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ABSTRACT PROPERTIES P SUCH THAT ANY SEMIGROUP WHICH IS A SEMILATTICE OF COMMUTATIVE SEMIGROUPS WITH P IS COMMUTATIVE

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Let $P_1(G)$ and $P_2(G)$ be abstract properties pertaining to commutative semigroups G in the sense of P. M. Cohn[3]. $P_1(G)$ is said to be weaker than or equal to $P_2(G)$ and denoted by $P_1(G) \geqq P_2(G)$ if and only if, for any commutative semigroup S, $P_1(G)$ is satisfied by S (i.e., $P_1(S)$ is true) whenever $P_2(G)$ is satisfied by S. If $P_1(G) \geqq P_2(G)$ and $P_2(G) \geqq P_1(G)$, then $P_1(G)$ and $P_2(G)$ are said to be equivalent and denoted by $P_1(G) \equiv P_2(G)$. If $P_1(G) \equiv P_2(G)$, we regard $P_1(G)$ and $P_2(G)$ as the same property. When S is a semigroup which is a semilattice of commutative semigroups S_ξ , $\xi \in \mathfrak{X}$, S is not necessarily commutative. However, there is an abstract property P(G) pertaining to commutative semigroups G such that any semigroup which is a semilattice of commutative semigroups with P(G) is commutative. Such an abstract property is called a fully c-invariant property (abbrev., f.c.i.-property). For example, it is well-known that the property P(G) " G is a group " is an f.c.i.-property. There is no greatest (i.e., weakest) f.c.i.-property with respect to the ordering relation defined above, but there is a maximal f.c.i.-property. Further, a maximal f.c.i.-property is not unique. The main purpose of this paper is to obtain maximal f.c.i.-properties, and some relevant results. All results are given without proofs.

§ 1. Introduction. A commutative idempotent semigroup Γ is called a <u>semilattice</u>. Define an ordering relation on Γ as follows :

(1. 1) $\alpha \leqq \beta$ if and only if $\alpha\beta = \beta\alpha = \beta$.

Then, it is obvious that Γ is a partially ordered set with respect to $\leqq$. If $\alpha \leqq \beta$ and $\alpha \neq \beta$, then we shall denote it by $\alpha < \beta$. If Γ

is a totally ordered set with respect to $\leqq$, then Γ is called a <u>chain</u>. Now, let $\{S_\gamma : \gamma \in \Gamma\}$ (Γ: a semilattice) be a collection of semigroups S_γ. Then, each S_γ is called the <u>γ-component</u> of this collection. If γ is not a minimal element of Γ (i.e., if there is an element $\alpha \in \Gamma$ such that $\alpha < \gamma$), then the corresponding S_γ is called a <u>multiple-component</u>. Let $S = \sum\{S_\gamma : \gamma \in \Gamma\}$ (hereafter, $\sum$ and $\dot{+}$ denote disjoint sum). If $\circ$ is multiplication in S such that

(1. 2) $S(\circ)$ is a semigroup, and each S_γ $(\gamma \in \Gamma)$ is embedded in $S(\circ)$, i.e., $x \circ y = xy$ for all $x, y \in S_\gamma$,

and

(1. 3) $S_\alpha \circ S_\beta \subset S_{\alpha\beta}$ for all $\alpha, \beta \in \Gamma$,

then the resulting system $S(\circ)$ is called a <u>composition of $\{S_\gamma : \gamma \in \Gamma\}$ (with respect to Γ)</u>. Further, next we shall generalize this concept as follows : Let $\{S_\xi : \xi \in \mathcal{X}\}$ ($\mathcal{X}$: a set) be a collection of semigroups S_ξ. Define multiplication $*$ in $\mathcal{X}$ and multiplication $\circ$ in $S = \sum\{S_\xi : \xi \in \mathcal{X}\}$ such that $\mathcal{X}(*)$ is a semilattice [chain] and $S(\circ)$ is a composition of $\{S_\xi : \xi \in \mathcal{X}(*)\}$. In this case, $S(\circ)$ is called a <u>semilattice [linear] composition of $\{S_\xi : \xi \in \mathcal{X}\}$</u>. Let $\{S_\gamma : \gamma \in \Gamma\}$ (Γ: a semilattice) be a collection of commutative semigroups S_γ. Then, sometimes there exists a composition $S(\circ)$ of $\{S_\gamma : \gamma \in \Gamma\}$ which is commutative. In this case, we shall call $S(\circ)$ a <u>commutative composition</u> of $\{S_\gamma : \gamma \in \Gamma\}$. Similarly if a semilattice [linear] composition $S(\circ)$ of a collection $\{S_\xi : \xi \in \mathcal{X}\}$ ($\mathcal{X}$: a set) of commutative semigroups S_ξ is commutative, then $S(\circ)$ is called a <u>commutative semilattice [linear] composition</u> of $\{S_\xi : \xi \in \mathcal{X}\}$. In general, for a given collection $\{S_\gamma : \gamma \in \Gamma\}$ (Γ: a semilattice) of semigroups S_γ, there is not necessarily a composition of $\{S_\gamma : \gamma \in \Gamma\}$ (see Yamada [4]) If there exists at least one composition of $\{S_\gamma : \gamma \in \Gamma\}$, then the collection $\{S_\gamma : \gamma \in \Gamma\}$ is said to be <u>composable</u>. If Γ is a chain, then it is well-known that $\{S_\gamma : \gamma \in \Gamma\}$ is necessarily composable (e.g., see Clifford [1]). For any given collection $\{S_\gamma : \gamma \in \Gamma\}$ (Γ : a semilattice) of commutative semigroups S_γ, a composition of $\{S_\gamma : \gamma \in \Gamma\}$ is (even if it exists) not necessarily commutative. This can be seen from the following simple example :

Let $\Gamma = \{\alpha, \beta\}$ ($\alpha\beta = \beta\alpha = \beta, \alpha \neq \beta$) be a chain, S_α a commutative semigroup, and S_β a null semigroup containing at least two elements. Let $S = S_\alpha \dot{+} S_\beta$, and define multiplication $\circ$ in S as follows :

$$x \circ y = \begin{cases} xy & \text{if } x, y \in S_\alpha \text{ or } \in S_\beta, \\ y & \text{if } x \in S_\alpha,\ y \in S_\beta, \\ 0 & \text{if } x \in S_\beta,\ y \in S_\alpha, \end{cases}$$

where 0 is the zero element of S_β. Then $S(\circ)$ is a non-commutative composition of $\{S_\alpha, S_\beta\}$ with respect to Γ. In § 2, we shall give a necessary and sufficient condition for a collection $\{S_\gamma : \gamma \in \Gamma\}$ (Γ : a semilattice) of commutative semigroups S_γ to be composable. Further, in the case where $\{S_\gamma : \gamma \in \Gamma\}$ is composable, we shall give a method of construction of all compositions of $\{S_\gamma : \gamma \in \Gamma\}$. We also give a necessary and sufficient condition for $\{S_\gamma : \gamma \in \Gamma\}$ that every composition of $\{S_\gamma : \gamma \in \Gamma\}$ (if it exists) be necessarily commutative. Let P(G) be a proposition pertaining to commutative semigroups G. As in Cohn [3], P(G) is said to be an <u>abstract property</u> (pertaining to commutative semigroups) if and only if P(G) is invariant under isomorphism, i.e.

(1. 4) for any commutative semigroups S_1, S_2 such that $S_1 \cong S_2$ (S_1 is isomorphic with S_2), $P(S_1)$ is true whenever $P(S_2)$ is true and vice-versa.

If P(S) is true for a commutative semigroup S, then we shall say that S <u>satisfies</u> P(G). In this case, we also say that S is a commutative semigroup <u>with</u> P(G). For example, the properties " G is a group " and " G is cancellative " pertaining to commutative semigroups G are abstract properties. Let $P_1(G)$ and $P_2(G)$ be abstract properties. Then $P_1(G)$ and $P_2(G)$ are said to be <u>equivalent</u> (denoted by $P_1(G) \equiv P_2(G)$) if the following is fulfilled :

(1. 5) For any commutative semigroup S, $P_1(S)$ is true if and only if $P_2(S)$ is true.

Hereafter, we shall consider $P_1(G)$, $P_2(G)$ as the same property if they are equivalent. Define an ordering relation on the set $\mathfrak{P}$ of abstract properties as follows : Let $P_1(G)$ and $P_2(G)$ be abstract properties. $P_1(G) \leqq P_2(G)$ if the following (1. 6) is fulfilled :

(1. 6) For every commutative semigroup S, $P_2(S)$ is true whenever $P_1(S)$ is true.

If $P_1(G) \leqq P_2(G)$ and $P_1(G) \not\equiv P_2(G)$, then the property $P_2(G)$ is said to be <u>weaker</u> than the property $P_1(G)$ and denoted by $P_1(G) < P_2(G)$. It is obvious that $\mathfrak{P}$ is a partially ordered set with respect to this relation $\leqq$ (when we regard properties $P_1(G)$ and $P_2(G)$ as the same property if $P_1(G) \equiv P_2(G)$).

Next, consider the following propositions concerning an abstract property P(G) :

(1. 7) For any collection $\{S_\gamma : \gamma \in \Gamma\}$ (Γ: a chain) of commutative semigroups S_γ, where each multiple-component S_α satisfies P(G), every composition of $\{S_\gamma : \gamma \in \Gamma\}$ is commutative.

(1. 8) For any collection $\{S_\gamma : \gamma \in \Gamma\}$ (Γ: a semilattice) of commutative semigroups S_γ, where each multiple-component S_α satisfies P(G), every composition of $\{S_\gamma : \gamma \in \Gamma\}$ (if it exists) is commutative.

If (1. 7) or (1. 8) is true for P(G), then P(G) is called a <u>linearly c-extensible property</u> (abbrev., l.c.e.-property) or a <u>fully c-extensible property</u> (abbrev., f.c.e.-property) respectively. For example, the abstract property " G is a group " pertaining to commutative semigroups G is an f.c.e.-property. By the definitions an f.c.e.-property is clearly an l.c.e.-property, but the converse is not true (see Remark below). In § 3, we shall show existence of the weakest l.c.e.-property and the weakest f.c.e.-property and try to determine these properties.

Next, consider also the following propositions concerning an abstract property P(G) :

(1. 9) For any collection $\{S_\xi ; \xi \in \lambda\}$ (λ : a set) of commutative semigroups S_ξ, where each S_ξ satisfies P(G), every linear composition of $\{S_\xi : \xi \in \lambda\}$ is commutative.

(1. 10) For any collection $\{S_\xi : \xi \in \lambda\}$ (λ : a set) of commutative semigroups S_ξ, where each S_ξ satisfies P(G), every semilattice composition of $\{S_\xi : \xi \in \lambda\}$ is commutative.

If (1. 9) or (1. 10) is true for P(G), then P(G) is called a <u>linearly c-invariant property</u> (abbrev., l.c.i.-property) or a <u>fully c-invariant property</u> (abbrev., f.c.i.-property) respectively. It is obvious from the definitions that an l.c.e. [f.c.e.] -property is an l.c.i. [f.c.i.] -property. In § 4, we shall show existence of maximal l.c.i.-properties and maximal f.c.i.-properties and determine some of them.

Remark. Let $P_u(G)$ be an abstract property as follows :

(1. 11) G is universal, i.e., $G^2 = G$.

Then it is easy to see that $P_u(G)$ is an l.c.e.-property (this will be shown later). Now, let T be a universal commutative semigroup which has a zero element and whose annihilator A contains a non-zero element. (Existence of such a semigroup T can be proved). Now, let $L_3 = \{\alpha, \beta, \gamma\}$ be a semilattice such that $\alpha < \gamma$, $\beta < \gamma$, $\alpha \nleqq \beta$ and $\beta \nleqq \alpha$. Let S_α and S_β be infinite cyclic semigroups generated by a and b respectively : $S_\alpha = (a)$ and $S_\beta = (b)$. Let $S_\gamma = T$. Then $S = S_\alpha \dotplus S_\beta \dotplus S_\gamma$ becomes a non-commutative composition of $\{S_\alpha, S_\beta, S_\gamma\}$ with respect to L_3 by multiplication $\circ$ defined as follows :

$$x \circ y = \begin{cases} xy & \text{if } x, y \in S_\alpha\ , \in S_\beta \text{ or } \in S_\gamma\ , \\ u & \text{if } x = a \text{ and } y = b, \\ 0 & \text{otherwise,} \end{cases}$$

where 0 is the zero element of T $(= S_\gamma)$ and u is a fixed non-zero element contained in A. Hence $P_u(G)$ is not an f.c.e.-property.

Notation. Throughout this paper, if $\{S_\xi : \xi \in \mathcal{X}\}$ is a collection of commutative semigroups S_ξ , we shall denote elements of S_ξ by small letters a_ξ , b_ξ , c_ξ etc. having ξ as their subscripts.

§ 2. Composition theorems. Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a semilattice) be a collection of commutative semigroups S_γ . For every pair (α, β) of $\alpha, \beta \in \Gamma$ with $\alpha \leqq \beta$, let $\mathfrak{M}(\alpha, \beta)$ be the set of mappings of S_α into S_β . Let $C(\alpha, \beta) = \{S_\xi : \xi \in \Gamma, \alpha\xi = \beta\}$. Clearly $S_\beta \in C(\alpha, \beta)$. For every $S_\xi \in C(\alpha, \beta)$, let ψ_ξ, φ_ξ be (not necessarily distinct) two mappings of S_ξ into $\mathfrak{M}(\alpha, \beta)$. Put $\psi_\xi(a_\xi) = \bar{a}_\xi^{(\alpha,\beta)}$ and $\varphi_\xi(a_\xi) = \tilde{a}_\xi^{(\alpha,\beta)}$. Let $\mathfrak{M}_L(\Omega) \equiv \mathfrak{M}_L(S_\gamma : \gamma \in \Gamma) = \{\bar{a}_\xi^{(\alpha,\beta)} : \alpha \leqq \beta, \alpha, \beta \in \Gamma, a_\xi \in S_\xi, S_\xi \in C(\alpha, \beta)\}$, and $\mathfrak{M}_R(\Omega) \equiv \mathfrak{M}_R(S_\gamma : \gamma \in \Gamma) = \{\tilde{a}_\xi^{(\alpha,\beta)} : \alpha \leqq \beta, \alpha, \beta \in \Gamma, a_\xi \in S_\xi, S_\xi \in C(\alpha, \beta)\}$.

If

(2. 1) $\mathfrak{M}(\Omega) \equiv \mathfrak{M}(S_\gamma : \gamma \in \Gamma) = \mathfrak{M}_L(\Omega) \dot{+} \mathfrak{M}_R(\Omega)$

satisfies the following condition (C), then $\mathfrak{M}(\Omega)$ is called a <u>set of composite factors on Ω</u> :

$$(C) \begin{cases} (1)\ \bar{a}_\alpha^{(\beta,\alpha\beta)} \tilde{c}_\gamma^{(\alpha\beta,\alpha\beta\gamma)} = \tilde{c}_\gamma^{(\beta,\beta\gamma)} \bar{a}_\alpha^{(\beta\gamma,\alpha\beta\gamma)}, \\ (2)\ \bar{a}_\alpha^{(\alpha,\alpha)} = \tilde{a}_\alpha^{(\alpha,\alpha)} = \text{the inner translation } \rho_{a_\alpha} \text{ on } S_\alpha \text{ induced by } a_\alpha , \\ (3)\ \bar{a}_\alpha^{(\beta,\alpha\beta)} \text{ and } \tilde{b}_\beta^{(\alpha,\alpha\beta)} \text{ are conjugate to each other in the following sense : } \bar{a}_\alpha^{(\beta,\alpha\beta)}(b_\beta) = \tilde{b}_\beta^{(\alpha,\alpha\beta)}(a_\alpha). \end{cases}$$

Theorem 1. Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a semilattice) be a collection of commutative semigroups S_γ .

(i) Ω is composable if and only if there exists a set of composite factors on Ω .

(ii) Let $\mathfrak{M}(\Omega)$ of (2. 1) be a set of composite factors on Ω. Then $S = \sum\{S_\gamma : \gamma \in \Gamma\}$ becomes a composition $S(\circ)$ of Ω by multiplication $\circ$ defined by

(2. 2) $a_\alpha \circ b_\beta = \overline{a}_\alpha^{(\beta, \alpha\beta)}(b_\beta) \ (= \widetilde{b}_\beta^{(\alpha,\alpha\beta)}(a_\alpha))$.

Further, every possible composition of Ω is found in this fashion.

The composition $S(\circ)$ in (ii) of Theorem 1 is called the composition of Ω induced by $\mathfrak{M}(\Omega)$.

Corollary 1. Let $\Omega=\{S_\gamma : \gamma \in \Gamma\}$ (Γ : a semilattice) be a collection of commutative semigroups S_γ, and $\mathfrak{M}(\Omega)$ of (2. 1) a set of composite factors on Ω. Then, the composition $S(\circ)$ of Ω induced by $\mathfrak{M}(\Omega)$ is non-commutative if and only if the following condition is satisfied :

(2. 3) $\overline{a}_\alpha^{(\beta,\alpha\beta)} \neq \widetilde{a}_\alpha^{(\beta,\alpha\beta)}$ for some $a_\alpha \in S_\alpha$, $\alpha,\beta \in \Gamma$.

Corollary 2. Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a semilattice) be a collection of commutative semigroups S_γ. Then, every composition of Ω is commutative if and only if there is no set, $\mathfrak{M}(\Omega)$ of (2. 1), of composite factors on Ω which satisfies the condition (2. 3).

Now, as a special case, we consider a collection $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ of commutative semigroups S_γ having a chain Γ as its index set. Let $\mathfrak{M}(\Omega)$ of (2. 1) be a set of composite factors on Ω.

Then, we have the following lemmas :

Lemma 1. For $\alpha,\beta\in\Gamma$ with $\alpha\leqq\beta$, each of $\overline{a}_\alpha^{(\beta,\alpha\beta)} (= \overline{a}_\alpha^{(\beta,\beta)})$ and $\widetilde{a}_\alpha^{(\beta,\alpha\beta)}$ $(= \widetilde{a}_\alpha^{(\beta,\beta)})$ is a translation on S_β.

Lemma 2. For $\alpha,\beta\in\Gamma$ with $\alpha\geqq\beta$, $\overline{a}_\alpha^{(\beta,\alpha)}(b_\beta) = \widetilde{b}_\beta^{(\alpha,\alpha)}(a_\alpha)$ and $\widetilde{b}_\alpha^{(\beta,\alpha)}(a_\beta) = \overline{a}_\beta^{(\alpha,\alpha)}(b_\alpha)$.

Putting $\overline{a}_\alpha^{(\beta,\beta)} = \rho_{a_\alpha,\beta}$ and $\widetilde{a}_\alpha^{(\beta,\beta)} = \sigma_{a_\alpha,\beta}$ for $\alpha\leqq\beta$, we have the following lemmas :

Lemma 3. $\rho_{a_\alpha,\alpha} = \sigma_{a_\alpha,\alpha} =$ the inner translation ρ_{a_α} on S_α induced by a_α.

Lemma 4. $\rho_{a_\alpha,\gamma}\sigma_{b_\beta,\gamma} = \sigma_{b_\beta,\gamma}\rho_{a_\alpha,\gamma}$ if $\alpha \leqq \gamma$, $\beta \leqq \gamma$.

Lemma 5.
$$\rho_{b_\beta,\gamma}\,\rho_{a_\alpha,\gamma} = \begin{cases} \rho_{\rho_{a_\alpha,\beta}(b_\beta),\gamma} & \text{if } \alpha \leqq \beta \leqq \gamma, \\ \rho_{\sigma_{b_\beta,\alpha}(a_\alpha),\gamma} & \text{if } \beta \leqq \alpha \leqq \gamma. \end{cases}$$

Lemma 6.
$$\sigma_{b_\beta,\gamma}\,\sigma_{a_\alpha,\gamma} = \begin{cases} \sigma_{\sigma_{a_\alpha,\beta}(b_\beta),\gamma} & \text{if } \alpha \leqq \beta \leqq \gamma, \\ \sigma_{\rho_{b_\beta,\alpha}(a_\alpha),\gamma} & \text{if } \beta \leqq \alpha \leqq \gamma. \end{cases}$$

By Lemmas 3 - 6, we obtain the following result : Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a chain) be a collection of commutative semigroups S_γ , $\mathfrak{M}(\Omega)$ of (2. 1) a set of composite factors on Ω. Let $S(\circ)$ be the composition of $\{S_\gamma : \gamma \in \Gamma\}$ induced by $\mathfrak{M}(\Omega)$.

Then, there exists a system

(2. 4) $\mathfrak{S}(\Omega) = \{\rho_{a_\alpha,\beta} : a_\alpha \in S_\alpha,\ \alpha \leqq \beta,\ \alpha,\beta \in \Gamma\} \dot{+} \{\sigma_{a_\alpha,\beta} : a_\alpha \in S_\alpha,\ \alpha \leqq \beta,\ \alpha,\beta \in \Gamma\}$, where $\rho_{a_\alpha,\beta}$ and $\sigma_{a_\alpha,\beta}$ are mappings of S_β into S_β , such that

$$(T)\quad \begin{cases} (1)\ \rho_{a_\alpha,\beta} \text{ and } \sigma_{a_\alpha,\beta} \text{ are translations on } S_\beta, \\ (2)\ \rho_{a_\alpha,\alpha} = \sigma_{a_\alpha,\alpha} = \text{the inner translation } \rho_{a_\alpha} \text{ induced by } a_\alpha \\ (3)\ \rho_{a_\alpha,\gamma}\sigma_{b_\beta,\gamma} = \sigma_{b_\beta,\gamma}\rho_{a_\alpha,\gamma}, \\ (4)\ \rho_{b_\beta,\gamma}\,\rho_{a_\alpha,\gamma} = \begin{cases} \rho_{\rho_{a_\alpha,\beta}(b_\beta),\gamma} & \text{if } \alpha \leqq \beta \leqq \gamma, \\ \rho_{\sigma_{b_\beta,\alpha}(a_\alpha),\gamma} & \text{if } \beta \leqq \alpha \leqq \gamma, \end{cases} \\ (5)\ \sigma_{b_\beta,\gamma}\,\sigma_{a_\alpha,\gamma} = \begin{cases} \sigma_{\sigma_{a_\alpha,\beta}(b_\beta),\gamma} & \text{if } \alpha \leqq \beta \leqq \gamma, \\ \sigma_{\rho_{b_\beta,\alpha}(a_\alpha),\gamma} & \text{if } \beta \leqq \alpha \leqq \gamma. \end{cases} \end{cases}$$

Further, the multiplication $\circ$ in $S(\circ)$ is represented by

$$(P)\quad a_\alpha \circ b_\beta = \begin{cases} \bar{a}_\alpha^{(\beta,\beta)}(b_\beta) = \rho_{a_\alpha,\beta}(b_\beta) & \text{if } \alpha \leqq \beta, \\ \tilde{b}_\beta^{(\alpha,\alpha)}(a_\alpha) = \sigma_{b_\beta,\alpha}(a_\alpha) & \text{if } \alpha \geqq \beta. \end{cases}$$

In general, let $\mathfrak{A}=\{S_\gamma : \gamma\in\Gamma\}$ (Γ : a chain) be a collection of commutative semigroups S_γ. For each pair (a_α, β), where $a_\alpha\in S_\alpha$, $\alpha\leqq\beta$ and $\alpha,\beta\in\Gamma$, let $\rho_{a_\alpha,\beta}$ and $\sigma_{a_\alpha,\beta}$ be (not necessarily distinct) two mappings of S_β into S_β. If $\mathfrak{S}(\mathfrak{A})=\{\rho_{a_\alpha,\beta} : a_\alpha\in S_\alpha, \alpha\leqq\beta, \alpha,\beta\in\Gamma\}\dot{+}\{\sigma_{a_\alpha,\beta} : a_\alpha\in S_\alpha, \alpha\leqq\beta, \alpha,\beta\in\Gamma\}$ satisfies (T) of (2. 4), then $\mathfrak{S}(\mathfrak{A})$ is called a <u>factor set of translations on $\mathfrak{A}$.</u> From this definition and the above-mentioned result, we can conclude as follows : Let $\mathfrak{A}=\{S_\gamma : \gamma\in\Gamma\}$ (Γ : a chain) be a collection of commutative semigroups S_γ. If $S(\circ)=\sum\{S_\gamma : \gamma\in\Gamma\}$ is a composition of $\mathfrak{A}=\{S_\gamma : \gamma\in\Gamma\}$, then there exists a factor set of translations on $\mathfrak{A}$, say $\mathfrak{S}(\mathfrak{A})=\{\rho_{a_\alpha,\beta} : a_\alpha\in S_\alpha, \alpha\leqq\beta, \alpha,\beta\in\Gamma\}\dot{+}\{\sigma_{a_\alpha,\beta} : a_\alpha\in S_\alpha, \alpha\leqq\beta, \alpha,\beta\in\Gamma\}$, and $\circ$ in $S(\circ)$ is represented by (P).

Conversely, let $\mathfrak{S}(\mathfrak{A})=\{\rho_{a_\alpha,\beta} : a_\alpha\in S_\alpha, \alpha\leqq\beta, \alpha,\beta\in\Gamma\}\dot{+}\{\sigma_{a_\alpha,\beta} : a_\alpha\in S_\alpha, \alpha\leqq\beta, \alpha,\beta\in\Gamma\}$ (Γ : a chain) be a factor set of translations on a collection $\mathfrak{A}=\{S_\gamma : \gamma\in\Gamma\}$ of commutative semigroups S_γ. Then, we can prove that $S=\sum\{S_\gamma : \gamma\in\Gamma\}$ becomes a composition of $\mathfrak{A}$ by multiplication $\circ$ given by (P).

Summerizing the results above, we obtain the following

Theorem 2. Let $\mathfrak{A}=\{S_\gamma : \gamma\in\Gamma\}$ (Γ : a chain) be a collection of commutative semigroups S_γ. Let $\mathfrak{S}(\mathfrak{A})$ of (2. 4) be a factor set of translations on $\mathfrak{A}$. Then $S=\sum\{S_\gamma : \gamma\in\Gamma\}$ becomes a composition $S(\circ)$ of $\mathfrak{A}$ by the multiplication $\circ$ defined by (P). Further, every composition of $\mathfrak{A}$ is found in this fashion.

This result is a special case of Theorem 2. 1 given by Yoshida [5]. $S(\circ)$ in Theorem 2 is called the composition of $\mathfrak{A}$ <u>induced</u> by $\mathfrak{S}(\mathfrak{A})$.

From Theorem 2, we obtain immediately the following

Corollary 1. Let $\mathfrak{A}=\{S_\gamma : \gamma\in\Gamma\}$ (Γ : a chain) be a collection of commutative semigroups S_γ, and $\mathfrak{S}(\mathfrak{A})$ of (2. 4) a factor set of translations on Ω. Then, the composition $S(\circ)$ of Ω induced by $\mathfrak{S}(\Omega)$ is non-commutative if and only if

(2. 5) $\quad \rho_{a_\alpha,\beta} \neq \sigma_{a_\alpha,\beta}$ for some $a_\alpha \in S_\alpha$, $\alpha, \beta \in \Gamma$, $\alpha < \beta$.

Moreover, the following is obvious from Corollary 1 :

Corollary 2. Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a chain) be a collection of commutative semigroups S_γ. Every composition of Ω is commutative if and only if there is no factor set, $\mathfrak{S}(\Omega)$ of (2. 4), of translations on Ω which satisfies (2. 5).

In the case where every composition of a collection $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a chain) of commutative semigroups S_γ is commutative, we have another construction theorem for the compositions of Ω which is somewhat simpler than Theorem 2 :

Theorem 3. Let $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a chain) be a collection of commutative semigroups S_γ every composition of which is commutative. Let $S = \sum \{S_\gamma : \gamma \in \Gamma\}$. For each pair (a_α, β), where $a_\alpha \in S_\alpha$, $\alpha, \beta \in \Gamma$ and $\alpha \leqq \beta$, let $\rho_{a_\alpha,\beta}$ be a mapping of S_β into S_β. Let $\overline{\mathfrak{S}}(\Omega) = \{\rho_{a_\alpha,\beta} : a_\alpha \in S_\alpha, \alpha \leqq \beta, \alpha, \beta \in \Gamma\}$.
If $\overline{\mathfrak{S}}(\Omega)$ satisfies the condition

$$(\overline{T}) \quad \begin{cases} (1) & \rho_{a_\alpha,\beta} \text{ is a translation on } S_\beta, \\ (2) & \rho_{a_\alpha,\alpha} = \text{the inner translation } \rho_{a_\alpha} \text{ on } S_\alpha \text{ induced by } a_\alpha, \\ (3) & \rho_{a_\alpha,\gamma}\rho_{b_\beta,\gamma} = \rho_{b_\beta,\gamma}\rho_{a_\alpha,\gamma} = \rho_{\rho_{a_\alpha,\beta}(b_\beta),\gamma} \quad \text{if } \alpha \leqq \beta \leqq \gamma, \end{cases}$$

then S becomes a composition $S(\circ)$ of Ω by the multiplication $\circ$ defined by

$$(\overline{P}) \quad a_\alpha \circ b_\beta = b_\beta \circ a_\alpha = \rho_{a_\alpha,\beta}(b_\beta) \quad \text{if } \alpha \leqq \beta.$$

Further, every composition of Ω is found in this fashion.

Next, we present some results concerning a factor set of translations on a collection $\Omega = \{S_\gamma : \gamma \in \Gamma\}$ (Γ : a chain) of commutative semigroups S_γ.

Lemma 7. Let $\Omega = \{ S_\gamma : \gamma \in \Gamma \}$ (Γ : a chain) be a collection of commutative semigroups S_γ. Let $\mathfrak{S}(\Omega)$ of (2. 4) be a factor set of translations on Ω which satisfies (T) in (2. 4). Then,

(i) for any $a_\alpha \in S_\alpha$, $\alpha, \beta \in \Gamma$ with $\alpha \leqq \beta$, $y_\beta \rho_{a_\alpha,\beta}(x_\beta) = \sigma_{a_\alpha,\beta}(y_\beta) x_\beta$, i.e., $\rho_{a_\alpha,\beta}$ and $\sigma_{a_\alpha,\beta}$ are linked, and

(ii) $\rho_{a_\alpha,\beta} \mid S_\beta^2 = \sigma_{a_\alpha,\beta} \mid S_\beta^2$.

A commutative semigroup S is said to be reductive if it satisfies the following abstract property $P_r(G)$:

Reductivity $P_r(G)$: $ax = bx$ for all $x \in G$ implies $a = b$.

Lemma 8. Let Ω and $\mathfrak{S}(\Omega)$ be as in Lemma 7. If each of the multiple-components of Ω is universal or reductive, then $\rho_{a_\alpha,\beta} = \sigma_{a_\alpha,\beta}$ for every $a_\alpha \in S_\alpha, \alpha, \beta \in \Gamma$ with $\alpha \leqq \beta$.

By using Lemma 8 and Corollary 2 to Theorem 2, we obtain

Corollary. Let $\Omega = \{ S_\gamma : \gamma \in \Gamma \}$ (Γ : a chain) be a collection of commutative semigroups S_γ. If each of the multiple-components of Ω is universal or reductive, then every composition of Ω is commutative.

Remark. This result will be more generalized in the next section.

§ 3. The weakest f.c.e. [l.c.e.] -property. In this section, we investigate l.c.e.-properties and f.c.e.-properties.

Let us consider the following abstract property $P_q(G)$ pertaining to commutative semigroups G :

(3. 1) There is no system $\{\sigma, \rho\}$ of distinct two translations on G such that (1) $\sigma\rho = \rho\sigma$ and (2) $\sigma \mid G^2 = \rho \mid G^2$.

This property $P_q(G)$ is called quasi-reductivity. As is shown later, reductivity implies quasi-reductivity. However, the converse is not true.

Lemma 9. Let $\{\sigma, \rho\}$ be a system of distinct two translations σ, ρ on a commutative semigroup S such that $\sigma\rho = \rho\sigma$ and $\sigma | S^2 = \rho | S^2$. Then there exist distinct two elements $x, y \in S$ and a prime element $t \in S$ such that (1) $xa = ya$ for all $a \in S$ and (2) $\sigma(t) = x$ and $\rho(t) = y$.

(Note : An element of $S \setminus S^2$ is called prime)

It is easily seen from Lemma 9 that reductivity implies quasi-reductivity.

Example. Let $S = \{a, a^2, \ldots, a^n\}$ be a cyclic semigroup of order n such that $n \geq 2$, $a^{n-1} \neq a^n$ and $aa^n = a^n$. Define mappings $\rho, \sigma : S \to S$ as follows : $\rho(a) = a^{n-1}$, $\rho(a^i) = a^n$ if $i > 1$; and $\sigma(a^i) = a^n$ for all i. Then ρ, σ are translations on S such that $\sigma\rho = \rho\sigma$ and $\sigma | S^2 = \rho | S^2$. Hence, of course, S is not quasi-reductive.

By using Lemma 9, we can prove the following theorem :

Theorem 4. $P_q(G)$ is the weakest l.c.e.-property.

In general, it is easy to see that if $P(G)$ and $P_1(G)$ are abstract properties such that $P_1(G) \leqq P(G)$ and if $P(G)$ is an l.c.e.-property, then $P_1(G)$ is also an l.c.e.-property. For abstract properties $P_1(G)$ and $P_2(G)$, denote the property " $P_1(G)$ or $P_2(G)$ " by $P_1(G) \vee P_2(G)$. It is obvious that $P_1(G) \leqq P_1(G) \vee P_2(G)$ and $P_2(G) \leqq P_1(G) \vee P_2(G)$. Now, it is easy to see that $P_r(G) \vee P_u(G) \leqq P_q(G)$. Since $P_r(G) \leqq P_r(G) \vee P_u(G)$ and $P_u(G) \leqq P_r(G) \vee P_u(G)$, we have $P_r(G) \leqq P_q(G)$ and $P_u(G) \leqq P_q(G)$. Since $P_q(G)$ is an l.c.e.-property, each of $P_r(G)$, $P_u(G)$ and $P_r(G) \vee P_u(G)$ is also an l.c.e.-property.

Thus, we have the following result as a corollary to Theorem 4 :

Corollary. Each of reductivity, universality and the property " reductive or universal" is an l.c.e.-property.

Remarks. (1) Moreover, the following is obvious from Theorem 4 :

Let $\mathcal{A} = \{S_\xi : \xi \in \mathcal{X}\}$ ($\mathcal{X}$: a set) be a collection of commutative semigroups S_ξ, where $P_q(S_\xi)$ is true for all $S_\xi \in \mathcal{A}$. Then, every linear composition of $\mathcal{A}$ is commutative.

(2) For a special collection $\mathcal{A} = \{S_\gamma : \gamma \in \Gamma\}$(Γ; a chain) of commutative semigroups S_γ, every composition of $\mathcal{A}$ is commutative even if there exists a multiple-component S_γ which does not satisfy $P_q(G)$. For example, let $L = \{0, 1\}$ be a chain with respect to the usual multiplication, $S_1 = \{e\}$ a semigroup consisting of a single element e and $S_0 = \{a, a^2, \ldots, a^{n-1}, a^n\}$ a cyclic semigroup of order n ($n > 2$) such that $a^{n-1} \neq a^n$ and $aa^n = a^n$. Then, it is easy to see from the above-mentioned example that $P_q(S_0)$ is not true. However, there is no non-commutative composition of $\{S_1, S_0\}$ with respect to L.

Hereafter, for any element x of a commutative semigroup S, the inner translation on S induced by x will be denoted by ρ_x.

Now, let us consider the following abstract property $P^*_r(G)$ pertaining to commutative semigroups G :

(3. 2) There is no system $\{u,v ; \xi, \eta\}$ of distinct two elements u,v of G and (not necessarily distinct) translations ξ, η on G such that (1) $\xi\eta = \eta\xi = \rho_u = \rho_v$ and (2) $\xi(u) = \xi(v)$ and $\eta(u) = \eta(v)$.

At first, we have

Lemma 10. Let $\{u,v ; \xi, \eta\}$ be a system of distinct two elements u,v of a commutative semigroup S and translations ξ, η on S, satisfying (1), (2) of (3. 2). Then, $uz = vz$ for all $z \in S$.

The following lemma was shown by R. Yoshida, though the result is not yet published :

Lemma 11. $P^*_r(G)$ is equivalent to $P_r(G)$.

By using Lemmas 10 and 11, we can prove the following theorem which is one of the main results of this paper :

Theorem 5. $P_r(G)$ is the weakest f.c.e.-property.

Corollary. Let $\Omega = \{S_\xi : \xi \in \mathcal{X}\}$ ($\mathcal{X}$: a set) be a collection of commutative semigroups S_ξ, where $P_r(S_\xi)$ is true for every $S_\xi \in \Omega$. Then, every semilattice composition of Ω is commutative.

Remarks. (1) It is easy to see that if $P_0(G)$ is an f.c.e. [l.c.e.]-property and if $P(G)$ is an abstract property such that $P(G) \leqq P_0(G)$, then $P(G)$ is also an f.c.e. [l.c.e.]-property. Let $\mathfrak{P}(Q) = \{P(G) : P(G)$ is an abstract property such that $P(G) \leqq P_q(G)\}$ and $\mathfrak{P}(R) = \{P(G) : P(G)$ is an abstract property such that $P(G) \leqq P_r(G)\}$. Then, $\mathfrak{P}(Q)$ and $\mathfrak{P}(R)$ are the set of all l.c.e.-properties and the set of all f.c.e.-properties respectively.

(2) As was shown in § 1, there exists a universal commutative semigroup S which has a zero element 0 and whose annihilator A contains a non-zero element. Since $P_q(S)$ is true and $P_r(S)$ is not true, $P_q(G) \not\equiv P_r(G)$. Hence, $P_q(G) > P_r(G)$. This also means that quasi-reductivity does not imply reductivity.

(3) Since $P_r(G)$ is weaker than each of separativity (see Clifford & Preston [2]) and cancellativity, the following results immediately follow from the above-mentioned Corollary :

(i) A semigroup which is a semilattice of commutative reductive semigroups is commutative and reductive.

(ii) A semigroup which is a semilattice of separative commutative semigroups is separative and commutative.

(iii) A semigroup which is a semilattice of cancellative commutative semigroups is separative and commutative.

The converse of this result also holds (see Clifford & Preston [2]) ; i.e., a separative commutative semigroup is a semilattice of cancellative commutative semigroups.

§5. Maximal f.c.i. [l.c.i.]-properties. Let $\mathcal{F}=\{P_\lambda(G) : \lambda\in\Lambda\}$ be the set of all f.c.i.-properties $P_\lambda(G)$. Then $\mathcal{F}$ is clearly a partially ordered set with respect to the ordering relation $\leqq$ defined by (1. 6). (Recall that equivalent two properties are regarded as the same property). Let $\mathcal{T}=\{P_\tau(G) : \tau\in\Lambda_0\}$ be any totally ordered subset of $\mathcal{F}$. Define an abstract property $T(G)$ as follows : $T(G)=\bigvee_{\tau\in\Lambda_0}P_\tau(G)$, i.e., $T(G)=$ the property " being at least one of $\{P_\tau(G) : \tau\in\Lambda_0\}$". Hence, a commutative semigroup S satisfies T(G) if and only if S satisfies at least one of the properties $\{P_\tau(G) : \tau\in\Lambda_0\}$. Now, let $\Omega=\{S_\xi : \xi\in\mathcal{X}\}$ ($\mathcal{X}$: a set) be a collection of commutative semigroups S_ξ such that every S_ξ satisfies T(G). Suppose that there exists a non-commutative semilattice composition $S(\circ)=\sum\{S_\xi : \xi\in\mathcal{X}(*)\}$ of Ω. Then there exist a, b such that $a\in S_\gamma$, $b\in S_\delta$, $\gamma,\delta\in\mathcal{X}$ and $a\circ b\neq b\circ a$. Clearly, both $a\circ b$ and $b\circ a$ are contained in $S_{\gamma*\delta}$. Put $S_\gamma\dot{+}S_\delta\dot{+}S_{\gamma*\delta}=M$. Then $M(\circ)$ is a subsemigroup of $S(\circ)$ and is non-commutative. Since $T(S_\gamma)$, $T(S_\delta)$ and $T(S_{\gamma*\delta})$ are all true, there exist $P_\alpha(G)$, $P_\beta(G)$ and $P_\varepsilon(G)$ of $\{P_\tau(G) : \tau\in\Lambda_0\}$ such that $P_\alpha(S_\gamma)$, $P_\beta(S_\delta)$ and $P_\varepsilon(S_{\gamma*\delta})$ are true. Let $P_\eta(G)$ be the weakest property in $\{P_\alpha(G), P_\beta(G), P_\varepsilon(G)\}$. Then $P_\eta(G)$ is of course an f.c.i.-property and $P_\eta(S_\gamma)$, $P_\eta(S_\delta)$, $P_\eta(S_{\gamma*\delta})$ are all true. Hence, the semilattice composition $M(\circ)$ of $\{S_\gamma, S_\delta, S_{\gamma*\delta}\}$ must be commutative. However, this is a contradiction since $M(\circ)$ was non-commutative. Consequently, every semilattice composition of Ω must be commutative. Therefore, T(G) is an f.c.i.-property and hence $T(G)\in\mathcal{F}$. Since $P_\tau(G)\leqq T(G)$ for all $\tau\in\Lambda_0$, T(G) is an upper bound of $\mathcal{T}$. Thus, $\mathcal{F}$ is an inductively ordered set. Hence, there exists a maximal f.c.i.-property in $\mathcal{F}$. Existence of maximal l.c.i.-properties is also proved by a similar method.

Hence, we have

Theorem 6. There exist a maximal l.c.i.-property and a maximal f.c.i.-property.

Corollary. For any f.c.i. [l.c.i.] -property $P(G)$, there exists a maximal f.c.i. [l.c.i.] -property $P_m(G)$ such that $P(G) \leqq P_m(G)$.

In fact, the following three theorems show that quasi-reductivity is a maximal l.c.i.-property and both universality and reductivity are maximal f.c.i.-properties :

Theorem 7. $P_q(G)$ is a maximal l.c.i.-property.

Theorem 8. $P_r(G)$ is a maximal f.c.i.-property.

Theorem 9. $P_u(G)$ is a maximal f.c.i.-property.

From Theorem 9, we also have immediately

Corollary. A semigroup which is a semilattice of universal commutative semigroups is universal and commutative.

Remark. Let $\mathcal{F}$ [$\mathcal{L}$] be the set of all f.c.i. [l.c.i.]-properties. For $P_1(G)$, $P_2(G) \in \mathcal{F}$ [$\mathcal{L}$], let us define an abstract property $P_1(G) \wedge P_2(G)$ as follows :

(4. 1) $P_1(G) \wedge P_2(G) =$ the property " being both $P_1(G)$ and $P_2(G)$".

Then, it is easy to see that $P_1(G) \wedge P_2(G) \in \mathcal{F}$ [$\mathcal{L}$] for any $P_1(G)$, $P_2(G) \in \mathcal{F}$ [$\mathcal{L}$] and $P_1(G) \wedge P_2(G)$ is the greatest lower bound of $P_1(G)$ and $P_2(G)$. Further, in fact $\mathcal{F}$ [$\mathcal{L}$] is a semilattice with respect to this operation $\wedge$.

Since $P_u(G)$ and $P_r(G)$ are non-equivalent maximal f.c.i.-properties, it is obvious that there is no greatest f.c.i.-property, i.e., there is no weakest f.c.i.-property $P_g(G)$ in the following sense :

(4. 2) $P_g(G) \geqq P(G)$ for any f.c.i.-property $P(G)$.

However, the author can not solve the following two problems and leaves them as open problems :-

Problem 1. Is there a maximal l.c.i.-property except $P_q(G)$? That is : Is $P_q(G)$ the greatest (weakest) l.c.i.-property ? Determine all of the maximal l.c.i.-properties.

Problem 2. Is there a maximal f.c.i.-property except $P_u(G)$ and $P_r(G)$? Determine all of the maximal f.c.i.-properties.

As a partial solution of Problem 1, we obtain the following result : Let C be an infinite cyclic semigroup : $C=\{a, a^2, \dots, a^n, \dots\}$. Let C^1 be the adjunction of an identity element to C : $C^1 = C \dot{+} \{1\}$. Let $\mathcal{L}^* = \{P(G) : P(G)$ is an abstract property such that $P(G) \leqq L(G)$, for some l.c.i.-property $L(G)$ satisfied by $C^1\}$. Then $\mathcal{L}^* \ni P_q(G)$, $P_r(G)$, $P_u(G)$, since each of $P_q(C^1)$, $P_u(C^1)$ and $P_r(C^1)$ is true and each of $P_q(G)$, $P_u(G)$ and $P_r(G)$ is an l.c.i.-property. Further, $\mathcal{L}^* \supset \bar{\mathcal{L}} = \{P(G) : P(G)$ is an l.c.i.-property which is comparable with $P_r(G)$ or $P_u(G)\}$. In fact, let $P(G)$ be a property of $\bar{\mathcal{L}}$. If $P(G) \leqq P_r(G)$ or $\leqq P_u(G)$, then $P(G) \in \mathcal{L}^*$ since each of $P_r(G)$ and $P_u(G)$ is an l.c.i.-property and is satisfied by C^1. If $P(G) > P_r(G)$ or $> P_u(G)$, then $P(C^1)$ is true since each of $P_r(C^1)$ and $P_u(C^1)$ is true. Since $P(C^1)$ is true and $P(G)$ is an l.c.i.-property, $P(G)$ is also contained in $\mathcal{L}^*$. In any cases, $P(G) \in \mathcal{L}^*$. Therefore, $\bar{\mathcal{L}} \subset \mathcal{L}^*$. Especially, cancellativity, separativity, regularity and the property " being a commutative semigroup G with 1 " are all contained in $\mathcal{L}^*$.

Now, we have

Theorem 10. $P_q(G)$ is the greatest (i.e., weakest) l.c.i.-property in $\mathcal{L}^*$.

References

1. A. H. Clifford : Naturally totally ordered commutative semigroups, Amer. J. Math. 126 (1954), 631 - 646.

2. A. H. Clifford and G. B. Preston : The algebraic theory of semigroups, vol. 1, Providence, R. I. (1961).

3. P. M. Cohn : Universal algebra, Harper and Row ; New York, Evanston and London (1965).

4. M. Yamada : Compositions of semigroups, Kōdai Math. Sem. Rep. 8 (1956), 107 - 111.

5. R. Yoshida : Ideal extensions of semigroups and compound semigroups Mem. Res. Inst. Sci. Engrg., Ritumeikan Univ. 14 (1965), 1 - 12.

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