From Symmetry to Painlevé type equations

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Recently, several series of interesting differential equations have been constructed by Yusuke Sasano[1]. These equations, which we call **Sasano systems**, have the following important features:

(1) They can be represented as coupled system of Painlevé equations.

(2) They have several symplectic coordinate systems on which the Hamiltonian is polynomial.

(3) They admit affine Weyl group symmetry as the Bäcklund transformations.

A typical example of Sasano system is the following Hamiltonian system with the Hamiltonian:

$$\begin{aligned} H = & H_{\rm VI}^{\alpha_3+\alpha_6,\alpha_3+2\alpha_4+\alpha_5,1-\alpha_0,\alpha_2(\alpha_1+\alpha_2)}(q_1,p_1,t) \\ & + H_{\rm VI}^{\alpha_6,\alpha_5,1-\alpha_3-\alpha_0,\alpha_4(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}(q_2,p_2,t) \\ & + \tfrac{2(q_1-t)q_2p_1(p_2(q_2-1)+\alpha_4)}{t(t-1)}, \end{aligned}$$

where $H_{\rm VI}^{a,b,c,d}(q,p,t)$ is the Hamiltonian of the sixth Painlevé equation:

$$t(t-1)H_{\rm VI}^{a,b,c,d}(q,p,t) = q(q-1)(q-t)p^2 - \{a(q-1)(q-t)+bq(q-t)-cq(q-1)\}p+dq.$$

This system has affine Weyl group symmetry of type $D_6^{(1)}$, with root parameters $\alpha_0,\ldots,\alpha_6$, $(\alpha_0+\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6=1)$. It is remarkable that the Bäcklund transformations obtained by Sasano are exactly the same as those obtained by Lie theoretic method before [2,3]. This fact suggests that Sasano systems have natural geometric origin and may be related with the Drinfeld-Sokolov hierarchy. This is indeed confirmed in some cases [4].

In this talk, I will make a brief introduction to Sasano system focusing on their symmetry aspects.

References

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