On R. Fuchs' problem and linear monodromy

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The Painlevé equations are given by isomonodromic deformations of a linear equation. In general we cannot calculate the monodromy data or the Stokes multipliers of the linear equation (*linear monodromy*) explicitly since the Riemann-Hilbert correspondence between linear equations and linear monodromy is a transcendental map.

Schlesinger and Garnier studied isomonodromic deformations in order to solve the Riemann-Hilbert problem. They considered that if we take a suitable deformation of a linear equation, we may calculate the linear monodromy. In recent development of the Painlevé analysis, **we can determine linear monodromy in special cases**. For a special solution $y(t)$ of the Painlevé equation, we can determine the linear monodromy of

$$\frac{d^2v}{dz^2} = Q(t, y(t), y'(t); z)v \tag{1}$$

For example, we can calculate the linear monodromy for the Boutroux solution of P1, Ablowitz-Segur solutions for P2. Jimbo studies local behavior of solutions of P6, which contain the linear monodromy in local asymptotic expansions.

Other examples are Umemura's classical solutions. Umemura gave a definition of *classical functions* by means of the differential Galois theory. The Painlevé equations have two types of classical solutions: one is algebraic, and the other is the Riccati type. For the Riccati type solution, the monodromy data is reducible if we take a suitable Bäcklund transformation. For algebraic solutions, R. Fuchs studied the following problem:

R. Fuchs' Problem (1910) Let $y(t)$ be an algebraic solution $y(t)$ of a Painlevé equation. Find a suitable transformation $x = x(z, t)$ such that the corresponding linear differential equation (1) is changed to the form without the deformation parameter t:

$$\frac{d^2u}{dx^2} = \tilde{Q}(x)u.$$

Here $v = \sqrt{dz/dx}\; u$.

He studies this problem for the Picard solutions of P6 and might consider this problem is true. But he did not mention P1-P5 (in 1910, isomonodromic deformations were not known except for P6). We remark that his paper

> "Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegene wesentlich singulären Stellen",
> *Math. Ann.* **70** (1911), 525–549.

was completely forgotten for long years.

In my talk, we will show that **R. Fuchs' Problem is true for P1-P5** up to the Bäcklund transformations. We obtain rational/algebraic solutions of P1-P5 and symmetric solutions of P1,P2,P4 by transformations of (degenerated) confluent hypergeometric equations. We can calculate the linear monodromy of algebraic solutions and symmetric solutions explicitly.

Here the **symmetric solutions** of P1,P2,P4

$$\begin{array}{ll} \text{P1} & y'' = 6y^2 + t, \\ \text{P2} & y'' = 2y^3 + ty + \alpha, \\ \text{P4} & y'' = \dfrac{1}{2y}y'^2 + \dfrac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \dfrac{\beta}{y} \end{array}$$

are solutions which are invariant by the action of symmetry

$$\begin{array}{ll} \text{P1} & y \to \zeta^3 y, \quad t \to \zeta t, \quad (\zeta^5 = 1) \\ \text{P2} & y \to \omega y, \quad t \to \omega^2 t, \quad (\omega^3 = 1) \\ \text{P4} & y \to -y, \quad t \to -t. \end{array}$$

Before we will consider R. Fuchs' Problem, **we revise coalescent diagram of the Painlevé equations**. In our new coalescent diagram, the Painlevé equations are classified into 5 or 8 or 10 types. 6 is just traditional! Our diagram contains the Flaschka-Newell form as the isomonodromic deformation of P34.

See `math.CA/0512243, math.CA/0601614` with S. Okumura for details.