Hypergeometric Solutions to the q-Painlevé Equation of Type $(A_1 + A_1')^{(1)}$

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1 Introduction

We consider the series of q-Painlevé equations characterized by the degeneration diagram of affine Weyl groups[1]:

$$E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow (A_2 + A_1)^{(1)} \rightarrow (A_1 + \underset{|\alpha|^2=14}{A_1})^{(1)} \tag{1}$$

The q-Painlevé equations arise as birational actions of translation subgroups of the corresponding affine Weyl groups. Other actions can be regarded as the Bäcklund transformations.

It is well-known that the differential Painlevé equations admit particular solutions expressible in terms of the hypergeometric functions for special values of parameters. In [2] the simplest hypergeometric solutions to the q-Painlevé equations in the diagram (1) have been constructed. The following is the degeneration diagram of corresponding hypergeometric functions:

$$\begin{array}{c}\text{balanced}\\ {}_{10}W_9\end{array} \rightarrow {}_8W_7 \rightarrow \begin{array}{c}\text{balanced}\\ {}_3\phi_2\end{array} \rightarrow {}_2\phi_1 \rightarrow {}_1\phi_1 \rightarrow \begin{array}{c} {}_1\phi_1\left(\begin{matrix} a \\ 0\end{matrix};q,z\right) \\ {}_1\phi_1\left(\begin{matrix} 0 \\ b\end{matrix};q,z\right)\end{array} \rightarrow {}_1\phi_1\left(\begin{matrix} 0 \\ -q\end{matrix};q,z\right)$$

In this talk, we consider the simplest case type $(A_1+A_1')^{(1)}$, and develop further study of hypergeometric solutions. The q-Painlevé equation is given by

$$(\overline{F}F-1)(F\underline{F}-1) = \frac{at^2F}{F+t}, \tag{2}$$

where $F = F(t)$, $\overline{F} = F(qt)$, $\underline{F} = F(t/q)$, and a is a parameter. Equation (2) admits a continuous limit to the Painlevé II equation and hence it is sometimes referred as a q-P_{II} equation. The purpose of this talk is to present the hypergeometric solutions with their determinant formula for eq.(2) and to discuss the continuous limit in detail. This is a joint work with T. Hamamoto and N. S. Witte.

2 Hypergeometric Solutions

It is possible to find the case of eq.(2) where it is specialized to the Riccati equation. Then linearizing the Riccati equation by the standard technique, it is reduced to a second-order linear difference equation. By constructing power-series solutions of the linear equation, we obtain the following hypergeometric solution to eq.(2):

Proposition 1. For $a = q$, eq.(2) admits the particular solution given by

$$F = \frac{\overline{\Phi}}{\Phi}, \quad \overline{\Phi} + t\Phi = \underline{\Phi}, \quad \Phi(t) = A\,{}_1\varphi_1\left(\begin{matrix} 0 \\ -q\end{matrix};q,-qt\right) + B e^{\pi i \frac{\log t}{\log q}}\,{}_1\varphi_1\left(\begin{matrix} 0 \\ -q\end{matrix};q,qt\right). \tag{3}$$

$${}_1\varphi_1\left(\begin{matrix} 0 \\ -q\end{matrix};q,\mp qt\right) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n(q;-q)_n}(\mp qt)^n. \tag{4}$$

It is known that eq.(2) admit the Bäcklund transformation T

$$T(a) = q^2 a, \quad T(t) = t, \quad T(F) = t\frac{qat\overline{F} + \overline{F}F - 1}{(\overline{F}F-1)(t\overline{F} + \overline{F}F - 1)}, \tag{5}$$

by which we obtain "higher" solutions from the "seed" solution in Proposition 1. Such solutions are given as rational functions in terms of Φ and $\overline{\Phi}$, but in fact the denominators and numerators are factorized, and each factor admits determinant formula as follows:

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Theorem 2.

$$F(t)=\begin{cases}\dfrac{1}{q^N}\,\dfrac{\tau_N(t)\tau_{N+1}(qt)}{\tau_N(qt)\tau_{N+1}(t)} & (N\geq 0),\\[2ex] -\dfrac{1}{q^{N+1}}\,\dfrac{\tau_N(t)\tau_{N+1}(qt)}{\tau_N(qt)\tau_{N+1}(t)} & (N<0),\end{cases}\qquad \tau_N(t)=\begin{cases}\det(\Phi(q^{-i+2j-1}t))_{i,j=1,\dots,N} & (N>0)\\ 1 & (N=0)\\ \det(\Phi(q^{i-2j}t))_{i,j=1,\dots,-N} & (N<0)\end{cases}$$

satisfies eq.(2) with $a=q^{2N+1}$.

3 Continuous Limit

Proposition 2. With the replacements

$$F=i\mathrm{e}^{-\delta w},\quad a=\mathrm{e}^{-\frac{\eta}{2}\delta^3},\quad q=\mathrm{e}^{-\frac{\delta^3}{2}},\quad t=-2i\mathrm{e}^{-\frac{s}{2}\delta^2}=-2iq^{\frac{s}{\delta}},\tag{6}$$

eq. (2) as $\delta\to 0$ has a limit to the Painlevé II equation ($\mathrm{P_{II}}$)

$$\frac{d^2w}{ds^2}=2w^3+2sw+\eta.\tag{7}$$

It is known that $\mathrm{P_{II}}$ admits the hypergeometric solution for $\eta=2N+1$ ($N\in\mathbb{Z}$)

$$w=-\frac{d}{ds}\log\frac{\kappa_{N+1}}{\kappa_N},\quad \kappa_N=\begin{cases}\det(v^{(i+j-2)}(s)))_{i,j=1,\dots,N} & (N>0)\\ 1 & (N=0)\\ \det(v^{(i+j-2)}(s))_{i,j=1,\dots,M} & (N=-M<0)\end{cases}\tag{8}$$

$$\frac{d^2v}{ds^2}=-sv,\quad v(s)=C\mathrm{Ai}(\mathrm{e}^{\frac{\pi i}{3}}s)+D\mathrm{Ai}(\mathrm{e}^{-\frac{\pi i}{3}}s),\tag{9}$$

where Ai is the Airy function. The above limit works well for the hypergeometric solutions, except that naive application of the above procedure to power-series solutions does not yield meaningful limit. This is because $q=1$ is the essential singularity of the hypergeometric functions when viewed as functions in (t,q). This difficulty can be overcome by constructing integral representations of the hypergeometric functions in eq.(3)[3], and applying a generalization of the saddle-point method[4] to obtain their asymptotic expansions for $q\sim 1$ ($\delta\sim 0$). As a result, we obtain the limit of the hypergeometric functions:

Theorem 3.

$$\begin{aligned}\Psi(t)&=\mathrm{e}^{-\frac{\pi i}{2}\frac{\log t}{\log q}}\Phi(t)=-A\mathrm{e}^{-\frac{\pi i}{2}\frac{\log t}{\log q}}{}_1\varphi_1\left(\begin{matrix}0\\-q\end{matrix};\,q,-qt\right)+B\mathrm{e}^{\frac{\pi i}{2}\frac{\log t}{\log q}}{}_1\varphi_1\left(\begin{matrix}0\\-q\end{matrix};\,q,qt\right)\\ &\sim 2\pi^{\frac{1}{2}}\,\delta^{-\frac{1}{2}}\left[A\mathrm{e}^{\frac{\pi^2}{2\delta^3}+\frac{\pi i}{12}}\mathrm{Ai}(\mathrm{e}^{\frac{\pi i}{3}}s)+B\mathrm{e}^{-\frac{\pi^2}{2\delta^3}-\frac{\pi i}{12}}\mathrm{Ai}(\mathrm{e}^{-\frac{\pi i}{3}}s)\right]\end{aligned}\tag{10}$$

References

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