数理解析研究所講究録
第 44 巻 1968 年 23-40

ON THE DISTRIBUTIONS OF THE MAXIMUM LATENT ROOTS AND TRACES OF TWO POSITIVE DEFINITE RANDOM MATRICES

TAKESI HAYAKAWA

1. Summary

Recently, T. Hayakawa [4] and T. Sugiyama [9] have given the density functions of the maximum latent roots of some positive definite random matrices which appeare in multivariate analysis. We give here some exact density functions of the maximum latent roots of a multivariate quadratic form in normal sample and of a non-central Wishart matrix with $\sum = I_m$. We also give a density function of traces of them. To derive these density functions we introduce a generalized Hermite polynomial, discuss some properties and give a generating function of generalized Hermite polynomials.

2. Notations and preliminary results

Let S be an $m\times m$ positive definite symmetric matrix. There exists a zonal polynomial $C_\kappa(S)$ which is given by A. T. James [6] corresponding to each partition $\kappa=(k_1,\cdots,k_m)$, $k_1\geqq\cdots\geqq k_m\geqq 0$ of integer k not more than m parts.

The following integrals are used in the sequel, which are fundamental properties of the zonal polynomials :

(1) $$\int C_{\kappa}(AHBH')\,d(H) = \frac{C_{\kappa}(A)\,C_{\kappa}(B)}{C_{\kappa}(I_m)} ,$$

(2) $$\int_{O(m)} (T_r XH)^k d(H) = \sum_{\kappa} \frac{(\frac{1}{2})k}{(\frac{n}{2})_{\kappa}} C_{\kappa}(\frac{1}{4}XX') ,$$

where the invariant orthogonal measure $d(H)$ is normalized to make the volume of the orthogonal group $O(m)$ unity, and A and B are $m \times m$ symmetric matrices, and

(3) $$(a)_{\kappa} = \prod_{i=1}^{m} (a - \frac{1}{2}(i-1))_{k_i} , \quad (a)_k = a(a+1)\cdots(a+k-1).$$

The most important Γ-type integral (Laplace transform) of zonal polynomial which was given by A. G. Constantine [1] is,

(4) $$\int_{R>0} \operatorname{etr}(-RZ)(\det R)^{\alpha-p} C_{\kappa}(R)\,dR = \Gamma_m(\alpha;\kappa)(\det Z)^{-\alpha} C_{\kappa}(Z^{-1}),$$

where

$$\Gamma_m(\alpha;\kappa) = \pi^{\frac{1}{4}m(m-1)} \prod_{i=1}^{m} \Gamma(\alpha + k_i - \frac{1}{2}(i-1))$$

and $$\operatorname{Re}(\alpha) + k_m > p-1 , \quad p = \frac{1}{2}(m+1).$$

If α is such that the gamma functions are defined, then the binomial type coefficient is

(5) $$(\alpha)_{\kappa} = \Gamma_m(\alpha;\kappa) / \Gamma_m(\alpha),$$

where

$$\Gamma_m(\alpha) = \pi^{\frac{1}{4}m(m-1)} \prod_{i=1}^{m} \Gamma(\alpha - \frac{1}{2}(i-1)).$$

Constantine [2] has defined the generalized Laguerre polynomials as follows. Let $A_\gamma(R)$ be a Bessel function, that is, for $\gamma > -1$,

(6) $$A_\gamma(R) = \frac{1}{\Gamma_m(\gamma+p)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(-R)}{(\gamma+p)_\kappa k!} = {}_0F_1(\gamma+p; -R) - \frac{1}{\Gamma_m(\gamma+p)},$$

then the generalized Laguerre polynomial corresponding to a partition κ of an $m\times m$ matrix S is defined as

(7) $$\mathrm{etr}(-S) L_\kappa^\gamma(S) = \int_{R>0} A_\gamma(RS)(\det S)^\gamma \,\mathrm{etr}(-R) C_\kappa(R)\, dR.$$

It should be noted that (7) is same as the γ-Hankel transform of a function $\mathrm{etr}(-R)C_\kappa(R)$, [Herz, 5]

Constantine has discussed some properties of a generalized Laquerre polynomial.

(i) The Laplace transform of a generalized Laguerre polynomial.

$$\int_{S>0} \mathrm{etr}(-RS)(\det S)^\gamma L_\kappa^\gamma(S)\, dS$$
$$= \Gamma_m(\gamma+p;\kappa)(\det R)^{-\gamma-p} C_\kappa(I-R^{-1})$$

(ii) $$L_\kappa^\gamma(0) = (\gamma+p)_\kappa C_\kappa(I_m).$$

(iii) $$|L_\kappa^\gamma(S)| \leqq (r+p)_\kappa C_\kappa(I_m)\,\mathrm{etr}(S).$$

(iv) $L_\kappa^\gamma(S)$'s are orthogonal polynomials with respect to the weight function $\mathrm{etr}(-S)(\det S)^\gamma$, that is,

$$\int_{S>0} \mathrm{etr}(-S)(\det S)^\gamma L_\kappa^\gamma(S) L_\tau^\gamma(S)\, dS$$
$$= \delta_{\kappa\tau}\delta_{kl}\, k!\, \Gamma_m(r+p;\kappa) C_\kappa(I_m),$$

where k and l are degrees of L_κ^γ and L_τ^γ, respectively.

(v) The generating function of the generalized Laguerre polynomials

$$\det(I-Z)^{-\gamma-p}\int_{O(m)} \operatorname{etr}(-SHZ(I-Z)^{-1}H')\,d(H) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{L_{\kappa}^{\gamma}(S)C_{\kappa}(Z)}{k!\,C_{\kappa}(I_m)}\,, \qquad \|Z\|<1\,,$$

where $\|Z\|$ means the maximum of the absolute values of the characteristic roots of Z.

3. The generalized Hermite polynomials

C. S. Herz [5] defined the generalized Hermite polynomial of matrix argument as

$$\operatorname{etr}(-TT')H_{\eta}(T) = \frac{(i)^d}{\pi^{\frac{1}{2}mn}}\int_U \operatorname{etr}(-2iTU')\operatorname{etr}(-UU')\eta(U)\,dU,$$

where U and T are $m\times n$ $(m\leq n)$ matrices and $\eta(U)$ is a homogeneous polynomial of degree d. Here we define the generalized Hermite polynomial (g. H. p) $H_{\kappa}(T)$ as

(8) $$\operatorname{etr}(-TT')H_{\kappa}(T) = \frac{(-1)^k}{\pi^{\frac{1}{2}mn}}\int_U \operatorname{etr}(-2iTU')\operatorname{etr}(-UU')C_{\kappa}(UU')\,dU,$$

where $C_{\kappa}(UU')$ is a zonal polynomial of degree k with a partition κ. It should be noted that (8) is a Fourier transform of a function $\operatorname{etr}(-UU')C_{\kappa}(-UU')$. Hence the Fourier inversion formula of a g. H. p. is

(9) $$\frac{1}{\pi^{\frac{mn}{2}}}\int_T \operatorname{etr}(2iTU')\operatorname{etr}(-TT')H_{\kappa}(T)\,dT = \operatorname{etr}(-UU')C_{\kappa}(-UU').$$

The following lemma which gives a relation between the Fourier transform and the Hankel transform is very important.

Lemma. Let $f(U) = f(UU')$ be a real valued function defined for a positive semi definite matrix UU' where U is an $m \times n$ $(m \leq n)$ matrix. If $f(UU')$ is square integrable over UU', that is,

$$\int_{R>0} |f(R)|^2 (\det R)^{\gamma} dR < \infty \quad , \gamma = \frac{n}{2} - p \, ,$$

then we have for a Fourier transform

$$\pi^{\frac{1}{2}mn} g(TT') = \int_U \operatorname{etr}(-2iTU') f(UU') dU,$$

the γ-Hankel transform of f, i. e.

$$g(TT') = \int_{R>0} A_{\gamma}(TT'R)(\det R)^{\gamma} f(R)\, dR.$$

Proof. This lemma is a special case of Herz [5]. We here show that the Fourier transform of $f(UU')$ can be reformulated by the Hankel transform of $f(UU')$.

$$\begin{aligned} g(TT') &= \frac{1}{\pi^{\frac{1}{2}mn}} \int_U \operatorname{etr}(-2i\,TU') f(UU') dU \\ &= \frac{1}{\pi^{\frac{1}{2}mn}} \int_U dU \int_{O(n)} \operatorname{etr}(-2i\,THU') f(UU') d(H) . \end{aligned}$$

From (2) and (6),

$$\begin{aligned} g(TT') &= \frac{\Gamma_m(\frac{n}{2})}{\pi^{\frac{mn}{2}}} \int_U A_{\frac{n}{2}-p}(TT'UU') f(UU') dU \\ &= \int_{R>0} A_{\frac{n}{2}-p}(TT'R)(\det R)^{\frac{n}{2}-p} f(R) dR . \end{aligned}$$

The second equality is shown by the Hsu's lemma. Hence the R. H. S. is a $(\frac{n}{2} - p)$ Hankel transform of $f(R)$, which completes the proof.

Theorem 1.

(10) $$H_{\kappa}(T) = (-1)^k L_{\kappa}^{\frac{n}{2}-p}(TT').$$

Proof. Let $f(UU') = \mathrm{etr}(-UU')C_{\varkappa}(-UU')$ in lemma, then from (7) and (8) we have (10) immediately.

Corollary 1.

(11) $$H_{\varkappa}(T) = H_{\varkappa}(H_1T) = H_{\varkappa}(TH_2),$$

where $H_1 \in O(m)$ and $H_2 \in O(n)$, respectively.

Proof. The invariance with respect to H_1 is clear from (8) by a simple calculation. The invariance with respect to H_2 is also clear from theorem 1.

Corollary 2.

(12) $$H_{\varkappa}(0) = (-1)^k \left(\frac{n}{2}\right)_{\varkappa} C_{\varkappa}(I_m)$$

Proof. (12) is obvious from theorem 1 and (ii) when $\gamma = \frac{n}{2} - p$. (12) is also given by a direct calculation of (8) when $T = 0$.

Remark : From corollary 2 we can consider that the g. H. p. $H_{\varkappa}(T)$ corresponds to the generalization of a univariate Hermite polynomial of even degree.

Corollary 3. The g. H. p!s are orthogonal functions with respect to a weight function etr $(-TT')$.

$$\int_T \mathrm{etr}(-TT')H_{\varkappa}(T)H_{\tau}(T)\,dT = \delta_{kl}\,\delta_{\varkappa\tau}\,\pi^{\frac{1}{2}mn}\,k!\left(\frac{n}{2}\right)_{\varkappa} C_{\varkappa}(I_m),$$

where $H_{\varkappa}(T)$ and $H_{\tau}(T)$ are g. H. p. corresponding to k degrees and

ℓ degrees zonal polynomials $C_\kappa(UU')$ and $C_\tau(UU')$, respectively.

Proof. From theorem 1 and (iv),

$$\begin{aligned}\int_T & \operatorname{etr}(-TT')\, H_\kappa(T) H_\tau(T)\, dT \\ &= (-1)^{k+\ell} \int_T \operatorname{etr}(-TT')\, L_k^{\frac{n}{2}-p}(TT')\, L_\tau^{\frac{n}{2}-p}(TT')\, dT \\ &= \frac{\pi^{\frac{1}{2}mn}}{\Gamma_m(\frac{m}{2})} (-1)^{k+\ell} \int_{Z>0} \operatorname{etr}(-Z)(\det Z)^{\frac{n}{2}-p}\, L_k^{\frac{n}{2}-p}(Z)\, L_k^{\frac{n}{2}-p}(Z)\, dZ \\ &= \delta_{k\ell}\, \delta_{\kappa\tau}\, \pi^{\frac{mn}{2}}\, k!\, (\tfrac{n}{2})_\kappa\, C_\kappa(I_m),\end{aligned}$$

where the second equality is shown by the Hsu's lemma.

Corollary 4.

(13) $$|H_\kappa(T)| \leqq (\tfrac{n}{2})_\kappa\, C_\kappa(I_m)\, \operatorname{etr}(TT').$$

Here we consider the generating function of the g. H. p.. In the univariate case the generating function is given by

$$\exp(-s^2 + 2ts) = \sum_{k=0}^{\infty} \frac{H_k(t)}{k!} s^k .$$

Herz [5] has given the generating function of the g. H. p.'s $H_\lambda(T)$ by the using the extension of a Hilbert schmit kernel of a generalized Weber function $\operatorname{etr}(-\frac{1}{2}TT')H_\lambda(T)$. Here we can give it in the following way.

Theorem 2. Let S and T be $m \times n$ $(m \leqq n)$ matrices, then the generating function of g. H. p.'s is given by

(14) $$\int_{O(m)}\int_{O(n)} \operatorname{etr}(-SS' + 2H_1TH_2S')\,d(H_1)\,d(H_2)$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{H_\kappa(T)\, C_\kappa(SS')}{k!(\frac{n}{2})_\kappa C_\kappa(I_m)}$$

where $H_1 \in O(m)$ and $H_2 \in O(n)$, respectively.

Proof. We prove this theorem by the uniqueness of a Fourier transform. Let multiply $\mathrm{etr}(+2iTM')\mathrm{etr}(-TT')$ on both side and integrate term by term over T. Using (1) and (2), the left hand side of (14) becomes

$$\begin{aligned}
\text{L. H. S.} &= \int_T \mathrm{etr}(+2iTM')\mathrm{etr}(-TT') \int\int_{O(m)\,O(n)} \mathrm{etr}(-SS' + 2H_1TH_2S')d(H_1)d(H_2) \\
&= \int_{O(m)}\int_{O(n)} \mathrm{etr}(-SS')\int_T \mathrm{etr}(-TT' + 2T(H_1'SH_2' + iM)')dT\,d(H_1)d(H_2) \\
&= \pi^{\frac{1}{2}mn}\,\mathrm{etr}(-MM')\int_{O(m)}\int_{O(n)} \mathrm{etr}(2iH_1'SH_2'M')\,d(H_1)d(H_2) \\
&= \pi^{\frac{1}{2}mn}\,\mathrm{etr}(-MM')\int_{O(m)} {}_0F_1(\tfrac{n}{2}\,;\,-H_1'SS'H_1MM')d(H_1) \\
&= \pi^{\frac{1}{2}mn}\,\mathrm{etr}(-MM')\;{}_0F_1^{(m)}(\tfrac{n}{2}\,;\,-SS',\,MM').
\end{aligned}$$

On the other hand, the right hand side becomes, using (9),

$$\begin{aligned}
\text{R. H. S.} &= \int_T \mathrm{etr}(2iTM')\,\mathrm{etr}(-TT')\sum_{k=0}^{\infty}\sum_{\kappa}\frac{H_\kappa(T)\,C_\kappa(SS')}{k!\,(\frac{n}{2})_\kappa\,C_\kappa(I_m)}\,dT \\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(SS')}{k!\,(\frac{n}{2})_\kappa C_\kappa(I_m)}\int_T \mathrm{etr}(2iTM')\mathrm{etr}(-TT')\,H_\kappa(T)dT \\
&= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(SS')}{k!\,(\frac{n}{2})_\kappa\,C_\kappa(I_m)}\pi^{\frac{1}{2}mn}\;\mathrm{etr}(-MM')C_\kappa(-MM') \\
&= \pi^{\frac{1}{2}mn}\,\mathrm{etr}(-MM')\;{}_0F_1^{(m)}(\frac{n}{2}\,;\,-SS',\,MM'),
\end{aligned}$$

which equals the previous expression, Q. E. D.

Remark : Corollary 1 is obvious from the orthogonal invariance of the orthogonal measure and theorem 2.

Corollary 5.

(15) $$\sum_{\kappa} H_{\kappa}(T) = (-1)^k \operatorname{etr}(TT') \frac{\Gamma(\frac{mn}{2}+k)}{\Gamma(\frac{mn}{2})} \ {}_1F_1(\frac{mn}{2}+k; \frac{mn}{2}; -\operatorname{Tr}TT')$$

(16) $$\sum_{\kappa} (\frac{n}{2})_{\kappa} C_{\kappa}(I_m) = (\frac{mn}{2})_k$$

Proof. To prove (15), we need the following equality which is given by Khatri [8]

(17) $$\int_{R>0} \operatorname{etr}(-R)(\det R)^{\alpha-p}(\operatorname{Tr}S)^k C_{\tau}(SR)dR$$
$$= \Gamma_m(\alpha;\tau) \frac{\Gamma(m\alpha+k+\ell)}{\Gamma(m\alpha+\ell)} C_{\tau}(S),$$

where $C_{\tau}(s)$ is a zonal polynomial of ℓ degrees with partition τ. Now from the definition of a g. H. p.

$$\sum_{\kappa} H_{\kappa}(T) = \frac{(-1)^k}{\pi^{\frac{1}{2}mn}} \operatorname{etr}(TT') \int_U \operatorname{etr}(-2iTU')\operatorname{etr}(-UU') \sum_{\kappa} C_{\kappa}(UU')dU$$
$$= \frac{(-1)^k}{\pi^{\frac{1}{2}mn}} \operatorname{etr}(TT') \int_U \int_{O(n)} \operatorname{etr}(-2iTHU')\operatorname{etr}(-UU')(\operatorname{Tr}UU')^k d(H)dU$$
$$= \frac{(-1)^k}{\pi^{\frac{1}{2}mn}} \operatorname{etr}(TT') \int_U {}_0F_1(\frac{n}{2}; -TT'UU')\operatorname{etr}(-UU')(\operatorname{Tr}UU')^k dU.$$

Applying the Hsu's lemma, we have

$$= \frac{(-1)^k}{\Gamma_m(\frac{n}{2})} \operatorname{etr}(TT') \int_{R>0} {}_0F_1(\frac{n}{2}; -TT'R)(\det R)^{\frac{n}{2}-p} \operatorname{etr}(-R)(\operatorname{Tr}R)^k dR$$
$$= \frac{(-1)^k}{\Gamma_m(\frac{n}{2})} \operatorname{etr}(TT') \sum_{\ell=0}^{\infty} \sum_{\tau} \frac{(-1)^{\ell}}{\ell!(\frac{n}{2})_{\tau}} \int_{R>0} \operatorname{etr}(-R)(\det R)^{\frac{n}{2}-p} (\operatorname{Tr}R)^k C_{\kappa}(TT'R)dR.$$

Hence from (17),

$$= (-1)^k \mathrm{etr}(TT') \sum_{\ell=0}^{\infty} \sum_{\tau} \frac{(-1)^{\ell}}{\ell!} \frac{\Gamma(\frac{mn}{2}+k+\ell)}{\Gamma(\frac{mn}{2}+\ell)} C_{\tau}(TT')$$

$$= (-1)^k \mathrm{etr}(TT') \frac{\Gamma(\frac{mn}{2}+k)}{\Gamma(\frac{mn}{2})} {}_1F_1(\frac{mn}{2}+k ; \frac{mn}{2} ; -\mathrm{Tr}\,TT')$$

where ${}_1F_1$ is a univariate hypergeometric function.

If we set $T = 0$ in (15), from (12) and ${}_1F_1 = 1$, we obtains

$$\sum_{\kappa} (\frac{n}{2})_{\kappa} C_{\kappa}(I_m) = (\frac{mn}{2})_k .$$

Corollary 6.

(18) $$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{H_{\kappa}(T)}{k!} = \frac{1}{2^{\frac{mn}{2}}} \mathrm{etr}\,(\frac{1}{2}TT')$$

(19) $$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{H_{\kappa}(T)}{k!(\frac{n}{2})_{\kappa}} = \mathrm{etr}\,(-I_m)\, {}_0F_1(\frac{n}{2} ; TT').$$

Proof. (18) can be shown from the definition of a generalized Hermite polynomials. (19) can be shown when we set $SS' = I_m$ in (14).

4. The density function of the maximum latent root

Here we give the useful transformation and related Bata type integrals.

Lemma. Let S be an $m \times m$ positive definite (p. d.) random symmetric matrix. We decompose S as follows,

$$S = H \begin{bmatrix} \lambda_1 & \\ & V \end{bmatrix} H',$$

where H is an $m \times m$ orthogonal matrix which has only $(m-1)$ independent variables and V is an $(m-1)\times(m-1)$ p. d. symmetric matrix which ranges $\lambda_1 I_{m-1} > V > 0$. Then the Jacobian of this transformation is given by

$$J(S \to \lambda_1, H, V) = \det(\lambda_1 I_{m-1} - V) \frac{1}{\sqrt{1-\sum_{i=x}^{m} h_i^2}}$$

Proof. See Hayakawa [4]

The following corallary is very important to give the density function of the maximum latent root.

Corollary 7.

$$(20) \quad \int_{I_m > W > 0} (\det W)^{\alpha-p} \det(I-W)\, C_\kappa\begin{pmatrix}1 & \\ & W\end{pmatrix} dW$$

$$= \frac{\Gamma_m(\alpha)\Gamma_m(p)}{\Gamma_m(\alpha+p)} \frac{\Gamma(\frac{m}{2})}{\pi^{\frac{m}{2}}} (\alpha m+k) \frac{(\alpha)_\kappa}{(\alpha+p)_\kappa} C_\kappa(I_m)$$

$$(21) \quad \int_{1>\omega_2>\cdots>\omega_m>0} \left(\prod_{i=2}^{m} \omega_i\right) \prod_{i=2}^{m}(1-\omega_i) \prod_{2\leq i<j\leq m} (\omega_i - \omega_j)\, C_\kappa\begin{pmatrix}1 & & \\ & \omega_2 & \\ & & \ddots \\ & & & \omega_m\end{pmatrix} \prod_{i=2}^{m} d\omega_i$$

$$= \frac{\Gamma_m(\alpha)\Gamma_m(p)}{\Gamma_m(\alpha+p)} \frac{\Gamma_m(\frac{m}{2})}{\pi^{\frac{m^2}{2}}} (\alpha m+k) \frac{(\alpha)_\kappa}{(\alpha+p)_\kappa} C_\kappa(I_m)$$

Proof, It is well known that the following integral holds,

$$(22) \quad \int_{I_m > S > 0} (\det S)^{\alpha-p} C_\kappa(S)\, dS = \frac{\Gamma_m(\alpha)\Gamma_m(p)}{\Gamma_m(\alpha+p)} \frac{(\alpha)_\kappa}{(\alpha+p)_\kappa} C_\kappa(I_m).$$

To prove (20), we have decompose S as

$$S = H \begin{bmatrix} \lambda_1 & \underline{0}' \\ \underline{0} & \lambda_1 W \end{bmatrix} H',$$

where W is an $(m-1)\times(m-1)$ p. d. symmetric matrix which ranges $I_{m-1} > W > 0$. Then the Jacobian is given by

$$J(S \to \lambda_1, W, H) = \lambda_1^{\frac{1}{2}(m-1)(m+2)} \det(I_{m-1} - W) \frac{1}{\sqrt{1-\sum_{i=2}^{m} h_i^2}}.$$

Hence the left hand side of (22) becomes

$$\text{L.H.S.} = \int_0^1 \lambda_1^{\alpha m+k-1} d\lambda_1 \int_{I_{m-1}>W>0} (\det W)^{\alpha-p} \det(I-W)\, C_\kappa\begin{pmatrix}1 & \\ & W\end{pmatrix} \frac{1}{\sqrt{1-\sum_{i=2}^{m} h_i^2}}\, dW\, dH$$

$$= \frac{1}{\alpha m+k} \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} \int_{I_{m-1}>W>0} (\det W)^{\alpha-p} \det(I-W)\; C_\kappa\begin{pmatrix}1 & \\ & W\end{pmatrix} dW,$$

since

$$\int_0^1 \lambda_1^{\alpha m+k-1} d\lambda_1 = \frac{1}{\alpha m+k} \quad \text{and} \quad \int_{\sum_{i=2}^{m} h_i^2 \le 1} \frac{dH}{\sqrt{1-\sum_{i=2}^{m} h_i^2}} = \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}$$

To prove (21) we decompose further $W = H_1 \Lambda_\omega H_1'$ where $H_1 \in O(m-1)$ and $\Lambda_\omega = \operatorname{diag}(\omega_2, \cdots, \omega_m)$, then the Jacobian is

$$J(W \to \Lambda_\omega, H_1) = \prod_{2 \le i \le j \le m} (\omega_i - \omega_j).$$

Hence inserting these results to (20) we obtain (21) since

$$\frac{1}{2^{m-1}} \int_{O(m-1)} dH = \frac{\pi^{\frac{1}{2}(m-1)^2}}{\Gamma_{m-1}(\frac{m-1}{2})}.$$

(21) is the same result as Sugiyama [9].

4.1. The density function of the maximum latent root of a multivariate quadratic form.

Let X be an $m \times n$ ($m \le n$) normal sample matrix with mean $\underline{0}$ and covariance Σ, and A be an $n \times n$ p.d. symmetric matrix. Hayakawa [3] has given the density function of a multivariate quadratic form $Z = XAX'$ as follows:

$$\frac{1}{\Gamma_m(\frac{n}{2})(\det 2\Sigma)^{\frac{n}{2}} (\det A)^{\frac{m}{2}}} (\det Z)^{\frac{n}{2}-p} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(-\frac{1}{2}\Sigma^{-1} Z) C_\kappa(A^{-1})}{k!\, C_\kappa(I_n)}.$$

The joint density function of the latent roots $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_m)$ of Z is given by

$$(23) \qquad \frac{\pi^{\frac{1}{2}m^2}}{\Gamma_m(\frac{n}{2})\Gamma_m(\frac{m}{2})(\det 2\Sigma)^{\frac{n}{2}}(\det A)^{\frac{m}{2}}}(\det \Lambda)^{\frac{n}{2}-p}\prod_{i<j}(\lambda_i - \lambda_j)$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\Lambda)C_\kappa(-\frac{1}{2}\Sigma^{-1})C_\kappa(A^{-1})}{k!\,C_\kappa(I_n)C_\kappa(I_m)} .$$

To give the density function of the maximum latent root λ_1, we set $\lambda_i = \lambda_1\omega_i$ $(i = 2, \cdots, m)$ in (23) and integrate it with respect to $1 > \omega_2 > \cdots > \omega_m > 0$.

$$\frac{\pi^{\frac{1}{2}m^2}}{\Gamma_m(\frac{n}{2})\Gamma_m(\frac{m}{2})(\det 2\Sigma)^{\frac{n}{2}}(\det A)^{\frac{m}{2}}}\lambda_1^{\frac{mn}{2}-1}\sum_{k=0}^{\infty}\frac{\lambda_1^k}{k!}$$

$$\sum_{\kappa}\frac{C_\kappa(-\frac{1}{2}\Sigma^{-1})C_\kappa(A^{-1})}{C_\kappa(I_n)C_\kappa(I_m)}\int_{1>\omega_2>\cdots>\omega_m>0}\left(\prod_{i=2}^{m}\omega_i\right)^{\frac{n}{2}-p}\prod_{i=2}^{m}(1-\omega_i)\prod_{2\le i\le j\le m}(\omega_i - \omega_j)\,C_\kappa\begin{pmatrix}1 & & & \\ & \omega_2 & & \\ & & \ddots & \\ & & & \omega_m\end{pmatrix}\prod_{i=2}^{m}d\omega_i .$$

Hence from (21), we have

$$(24) \qquad \frac{\Gamma_m(p)}{\Gamma_m(\frac{n}{2}+p)(\det 2\Sigma)^{\frac{n}{2}}(\det A)^{\frac{m}{2}}}\lambda_1^{\frac{1}{2}mn-1}$$

$$\cdot\sum_{k=0}^{\infty}\frac{(\frac{mn}{2}+k)}{k!}\lambda_1^k\sum_{\kappa}\frac{(\frac{n}{2})_\kappa}{(\frac{n}{2}+p)_\kappa}\frac{C_\kappa(-\frac{1}{2}\Sigma^{-1})C_\kappa(A^{-1})}{C_\kappa(I_m)} .$$

Theorem 3. Let X be distributed with a density function

$$\frac{1}{\pi^{\frac{1}{2}mn}(\det 2\Sigma)^{\frac{n}{2}}}\quad \mathrm{etr}(-\frac{1}{2}\Sigma^{-1}XX'),$$

then the density function of the maximum latent root of XAX' is given by (24).

Corollary 8. The cdf of the maximum latent root λ_1 of the multivariate quadratic form XAX' is given by

$$\text{(25)} \qquad \Pr(\lambda_1 < x) = \frac{\Gamma_m(p)}{\Gamma_m(\frac{n}{2}+p)(\det 2\Sigma)^{\frac{n}{2}}(\det A)^{\frac{m}{2}}} x^{\frac{1}{2}mn}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k \sum_{\kappa} \frac{(\frac{n}{2})_\kappa}{(\frac{n}{2}+p)_\kappa} \frac{C_\kappa(-\frac{1}{2}\Sigma^{-1})C_\kappa(A^{-1})}{C_\kappa(I_n)} .$$

Corollary 9. If we set $A = I_n$ in (24) then we have the density function of the maximum latent root λ_1 of a central Wishart matrix, which is the same as Sugiyama [9].

4. 2. The density function of the maximum latent root of a non-central Wishart matrix with $\Sigma = I_m$.

The joint density function of the latent roots $\Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_m)$ of a non-central Wishart matrix with $\Sigma = I_m$ was given by

$$\text{(26)} \qquad \frac{\pi^{\frac{1}{2}m^2}}{2^{\frac{1}{2}mn}\Gamma_m(\frac{n}{2})\Gamma_m(\frac{m}{2})} \operatorname{etr}(-\tfrac{1}{2}MM')\operatorname{etr}(-\tfrac{1}{2}\Lambda)(\det\Lambda)^{\frac{n}{2}-p} \prod_{i<j}(\lambda_i - \lambda_j)$$

$$\sum_{k=0}^{\infty}\sum_{\kappa} \frac{C_\kappa(-\frac{1}{2}\Lambda)C_\kappa(\frac{1}{2}MM')}{k!(\frac{m}{2})_\kappa C_\kappa(I_m)} .$$

However, (26) is not an appropriate form to derive the density function of the maximum latent root λ_1 when we use (21). Hence we give an another form of the joint density function of the latent roots Λ which is appropriate to discuss.

Lemma. Let X be distributed with a density function

$$\text{(27)} \qquad \frac{1}{(2\pi)^{\frac{1}{2}mn}} \operatorname{etr}(-\frac{1}{2}XX' + XM') \operatorname{etr}(-\frac{1}{2}MM'),$$

then the joint density function of the latent roots Λ of XX' is given by

$$(28) \qquad \frac{\pi^{\frac{1}{2}m^2}}{2^{\frac{1}{2}mn}\Gamma_m(\frac{n}{2})\Gamma_m(\frac{m}{2})} \operatorname{etr}(-\frac{1}{2}MM')(\det\Lambda)^{\frac{n}{2}-p}\prod_{i<j}(\lambda_i-\lambda_j)$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{H_\kappa(\frac{1}{\sqrt{2}}M)\,C_\kappa(\frac{1}{2}\Lambda)}{k!\,(\frac{n}{2})_\kappa C_\kappa(I_m)}.$$

Proof. We here decompose X in (27) as follows

$$(29) \qquad X = H_1\Lambda^{\frac{1}{2}}L'$$

where H_1 is an orthogonal matrix with positive element in the first column and $\Lambda^{\frac{1}{2}}$ is a diagonal matrix of square roots of latent roots $\lambda_1,\cdots,\lambda_m$ of XX' in descending order and L is a $n\times m$ Stiefel matrix satisfying $L'L = I_m$. The Jacobian of this transform is

$$dX = \frac{2^m\pi^{\frac{1}{2}(m+n)n}}{\Gamma_m(\frac{m}{2})\Gamma_m(\frac{n}{2})}(\det\Lambda)^{\frac{n}{2}-p}\prod_{i<j}(\lambda_i-\lambda_j)\,d\Lambda\,d(H_1)\,d(L)$$

where $d(L)$ is a normalized Stiefel invariant measure with volume unity. Thus inserting (29) to (28), we obtain the joint density function of Λ, H_1 and L.

$$(30) \qquad \frac{\operatorname{etr}(-\frac{1}{2}MM')\,2^m\pi^{\frac{1}{2}m^2}}{2^{\frac{1}{2}mn}\Gamma_m(\frac{n}{2})\Gamma_m(\frac{n}{2})}(\det\Lambda)^{\frac{n}{2}-p}\prod_{i<j}(\lambda_i-\lambda_j)$$

$$\operatorname{etr}(-\frac{1}{2}\Lambda^{\frac{1}{2}}L'L\Lambda^{\frac{1}{2}} + H_1\Lambda^{\frac{1}{2}}L'M').$$

Hence we only integrate (30) with respect to H_1 and L. If we set $L \to H_2'L$, $H_2 \in O(n)$, then $L'H_2H_2'L = L'L = I_m$ and the Stiefel invariant measure $d(L)$ is unchanged.

Thus

$$\frac{1}{2^m}\int_{O(m)}\int_{L'L=I_m}\operatorname{etr}(-\frac{1}{2}\Lambda^{\frac{1}{2}}L'L\Lambda^{\frac{1}{2}} + H_1\Lambda^{\frac{1}{2}}L'M')\,d(H)\,d(L)$$

$$= \int_{L'L=I_m} d(L) \frac{1}{2^m} \int_{O(m)} \int_{O(n)} \operatorname{etr}\left(-\frac{1}{2}\Lambda^{\frac{1}{2}} L'L \Lambda^{\frac{1}{2}} + H_1 \Lambda^{\frac{1}{2}} L' H_2 M'\right) d(H_1) d(H_2) .$$

Hence we can see that the integral with respect to H_1 and H_2 is the same form as theorem 2 if we set $S = \frac{1}{\sqrt{2}} \Lambda^{\frac{1}{2}} L'$ and $T = \frac{1}{\sqrt{2}} M$ in (14), thus

$$\int_{L'L=I_m} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{H_\kappa(\frac{M}{\sqrt{2}})\, C_\kappa(\frac{1}{2}\Lambda)}{k!\,(\frac{n}{2})_\kappa C_\kappa(I_m)} d(L)$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{H_\kappa(\frac{M}{\sqrt{2}})\, C_\kappa(\frac{1}{2}\Lambda)}{k!\,(\frac{n}{2})_\kappa C_\kappa(I_m)} ,$$

which completes the proof.

Now we derive the density function of maximum latent root λ_1. If we set $\lambda_i = \lambda_1 \omega_i \ (i=2,\cdots,m)$ in (28) and integrate it with respect to $1 > \omega_2 > \cdots > \omega_m > 0$ by the use of (21), then we have

$$(31) \qquad \frac{\Gamma_m(p)}{2^{\frac{1}{2}mn} \Gamma_m(\frac{n}{2}+p)} \operatorname{etr}\left(-\frac{1}{2}MM'\right) \lambda_1^{\frac{1}{2}mn-1} \sum_{k=0}^{\infty} \frac{(\frac{mn}{2}+k)}{k!} \left(\frac{\lambda_1}{2}\right)^k \sum_{\kappa} \frac{H_\kappa(\frac{1}{\sqrt{2}} M)}{(\frac{n}{2}+p)_\kappa} .$$

Theorem 4. The density function of the maximum latent root λ_1 of a non-central Wishart matrix of n degrees of freedom with $\Sigma = I_m$ is given by (31).

Note : Hayakawa [4] has given the density function of the maximum latent root λ_1 by the use of two expansions of zonal polynomial such that

$$C_\kappa(A \dot{+} B) = \sum_{\sigma,\tau} a_{\sigma\tau}^{\kappa} C_\sigma(A) C_\tau(B) ,$$

and

$$C_\kappa(A) C_\tau(A) = \sum_{\sigma} b_{\kappa\tau}^{\sigma} C_\sigma(A).$$

However (31) is more exact form than Hayakawa [4,(33)].

5. On the distribution of a trace of a non-central Wishart matrix with $\Sigma = I_m$.

Let Λ be distributed as (28). We derive the density function of $T = \mathrm{Tr}\, XX'$ $= \mathrm{Tr}\, \Lambda$ from the inversion formula of a Laplace transform $\varphi(t)$.

Let $\varphi(t) = E(\mathrm{etr}(-t\Lambda))$

$$= \frac{\pi^{\frac{1}{2}m^2}}{2^{\frac{mn}{2}}\Gamma_m(\frac{m}{2})\Gamma_m(\frac{n}{2})} \mathrm{etr}(-\frac{1}{2}MM') \sum_{k=0}^{\infty}\sum_{\kappa} \frac{H_\kappa(\frac{M}{\sqrt{2}})(\frac{1}{2})^k}{k!(\frac{n}{2})_k C_\kappa(I_m)}$$

$$\int_{\lambda_1>\cdots>\lambda_m>0} \mathrm{etr}(-t\Lambda)(\det\Lambda)^{\frac{n}{2}-p} \prod_{i<j}(\lambda_i-\lambda_j) C_\kappa(\Lambda)\, d\Lambda$$

$$= \frac{1}{2^{\frac{1}{2}mn}} \mathrm{etr}(-\frac{1}{2}MM') \sum_{k=0}^{\infty}\sum_{\kappa} \frac{H_\kappa(\frac{M}{\sqrt{2}})}{k!} (\frac{1}{2})^k t^{-\frac{1}{2}mn-k}.$$

If we use the well known formula

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{tT} t^{-\frac{mn}{2}-k} dt = \frac{T^{\frac{1}{2}mn+k-1}}{\Gamma(\frac{mn}{2}+k)},$$

then the density of T is given by

(32) $$\frac{1}{2^{\frac{1}{2}mn}\Gamma(\frac{mn}{2})} \mathrm{etr}(-\frac{1}{2}MM')\, T^{\frac{1}{2}mn-1} \sum_{k=0}^{\infty} \frac{1}{k!(\frac{mn}{2})_k} (\frac{T}{2})^k \sum_{\kappa} H_\kappa(\frac{M}{\sqrt{2}}),$$

since $\Gamma(\frac{mn}{2}+k) = (\frac{mn}{2})_k \Gamma(\frac{mn}{2})$.

Theorem 5. Let Λ be distributed with the density function (28), then the density function of $T = \mathrm{Tr}\,\Lambda$ is given by (32).

Corollary 10. If we set $M = 0$ in (32), then we obtain the density function of χ^2 with mn degrees of freedom since

$$\sum_{\kappa} \left(\frac{n}{2}\right)_{\kappa} C_{\kappa}(I_m) = \left(\frac{mn}{2}\right)_k .$$

References

[1] Constantine, A.G., "Some non-central distribution problems in multivariate analysis", Ann. Math. Statist. 34(1963), 1270-1285.

[2] Constantine, A.G., "The distribution of Hotelling's generalized T_0^2 ", Ann. Math. Statist., 37(1966), 215-225.

[3] Hayakawa, T., "On the distribution of a quadratic form in a multivariate normal sample", Ann. Inst. Statist. Math., 18(1966), 191-201.

[4] Hayakawa, T., "On the distribution of the maximum latent root of a positive definite symmetric random matrix", Ann. Inst. Statist. Math. 19(1967), 1-17.

[5] Herz, C.S., "Bessel functions of matrix argument", Ann. Math., 61(1955) 474-523.

[6] James, A.T., "Zonal polynomials of the real positive definite symmetric matrices," Ann. Math., 32(1961), 456-469.

[7] James, A.T., "Distributions of matrix variates and latent roots derived from normal samples", Ann. Math. Statsist. 35(1964), 475-501.

[8] Khatri, C.G., "On certain distribution problems based on positive definite quadratic functions in normal vectors", Ann. Math. Statist., 37(1966), 468-479.

[9] Sugiyama, T. "On the distribution of largest latent root of the covariance matrix", Ann. Math. Statist., 38(1967), 1148-1151