

特異積分方程式の数値解法

(除く. 固有値問題, Volterra型, 微積分方程式, 純理論)

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§1 方程式の分類

$$\begin{cases} \text{第1種} & \int K(s,t)f(t)dt = g(s). \\ \text{第2種} & f + \int K(s,t)f(t)dt = g(s). \\ \text{同次} & \lambda \int Kf = f. \end{cases}$$

$$\begin{cases} \text{regular} & \text{核 } K: \text{自乗積分可能.} \\ \text{singular} & K: \text{自乗積分発散, etc.} \end{cases}$$

例 $K = H(s,t)(s-t)^{-n}$, $K = H(s,t)\cot(s-t)$, $? K = \ln|s-t|$

§2 教科書

実用的

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- 19) Gakhov, F.D. "Boundary value problems." Pergamon £ 4-0-0. 1963.
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(方程式 $J(\tau) = \frac{1}{2} \int_0^{\infty} J(t) E_1(|t-\tau|) dt$ について)

§ 3 積分方程式の解法

1) 厳密解, 解析的方法. (参考: 教科書, 文献 6. 48)-76.)

例. Abel's eq. $\int_0^s \frac{f(t)}{\sqrt{s-t}} dt = g(s).$

解: $f(t) = \frac{1}{\pi} \frac{d}{dt} \int_0^t \frac{g(s)}{\sqrt{t-s}} ds.$

Cauchy 核 をもつ eq.

$$\frac{1}{2\pi} \int_{-1}^1 \frac{f(t)}{s-t} dt = g(s).$$

$$\text{解: } f(t) = \frac{1}{\pi} \frac{1}{\sqrt{1-t^2}} \left[C - \int_{-1}^1 \frac{g(s)}{t-s} \sqrt{1-s^2} ds \right]$$

Wiener-Hopf の方程式

2) 近似解法 (正規核及び特異核)

(i) 特異核を正規な核に変換して, 通常の近似法を用いる。(参考: §2 の教科書, 文献 5 の 44) - 47.)

例 Hamel (教科書 5) P140.

Bückner (教科書 3) pp 443 - 444).

(ii) 核を縮退核で近似。(参考: §2 の教科書 1), 3), 4), 文献 29) - 31).)

例 Bateman の方法, 直交関数展開, Galerkin 法.

(iii) 逐次近近似法 iterative methods.

(参考: 教科書 1), 3), 4) etc. 文献 1. の 1) - 23).)

例 第 2 種方程式に対して

Neuman-Liouville, Schmidt, Wiarda,

Bückner, Schönberg, Bellman, Wagner, ...

第 1 種方程式に対して

Berg, ...

(iv) 変分法

(v) Analog methods, 数値積分公式または補間公式を使う。(参考: 教科書, 中田 1) pp 28-32, P40, 日高 2), 文献 4 の 32) - 43).)

$$\int K(s, t) f(t) dt \doteq \sum K_{ij} f(t_j).$$

特異方程式に対して,

(イ) Kryloff の方法 (中田 1) p40).

$$K(s, t) = \frac{d}{dt} H(s, t) \text{ とすると}$$

$$\int K f dt = \int f dH \doteq \sum f(t_n) \Delta H$$

(ロ) Nyström の方法 (日高 2)).

補間公式を用いる。煩雑。

(ハ) Sofronov (文献 35) または Math. Rev. 19 (1958) p. 66.)

(ニ) 簡便な方法 (中田 1) p31).

$$\lim_{t \rightarrow s} K(s, t)(t-s) = 0 \text{ のとき}$$

$$\int K(s, t) f(t) dt = \int K(s, t) \{f(t) - f(s)\} dt + f(s) \int K(s, t) dt.$$

右辺の初項は通常の数値積分公式を使う。第2項は K が解析的に積分できないときは,

$$K = \mathcal{K} + (K - \mathcal{K})$$

とすると (\mathcal{K} : K の principal part, 初等関数)

$$\int K dt = \int \mathcal{K} dt + \int (K - \mathcal{K}) dt$$

(ホ) Young, (Aso),

$$K = \mathcal{K}_0 P + (K - \mathcal{K}_0 P),$$

P : 多項式 (または正則関数)

数値積分公式 $\int K_0 F(t) dt = \sum_j w_j(s) F(t_j)$.

$$\int K f dt = \int K_0 P f dt + \int (K - K_0 P) f dt.$$

$K_0 P$ の選択に注意.

{ 特異点近傍で K のよい近似.
特異点から離れた所でも K との差は大きくならない.

失敗例 ----- § 4.

(vi) Monte Carlo 法

§ 4. 計算失敗例.

流れに垂直な平板を過きる粘性流 (Oseen 近似)

$$l = R/4, \quad (k = U/2l) = 1, \quad R = 2lU/D = 4kl.$$

$$\text{Eq. } \int_{-l}^l [K_0(y-\eta)F(y) + iH(y-\eta)\bar{F}(y)] dy = 4\pi U.$$

$$\text{ただし } H(Y) = (1/Y) - K_1(Y),$$

K_0, K_1 : 変形ベッセル関数.

$F(y)$ は $y = \pm l$ で $1/\sqrt{l^2 - y^2}$ なる形の特異性を持つ。
よって

$$\text{変換: } y = l \sin t, \quad \eta = l \sin \tau,$$

$$f(t) = l \cos t \cdot F \equiv f_1 + if_2.$$

数値解 — § 3 (v)(ホ) の方法による。

数値積分公式 $\int_{-1}^1 f(x) \ln|x-\xi_m| dx = \sum_n w_n(\xi_m) f(x_n),$

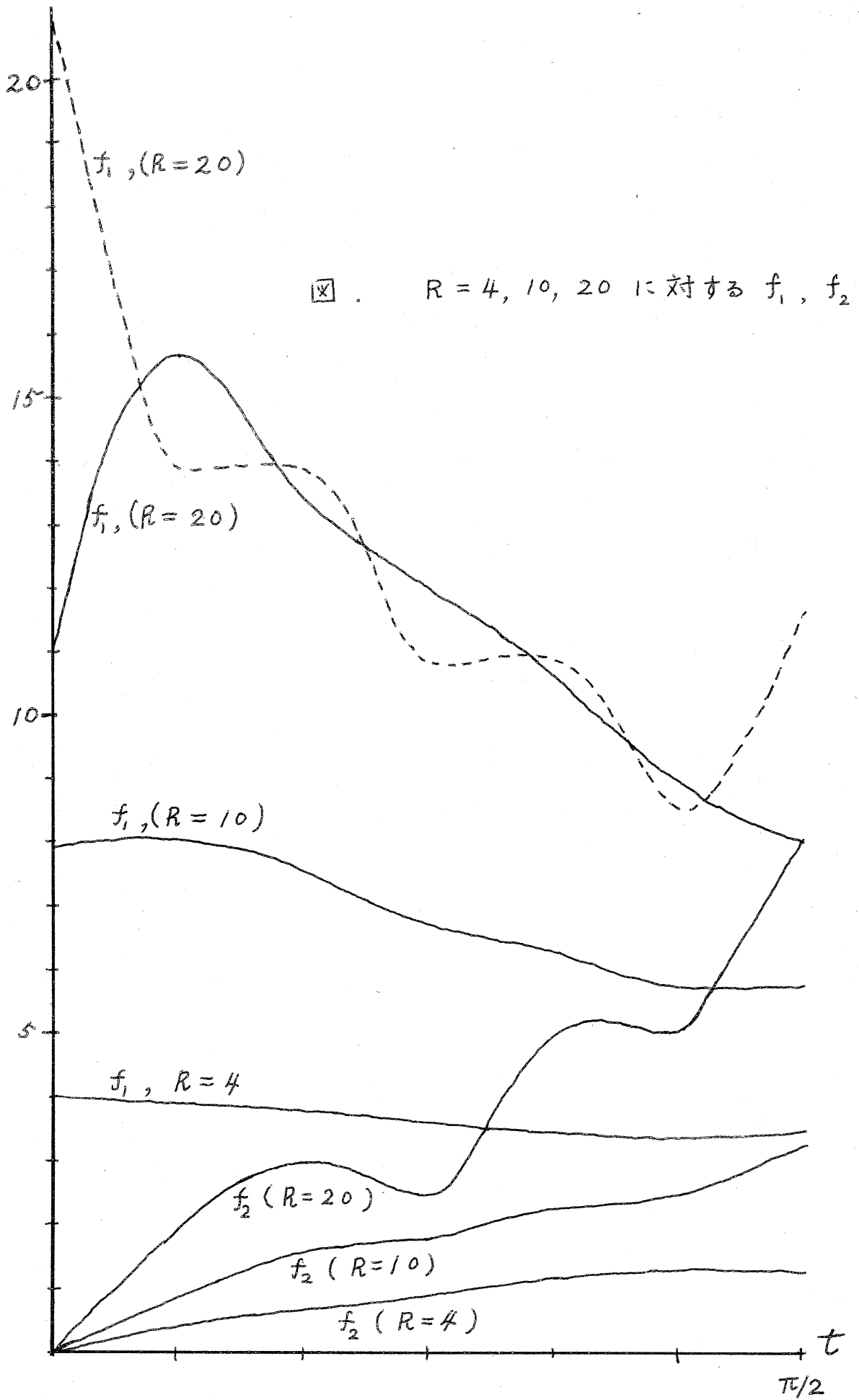
係数 $w_n(\xi_m)$ はあらかじめ求められている。

$f_1(t), f_2(t)$ はそれぞれ偶, 奇関数であるので積分域を $[0, \pi/2]$ とする。更に変換 $(t, \eta \rightarrow x, \xi)$ して $[-1, 1]$ とする。

$$K_0(y-\eta) = \ln 2 - \gamma - \ln\left(\frac{\pi}{4} l \cos \tau\right) \\ - \ln|x-\xi| \left[1 + \frac{1}{4} l^2 \left\{ \cos^2 \tau (t-\tau)^2 - \cos \tau \sin \tau (t-\tau)^3 \right\} \right] \\ + \dots$$

$$H(y-\eta) = \dots \square \dots$$

そこで K_0, P として上式の \square まで, または終までを採用する。前者による計算結果は図の実線, 後者によるものは点線をもって示されている。区間 $[-1, 1]$ は 6 等分した。後者の方が, 労多くしてかえって悪い結果に導いた。ただし, 低 Reynolds 数では, 両者は一致した。 Y が大なるとき $K_0(Y) \rightarrow 0$ であるのに, $\ln|Y| \cdot Y^3$ 等はかなり大となり, $\int (K_0 - K_0, P) f dt$ の評価において誤差を大きくしたものと思われる。



5.文献

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$$\left\{ \begin{aligned} f_{n+1} &= f_n - E(g, f_n, K) - j E(g, f_n, K), \quad E(g, f, K) \equiv g - f - \int K f. \\ j &= 0 \quad \text{ととると, Neuman series.} \\ j &= \text{const.} \neq 0, \quad \text{Wiarda, Bückner.} \\ j &= -\bar{K}(s) / [1 + (b-a)\bar{K}], \quad \text{C. Wagner の方法,} \\ j &= -1 \quad \bar{K} \equiv \int_a^b K(s, t) dt / (b-a), \quad \bar{\bar{K}} \equiv \int \bar{K}(s) ds / (b-a). \end{aligned} \right.$$

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eq. $\int_a^b Kf = g$, K : 対稱, positive definite.

solut. $f_n = f_{n-1} + \lambda [g - \int Kf_{n-1}]$, $0 < \lambda < 2\lambda_1$, (λ_1 : 最小固有値)

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eq. $\int_a^b \int_a^b f(\xi, \eta) [(x-\xi)^2 + (y-\eta)^2]^{-1/2} d\xi d\eta = g(x, y),$

maximum of

$$I(F) = \lambda \iint F dx dy - \iiint F(\xi, \eta) F(x, y) [(x-\xi)^2 + (y-\eta)^2]^{-1/2} dx dy d\xi d\eta,$$

where $F = \sum \sum \alpha_{rs} f_{rs}(x, y).$

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$$\begin{cases} K(s, t) \rightarrow H_n(s, t) \\ H_n(s, t) = \sum \sum \alpha_{ij} K(s, t_j) K(s_i, t) / W_n, \quad W_n = \det(K(s_i, t_j)). \end{cases}$$

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$$f(s) - \lambda \int_a^b K f dt = g(s), \quad \text{求積法による} \int_a^b F(t) dt \approx \sum_{j=1}^n A_j F(t_j).$$

$$\therefore \int K(s_i, t) f(t) dt \approx \sum L_{ij} f_j, \quad \text{近似解のベクトルを } \tilde{f} = (f_j) \text{ とする.}$$

$$\text{ベクトル } p \equiv (I - \lambda L) \tilde{f} - g, \quad (I: \text{単位行列})$$

$$\text{Put } \varepsilon_f(s) \equiv \int K_f - \sum A_j K(s, t_j) g(t_j), \quad \varepsilon(s, t) \equiv \int K K - \sum A_j K(s, t_j) K(t_j, t)$$

$$\varepsilon_i \equiv \max_{a \leq s \leq b} \sum_j |A_j \varepsilon(s, t_j)|, \quad \varepsilon \equiv \max_{1 \leq i \leq n} \int_a^b |\varepsilon(t_i, t)| dt,$$

$$K_i \equiv \max_{a \leq s \leq b} \sum_j |A_j K(s, t_j)|, \quad R(s, t, \lambda) \equiv K(s, t) \text{ の}$$

$$\max \int |R(s, t, \lambda)| dt \leq P, \quad B \equiv 1 + |\lambda| P, \quad q \equiv \lambda^2 \varepsilon, \quad B \leq 1,$$

$$\|f - \tilde{f}\| \leq (|\lambda| \|\varepsilon_f\| + |\lambda|^2 B \|f\| \varepsilon + \|P\|) (1 + |\lambda| K_i B) / (1 - q).$$

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$$\left\{ \begin{array}{l} \text{eq. } \frac{1}{\pi} \left[\int_{-1}^{-k} + \int_{1}^k \right] \frac{\varphi(y)}{y-x} dy = f(x), \quad k < x < 1. \\ \text{sol. } \varphi = g - \left\{ \frac{1}{\pi} \int_{-k}^k [(1-y^2)(k^2-y^2)]^{1/2} \frac{g(y)}{y-x} dy + C_1 x + C_2 \right\}, \\ \text{where } C_1, C_2 : \text{ arbit. consts.} \\ g(x) = -\frac{1}{\pi} \left(\int_{-1}^{-k} + \int_k^1 \right) \left(\frac{1-y^2}{1-x^2} \right)^{1/2} \frac{f(y)}{y-x} dx. \end{array} \right.$$

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$$\text{eq. } a(s)f(s) - \lambda \int_{-1}^1 \frac{f(t)}{t-s} dt = g(s).$$

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$$\text{eq. } 53) \int_{-1}^1 f(t)(1-t^2)^{-1/2} \ln |t-s| dt = \lambda g(s), \quad s \in \mathbb{R}$$

$$\frac{d^2}{ds^2} \int_{-1}^1 f(t)(1-t^2)^{1/2} \ln |t-s| dt = \lambda g(s).$$

$$54) f(s) = -\frac{1}{2\pi} \int_0^\pi g(t) \ln \frac{1-\cos(s+t)}{1-\cos(s-t)} dt$$

$$g(t) = \frac{1}{2\pi} \int_0^\pi f''(s) \ln \frac{1-\cos(s+t)}{1-\cos(s-t)} ds.$$

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eq. $g(x) = \int_{-1}^1 f(y) Z_0(k|x-y|) dy,$
 $g(x) = \int_{-1}^1 f(y) \frac{|x-y|}{x-y} Z_1(k|x-y|) dy,$
 $Z_n(\alpha) = AJ_n(\alpha) + BN_n(\alpha).$

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eq. $f(s) = \frac{1}{2} s \int_0^\infty E_n(1s-t) f(t) dt + g(s),$

E_n : the n-th expon. integ.

58)-72) Dual eqs. triple eqs.

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eq. $\begin{cases} f(s) + \int_0^\infty K(s-t) f(t) dt = g(s) & s > 0 \\ f(s) + \int_0^\infty K(s+t) f(t) dt = g(s) & s < 0 \end{cases}$
 $K(x) = O(\ln|x|) \quad (x \rightarrow 0).$

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Izv.Akad.Nauk SSSR Ser.Mat.20(1956) 33-52 (MR

eq. $f(s) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty k_1(s-t) f(t) dt = g(s) \quad (0 < s < \infty)$

$f(s) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty k_2(s-t) f(t) dt = g(s) \quad (-\infty < s < 0).$

solut. $f(s) = \frac{1}{\sqrt{2\pi}} \int_{ib_1 - \infty}^{ib_1 + \infty} \phi^+(\zeta) e^{-is\zeta} d\zeta - \frac{1}{\sqrt{2\pi}} \int_{ia_2 - \infty}^{ia_2 + \infty} \phi^-(\zeta) e^{-is\zeta} d\zeta.$

$\phi(\zeta)$: Riemann 境界値問題の適当な解.

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$$\text{eq. } \begin{cases} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_1(\lambda-t)f(t)dt = g(\lambda) & 0 < \lambda < \infty \\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_2(\lambda-t)f(t)dt = g(\lambda) & -\infty < \lambda < 0. \end{cases}$$

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