An Ergodic Study of Painlevé VI

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Abstract

An ergodic study of Painlevé VI is developed. The chaotic nature of its Poincaré return map is established for almost all loops. The exponential growth of the numbers of periodic solutions is also shown. Principal ingredients of the arguments are a moduli-theoretical formulation of Painlevé VI, a Riemann-Hilbert correspondence, the dynamical system of a birational map on a cubic surface, and the Lefschetz fixed point formula.

Most researches into Painlevé equations so far have been from the viewpoint of integrable systems. However we show that Painlevé VI is a *chaotic* dynamical system in a definitive sense.

The sixth Painlevé dynamics $\mathrm{P}_{\mathrm{VI}}(\kappa)$ is a holomorphic uniform foliation on a fibration

$$\pi_\kappa : M(\kappa) \to Z := \mathbb{P}^1 - \{0, 1, \infty\}$$

of certain smooth quasi-projective rational surfaces, where $M(\kappa)$ is realized as a moduli space of stable parabolic connections. The fiber $M_z(\kappa)$ over $z \in Z$ is called the space of initial conditions at time z. Due to the Painlevé property, each loop $\gamma \in \pi_1(Z, z)$ admits global horizontal lifts along the foliation and induces an automorphism, called the Poincaré return map along γ,

$$\gamma_* : M_z(\kappa) \to M_z(\kappa) \qquad \text{(see Figure 1)} \tag{1}$$

Poincaré return map (1) is chaotic along every non-elementary loop $\gamma \in \pi_1(Z, z)$

where the terms "chaotic" and "non-elementary loop" are used in the following senses.

Definition 1 The dynamical system of a holomorphic map $f : S \to S$ on a complex surface S (in our case, $S = M_z(\kappa)$ and $f = \gamma_*$) is said to be *chaotic* if there exists an f-invariant Borel probability measure μ on S such that the following conditions are satisfied:

(C1) f has a positive entropy $h_\mu(f) > 0$ with respect to the measure μ.

(C2) f is mixing with respect to the measure μ, that is, $\mu(f^{-n}(A) \cap B) \to \mu(A)\mu(B)$ as $n \to \infty$ for any Borel subsets A and B of S. In particular, f is ergodic with respect to μ.

(C3) μ is a hyperbolic measure of saddle type, that is, the two Lyapunov exponents $L_\pm(f)$ of f with respect to the ergodic measure μ satisfy $L_-(f) < 0 < L_+(f)$. Moreover, μ has product structure with respect to local stable and unstable manifolds.

(C4) hyperbolic periodic points of f are dense in the support of μ.

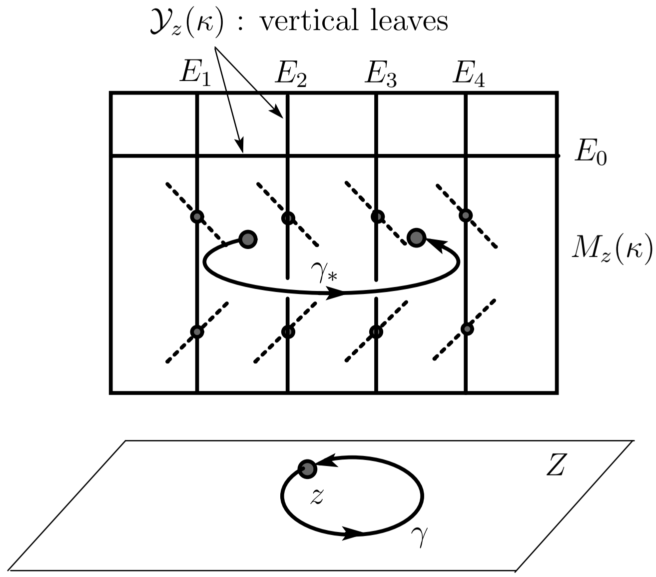

Figure 1: Poincaré return map $\gamma_* : M_z(\kappa) \circlearrowleft$ along a loop $\gamma \in \pi_1(Z, z)$

Definition 2 A loop $\gamma \in \pi_1(Z, z)$ is said to be *elementary* if γ is conjugate to the loop γ_i^m for some $i \in \{1, 2, 3\}$ and $m \in \mathbb{Z}$ (see Figure 2). Otherwise, γ is said to be *non-elementary*.

Theorem 3 *For any non-elementary loop $\gamma \in \pi_1(Z, z)$ there exists a natural γ_*-invariant Borel probability measure μ_γ such that all the conditions in Definition* 1 *are satisfied. Moreover,*

(1) **measure-theoretic entropy:** $h_{\mu_\gamma}(\gamma_*) = \log \lambda(\gamma)$,

(2) **number of periodic points:** $\#\mathrm{Per}_N(\gamma; \kappa) = \lambda(\gamma)^N + \lambda(\gamma)^{-N} + 4$, *with*

$$\mathrm{Per}_N(\gamma; \kappa) := \{\, Q \in M_z(\kappa) \,:\, \gamma_*^N Q = Q \,\} \qquad (\textit{period } N).$$

Here we have an algorithm to calculate $\lambda(\gamma)$ exactly in terms of the reduced word for γ.

$$\lambda(\gamma) \geq 3 + 2\sqrt{2} \qquad (\textit{equality} \Longleftrightarrow \gamma \textit{ is an "eight-loop"}).$$

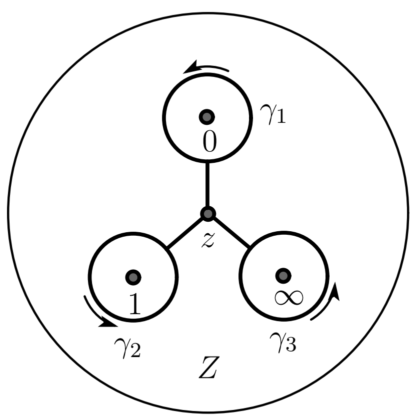

Figure 2: Three basic loops γ_1, γ_2, γ_3 in $Z = \mathbb{P}^1 - \{0, 1, \infty\}$