A Matching Procedure for the Sixth Painlevé Equation

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In the context of the isomonodromy deformation method, we present a constructive procedure (a matching procedure) to obtain the critical behavior of Painlevé VI transcendents and solve the connection problem. This procedure yields two and one parameter families of solutions, including logarithmic behaviors and three classes of solutions with Taylor expansion at a critical point. The matching procedure was developed by A.Kitaev for the fifth Painlevé equation.

Let us call $y=y(x)$ a solution of PVI. We consider its associated system of isomonodromy deformations:

$$\frac{d\Psi}{d\lambda}=\left[\frac{A_0(x,\theta)}{\lambda}+\frac{A_x(x,\theta)}{\lambda-x}+\frac{A_1(x,\theta)}{\lambda-1}\right]\Psi, \tag{1}$$

$$A_0+A_1+A_x=\begin{pmatrix}-\frac{\theta_\infty}{2} & 0\\ 0 & \frac{\theta_\infty}{2}\end{pmatrix}=:-A_\infty. \qquad \text{Eigenvalues }(A_i)=\pm\frac{1}{2}\theta_i, \quad i=0,1,x;$$

The matrix elements are certain rational functions of x, y, dy/dx and $\int y\ dx$. Conversely, a solution is $y(x)=x\ (A_0)_{12}\big\{x\left[(A_0)_{12}+(A_1)_{12}\right]-(A_1)_{12}\big\}^{-1}$.

For $x\to 0$, we divide the λ-plane into two domains. In the "outside" domain, defined for λ sufficiently big, namely $|\lambda|\geq|x|^{\delta_{OUT}}$, $\delta_{OUT}>0$, we approximate (1) by:

$$\frac{d\Psi_{OUT}}{d\lambda}=\left[\frac{A_0+A_x}{\lambda}+\frac{A_x}{\lambda}\sum_{n=1}^{N_{OUT}}\left(\frac{x}{\lambda}\right)^n+\frac{A_1}{\lambda-1}\right]\ \Psi_{OUT}. \tag{2}$$

In the "inside" domain, defined for λ comparable with x, namely $|\lambda|\leq|x|^{\delta_{IN}}$, $\delta_{IN}>0$, we approximate (1) by:

$$\frac{d\Psi_{IN}}{d\lambda}=\left[\frac{A_0}{\lambda}+\frac{A_x}{\lambda-x}-A_1\sum_{n=0}^{N_{IN}}\lambda^n\right]\ \Psi_{IN}. \tag{3}$$

Here, N_{IN} and N_{OUT} are suitable integers.

Then, we compute two fundamental matrix solutions $\Psi_{OUT}(\lambda,x)$, $\Psi_{IN}(\lambda,x)$, and we match them in the region of overlap, provided this is not empty:

$$\Psi_{OUT}(\lambda,x)\sim\Psi_{IN}(\lambda,x), \qquad |x|^{\delta_{OUT}}\leq|\lambda|\leq|x|^{\delta_{IN}}, \quad x\to 0 \tag{4}$$

This relation is to be intended in the sense that the leading terms of the local behavior of Ψ_{OUT} and Ψ_{IN} for $x\to 0$ must be equal. As a result, we obtain the leading terms of the behavior of $y(x)$ for $x\to 0$, without any *a priori* assumption.

The computation of the monodromy data of the systems (2), (3) and a careful matching of Ψ_{IN} and Ψ_{OUT} with a fundamental solution Ψ of (1), yield the monodromy data of (1) associated with the solution $y(x)$.

From the matching procedure we obtain different kinds of behaviors, depending on the choice of N_{OUT} and N_{IN}:

1) A well known two parameter family of solutions, with parameters $(\mathbf{a},\sigma)\in\mathbf{C}^2$, $\sigma\neq 0$, $|\Re\sigma|<1$:

$$y(x)\sim\frac{1}{\mathbf{a}}\ \frac{[\sigma^2-(\theta_0+\theta_x)^2][(\theta_0-\theta_x)^2-\sigma^2]}{16\sigma^3}\ x^{1-\sigma}\ +\frac{\theta_0^2-\theta_x^2+\sigma^2}{2\sigma^2}\ x\ -\ \frac{\mathbf{a}}{\sigma}\ x^{1+\sigma}, \quad \mathbf{a}\neq 0.$$

2) A one-parameter family of solutions, depending on a parameter $\mathbf{a}\in\mathbf{C}$:

$$y(x)=y_0(x)+y_1(x)\ \mathbf{a}x^\sigma\ +y_2(x)\ \left(\mathbf{a}x^\sigma\right)^2\ +...\ =\sum_{N=0}^{\infty}y_N(x)\ \left(\mathbf{a}x\right)^{N\sigma}, \quad x\to 0; \tag{5}$$

The $y_N(x)$'s are Taylor series $y_N(x)=\sum_{k=0}^\infty b_{k,N}(\theta_1,\theta_\infty,\theta_0,\theta_x)\ x^k$. $y_0(x)$ is (7) and $\sigma=\pm(\theta_\infty\pm\theta_1-1)\neq 0$. The $b_{k,N}(\theta_1,\theta_\infty,\theta_0,\theta_x)$'s are certain rational functions of their argument.

3) Logarithmic one-parameter families of solutions, with parameter $\mathbf{a} \in \mathbf{C}$:

$$y(x) \sim \begin{cases} x\left(\frac{\theta_x^2-\theta_0^2}{4}\log^2 x - 2\left(\mathbf{a}+\frac{\theta_0}{2}\right)\log x + \frac{4\ \mathbf{a}(\mathbf{a}+\theta_0)}{\theta_x^2-\theta_0^2}\right), & \theta_0 \neq \pm\theta_x, \\ x\ (\mathbf{a}\ \pm\ \theta_0\ \ln x), & \theta_0 = \pm\theta_x. \end{cases} \qquad x \to 0 \tag{6}$$

4) Three classes of Taylor expansions at $x = 0$. The first class is represented by one of the following series, defined for $\theta_\infty \neq 1$ and $\theta_1 - \theta_\infty \notin \mathbf{Z}$, or for $\theta_\infty \neq 1$ and $\theta_1 + \theta_\infty \notin \mathbf{Z}$ respectively:

$$y(x) = \frac{1 \pm \theta_1 - \theta_\infty}{1-\theta_\infty} + \sum_{n=2}^{\infty} b_n(\pm\theta_1, \theta_\infty, \theta_0, \theta_x)\ x^n, \tag{7}$$

The coefficients are certain rational functions of $\theta_0, \theta_\infty, \theta_0, \theta_x$. No parameter appears.

The second class is represented by a solution defined for $\theta_1 = \theta_\infty \neq 1$, $\theta_0 = \pm\theta_x$. This solution depends on a parameter $\mathbf{a} \in \mathbf{C}$:

$$y(x) = \frac{1}{1-\theta_\infty} + \mathbf{a}x + \sum_{n=2}^{\infty} b_n(\mathbf{a}; \theta_0, \theta_\infty)x^n. \tag{8}$$

The third class is represented by a solution defined for $\theta_\infty = 1$, $\theta_1 = 0$. Also this solution depends on a parameter $\mathbf{a} \in \mathbf{C}$:

$$y(x) = \mathbf{a} + \frac{1-\mathbf{a}}{2}(1+\theta_0^2-\theta_x^2)\ x + \sum_{n=2}^{\infty} b_n(a; \theta_0; \theta_x)x^n. \tag{9}$$

The coefficients are certain rational functions of their argument.

The symmetries of PVI (birational transformations) can be applied to the above solutions. For example, solutions (7), (8), (9) are the representatives of the three equivalent classes including all the solutions with Taylor expansion at a critical point.

The parameters $\mathbf{a}$ and σ can be computed in terms of the monodromy data. In this way, the connection problem is solved.