The anti-self-dual Yang-Mills equation and the third Painlevé equation

Tetsu MASUDA (Kobe University)

The anti-self-dual Yang-Mills (ASDYM) equation is given by

$$\partial_z A_w - \partial_w A_z + [A_z, A_w] = 0,$$
$$\partial_{\tilde{z}} A_{\tilde{w}} - \partial_{\tilde{w}} A_{\tilde{z}} + [A_{\tilde{z}}, A_{\tilde{w}}] = 0,$$
$$\partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z - \partial_w A_{\tilde{w}} + \partial_{\tilde{w}} A_w + [A_z, A_{\tilde{z}}] - [A_w, A_{\tilde{w}}] = 0,$$

where $A_* = A_*(\tilde{z}, \tilde{w}, z, w)$ are $\mathfrak{sl}(2, \mathbb{C})$-valued functions. Both the ASDYM equation and the six Painlevé equations play a key role in the theory of integrable systems. Mason and Woodhouse have shown that the ASDYM equation can be reduced to the Painlevé equations under certain three dimensional Abelian groups of conformal symmetries [1].

Corrigan *et. al.* [2] have constructed a family of solutions to Yang's equation,

$$\partial_w \left(J^{-1} \partial_{\tilde{w}} J\right) - \partial_z \left(J^{-1} \partial_{\tilde{z}} J\right) = 0, \quad J \in SL(2, \mathbb{C}),$$

which is equivalent to the ASDYM equation. These solutions can be expressed in terms of Hankel determinants whose entries satisfy the Laplace equation.

On the other hand, it is known that the classical solutions to the Painlevé equations admit determinant expressions. In particular, the classical transcendental solutions can be expressed in terms of two-directional Wronskians whose entries satisfy (confluent) hypergeometric differential equations [3].

It is meaningful to investigate the reduction process from the ASDYM equation to the Painlevé equations with respect to special solutions and their τ-functions. The aim of my talk is to construct a family of solutions to the ASDYM equation and Yang's equation that corresponds to the classical transcendental solutions of the third Painlevé equation,

$$\frac{d^2 y}{d\rho^2} = \frac{1}{y}\left(\frac{dy}{d\rho}\right)^2 - \frac{1}{\rho}\frac{dy}{d\rho} - \frac{4}{\rho}[\eta_\infty \theta_\infty y^2 + \eta_0(\theta_0 + 1)] + 4\eta_\infty^2 y^3 - \frac{4\eta_0^2}{y}.$$

The main result is stated as follows.

Theorem Define a sequence of functions φ_j $(j \in \mathbb{Z})$ by

$$\varphi_j = (-2\eta_0)^{-j}\varphi^{[\nu+1-j]}, \quad \varphi^{[\nu]} = 2\tilde{w}^{\nu} e^{-\eta_0 z + \eta_\infty \tilde{z}} \rho^{-\nu} \psi_\nu \qquad (\rho^2 = w\tilde{w}),$$

where $\psi_\nu = \psi_\nu(\rho)$ is a general solution to the following linear differential equation

$$\psi_\nu'' + (4\eta_0\eta_\infty\rho^2 - (\nu+1)^2)\psi_\nu = 0, \quad ' = \rho\frac{d}{d\rho}.$$

(Note that this is essentially Bessel's differential equation.) Let τ_n^m $(m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0})$ be

$$\tau_n^m = \begin{vmatrix} \varphi_{m-n+1} & \varphi_{m-n+2} & \cdots & \varphi_m \\ \varphi_{m-n+2} & \varphi_{m-n+3} & \cdots & \varphi_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_m & \varphi_{m+1} & \cdots & \varphi_{m+n-1} \end{vmatrix}.$$

Then

$$J = \frac{1}{\tau_n^m}\begin{pmatrix} \tau_n^{m-1} & \tau_{n+1}^m \\ \tau_{n-1}^m & \tau_n^{m+1} \end{pmatrix},$$

gives rise to a family of solutions to Yang's equation that corresponds to the classical transcendental solutions of the third Painlevé equation.

References

[1] L. J. Mason and N. M. J. Woodhouse, Integrability, Self-duality and Twister Theory, Oxford University Press, 1996.

[2] E. F. Corrigan, D. B. Fairlie, R. G. Yates and P. Goddard, The construction of self-dual solutions to $SU(2)$ gauge theory, Commun. Math. Phys. **58** (1978) 223-240.

[3] T. Masuda, Classical transcendental solutions of the Painlevé equations and their degeneration, Tohoku Math. J. **56** (2004) 467-490.