Moving frames and foliations

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The method of moving frame was invented and developed by E. Cartan for the theory of "infinite Lie groups" (which I prefer to call "Lie groupoids"). The main application was the study of the so-called "problème d'équivalence". Here, I present another application, to the theory of foliations, and more precisely to the non-linear differential Galois theory.

Let X be a smooth algebraic variety over $\mathbb{C}$, and let $\mathrm{Aut}(X)$ be the groupoid of germs of (analytic) maps of X into itself. Roughly speaking, a Lie groupoid L on X is an (algebraic) system of partial differential equations on X whose analytic solutions are a subgroupoid of $\mathrm{Aut}(X)$. Of course, the precise definition has to be given in terms of jet spaces.

If we restrict X to a suitable Zariski open set, one can prove that L has good regularity properties (= those which are usually assumed in differential geometry). Therefore, one has a description of L in terms of "moving frames", i.e. in terms of frame bundles + a fundamental form, as explained in the literature, e.g. Singer-Sternberg, or Morimoto.

Now, a Lie groupoid has a Lie algebra, given by the linearized equations along the identity; this is what I call "a D-Lie algebra", i.e. a linear system of p.d.e. on TX, such that the bracket of two solutions is again a solution. But the converse is not true (as an example, a Lie algebra over $\mathbb{C}$ does not come necessarily from an algebraic group). So given a D-Lie algebra A, one can consider its "envelope", i.e. the smallest Lie groupoid whose Lie algebra contains A. The most interesting case is the case of an (algebraic) foliation of X, which is obviously a D-Lie algebra (the solutions being the vector fields tangent to the foliation). I call its envelope its "Galois groupoid". In the case of linear equations, one sees that one recuperates in that way the usual Picard-Vessiot group.

Here again, the most convenient way to analyse the situation is the use of the method of moving frames of Cartan. In this context, it is very much related with the -Godbillon -Vey sequence. In the case of differential equations, this is also equivalent to consider the variational equations of any order, and their Picard-Vessiot theory.