C. R. Acad. Sci. Paris, t. 315, Série I, p. 375-380, 1992

Théorie des groupes/*Group Theory*

Vertex models and crystals

Seok-Jin Kang, Masaki Kashiwara, Kailash C. Misra, Tetsuji Miwa, Toshiki Nakashima and Atsushi Nakayashiki

Abstract – The one point functions of the vertex models associated with quantum affine Lie algebras are computed by using the path description of the crystal bases of integral highest weight modules.

Modèles de vertex et cristaux

Résumé – *Les probabilités d'état local des modèles de vertex associés aux algèbres affines quantiques sont calculées en utilisant la description des bases cristallines des modules intégrables de plus haut poids en termes de chemins.*

Version française abrégée – La notion d'algèbre enveloppante quantique $U_q(\mathfrak{g})$ a son origine dans la théorie des modèles exactement solubles. Le paramètre q qui apparaît dans $U_q(\mathfrak{g})$ est la température et $q=0$ correspond à la température du zéro absolu. La considération de $q=0$ a conduit l'un des auteurs à introduire la notion de base cristalline, une base d'un $U_q(\mathfrak{g})$-module en $q=0$. D'autre part les probabilités d'état local jouent un rôle important dans la théorie des modèles exactement solubles. Elles peuvent être calculées par la méthode des matrices de transfert de coin inventée par Baxter et elles deviennent ainsi des sommes infinies le long des chemins. De plus cette somme s'exprime souvent en termes des caractères des $U_q(\mathfrak{g})$-modules de plus haut poids. Dans cette Note on clarifie les relations entre bases cristallines et chemins, ce qui permet le calcul explicite des probabilités d'état local. Soient $\mathfrak{g}$ une algèbre de Lie affine indécomposable, $\mathfrak{t}$ sa sous-algèbre de Cartan, $\{\alpha_i\}_{i\in I}$ son système de racines simples, $\{h_i\}_{i\in I}$ celui des coracines simples et $\{\Lambda_i\}_{i\in I}$ celui des poids fondamentaux. Soient c le centre de $\mathfrak{t}$, δ la racine imaginaire et ρ le poids tel que $\langle h_i, \rho\rangle=1$. On prend $i_0\in I$ tel que $\delta-\alpha_{i_0}$ soit une combinaison linéaire des $\alpha_i (i\neq i_0)$. On note par $U'_q(\mathfrak{g})$ l'algèbre enveloppante quantique associée à $\mathfrak{g}$ dont la partie torique est engendrée par les q^{h_i}. Soit V un $U'_q(\mathfrak{g})$-module de dimension finie qui possède une base cristalline (L, B).

Pour $b\in B$ et $i\in I$, on note $\varepsilon_i(b)=\max\{n\geqq 0;\, \tilde{e}_i^n b\neq 0\}$, $\varphi_i(b)=\max\{n\geqq 0;\, \tilde{f}_i^n b\neq 0\}$ et $\varepsilon(b)=\sum\varepsilon_i(b)\Lambda_i$, $\varphi(b)=\sum\varphi_i(b)\Lambda_i$. Alors $\varphi(b)-\varepsilon(b)$ est égal au poids $\mathrm{wt}(b)$ de b et $\langle c, \varphi(b)\rangle=\langle c, \varepsilon(b)\rangle$. On dit que B est parfaite de niveau $l\in E_{\geq 0}$ si elle satisfait les conditions :

(1) $\langle c, \varepsilon(b)\rangle\geqq l$ pour tout b dans B.

(2) $B\otimes B$ est connexe.

(3) Les applications ε et φ de

$$B_l=\{b\in B;\, \langle c, \varepsilon(b)\rangle=l\} \quad \text{dans} \quad (P_{cl}^+)_l=\{\lambda\in\sum_i \mathbf{Z}_{\geqq 0}\Lambda_i;\, \langle c, \lambda\rangle=l\}$$

sont bijectives.

(4) Il existe un poids λ_0 tel que $\#(B_{\lambda_0})=1$ et $\mathrm{wt}(B)\subset\lambda_0+\sum_{i\neq i_0}\mathbf{Z}_{\leqq 0}\alpha_i$.

Pour une telle base cristalline B, on note par $b(\lambda)$ l'image inverse de $\lambda\in(P_{cl}^+)_l$ par φ et on définit l'isomorphisme σ de $(P_{cl}^+)_l$ par $\varepsilon(b(\lambda))=\sigma\lambda$. Alors pour tout $k\gg 0$ il existe un et un seul isomorphisme $\psi_k: B(\lambda)\to B(\sigma^k\lambda)\otimes B^{\otimes k}$ tel que ψ_k commute avec les $\tilde{e}_i$ et les

Note présentée par Michel Duflo.

$\tilde{f}_i$ et que $\psi_k(u_\lambda)=u_{\sigma^k \lambda}\otimes b_k\otimes\ldots\otimes b_1$. Ici, $b_n=b(\sigma^{n-1}\lambda)$, $B(\lambda)$ est la base cristalline du $U'_q(\mathfrak{g})$-module de plus haut poids λ et u_λ est son élément de plus haut poids. Pour $\lambda\in(P^+_{cl})_l$, un λ-chemin est par définition une suite $\{p(n)\}_{n\geqq 1}$ dans B telle que $p(n)$ soit égal à b_n pour $n\gg 0$.

PROPOSITION. — *Soit* $\mathscr{P}(\lambda; B)$ *l'ensemble des* λ-*chemins.*

(i) $B(\lambda)$ *et* $\mathscr{P}(\lambda, B)$ *sont isomorphes par la correspondance* $b\leftrightarrow\{p(n)\}_{n\geqq 1}$ *donnée par* $\psi_k(b)=u_{\sigma^k \lambda}\otimes p(k)\otimes\ldots\otimes p(1)$ *pour* $k\gg 0$.

(ii) *Si* b *et* $\{p(n)\}$ *se correspondent, alors* $\mathrm{wt}(b)=\lambda+\sum(\mathrm{wt}(p(n))-\mathrm{wt}(b_n))-\omega(p)\delta$. *Ici* $\omega(p)=\sum_{k\geqq 1} k(H(p(k+1), p(k))-H(b_{k+1}, b_k))$, *pour une fonction* H *sur* $B\times B$ *à valeurs entières.*

Pour x, on note par V_x le $U'_q(\mathfrak{g})$-module V sur lequel les actions de e_{i_0} et f_{i_0} sont remplacées par xe_{i_0} et $x^{-1}f_{i_0}$. Si B est parfaite, il existe une et une seule R-matrice $R(x/y): V_x\otimes V_y\to V_y\otimes V_x$ à une constante près. De plus les valeurs propres de $R(x/y)$ en $q=0$ sont données par $(y/x)^{H(b, b')}$ pour b, $b'\in B$. La méthode des matrices transfert de coin permet d'exprimer les probabilités d'état local $P(a)$ du modèle donné par cette R-matrice comme suit : $P(a)=G(a)/\sum_{a'} G(a')$. Ici $G(a)=\sum_p q^{-4(\rho, a-\omega(p)\delta)}$ où p parcourt l'ensemble des λ-chemins de poids dans $a+\mathbf{Z}\delta$. Alors la proposition précédente entraîne que $G(a)$ est égale à la fonction de corde : $\sum \dim V(\lambda)_{a-i\delta}\, q^{-4(\rho, a-i\delta)}$.

---

1. CRYSTALS. — 1.1. *Quantized universal enveloping algebra.* — Consider (i) a free $\mathbf{Z}$-module P of finite rank and its dual P*, (ii) a finite set I, and $\alpha_i\in P$ and $h_i\in P^*$ for $i\in I$, (iii) a $\mathbf{Q}$-valued symmetric bilinear form (,) on P. We suppose (i) $\langle h_i, \alpha_j\rangle$ is a generalized Cartan matrix, (ii) $(\alpha_i, \alpha_i)\in\mathbf{Z}_{>0}$ for any i, (iii) $\langle h_i, \lambda\rangle=2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for any i and λ.

Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra ([2], [4]) associated with these data. It is the $\mathbf{Q}(q)$-algebra generated by the symbols e_i, f_i $(i\in I)$ and q^h $(h\in P^*)$ which satisfy the well-known defining relations. In this Note we follow [7] for the basic definitions and notations.

Denote by $U_q(\mathfrak{g}_{\{i\}})$ the $\mathbf{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i and q^h $(h\in P^*)$.

A $U_q(\mathfrak{g})$-module M is called *integrable* if it satisfies:

(i) $M=\bigoplus_{\lambda\in P} M_\lambda$, where $M_\lambda=\{u\in M\mid q^h u=q^{\langle h, \lambda\rangle}u \text{ for any } h\in P^*\}$,

(ii) $\dim M_\lambda<\infty$,

(iii) M is a union of finite-dimensional $U_q(\mathfrak{g}_{\{i\}})$-modules for any $i\in I$.

We define the tensor product of $U_q(\mathfrak{g})$-modules by the following comultiplication:

$$\Delta(e_i)=e_i\otimes t_i^{-t}+1\otimes e_i,\qquad \Delta(f_i)=f_i\otimes 1+t_i\otimes f_i,\qquad \Delta(t_i)=t_i\otimes t_i.$$

1.2. *Crystal.* — Consider a set B endowed with maps $\tilde{e}_i, \tilde{f}_i: B\to B\sqcup\{0\}$ $(i\in I)$. It is called a *crystal* if and only if it satisfies

(i) $b'=\tilde{f}_i b$ if and only $b=\tilde{e}_i b'$ for b, $b'\in B$ and $i\in I$,

(ii) $\varepsilon_i(b)$, $\varphi_i(b)<\infty$ for $b\in B$ and $i\in I$,

where $\varepsilon_i(b)=\max\{n\mid \tilde{e}_i^n b\neq 0\}$ and $\varphi_i(b)=\max\{n\mid \tilde{f}_i^n b\neq 0\}$.

A crystal B is called a P-*weighted crystal* if it has a decomposition $B=\bigsqcup_{\lambda\in P} B_\lambda$, and for any $i\in I$ and $b\in B_\lambda$, $\tilde{e}_i b\in B_{\lambda+\alpha_i}\sqcup\{0\}$, $\tilde{f}_i b\in B_{\lambda-\alpha_i}\sqcup\{0\}$, the equality $\varphi_i(b)-\varepsilon_i(b)=\langle h_i, \lambda\rangle$ holds.

We define the tensor product $B_1 \otimes B_2$ of crystals B_1 and B_2 as follows. As a set $B_1 \otimes B_2$ is equal to the Cartesian product $B_1 \times B_2$.

The maps $\tilde{e}_i, \tilde{f}_i : B_1 \otimes B_2 \to B_1 \otimes B_2 \perp\!\!\!\perp \{0\}$ are given by

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i(b_2) & \text{if } \varphi_i(b_1) \leqq \varepsilon_i(b_2), \end{cases}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geqq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2). \end{cases}$$

If both B_1 and B_2 are P-weighted crystals, then so is $B_1 \otimes B_2$.

1.3. *Crystal associated with representation.* — Let M be an integrable $U_q(\mathfrak{g})$-module, and u a weight vector belonging to M_λ. For each $i \in I$, we have a unique decomposition

$$u = \sum f_i^{(n)} u_n, \qquad \text{where} \quad u_n \in M_{\lambda + n\alpha_i} \cap \operatorname{Ker} e_i.$$

(Here $f_i^{(n)} = f_i^n/[n]!$ as in [7].)

Define $\tilde{e}_i, \tilde{f}_i \in \operatorname{End}_{\mathbf{Q}(q)}(M)$ by $\tilde{e}_i u = \sum f_i^{(n-1)} u_n$ and $\tilde{f}_i u = \sum f_i^{(n+1)} u_n$.

Let A be the subring of $\mathbf{Q}(q)$ consisting of $f \in \mathbf{Q}(q)$ that is regular at $q = 0$.

A *crystal base* of an integrable $U_q(\mathfrak{g})$-module M is a pair (L, B) such that

(1.1) L is a free A-submodule of M satisfying $M \cong \mathbf{Q}(q) \otimes_A L$,

(1.2) $L = \bigoplus_{\lambda \in P} L_\lambda$, where $L_\lambda = L \cap M_\lambda$,

(1.3) $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for $i \in I$,

(1.4) B is a $\mathbf{Q}$-base of L/qL,

(1.5) B is a P-weighted crystal with $\tilde{e}_i, \tilde{f}_i$ induced from those acting on L.

Suppose that $\lambda \in P_+ = \{\lambda \in P \mid \langle h_i, \lambda \rangle \geqq 0\}$. We denote by $V(\lambda)$ the irreducible $U_q(\mathfrak{g})$-module with highest weight λ. Let u_λ be a highest weight vector of $V(\lambda)$, and let $L(\lambda)$ be the smallest A-module containing u_λ stable under $\tilde{f}_i$'s. Set

$$B(\lambda) = \{ b \in L(\lambda)/qL(\lambda) \mid b = \tilde{f}_{i_1} \ldots \tilde{f}_{i_l} u_\lambda \bmod qL(\lambda) \} \setminus \{0\}.$$

Then $(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$ [7].

Sometimes we replace the condition (1.4) with

$$B_{ps} = B' \perp\!\!\!\perp (-B') \text{ where } B' \text{ is a } \mathbf{Q}\text{-base of } L/qL,$$

and define B in (1.5) to be $B_{ps}/\{\pm 1\}$. We then call (L, B_{ps}) a *crystal pseudo-base.*

2. Perfect crystals and paths. — 2.1. *Classical and affine crystals.* — Let $\mathfrak{g}$ be an affine Lie algebra and let $\mathfrak{t}$ be its Cartan subalgebra. We assume that $\{\alpha_i \mid i \in I\} \subset \mathfrak{t}^*$ and $\{h_i \mid i \in I\} \subset \mathfrak{t}$ are linearly independent and $\dim \mathfrak{t} = \sharp\, I + 1$. Let $\delta \in \sum_i \mathbf{Z}_{\geqq 0} \alpha_i$ be the generator of null roots, and let $c \in \sum_i \mathbf{Z}_{\geqq 0} h_i$ be the generator of the center. Set $\mathfrak{t}_{cl} = \oplus \mathbf{Q} h_i \subset \mathfrak{t}$ and $\mathfrak{t}_{cl}^* = (\bigoplus_i \mathbf{Q} h_i)^*$. Let $cl : \mathfrak{t}^* \to \mathfrak{t}_{cl}^*$ be the canonical morphism. We have an exact sequence $0 \to \mathbf{Q}\delta \to \mathfrak{t}^* \xrightarrow{cl} \mathfrak{t}_{cl}^* \to 0$.

Fix $i_0 \in I$ such that $\delta - \alpha_{i_0} \in \sum_{i \neq i_0} \mathbf{Z} \alpha_i$. For simplicity of notation we write 0 for i_0. Choose and fix a map $af : \mathfrak{t}_{cl}^* \to \mathfrak{t}^*$ satisfying $cl \circ af = id$ and $af \circ cl(\alpha_i) = \alpha_i$ for $i \neq 0$.

Let Λ_i be the element of $af(\mathfrak{t}_{cl}^*) \subset \mathfrak{t}^*$ such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}$. We take $P = \sum_i \mathbf{Z} \Lambda_i + \mathbf{Z}\delta \subset \mathfrak{t}^*$ and we set $P_{cl} = cl(P) \subset \mathfrak{t}_{cl}^*$.

Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra associated with P, and $U'_q(\mathfrak{g})$ the quantized universal enveloping algebra associated with P_{cl}. A P-weighted crystal is called an *affine crystal* and a P_{cl}-weighted crystal is called a *classical crystal.*

2.2. R-*matrix and energy function.* — Let V be a finite-dimensional $U'_q(\mathfrak{g})$-module. Set $V_x = \mathbf{Q}[x, x^{-1}] \otimes V$, and define the actions of e_i, f_i and t_i on V_x by adapting those on V as $x^{\delta_{i0}} e_i$, $x^{-\delta_{i0}} f_i$ and t_i, respectively. The R matrix for V is a non-zero $\mathbf{Q}[x, x^{-1}, y, y^{-1}] \otimes U_q(\mathfrak{g})$-linear map $R(x/y, q): V_x \otimes V_y \to V_y \otimes V_x$ satisfying the Yang-Baxter equation [4].

Suppose that V has a crystal pseudo-base (L, B_{ps}). Let $B = B_{ps}/\{\pm 1\}$ be the associated crystal. Suppose also that

(2.1) $B \otimes B$ is connected

(2.2) there exists $\lambda_0 \in P_{cl}$ such that $\sharp (B \otimes B)_{2\lambda_0} = 1$.

Then, there exists an R-matrix.

A **Z**-valued function H on $B \otimes B$ is called an *energy function* on B if for any $i \in I$ and $b \otimes b' \in B \otimes B$ such that $\tilde{e}_i(b \otimes b') \neq 0$ we have

$$\begin{aligned} H(\tilde{e}_i(b \otimes b')) &= H(b \otimes b') && \text{if } i \neq 0, \\ &= H(b \otimes b') + 1 && \text{if } i = 0 \text{ and } \varphi_0(b) \geqq \varepsilon_0(b'), \\ &= H(b \otimes b') - 1 && \text{if } i = 0 \text{ and } \varphi_0(b) < \varepsilon_0(b'). \end{aligned}$$

Take a base $\{u_j\}$ of V whose modulo classes form B. One can show that the $q \to 0$ limit of the R-matrix is diagonal with respect to $u_j \otimes u_k$ and, after a suitable normalization, is expressed by the energy function:

$$\lim_{q \to 0} R = \operatorname{diag}((y/x)^{H(u_j \otimes u_k)}).$$

2.3. *Perfect crystal and paths.* — Let B be a classical crystal. For $b \in B$, we set $\varepsilon(b) = \sum \varepsilon_i(b) \Lambda_i$ and $\varphi(b) = \sum \varphi_i(b) \Lambda_i$. We set $(P_+)_l = \{\lambda \in \sum \mathbf{Z}_{\geqq 0} \Lambda_i \mid \langle c, \lambda \rangle = l\}$ for $l \in \mathbf{Z}_{\geqq 0}$. We call B *perfect of level l* if

(i) it is associated with a finite-dimensional $U'_q(\mathfrak{g})$-module V which has a crystal pseudo-base (L, B_{ps}) such that (2.1-2) hold,

(ii) for any b, we have $\langle c, \varepsilon(b) \rangle \geqq l$,

(iii) the maps ε and φ from $B_l = \{b \mid \langle c, \varepsilon(b) \rangle = l\}$ to $(P_+)_l$ are bijective.

For a technical reason we now assume that $\mathfrak{g}$ is of rank $\geqq 3$.

Let $B(\lambda)$ be the affine crystal with highest weight λ, and denote by u_λ the highest weight element of $B(\lambda)$.

THEOREM 1. — *Let* B *be a perfect crystal of level l with an energy function* H, *and let* $\lambda \in (P^+_{cl})_l$ *be a dominant integral weight of level l. Let* b_0 *be the unique element of* B *such that* $\varepsilon(b_0) = \lambda$. *Then we have isomorphism of classical crystals*

$$B(\lambda) \otimes B \cong B(\lambda + \mathrm{wt}(b_0))$$

given by $u_\lambda \otimes b_0 \mapsto u_{\lambda + \mathrm{wt}(b_0)}$.

For $\lambda \in (P^+_{cl})_l$, let $b(\lambda)$ be the unique element of B such that $\varphi(b(\lambda)) = \lambda$. We define the isomorphism σ of $(P^+_{cl})_l$ by $\varepsilon(b(\lambda)) = \sigma\lambda$. We set $b_k = b(\sigma^{k-1}\lambda)$ and $\lambda_k = \sigma^k \lambda$ for $k \geqq 1$. Then by applying Theorem 1 repeatedly, we obtain an isomorphism of classical crystals

$$\psi_k: \quad B(\lambda) \cong B(\lambda_k) \otimes B^{\otimes k}$$

given by $u_\lambda \mapsto u_{\lambda_k} \otimes b_k \otimes \ldots \otimes b_1$.

The sequence $(b_1, b_2, \ldots)$ is called the *ground-state* path of weight λ. A λ-*path* in B is, by definition, a sequence $p=(p(n))_{n \geqq 1}$ in B such that $p(n)=b_n$ for $n \gg 0$. Let us denote by $\mathscr{P}(\lambda, \mathrm{B})$ the set of λ-paths. In the following theorem, we realize the crystal $\mathrm{B}(\lambda)$ as the set $\mathscr{P}(\lambda, \mathrm{B})$ of λ-paths.

THEOREM 2. — $\mathrm{B}(\lambda)$ *is isomorphic to* $\mathscr{P}(\lambda, \mathrm{B})$ *by* $\mathrm{B}(\lambda) \ni b \mapsto p \in \mathscr{P}(\lambda, \mathrm{B})$ *where* $\psi_k(b)=u_{\lambda_k} \otimes p(k) \otimes \ldots \otimes p(1)$ *for* $k \gg 0$.

The weight of b is given by the following formula:

$$\mathrm{wt}(b)=\lambda+\sum_{k=1}^{\infty}(\mathrm{af}(\mathrm{wt}\, p(k))-\mathrm{af}(\mathrm{wt}(b_k)))-\left(\sum_{k=1}^{\infty} k(\mathrm{H}(p(k+1) \otimes p(k))-\mathrm{H}(b_{k+1} \otimes b_k))\right)\delta.$$

The maps $\tilde{f}_i$ and $\tilde{e}_i$ on $\mathscr{P}(\lambda, \mathrm{B})$ have also a simple description similar to those given in [11], [5] for $\mathfrak{g}=\mathfrak{sl}(n)$ and in [10] for $\mathfrak{g}=\mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n$.

3. VERTEX MODEL AND 1 POINT FUNCTION. — 3.1. *Dual variable.* — Let V and B be as in the previous section. Choose and fix a base $\mathscr{W}=\{v(b) \mid b \in \mathrm{B}\}$ of V consisting of weight vectors such that $v(b) \bmod q\mathrm{L}=b$. Define the Boltzmann weight of a vertex model on a 2 dimensional square lattice $\mathscr{L}$ as

$$b_1 \overset{b_2}{\underset{b_3}{+}} b_4=\mathrm{R}(b_1, b_3 \mid b_2, b_4; x, q)$$

where $\mathrm{R}(x, q)(v(b_2) \otimes v(b_4))=\sum_{b_1, b_3} \mathrm{R}(b_1, b_3 \mid b_2, b_4; x, q)\, v(b_1) \otimes v(b_3)$. We consider the model in the region $|q|<1$ and $|x|<1$.

Choose and fix a dominant integral classical weight λ of level l. Let $(b_1, b_2, \ldots)$ be the corresponding ground-state path. We define b_n for $n \in \mathbf{Z}_{\leqq 0}$ by using periodicity. We number the set of vertices $\mathscr{V}$, the set of edges $\mathscr{E}$ and the set of faces $\mathscr{F}$ (of $\mathscr{L}$) by $((1/2)+\mathbf{Z}) \times ((1/2)+\mathbf{Z})$, $(((1/2)+\mathbf{Z}) \times \mathbf{Z}) \perp\!\!\!\perp (\mathbf{Z} \times ((1/2)+\mathbf{Z}))$ and $\mathbf{Z} \times \mathbf{Z}$, respectively. An edge configuration is a map $\mathrm{E}: \mathscr{E} \to \mathrm{B}$. The edge configuration E_{gr} given by $\mathrm{E}_{\mathrm{gr}}(i, j)=b_{i-j+(1/2)}$ is called the ground-state configuration. We set

$$\mathscr{C}(\lambda)=\{\mathrm{E} \mid \mathrm{E}(i, j)=\mathrm{E}_{\mathrm{gr}}(i, j) \text{ for all but finite } (i, j)\}.$$

A face configuration corresponding to $\mathrm{E} \in \mathscr{C}(\lambda)$ is a map $\mathrm{F}: \mathscr{F} \to \mathrm{P}_{\mathrm{cl}}$ such that

$$\mathrm{F}(i+(1/2), j)-\mathrm{F}(i-(1/2), j)=\mathrm{wt}(\mathrm{E}(i, j)), \qquad \mathrm{F}(i, j+(1/2))-\mathrm{F}(i, j-(1/2))=\mathrm{wt}(\mathrm{E}(i, j)).$$

This is uniquely determined up to an overall constant weight. We fix it by requiring that $\mathrm{F}(0, 0)=0$ for $\mathrm{E}=\mathrm{E}_{\mathrm{gr}}$. The 1 point function $\mathrm{P}(\mu \mid \lambda)$ (at $(0, 0)$) is the probability of the occurence of $\mathrm{F}(0, 0)=\mu$ in the sector defined by λ. Formally it is defined as

$$\mathrm{P}(\mu \mid \lambda)=\frac{\sum_{\mathrm{E} \in \mathscr{C}(\lambda)} \delta_{\mathrm{F}(0, 0)=\mu} \prod_{(i, j) \in \mathscr{V}} \mathrm{R}}{\sum_{\mathrm{E} \in \mathscr{C}(\lambda)} \prod_{(i, j) \in \mathscr{V}} \mathrm{R}}.$$

Here R is a shorthand for

$$\mathrm{R}(\mathrm{E}(i-(1/2), j), \mathrm{E}(i, j-(1/2)) \mid \mathrm{E}(i, j+(1/2)), \mathrm{E}(i+(1/2), j); x, q).$$

By applying the corner transfer matrix method [1], and then Theorem 2, we get

THEOREM 3. — *Suppose that* B *is a perfect crystal of level l. Then we have*

$$P(\mu \mid \lambda) = \frac{\sum_{i} \dim V(\lambda)_{\mu - i\delta}\, q^{-4(\rho, \mu - i\delta)}}{\sum_{\xi} \dim V(\lambda)_{\xi}\, q^{-4(\rho, \xi)}}.$$

This theorem generalizes partial results obtained in [3], [11], [5], [12].

3.2. *Perfect crystals.* — If B_1 and B_2 are perfect crystals of the same level, then so is $B_1 \otimes B_2$. The following is the Table of perfect crystals B of level l we have so far found (except for those obtained by tensor product). The last column in each row shows the crystal obtained from B by removing all the color 0-arrows.

Affine Lie algebra $\mathfrak{g}$	Classical part $\mathfrak{g}_{(0)}$	Crystal without 0-arrows
$A_n^{(1)}\ (n \geqq 2)$	A_n	$B(l\Lambda_k)\ (k=1, \ldots, n)$
$B_n^{(1)}\ (n \geqq 3)$	B_n	$B(l\Lambda_1)$
$C_n^{(1)}\ (n \geqq 2)$	C_n	$B(l\Lambda_n)$
$D_n^{(1)}\ (n \geqq 4)$	D_n	$B(l\Lambda_1)$, $B(l\Lambda_{n-1})$, $B(l\Lambda_n)$
$A_{2n}^{(2)}\ (n \geqq 2)$	B_n	$B(l\Lambda_1) \oplus B((l-2)\Lambda_1) \oplus \ldots$
$A_{2n-1}^{(2)}\ (n \geqq 3)$	C_n	$B(l\Lambda_1)$
$D_{n+1}^{(2)}\ (n \geqq 2)$	B_n	$B(l\Lambda_1) \oplus B((l-1)\Lambda_1) \oplus \ldots \oplus B(0)$

We thank Michio Jimbo for his interest in this work and valuable discussions. K. C. M. and S. J. K. are grateful for the hospitality by R.I.M.S. during their visit in 1991. T. M. is grateful for the hospitality by Department of Mathematics, N.C.S.U. during his visit 1990-1991 winter. K. C. M. gratefully acknowledges support from N.S.A./M.S.P. Grant ♯ MDA 90-H-4039.

Note remise le 10 décembre 1991, acceptée le 18 décembre 1991.

REFERENCES

[1] R. J. BAXTER, *Exactly solved models in statistical mechanics*, Academic Press, London, 1982.
[2] V. G. DRINFELD, *Proc. I.C.M. Berkeley*, 1986, pp. 798-820.
[3] E. DATE, M. JIMBO, A. KUNIBA, T. MIWA and M. OKADO, *Lett. Math. Phys.*, 17, 1989, pp. 69-77.
[4] M. JIMBO, *Internat. J. Modern Phys.*, *A*, 4 (15), 1989, pp. 3759-3777.
[5] M. JIMBO, K. C. MISRA, T. MIWA and M. OKADO, *Comm. Math. Phys.*, 136, 1991, pp. 543-566.
[6] M. KASHIWARA, *Comm. Math. Phys.*, 133, 1990, pp. 249-260.
[7] M. KASHIWARA, *C. R. Acad. Sci. Paris*, 311, Series I, 1990, pp. 277-280.
[8] M. KASHIWARA, *Proc. I.C.M. Kyoto*, 1990.
[9] M. KASHIWARA, *R.I.M.S. preprint*, No. 728, 1990.
[10] M. KASHIWARA and T. NAKASHIMA, *R.I.M.S. preprint*, No. 767, 1991.
[11] K. C. MISRA and T. MIWA, *Comm. Math. Phys.*, 134, 1990, pp. 79-88.
[12] M. OKADO, *Comm. Math. Phys.*, 134, 1990, pp. 467-486.

S.-J. K. and K. C. M. : *Department of Mathematics, North Carolina State University, Raleigh, NC* 27695-8205, *U.S.A;*

T. M. and T. N. : *Research Institute for Mathematical Sciences, Kyoto University, Kyoto* 606, *Japan;*

A. N. : *The Graduate School of Science and Technology, Kobe University, Rokkodai, Kobe* 657, *Japan.*