Lax form of q-Painlevé equation associated to $A_2^{(1)}$-surface

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Abstract

There are many discrete analog of the Painlevé differential equations. A classification of discrete Painlevé equations with a view of theory of rational surfaces is also known.

surface	$A_0^{(1)}$	$A_0^{(1)*}$	$A_1^{(1)}$	$A_2^{(1)}$	$A_3^{(1)}$	$A_4^{(1)}$	$A_5^{(1)}$	$A_6^{(1)}$	$A_7^{(1)}$	$A_7^{(1)\prime}$	$A_8^{(1)}$
symmetry	$E_8^{(1)}$	$E_8^{(1)}$	$E_7^{(1)}$	$E_6^{(1)}$	$D_5^{(1)}$	$A_4^{(1)}$	$(A_2+A_1)^{(1)}$	$(A_1+A_1)^{(1)}$	$A_1^{(1)}$	$A_1^{(1)}$	–

$A_0^{(1)**}$	$A_1^{(1)*}$	$A_2^{(1)*}$	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$
$E_8^{(1)}$	$E_7^{(1)}$	$E_6^{(1)}$	$D_4^{(1)}$	$A_3^{(1)}$	$(A_1+A_1)^{(1)}$	$A_1^{(1)}$	–	$A_2^{(1)}$	$A_1^{(1)}$	–

However, Lax forms of some discrete Painlevé equations have been not obtained yet. The aim of this talk is to present a Lax pair of q-Painlevé equation associated to $A_2^{(1)}$-surface.

The deformation theory of linear differential equations is one of the most important aspects for study of the Painlevé equations. R. Fuchs found a relation between monodromy preserving deformation and the sixth Painlevé equation. This result of R. Fuchs was generalized afterwards by R. Garnier and L. Schlesinger. A result of R. Garnier is connected to deformation theory of 2nd order linear equation with irregular singularities. He obtained the other five Painelvé equations from this consideration. L. Schlesinger consider the isomonodromic deformation of $m \times m$-linear system of 1st order differential equations with regular singularities. At a later time M. Jimbo, T. Miwa, and K. Ueno established a general theory of monodromy preserving deformation for the matrix system of 1st order differential equations with regular and irregular singularities. In their theory the Painlevé equations are written by the form of a compatibility condition between 2×2-linear system and an associated deformation system. We call this description "Lax form" of the Painlevé equations.

We see some merits that we could express the Painlevé equations in their Lax form. First of all, linear differential equations are easy to be identified with their data of singularities;

in particular, the classification of the Painlevé equations corresponds with coalescence of singularities of linear differential equations.

Getting back to the discrete case, we consider a discrete analog of monodromy preserving deformation. As concerns difference Painlevé equations of $D_l^{(1)}$ and $E_l^{(1)}$ types, they possess the common rational surfaces with the Painlevé differential equations as their spaces of initial conditions. Difference equations can be regarded as contiguity relations of the Painlevé differential equations. We can lift up these relations to associated linear equations; we see them as discrete deformation of linear differential equations and also as coming from compatibilities of two discrete deformations of linear differential equations.

Although the difference equations of types $A_0^{(1)**}$, $A_1^{(1)*}$, and $A_2^{(1)*}$ do not correspond to any Painlevé differential equation, the author believes that they should correspond to the Garnier or degenerated Garnier systems; they should be written in the form of Schlesinger transformations, which is generally studied in M. Jimbo and T. Miwa's paper ([2]).

Therefore, if we want different one from M. Jimbo and T. Miwa's, the author thinks, that would be elliptic-difference or q-difference. In the paper of M. Jimbo and the author, they studied q-analog of R. Fuchs' result, that is, a deformation theory of linear q-difference equation ([3]). Recently, q-analog of Garnier system, which is a higher dimensional extension of R. Fuchs' result, was also studied ([4]).

Just like differential case, they are quite natural and general situation; so q-Painlevé equation should be included among q-Garnier system or its degenerations if they could be written in a 2×2-Lax form. However, while the most generic Painlevé equation, the sixth, coincides with the $N = 1$ Garnier system, the q-Garnier system with $N = 1$ coincides with q-Painlevé equation of $A_3^{(1)}$ type; more generic equations, $A_0^{(1)*}$, $A_1^{(1)}$, and $A_2^{(1)}$, do not appear.

In this talk, we see that q-Painlevé equation of $A_2^{(1)}$ type appear as a particular case of q-Garnier system with $N = 2$. This construction owes much to calculations in D. Arinkin and A. Borodin's paper ([1]). The same problem for $A_0^{(1)*}$ and $A_1^{(1)}$ still remains open.

References

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