$. Hence $Bs_iBwx_0 = Bs_iBs_iBwx_0$. Since $Bs_iBs_iB = P_i$, $Bs_iBwx_0 = P_iwx_0$ and it contains wx_0 and s_iwx_0.

Lemma 4.5.3. *$wBx_0 \subset \bigcup_{w' \leq w} Bw'x_0$, where $\leq$ is the Bruhat order (the order generated by $s_{i_1} \cdots s_{i_{k-1}} s_{i_{k+1}} \cdots s_{i_l} \leq s_{i_r}$ for a reduced expression $s_{i_1} \cdots s_{i_r}$).*

Proof. We shall prove by the induction of $l(w)$. If $l(w) = 0$, it is trivial. Otherwise, set $w = s_i w'$ with $l(w) = 1 + l(w')$. Then by the hypothesis of the induction, $wBx_0 \subset \bigcup_{w'' \leq w'} s_i Bw''x_0 \subset \bigcup_{w'' \leq w'} Bs_i w''x_0 \cup Bw''x_0 \subset \bigcup_{w'' \leq w} Bw''x_0$.

Lemma 4.5.4.

(i) $Bwx_0 \cap \mathrm{Grass}_{\hat{n}}(\hat{\mathfrak{g}}) = \emptyset$ *if* $w \neq 1$.
(ii) $wBx_0 \cap \mathrm{Grass}_{\hat{n}}(\hat{\mathfrak{g}}) \subset Bx_0$.

Proof. (i) Let $g \in B$ and assume that $gwb_- \rightrightarrows \hat{\mathfrak{g}}/\hat{n}$. Then $wb_- \rightrightarrows \hat{\mathfrak{g}}/\hat{n}$. Hence $w\Delta_- = \Delta_-$, which implies $w = 1$.
(ii) follows from (i) and the preceding lemma.

Corollary 4.5.5. *$X = \bigcup_{w \in W'} wBx_0$ is a subscheme of* $\mathrm{Grass}(\hat{\mathfrak{g}})$ *and wBx_0 is open in X for any $w \in W$.*
This easily follows from $X \cap \mathrm{Grass}_{\hat{n}}(\hat{\mathfrak{g}}) = Bx_0$.

Definition 4.5.6. We call X the flag variety of $\mathfrak{g}$.
Since $\mathrm{Grass}(\hat{\mathfrak{g}})$ is a separated scheme, X is also a separated scheme, and $\{wBx_0\}$ is an open affine covering of X. Note that X is not quasi-compact if W is an infinite group. I do not know whether X is a closed subscheme of $\mathrm{Grass}(\hat{\mathfrak{g}})$ or not.

Lemma 4.5.7. *Bwx_0 is a closed subscheme of wBx_0 and we have a commutative diagram:*

$$\begin{array}{ccc} Bwx_0 & \hookrightarrow & wBx_0 \\ \uparrow\!\!\!\wr & & \uparrow\!\!\!\wr \\ \hat{n} \cap {}^{w^{-1}}\hat{n} & \hookrightarrow & \hat{n} \end{array} \tag{4.5.5}$$

Proof. We have $U = (U \cap^w U) \times (U \cap^w U_-)$. Since $(U \cap^w U_-)x_0 = x_0$, we have $Uwx_0 = (U \cap^w U)wx_0 = w({}^{w^{-1}}U \cap U)x_0$. Then the lemma follows from Lemma 4.4.1.

Corollary 4.5.8. *Bwx_0 is affine and codimension $l(w)$ in X.*

Proposition 4.5.9. *$X(\mathbf{C}) = \bigsqcup_{w \in W} Bwx_0$.*

Proof. By Lemma 4.5.3, it is enough to show $Bwx_0 = Bw'x_0$ implies $w = w'$.

We have $wx_0 \in Bw'x_0 \subset w'Bx_0$. Hence $w'^{-1}wx_0 \subset Bw'^{-1}wx_0 \cap Bx_0$. Then Lemma 4.5.4 implies $w' = w$.

LEMMA 4.5.10. *Let $w_1, w_2 \in W$ and assume $l(w_1s_iw_2) = l(w_1) + l(w_2) + 1$. Then $Bw_1s_iw_2x_0 \subset \overline{Bw_1w_2x_0}$.*

Proof. Since $l(w_1s_i) > l(w_1)$, we have $w_1\alpha_i \in \Delta_+$, and hence $G_i \cap {}^{w_1^{-1}}B \subset G_i \cap B$. Since $l(s_iw_2) > l(w_2)$, $w_2^{-1}\alpha_i \in \Delta_+$ and hence $G_i \cap {}^{w_2}B_- \subset G_i \cap B_-$. Since $(G_i \cap B)\cdot(G_i \cap B_-)$ is dense in G_i, we obtain

$$Bw_1s_{i_0}w_2x_0 \subset Bw_1G_iw_2x_0 \subset \overline{Bw_1(G_i \cap {}^{w_1^{-1}}B)(G_i \cap {}^{w_2}B_-)w_2x_0}$$
$$= \overline{Bw_1w_2x_0}.$$

PROPOSITION 4.5.11. $\overline{Bwx_0} = \bigcup_{w'\geq w} Bw'x_0$.

Proof. We shall prove first $\overline{Bwx_0} \supset Bw'x_0$ if $w' \geq w$ by the induction of $l(w')$. If $l(w') = 0$, then $w = w' \doteq e$ and this is evident. If $l(w') > 0$, there is $w_1, w_2 \in W$ and i such that $w' = w_1s_iw_2$, $w_1w_2 \geq w$ and $l(w') = l(w_1) + l(w_2) + 1$. Hence $Bw'x_0 \subset \overline{Bw_1w_2x_0} \subset \overline{Bwx_0}$.

Now, we shall prove the converse inclusion.

In order to see this, we shall prove that $\overline{Bwx_0} \supset Bw'x_0$ implies $w \leq w'$ by the induction of $l(w')$. If $l(w') = 0$, $w \neq 1$ implies $Bwx_0 \cap Bx_0 = \emptyset$. Hence $\overline{Bwx_0} \cap Bx_0 = \emptyset$. Assume that $l(w') > 0$. Then there is i such that $l(s_iw') < l(w')$. Thus we have $\overline{Bs_iBwx_0} \supset Bs_iBw'x_0 = Bs_iw'x_0$ by Lemma 4.5.2.

If $l(s_iw) < l(w)$, then by Lemma 4.5.2, $\overline{Bs_iwx_0} = \overline{Bs_iBwx_0} \supset Bs_iw'x_0$ and hence $s_iw' \geq s_iw$, which implies $w' \geq w$.

If $l(s_iw) > l(w)$, then $\overline{Bs_iBwx_0} = \overline{Bwx_0} \supset Bs_iw'x_0$ and hence $w' \geq s_iw' \geq w$.

PROPOSITION 4.5.12. $BwBx_0 = \bigcup_{w'\leq w} Bw'x_0$.

Proof. By Lemma 4.5.3, it is enough to show $BwBx_0 \supset Bw'x_0$ implies $w \geq w'$, or equivalently

$$(4.5.8) \qquad wBx_0 \cap Bw'x_0 \neq \emptyset \quad \text{implies} \quad w \geq w'.$$

We shall prove this by the induction on $l(w)$. If $l(w) = 0$, this is already proven. Assume $l(w) > 0$. Then there exists i such that $w'' = s_i w$ satisfies $l(w'') < l(w)$. Then $wBx_0 \cap Bw'x_0 \neq \emptyset$ implies $w''Bx_0 \cap Bs_iBw'x_0 \neq \emptyset$.

If $l(s_iw') < l(w')$, Lemma 4.5.2 implies $w''Bx_0 \cap Bs_iw'x_0 \neq \emptyset$. Hence the hypothesis of the induction implies $w'' \geq s_iw'$, which gives $w \geq w'$. If $l(s_iw') > l(w')$, then $w''Bx_0 \cap (Bs_iw'x_0 \cup Bw'x_0) \neq \emptyset$.

Hence $w' \geq s_iw'$ or $w'' \geq w'$. Hence in the both cases, we have $w \geq w'$.

Corollary 4.5.13. $BwBx_0 = \bigcup_{w' \leq w} w'Bx_0$.

Proof. If $w' \leq w$, $w'Bx_0 \subset \bigcup_{w'' \leq w'} Bw''x_0 \subset BwBx_0$. The inverse inclusion follows from $w'Bx_0 \supset Bw'x_0$ (Lemma 4.5.7).

Remark 4.5.14. For $w, w' \in W$, we have

$$\overline{Bwx_0} \cap w'Bx_0 \cong (U \cap {}^{w'}U) \times (\overline{Bwx_0} \cap w'(B \cap {}^{w'^{-1}}B_-)x_0)$$

because $w'Bx_0 = (U \cap {}^{w'}U) \times w'(B \cap {}^{w'^{-1}}B_-)x_0$ and $\overline{Bwx_0}$ is invariant by $U \cap {}^{w'}U$. Then $\overline{Bwx_0} \cap w'(B \cap {}^{w'^{-1}}B_-)x_0$ is a finite-dimensional variety. Thus, $\overline{Bwx_0}$ is locally finite-dimensional or the product of a finite-dimensional variety and $\mathbf{A}^{\infty}$.

Proposition 4.5.15. *X is irreducible.*

Proof. Since $X = \bigcup wBx_0$ is an open covering by irreducible subsets, it is enough to show $wBx_0 \cap w'Bx_0 \neq \emptyset$ for any w, w'. This follows from $Bw'^{-1}wBx_0 \supset Bx_0$ (Proposition 4.5.12).

5. The second construction of the flag variety.

5.1. Following Kac-Peterson [K-P], we shall first define the ring of regular functions. Recall that $U(\mathfrak{g})^*$ has the structure of two-sided $\mathfrak{g}$-modules (Section 3.3).

Definition 5.1.1. $A(\mathfrak{g}, P) = \bigoplus_{\mu \in P} \{\varphi \in U(\mathfrak{g})^*;\ \varphi$ satisfies the following conditions (5.1.1) and (5.1.2)$\}$.

(5.1.1) φ is finite with respect to the left action of p_i and the right action of p_i for all i.

(5.1.2) φ is a weight vector of weight μ with respect to the left action of t.

LEMMA 5.1.2. *$A(\mathcal{G}, P)$ is a subring of $U(\mathcal{G})^*$.*

This easily follows from the fact that $\delta : U(\mathcal{G}) \to U(\mathcal{G}) \otimes U(\mathcal{G})$ is p_i-linear with respect to the left and right actions.

Definition 5.1.3. We define G_∞ as $\mathrm{Spec}(A(\mathcal{G}, P))$.

LEMMA 5.1.4. *Let V be a p_i-module, and $v \in V$.*

(i) *If v is b-finite, then $f_i v$ is also b-finite.*
(ii) *If v is b-finite and $f_i^N v = 0$ for $N \gg 0$, then v is p_i-finite.*

Proof. Since $[b, f_i] \subset p_i = b + \mathbf{C}f_i$, we have

$$U(b)f_i \subset U(b) + f_i U(b). \tag{5.1.3}$$

This shows (i). If $f_i^N v = 0$, then $U(p_i)v = \Sigma_{k<N} U(b) f_i^k v$, which shows (ii).

LEMMA 5.1.5. *Let V be a $\mathcal{G}$-module. Then, for any $i \in I$, the set of p_i-finite vectors is a sub-$\mathcal{G}$-module.*

Proof. It is enough to show that if v is a p_i-finite vector then $f_j v$ is also p_i-finite vector for $j \neq i$. By the preceding lemma, $f_j v$ is b-finite. Hence it is enough to show $f_i^N f_j v = 0$ for $N \gg 0$. But this follows from (3.1.7) and $f_i^N f_j v = \Sigma_k \binom{N}{k}((ad f_i)^k f_j) f_i^{N-k} v$.

LEMMA 5.1.6. *For any $\lambda \in t^0$, $\lambda + N\alpha_i$ is not a weight of $U(n_i)$ except finitely many $N \in \mathbf{Z}$.*

Proof. We may assume that λ is a weight of $U(n_i)$ and I is finite. For $\lambda = \Sigma\, m_j \alpha_j \in \bigoplus_j \mathbf{Z}\alpha_j$, set $|\lambda|' = \Sigma_{j \neq i} m_j$. Then if α is a weight of n_i, then $|\alpha|' > 0$. Now assume $\lambda + N\alpha_i$ is a weight of $U(n_i)$. Then

$$\lambda + N\alpha_i = \sum_{\nu=1}^{r} \gamma_\nu$$

where γ_ν are weights of n_i. Hence $|\lambda|' = \Sigma_{\nu=1}^r |\gamma_\nu|'$. Hence $r \leq |\lambda|'$ and $|\gamma_\nu|' \leq |\lambda|'$. Since for any root β, there is only finitely many roots of the form $\beta + N\alpha_i$, there are only finitely many possibilities for γ_ν. Thus we obtain the result.

LEMMA 5.1.7.

(i) $[n_i, f_i] \subset n_i$.

(ii) $(ad f_i)$ *acts locally nilpotently on* $U(n_i)$.

(iii) *For any two-sided ideal* I *of* $U(n_i)$ *such that* $[t, I] \subset I$ *and* $\dim(U(n_i)/I) < 0$, *there exists* N *such that*

(a) $(ad f_i)^m U(n_i) \in I$ *for* $m \geq N$.

(b) $f_i^{N+m} U(n_i) \subset I\mathbf{C}[f_i] + U(n_i)\mathbf{C}[f_i] f_i^m$ *for* $m \geq 0$.

Proof.

(i) follows from $(\Delta_+ - \alpha_i) \cap \Delta \subset \Delta_+ \backslash \{\alpha_i\}$.

(ii) follows from the fact that weights of $U(n_i)$ belong to $\Sigma\, \mathbf{Z}_{\geq 0}\alpha_j$.

(iii) In order to see (a), it is enough to show, for any weight β of $U(n_i)$, $\beta + N\alpha_i$ is not a weight of $U(n_i)$ if $N \gg 0$. This follows from Lemma 5.1.6. (b) follows from (a) and $f_i^{N+m} U(n_i) \subset \Sigma((ad f_i)^k U(n_i)) f_i^{N+m-k}$.

LEMMA 5.1.8. *If* $\varphi \in U(\mathfrak{g})^*$ *is left* b*-finite and right* p_i^-*-finite, then* φ *is left* p_i*-finite.*

Proof. By Lemma 5.1.4, it is enough to show

$$L(f_i)^N \varphi = 0 \quad \text{for} \quad N \gg 0. \tag{5.1.5}$$

There exists a two-sided ideal I of $U(b)$ such that $\varphi(IU(\mathfrak{g})) = 0$ and $\dim U(b)/I < \infty$. Then by the preceding lemma, there exists N such that

$$f_i^{N+m} U(n_i) \subset IU(\mathfrak{g}) + U(n_i) f_i^m U(p_i^-) \quad \text{for} \quad m \geq 0.$$

Since $U(\mathfrak{g}) = U(n_i)U(p_i^-)$, we have

$$\begin{aligned} \varphi(f_i^{N+m} U(\mathfrak{g})) &\subset \varphi(IU(\mathfrak{g}) + U(n_i) f_i^m U(p_i^-)) \\ &\subset \{R(f_i)^m R(U(p_i^-))\varphi\}(U(\mathfrak{g})) = 0 \end{aligned}$$

for $m \gg 0$.

PROPOSITION 5.1.9. $\mathcal{O}(G_\infty)$ *is a two-sided sub-*$\mathfrak{g}$*-module of* $U(\mathfrak{g})^*$.
This follows immediately from Lemma 5.1.5.
Let $e \in G_\infty$ be the point given by $U(\mathfrak{g}) \to U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{g} \cong \mathbf{C}$.

THEOREM 5.1.10.

(i) P_i *acts on* G_∞ *from the left and* P_i^- *acts on* G_∞ *from the right.*

(ii) *The action of* B *on* G_∞ *induced from the one of* P_i *does not depend on* i.

(iii) *For* $g \in G_i$, $ge = eg$.

Proof. The multiplication homomorphism $\mu_i : U(p_i) \otimes U(\mathcal{G}) \to U(\mathcal{G})$ gives a $\varphi : U(\mathcal{G})^* \to (U(p_i) \otimes U(\mathcal{G}))^*$. We shall show that

$$\varphi(\mathcal{O}(G_\infty)) \subset \mathcal{O}(P_i) \otimes \mathcal{O}(G_\infty). \tag{5.1.6}$$

Then φ is a ring homomorphism and defines $P_i \times G_\infty \to G_\infty$. It is easy to check this is an action of P_i. Similarly $U(\mathcal{G}) \otimes U(p_i^-) \to U(\mathcal{G})$ defines $G_\infty \times P_i^- \to G_\infty$ and it gives the right action of P_i^- on G_∞. The rest is easy to check. Now, we shall show (5.1.6).

Let $f \in \mathcal{O}(G_\infty)$. Then by the definition, there exists a two-sided ideal I of $U(p_i)$ such that $f(IU(\mathcal{G})) = 0$, $U(p_i)/I$ is finite-dimensional and that t acts semisimply and the weights belong to P.

Hence $f \circ \mu_i : U(p_i) \otimes U(\mathcal{G}) \to \mathbf{C}$ splits to $U(p_i) \otimes U(\mathcal{G}) \to (U(p_i)/I) \otimes U(\mathcal{G})$. Hence f belongs to $(U(p_i)/I)^* \otimes U(\mathcal{G})^* \subset \mathcal{O}(P_i) \otimes U(\mathcal{G})^*$. Write $f = \Sigma\, \varphi_k \otimes \psi_k$ with $\varphi_k \in \mathcal{O}(P_i)$ and $\psi_k \in U(\mathcal{G})^*$, such that $\{\varphi_k\}$ is linearly independent. Then there are $R_k \in U(p_i)$ such that $\varphi_k(R_{k'}) = \delta_{kk'}$. Then $\psi_k(P) = f(R_k P)$ for any $P \in U(\mathcal{G})$. Hence $\psi_k \in \mathcal{O}(G_\infty)$ by Proposition 5.1.9.

5.2. For $\Lambda \in t^0$, let us denote $K_\Lambda \in U(\mathcal{G})^*$ given by

$$K_\Lambda : U(\mathcal{G}) \xleftarrow{\sim} U(n) \otimes U(t) \otimes U(n_-) \longrightarrow U(t) \xrightarrow{-\Lambda} \mathbf{C} \tag{5.2.1}$$

where the middle arrow is given by $U(n) \to U(n)/U(n)n \xleftarrow{\sim} \mathbf{C}$ and $U(n_-) \to U(n_-)/U(n_-)n_- \xleftarrow{\sim} \mathbf{C}$ and the last arrow is given by $h \mapsto -\Lambda(h)$. We have in the ring $U(\mathcal{G})^*$

$$K_{\Lambda_1} \cdot K_{\Lambda_2} = K_{\Lambda_1 + \Lambda_2} \quad \text{for} \quad \Lambda_1, \Lambda_2 \in t^0. \tag{5.2.2}$$

$$L(h)K_\Lambda = \langle \Lambda, h \rangle K_\Lambda \quad \text{and} \quad R(h)K_\Lambda = -\langle \Lambda, h \rangle K_\Lambda \quad \text{for} \quad h \in t, \quad \Lambda \in t^0. \tag{5.2.3}$$

LEMMA 5.2.1. *Let* $\varphi \in U(\mathcal{G})^*$ *be a left* b*-finite and right* b_-*-finite element,* a, b *nonnegative integers. Assume that*

$$R(f_i)^{1+a} R(U(n_-))\varphi = 0. \tag{5.2.4}$$

(5.2.5) *Either* $R(e_i)^{1+b}(R(U(n_-))\varphi|_{U(b)}) = 0$ *or* $L(e_i)^{1+b}L(U(n))\varphi = 0$.

(5.2.6) *Assume that* t *acts, by* R, *semisimply on* $(R(U(b_-))\varphi)|_{U(b)} \subset U(b)^*$ *and its weight* Λ *satisfies* $\Lambda(h_i) \leq -a - b$ *and* $\Lambda(h_i) \in \mathbf{Z}$.

Then φ *is* p_i*-finite.*

Proof. Let N be an integer such that $N \geq 1 - \Lambda(h_i)$ for any weight Λ of $R(U(b_-))\varphi|_{U(b)}$. By Lemma 5.1.4, it is enough to show

$$L(f_i)^{N+m}\varphi = 0 \quad \text{if} \quad m \gg 0. \tag{5.2.7}$$

Let I be the ideal of $U(b)$ given by $\{P \in U(b); L(P)\varphi = 0\}$. Then by Lemma 5.1.7 we have $f_i^{N+m}U(\mathfrak{g}) \subset U(n_i)f_i^N\mathbf{C}[e_i]U(b_-) + IU(\mathfrak{g})$. We have

$$f_i^N e_i^k = \Sigma \frac{N!k!}{(N-\nu)!(k-\nu)!} e_i^{k-\nu}(-h_i - N - k + 2\nu;\, \nu)f_i^{N-\nu} \tag{5.2.8}$$

where $(x; n) = x(x-1)\cdots(x-n+1)/n!$.

We obtain

$$\varphi(f_i^{N+m}U(\mathfrak{g})) \subset \sum_{0\leq\nu\leq k,N} \varphi(U(n_i)e_i^{k-\nu}(-h_i - N - k + 2\nu;\, \nu)f_i^{N-\nu}U(b_-)). \tag{5.2.9}$$

Hence it is enough to show

$$\varphi(U(n_i)e_i^{k-\nu}(-h_i - N - k + 2\nu;\, \nu)U(t)f_i^{N-\nu}U(n_-)) = 0 \quad \text{for} \quad 0 \leq \nu \leq k, N. \tag{5.2.10}$$

If $N - \nu \geq 1 + a$, (5.2.10) holds by (5.2.4). If $k - \nu \geq 1 + b$, (5.2.10) holds by (5.2.5). Hence we may assume $0 \leq N - \nu \leq a$ and $0 \leq k - \nu \leq b$. Then in this case, it is enough to show

$$(R((-h_i - N - k - 2\nu;\, \nu))R(U(b_-))\varphi)|_{U(b)} = 0. \tag{5.2.11}$$

This is true, if for any weight Λ of $R(U(b_-))\varphi|_{U(b)}$ satisfies

$$0 \leq -\Lambda(h_i) - N - k + 2\nu \leq \nu - 1.$$

This is true if $N \geq 1 - \Lambda(h_i)$, $0 \leq N - \nu \leq a$ and $0 \leq k - \nu \leq b$.

COROLLARY 5.2.2. $K_\Lambda \in \mathcal{O}(G_\infty)$ *if* $\Lambda \in P_+$.
In fact, we can apply the preceding lemma with $a = b = 0$.

5.3. For a subset J of I, we set

(5.3.1) $$\Delta_J = \Delta \cap (\sum_{j \in J} \mathbf{Z}\alpha_j) \quad \text{and} \quad \Delta_J^{\pm} = \Delta^{\pm} \cap \Delta_J,$$

(5.3.2) $$\mathcal{G}_J = t \oplus \bigoplus_{\alpha \in \Delta_J} \mathcal{G}_\alpha; \qquad n_J^{\pm} = \bigoplus_{\alpha \in \Delta_{\pm} \setminus \Delta_J} \mathcal{G}_\alpha.$$

Then $\mathcal{G} = n_J^+ \oplus \mathcal{G}_J \oplus n_J^-$ and $U(\mathcal{G}) \leftarrow U(n_J^+) \otimes U(\mathcal{G}_J) \otimes U(n_J^-)$.
We have

(5.3.3) $$[\mathcal{G}_J + n_J^+, n_J^+] \subset n_J^+.$$

Since $\mathcal{G}_J$ is also a Kac-Moody algebra, we set $G_{J\infty}$ the corresponding variety Spec$(A(\mathcal{G}_J, P))$. We also set U_J, $U_J^{\mp}$ the subgroups of U and U^- with the Lie algebra $\hat{n}_J^+$ and $\hat{n}_J^-$. Set

(5.3.4) $A_J = \bigoplus_{\mu \in P} \{\varphi \in U(\mathcal{G})^*;\ \varphi$ is a weight vector of weight μ with respect to the left action of t and φ is left p_j-finite and right p_j^--finite for any $j \in J$ and φ is left b-finite and right b_--finite$\}$.

Then we can easily show that

(5.3.5) A_J is a subring of $U(\mathcal{G})^*$ and a two-sided sub-$\mathcal{G}$-module of $U(\mathcal{G})^*$.

LEMMA 5.3.4. $A_J \cong \mathcal{O}(U_J) \otimes \mathcal{O}(G_J) \otimes \mathcal{O}(U_J^-)$.

Proof. We have

(5.3.6) $$\mathcal{O}(U_J) \otimes \mathcal{O}(G_J) \otimes \mathcal{O}(U_J^-) \subset (U(n_J^+) \otimes U(\mathcal{G}_J) \otimes U(n_J^-))^* \cong (U(\mathcal{G}))^*.$$

We shall show first $A_J \subset \mathcal{O}(U_J) \otimes \mathcal{O}(G_J) \otimes \mathcal{O}(U_J^-)$. For $f \in A_J$, let $\mathfrak{a}$ be the annihilator in $U(b)$ of $L(U(b))f$. Then $f : U(\mathcal{G}) \to \mathbf{C}$ splits into $U(\mathcal{G}) \leftarrow U(n_J) \otimes U(\mathcal{G}_J) \otimes U(n_J^-) \to (U(n_J)/(\mathfrak{a} \cap U(n_J))) \otimes U(\mathcal{G}_J) \otimes U(n_J^-)$. Hence f belongs to $\mathcal{O}(U_J) \otimes (U(\mathcal{G}_J) \otimes U(n_J^-))^*$. Similarly f belongs to

$(U(n_J) \otimes U(\mathfrak{g}_J)) \otimes \mathcal{O}(U_J)$, and hence to the intersection $\mathcal{O}(U_J) \otimes U(\mathfrak{g}_J)^* \otimes \mathcal{O}(U_J^-)$. Write $f = \Sigma_{k=1}^N \varphi_k \otimes \psi_k \otimes \xi_k$ with $\varphi_k \in \mathcal{O}(U_J)$, $\psi_k \in U(\mathfrak{g}_J)^*$, $\xi_k \in \mathcal{O}(U_J^-)$. We take an expression such that N is minimal among them. Then there are $S_k^\nu \in U(n_J)$ and $R_k^\nu \in U(n_J^-)$ such that $\varphi_k(S_{k'}^\nu)\psi_k(R_{k'}^\nu) = \delta_{kk'}$. Hence $\psi_k(P) = f(S_k^\nu P R_k^\nu)$. Since A_J is a two-sided $\mathfrak{g}$-module, ψ_k belongs to $\mathcal{O}(G_J)$.

We shall prove the converse inclusion $A_J \supset \mathcal{O}(U_J) \otimes \mathcal{O}(G_J) \otimes \mathcal{O}(U_J)$. In order to see this, it is enough to show that any element in $\mathcal{O}(U_J) \otimes \mathcal{O}(G_J) \subset (U(n_J \oplus \mathfrak{g}_J))^*$ is b-finite and p_j-finite for any $j \in J$. For any $\varphi \in \mathcal{O}(U_J)$, there exists a two-sided ideal $\mathfrak{a}$ of $U(n_J)$ such that $[b, \mathfrak{a}] \subset \mathfrak{a}$, dim $U(n_J)/\mathfrak{a}$ and $\varphi(\mathfrak{a}) = 0$. For any $\psi \in \mathcal{O}(G_J)$, there exists an ideal k of $U(\mathfrak{g}_J \cap b)$ such that $\dim(U(\mathfrak{g}_J \cap b)/k) < \infty$ and $\psi(k) = 0$. Since $bU(n_J) \subset U(n_J) + U(n_J)(b \cap \mathfrak{g}_J)$, $U(n_J) \otimes k + \mathfrak{a} \otimes U(\mathfrak{g}_J)$ is a left b-module. Since $\varphi \otimes \psi$ decomposes into

$$U(n_J) \otimes U(\mathfrak{g}_J) \to U(n_J + \mathfrak{g}_J)/(U(n_J) \otimes kU(\mathfrak{g}_J) + \mathfrak{a} \otimes U(\mathfrak{g}_J))$$

$$\cong (U(n_J)/\mathfrak{a}) \otimes (U(\mathfrak{g}_J)/kU(\mathfrak{g}_J)),$$

$\varphi \otimes \psi$ is b-finite.

We have

$$(adf_i)^N U(n_J) \subset \mathfrak{a} \quad \text{for} \quad N \gg 0 \quad \text{for} \quad i \in J.$$

In fact, this follows from the fact that for any $\lambda \in t^0$, $\lambda + m\alpha_i$ is a weight of $U(n_J)$ except finitely many integer m (Lemma 5.1.6). Hence $\varphi \otimes \psi$ is f_i-finite. Thus, $\varphi \otimes \psi$ is p_i-finite for any $i \in J$. Since $\varphi \otimes \psi$ is b_--finite, we obtain $\varphi \otimes \psi \in A_J$.

PROPOSITION 5.3.5. ([K-P]). *$A_J = \mathcal{O}(G_\infty)[K_\Lambda^{-1}; \Lambda \in P_+, h_j(\Lambda) = 0$ for $j \in J]$.*

Proof. Since K_Λ is invertible in $\mathcal{O}(G_{J\infty})$ if $h_j(\Lambda) = 0$ for $j \in \Lambda$, we have

$$A_J \supset \mathcal{O}(G_\infty)[K_\Lambda^{-1}; \Lambda \in P_+, h_j(\Lambda) = 0 \text{ for } j \in J].$$

Now, we shall show the converse inclusion.

Let $\varphi \in A_J$. Then there exists $a > 0$ such that $R(n_-)^{1+a}\varphi = L(n)^{1+a}\varphi = 0$. Let S be the set of weights of $R(U(b_-))\varphi$ with respect to the right action of t. Taking a sufficiently large, we may assume that $\langle \lambda, h_i \rangle \leq a$ for any $i \in I$ and $\lambda \in S$. Moreover, there exists a finite set K of I such that $R(e_i)\varphi = L(e_i)\varphi = 0$, $\langle \lambda, h_i \rangle = 0$ for any $i \in I \backslash K$ and $\lambda \in S$.

Now, let $\Lambda \in P_+$ be such that $h_j(\Lambda) = 0$ for $j \in J$ and $h_j(\Lambda) \geq a$ for $j \in K \backslash J$. Then $\varphi \cdot K_\Lambda$ is p_j-finite for $j \in J$ and p_j-finite for $j \in I \backslash J$ by Lemma 5.2.1. Hence $\varphi K_\Lambda \in \mathcal{O}(G_\infty)$.

5.4. By Proposition 5.3.5, for finite subsets J and J' with $J \subset J'$, $\mathrm{Spec}(A_J)$ is an open subscheme of $\mathrm{Spec}(A_{J'})$. We set $G_{\infty f} = \bigcup_J U_J \times G_J \times U_J^-$ where J ranges through finite subsets of I. Then $G_{\infty f}$ is an irreducible separated scheme, and $U \times T \times U_-$ is an open subscheme of $G_{\infty f}$. The groups P_i and P_i^- act on $G_{\infty f}$ from the left and the right, respectively.

Definition 5.4.1. Let G be the smallest open subset of $G_{\infty f}$ containing $U \times T \times U_-$ closed by the left and right actions of G_i $(i \in I)$.

5.5. Hence G is invariant by the left action of P_i, and the right action of P_{i-}. Since $G_{\infty f}$ is irreducible, G is also irreducible. In Section 6, we shall study more precisely the structure of $G_{\infty f}$ in the symmetrisable case.

5.6. Since G_i acts on G_∞, $G_{\infty f}$ and G, $s_i' \in G_i$ acts on them. Then we have the braid condition (3.1.13). In fact, if $i, j \in I$ satisfies $\langle h_i, \alpha_j \rangle \langle h_j, \alpha_i \rangle \leq 3$, then the semisimple part of $G_{\{i,j\}}$ is a finite-dimensional group. Thus we can apply the braid condition for finite-dimensional Lie group and hence s_i' and s_j' satisfy the braid condition in $G_{\{i,j\}}$. Since we can check easily that $G_{\{i,j\}}$ acts on G_∞, $G_{\infty f}$ and G, we obtain (3.1.13). Thus the braid group W' acts on G, $G_{\infty f}$ and G_∞.

Let us embed W into W' by $w \mapsto s_{i_1}' \cdots s_{i_l}'$ where $w = s_{i_1} \cdots s_{i_l}$ is a reduced expression of w.

LEMMA 5.6.1. $G = \bigcup_{w \in W} w(U \times T \times U_-)$
$= \bigcup_{w \in W} (U \times T \times U_-)w.$

In fact, we have $P_i^- = G_i \ \ U_-$, and $(U \times T \times U_-)P_i^- = Ue \cdot P_i^- = UG_ie \cdot U_- = P_ie \cdot U_-$. Since $P_i \subset s_iBG_i^- \cup BG_i^-$, we have $P_ie \cdot U_- \subset s_iBeU_- \cup Be \cdot U_-$. Thus $\bigcup_{w \in W} w(U \times T \times U_-)$ is invariant by P_{i-}. Hence if A (resp. A') is the smallest open subset containing $U \times T \times U_-$ and invariant by P_i (resp. P_{i-}) for any i, we have $A \supset \bigcup_{w \in W} w(U \times T \times U_-) \supset A'$. Similarly $A \subset A'$. Hence $A = A' = \bigcup_{w \in W} w(U \times T \times U_-)$.

5.7. In general, let X be a scheme and G a group scheme acting on X. We say that G acts locally freely on X if any point has a G-stable open neighborhood which is isomorphic to $G \times U$ for some scheme U. In this case, the quotient X/G in the Zariski topology is representable by a scheme. Note that X/G is not necessarily separated even if X is separated.

5.8. Now, B_- acts on G locally freely. Hence G/B_- is a scheme and covered by open affine subsets $wU \times B_-/B_-$. Note that we have not yet shown that G/B_- is a separated scheme.

PROPOSITION 5.8.1. *$X \cong G/B_-$. Here X is the flag variety defined in Section* 4.

Proof. We have $G/B_- = \bigcup_{w \in W} wUB_-/B_-$ and $X = \bigcup_{w \in W} wUx_0$. We define for $w \in W'$, the morphism

$$\varphi_w : wUB_- \to wUx_0 \quad \text{by} \quad wgb_- \mapsto wg.$$

We shall show

$$\varphi_w = \varphi_{w'} \quad \text{on} \quad wUB_- \cap w'UB_-. \tag{5.8.1}$$

This follows from the case where $w' = 1$. If $w = 1$, this is trivial. If $w = s_i'^{\pm 1}$, then this is trivial because φ_w and φ_1 are the restrictions of $P_ieU_i^- \to X$ given by $geg' \mapsto gx_0$ $(g \in P_i, g' \in U_i^-)$.

Arguing by induction on the length of w, we may assume $w = s_i'^{\pm 1}w''$ and

$$\varphi_{w''}|_{w''UeB_- \cap UeB_-} = \varphi_1|_{w''UeB_- \cap UeB_-}$$

and hence

$$\varphi_w|_{wUeB_- \cap s_i'^{\pm 1}UeB_-} = \varphi_{s_i'^{\pm 1}}|_{wUeB_- \cap s_i'^{\pm 1}UeB_-}.$$

Hence φ_w and φ_1 coincide on $wUeB_- \cap s_i'^{\pm 1}UeB_- \cap UeB_-$. Since $wUeB_- \cap s_i'^{\pm 1}UeB_- \cap UeB_-$ is open dense in $wUB_- \cap w'UB_-$ and X is separated, we have (5.8.1).

Thus, we can construct $\varphi : G \to X$ such that $\varphi|_{wUeB_-} = \varphi_w$. Taking the quotient, we obtain $\tilde{\varphi} : G/B_- \to X$.

By the definition, $\tilde{\varphi}$ is W'-equivariant. Also, $\tilde{\varphi}$ is B-equivariant. This is because $\varphi|_{BeB_-}$ is B-equivariant and BeB_- is open dense in G.

Since $\tilde{\varphi}$ is clearly a local isomorphism and surjective, it is enough to show that $\tilde{\varphi}$ is injective. In order to see this, we shall prove that, for two $\mathbf{C}$-valued points g, g' of G/B_-, $\varphi(g) = \varphi(g')$ implies $g = g'$. Since φ is W'-equivariant, we may assume $g \in BeB_-/B_-$. Since φ is B-equivariant, we may assume $g = e \bmod B_-$. Assume $g' \in wUeb_-/B_-$ for $w \in W$. Write $g' = wuB_-/B_-$ for $u \in U$. Then $\varphi(g) = \varphi(g')$ implies $x_0 = wux_0$. Hence Proposition 4.5.9 implies $w = 1$ and Lemma 4.4.1 implies $u = 1$. Hence $g = g'$.

6. Symmetrisable case.

6.1. In Section 6, we shall assume that the set I of simple roots is finite and the Kac-Moody Lie algebra is symmetrisable. Then by Gabber-Kac [G-K], any integrable $U(\mathcal{G})$-module generated by a highest weight vector is semisimple. For $\Lambda \in P_+$, let L_Λ be the irreducible $\mathcal{G}$-module with highest weight Λ. Then we have

LEMMA 6.1.1. ([K-P]). $A(\mathcal{G}, P) = \mathcal{O}(G_\infty) \cong \bigoplus_{\Lambda \in P_+} L_\Lambda \otimes L_\Lambda^*$.

6.2. We shall assume further that any irreducible finite-dimensional representation of $\mathcal{G}$ is one-dimensional. This is equivalent to saying that any connected component of the Dynkin diagram of $\mathcal{G}$ is not finite-dimensional. In this case, letting $P_0 = \{\Lambda \in P; \langle \Lambda, h_j \rangle = 0 \text{ for any } j\}$, any irreducible finite-dimensional representation is $\mathbf{C}$ with weight $\Lambda \in P_0$.

LEMMA 6.2.1. $\bigoplus_{\Lambda \in P_+ \setminus P_0} (L_\Lambda \otimes L_\Lambda^*)$ *is an ideal of* $A(\mathcal{G}, P)$.

Proof. For $\Lambda_1, \Lambda_2 \in P_+ \setminus P_0$,

$$(L_{\Lambda_1} \otimes L_{\Lambda_1}^*) \cdot (L_{\Lambda_2} \otimes L_{\Lambda_2}^*) \subset \sum_{\Lambda} L_\Lambda \otimes L_\Lambda^*$$

where Λ ranges over the set Λ with $L_\Lambda \subset L_{\Lambda_1} \otimes L_{\Lambda_2}$. If $\Lambda \in P_0$ and $L_\Lambda \subset L_{\Lambda_1} \otimes L_{\Lambda_2}$, then we have a homomorphism $L_{\Lambda_1}^* \otimes L_\Lambda \to L_{\Lambda_2}$. Therefore L_{Λ_2} has a lowest weight vector, which implies L_{Λ_2} is finite-dimensional. Hence $\Lambda_2 \in P_0$, which is a contradiction.

Definition 6.2.2. Let us define $\infty \in G_\infty$ by

$$A(\mathcal{G}, P) \to A(\mathcal{G}, P)/(\sum_{\Lambda \in P_+ \setminus P_0} L_\Lambda \otimes L_\Lambda^*) \rightleftarrows \bigoplus_{\Lambda \in P_0} \mathbf{C}K_\Gamma \to \mathbf{C}$$

where the last arrow is given by $K_\Lambda \mapsto 1$.

Note that

$$T \cdot \infty \cong \operatorname{Spec}(\mathbf{C}[K_\Lambda; \Lambda \in P_0]) \tag{6.2.1}$$

$$P_i\infty = \infty P_i^- = T \cdot \infty \quad \text{for any } i. \tag{6.2.2}$$

6.3. Proposition 6.3.1.

$$G_\infty \setminus T \cdot \infty = \bigcup_{\substack{w \in W' \\ J \neq I}} w(U_J \times G_J \times U_J^-) = \bigcup_{\substack{w \in W' \\ J \neq I}} (U_J \times G_J \times U_J^-)w.$$

Proof. The last identity can be proven as in the proof of Lemma 5.6.1. For $v \in L_\Lambda$, $w \in L_\Lambda^*$, let us denote by $\langle v, gw \rangle$ the corresponding function on $g \in G_\infty$. Now, let g be an element of $G_\infty \setminus T \cdot \infty$. Let us denote by G_f the subgroup of $\operatorname{Aut}(L_+)$ generated by the G_i. By the assumption, there is $\Lambda \in P_+ \setminus P_0$ and $v \in L_\Lambda$, $w \in L_\Lambda^*$ such that $\langle v, gw \rangle \neq 0$. Then $\{v' \in L_\Lambda, \langle G_f v', gw \rangle = 0\}$ is a $\mathcal{G}$-module. Hence, it is zero. Therefore, if we denote by v_Λ the highest weight vector of L_Λ, then $\langle G_f v_\Lambda, gw \rangle \neq 0$. Hence there exists $g_0 \in G_f$ such that $\langle v_\Lambda, g_0^{-1} gw \rangle \neq 0$. Since $\bigcup w(U_J \times G_J \times U_J^-)$ is invariant by G_f, we may assume from the beginning $\langle v_\Lambda, gw \rangle \neq 0$.

Similarly, $\{w'; \langle v_\Lambda, gG_f w' \rangle = 0\}$ is $\mathcal{G}$-invariant and hence it is zero. Therefore if $v_{-\Lambda}$ is the lowest weight vector of L_Λ^* such that $\langle v_\Lambda, v_{-\Lambda} \rangle = 1$, then $\langle v_\Lambda, gG_f v_{-\Lambda} \rangle \neq 0$. Hence replacing g with an element in gG_f, we may assume $\langle v_\Lambda, gv_{-\Lambda} \rangle \neq 0$. Since $K_\Lambda(g) = \langle v_\Lambda, gv_{-\Lambda} \rangle \neq 0$, g belongs to $U_{I \setminus \{j\}} \times G_{I \setminus \{j\}} \times U_{I \setminus \{j\}}^-$ for $j \in I$ with $\langle h_j, \Lambda \rangle \neq 0$, by Proposition 5.3.5.

7. Example.

7.1. We shall give here one example $A_\infty^{(1)}$. Let I be $\mathbf{Z}$, $P = \bigoplus_{i \in I} \mathbf{Z}\Lambda_i$, $\alpha_i = 2\Lambda_i - \Lambda_{i+1} - \Lambda_{i-1}$ and $h_i \in t$ is given by $\langle h_i, \Lambda_j \rangle = \delta_{i,j}$.

Let $V' = \mathbf{C}^\mathbf{Z} = \Pi_{i \in \mathbf{Z}} \mathbf{C}v_i$, $V_{\leq q} = \Pi_{i \leq q} \mathbf{C}v_i \subset V'$ for $q \in \mathbf{Z}$ and $V = \bigcup V_{\leq q}$. Let us define $g \to \operatorname{End}(V)$ by

$$t \ni h : \sum a_i v_i \mapsto \sum (\Lambda_i(h) - \Lambda_{i-1}(h)) a_i v_i$$

$$e_i : \sum a_j v_j \mapsto a_{i+1} v_i$$

$$f_i : \sum a_j v_j \mapsto a_i v_{i+1}.$$

For $p \leq q$, let $\mathrm{GL}_{p,q}(\infty)$ be the subgroup of $\mathrm{GL}(V)$ given by

$$\{g \in \mathrm{End}(V); g|_{V_{\leq k}} \subset V_{\leq k} \text{ for } k < p \text{ or } k \geq q \text{ and } g|_{V_{\leq k}/V_{\leq k-1}} \text{ is invertible for } k < p \text{ or } k > q \text{ and } g|_{V_{\leq q}/V_{\leq p-1}} \text{ is invertible}\}.$$

This is an affine group scheme. With matrix expression, $\mathrm{GL}_{p,q}(\infty) = \{(g_{ij}); g_{ij} = 0 \text{ for } j < i \text{ and } j < p, j < i \text{ and } i \geq q, g_{ii} \text{ invertible for } i < p \text{ or } i > q \text{ and } \det((g_{ij})_{p \leq i,j \leq q}) \text{ is invertible}\}$. We define the affine group scheme $\mathrm{GL}_{p,q}(\infty)$ by

$$\widetilde{\mathrm{GL}}_{p,q}(\infty) = \mathrm{GL}_{p,q}(\infty) \times \mathbf{C}^*.$$

We define for $p' \leq p \leq q \leq q'$ $\widetilde{\mathrm{GL}}_{p,q}(\infty) \to \widetilde{\mathrm{GL}}_{p',q'}(\infty)$ by

$$(g, c) \mapsto (g, c \det(g|_{V_{\leq q'}/V_{\leq q}})).$$

Then for $p'' \leq p' \leq p \leq q \leq q' \leq q''$,

$$\begin{array}{ccc} \widetilde{\mathrm{GL}}_{p,q}(\infty) & \longrightarrow & \widetilde{\mathrm{GL}}_{p',q'}(\infty) \\ & \searrow & \downarrow \\ & & \widetilde{\mathrm{GL}}_{p'',q''}(\infty) \end{array}$$

commutes. We set

$$\widetilde{\mathrm{GL}}(\infty) = \varinjlim_{(p,q)} \widetilde{\mathrm{GL}}_{p,q}(\infty), \qquad \mathrm{GL}(\infty) = \varinjlim_{(p,q)} \mathrm{GL}_{p,q}(\infty).$$

Then $\widetilde{\mathrm{GL}}(\infty)$ and $\mathrm{GL}(\infty)$ are ind-objects in the category of schemes with group structure. The group $\widetilde{\mathrm{GL}}_{p,q}(\infty)$ coincides with $U_J \times G_J$ where $J = \{i \in \mathbf{Z}; p \leq i \leq q\}$. Note that we have an exact sequence

$$1 \to \mathbf{C}^* \to \widetilde{\mathrm{GL}}(\infty) \to \mathrm{GL}(\infty) \to 1,$$

which does not split.

In this case, the flag variety is, under the notation in Corollary 2.2.3, $\{(W_i)_{i\in\mathbf{Z}};\ W_i \in \mathrm{Grass}^i(V),\ W_i \subset W_{i+1}\}$.

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