数理解析研究所講究録
第48巻 1968年 53-73

Approximate Methods for Nonlinear Stochastic Control Processes

by

Yoshifumi Sunahara

Department of Applied Mathematics & Physics
Kyoto University

(March 26, 1968)

1. Introduction and symbolic conventions

In spite of the systematic establishment of linear control theory and the numerous contributions made to it by many scientists, the phenomena of actual control systems have yielded a wide variety of mathematical problems which can not be solved. It is, for instance, pretty certain to say that many control engineers were mystified by the fact that a control system, well designed by linear control theory, fluctuates with constant amplitude and frequency around its balancing level. This phenomenon, which can not be solved by linear control theory, has been called the limit cycle, and it is due to nonlinear transfer characteristics existing within the feedback control circuit. Consequently, we are forced to take into account an extended region of the input magnitude. Our analytical field of vision must be extended to various kinds of nonlinear characteristics which are always exhibited in control systems, whether intentionally placed there or not. Thus, control theory and its practice were placed on a new situation in which the characteristics of dynamical systems to be controlled were nonlinear and these undergo drastic changes in the stochastic sense, according to time and environment.

In addition to the situation mentioned above, as the uses of automatic control have multiplied in industrial processes, in manufacturing, in the steering and operation of modern weapons and in setting the flight of artificial satellites, the demands placed on control systems have also severely taxed the designers and forced them to analyse the dynamic behavior of increasingly complex systems, more precisely, to improve the control performance. In compliance with recent trends mentioned here, a modern approach to design and synthesis of control systems was formulated, based on the basic notion of state representation of existing systems. The advent of high-speed digital computers revolutionized the basic concept of design of control systems. The design philosophy made a change to the determination of an extremum of a functional from the classical approaches whose ability is limited to solve the problems of finding optimal parameters which are adjustable in a configuration of control systems. The digital computer in the control system usually performs the functions of monitoring, data processing and optimal control, as illustrated in Fig.1.

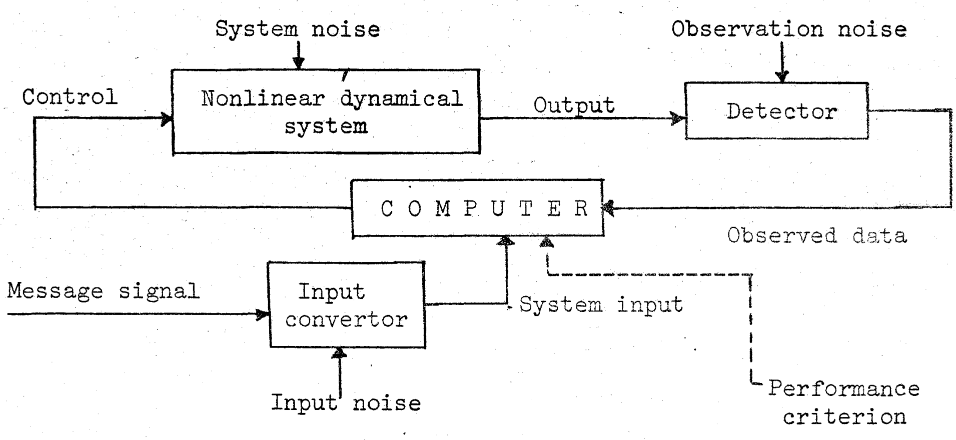

Fig.1. Illustration of a configuration of control systems

Thus, analytical methods of designing optimal control for a dynamical system characterized by uncertainties and a large amount of fluctuations require a new concept of stochastic theory in the non-stationary sense. The general problem to be solved is the control of a noisy nonlinear dynamical system in some optimal fashion, given only partial and noisy observations of the system state and, possibly, only an incomplete knowledge of the system. Fel'dbaum has shown in his works [1] that the optimal control law under these conditions is a functional on all available data by a direct application of the method of dynamic programming [2], and that computing algorithms of the optimal control are available in real time.

In recent years, however, a more elegant and precise technique is in the process of making a widespread development showing an abstract beauty combined the theory of dynamic programming optimization with that of Markovian stochastics. It has already been verified in the Markovian framework that, if the dynamical characteristics of a system to be controlled are idealized to be linear, then a separative discussion of stochastic optimization can be developed from the version of state estimation, and then the configuration shown in Fig.1 is reduced to that in Fig.2. [3], [4]

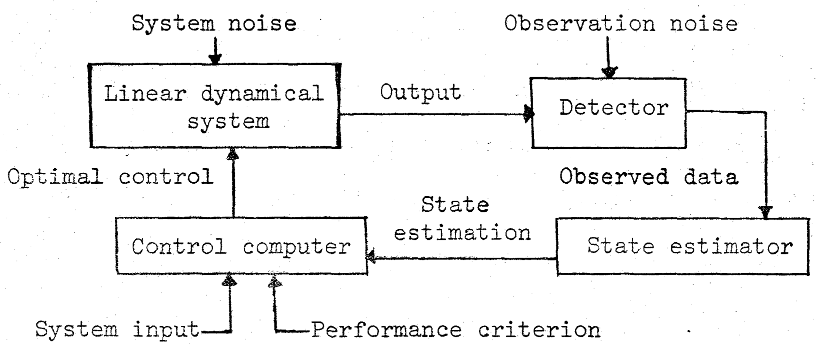

Fig.2. Schematic configuration of the optimal linear control system under noisy observations

Is the configuration in Fig.2 kept in an invariance fashion of the stochastic optimization for nonlinear stochastic systems? The answer is "No", and, more concretely, the correct version of this answer is in a black vail. Unfortunately, in spite of the fact that there exists a large amount of accumulated knowledge concerning the solution of nonlinear differential equations of some special types, we should point out that our knowledge of nonlinear stochastic systems behavior is extremely limited as compared with the situation in the field of linear counterparts. Thus, the establishment of possible methods of approximation for nonlinear stochastic processes is strongly being required to solve various problems in both qualitative and quantitative aspects. Intuitively, a possible way of approximations is the development of quasi-linearization technique, keeping in touch with the context of linear control theory. From this point of view, let us observe once again the configuration of optimal linear control systems. This indicates that, under noisy observations, the state estimation procedure should be performed before generating the optimal control. In other words, our first effort should be devoted to the physical realization of optimal nonlinear estimators with which this short note is mainly concerned.

Throughout this short note, we use the same symbols for the true stochastic processes and for the quasi-linear stochastic processes which are the approximations to the true one by the method described later. The symbol, Y_t, denotes the smallest σ-algebra of ω sets with respect to which random variables $y(\tau)$ with $\tau \leqq t$ are measurable, where ω is the generic point of the probability space Ω. The conditional expectation of a random variable conditioned by Y_t is simply expressed by "$\hat{\cdot}$" such that $\hat{x}(t|\tau) = E\{x(t)|Y_\tau\}$ where $\tau \leqq t$. Vector and matrix notations follow the usual manner. If M is a matrix, then M' denotes its transpose, and then $|M|$ denotes the determinant of the matrix M. For the convenience of the present description, the principal symbols used here are listed below:

t: time variable, particularly the present time

t_0: the initial time at which observations start

$x(t)$ and $y(t)$: n-dimensional vector stochastic processes representing the system states and the observations respectively

$w(t)$ and $v(t)$: d_1- and d_2-dimensional Brownian motion processes respectively

$Q(t)$ and $R(t)$: $n \times d_1$ and $n \times d_2$ matrices whose components depend on t

$f[t, x(t)]$ and $g[t, x(t)]$: n-dimensional vector valued nonlinear functions respectively

$x(t|t)$: optimal estimate of $x(t)$ conditioned by Y_t, i.e., $x(t|t) \triangleq E\{x(t)|Y_t\}$

$P(t|t)$: error covariance matrix in optimal estimate of $x(t)$ conditioned by

Y_t, i.e., $P(t|t) \triangleq \text{cov.} \{x(t)|Y_t\}$

2. Mathematical models

A broad class of dynamical systems encountered in control engineering is characterized by a multiplicity of inputs and corresponding outputs. Guided by a well-known concept of state space representation, the dynamics of an important class of dynamical systems can be described by a nonlinear vector differential equation,

$$\frac{dx(t, \omega)}{dt} = f[t, x(t, \omega)] - C(t)u(t) + Q(t)\gamma(t, \omega) \tag{2.1}$$

where $C(t)$ is an n x n matrix whose components depend on t, $u(t)$ is an n-dimensional control vector, and where $\gamma(t, \omega)$ denotes an n-dimensional Gaussian white random disturbance.

The development of the present discussion requires that, until further notice, we set the control $u(t)$ equal to zero in Eq. (2.1). When $u(t) = 0$, the same symbol $x(t)$ is used, disregarding the necessity of changing it and omitting the symbol ω, i.e.,

$$\frac{dx(t)}{dt} = f[t, x(t)] + Q(t)\gamma(t) \tag{2.2}$$

It should further be noted in Eq. (2.2) that, when $\gamma(t) = 0$, the variable $x(t)$ becomes deterministic.

To make Eq. (2.2) more precise in the Markovian framework, we shall write

$$dx(t) = f[t, x(t)]dt + Q(t)dw(t) \tag{2.3}$$

where the d_1-dimensional Brownian motion process, $w(t)$, has been introduced here along the relation between a Brownian motion process and a white noise or a sufficiently wide (but finite) band Gaussian random process $\gamma(t)$, (for more detail see [5], [6])

$$w(t) = \int^t \gamma(s)ds \tag{2.4}$$

We suppose that observations are made at the output of the nonlinear system with additive Gaussian disturbance. The observation process $y(t)$ is the n-dimensional vector random process determined by

$$dy(t) = h[t, x(t)]dt + R(t)dv(t) \tag{2.5}$$

where we assume that the system noise $w(t)$ and the observation noise $v(t)$ are mutually independent.

3. Fundamental hypotheses and problem statement

For the purpose of security in the mathematical development, the following assumptions are made with respect to Eqs. (2.3) and (2.5):

H-1: The components of the functions $f[\cdot,\cdot]$ and $h[\cdot,\cdot]$ are Baire functions with respect to the pair (t, ξ) for $t_0 \leqq t \leqq T$ and $-\infty < \xi < \infty$, where $x(t, \omega) = \xi$

H-2: The functions $f[\cdot,\cdot]$ and $h[\cdot,\cdot]$ satisfy a uniform Lipschitz conditions in the variable ξ and are bounded respectively by

$$\| f(t, \xi) \| \leqq K_1 (1 + \xi'\xi)^{1/2} \tag{3.1}$$

and

$$\| h(t, \xi) \| = K_1'(1 + \xi'\xi)^{1/2} \tag{3.2}$$

where K_1 and K_1' are real positive constants and are independent of both t and ξ respectively

H-3: $x(t_0)$ is a random variable independent of the $w(t)$-process

H-4: All parameter matrices are measurable and bounded on the finite time interval $[t_0, T]$

H-5: $\{R(t)R(t)'\}^{-1}$ exists and this is bounded on $[t_0, T]$

The problem is to find the minimal variance estimate of the state $x(t)$, provided that the process $y(s)$ for $t_0 \leqq s \leqq t$ is acquired as the observation process, where $y(t_0) = 0$.

4. Differential equation of the optimal filter

It is not difficult to show that the minimal variance estimate of a random variable given a related quantity is simply the conditional expectation so that

$$\hat{x}(t|t) = E\{x(t)|Y_t\} = \int_{E_n} x(t)p\{x(t)|Y_t\}dx(t) \tag{4.1}$$

and the problem is to find an equation for the conditional probability density function $p\{x(t)|Y_t\}$, where, in (4.1), the symbol E_n means the n-dimensional Euclidean space.

Invoking Bayes' formula and the related calculus, it follows that

$$p\{x(t)|Y_t\} = \frac{E\{\exp\phi|x\}p(x)}{E\{\exp\phi\}} \tag{4.2}$$

where

$$d\phi = \frac{1}{2}h'(RR')^{-1}hdt + h'R^{-1}dv \tag{4.3}$$

where $p(x)$ is the probability density function of the $x(t)$-process. The version of dp is given by [7]~[9]

$$dp = L^*[p]dt + (h-\hat{h})'(RR')^{-1}(dy - \hat{h}dt) \tag{4.4}$$

where

$$\hat{h} = \hat{h}(t|t) = E\{h[t, x(t)]|Y_t\} \tag{4.5}$$

and L^* denotes the formal adjoint of the diffusion operator [10]

$$L = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} [GG']_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \tag{4.6}$$

In (4.6) the suffix i indicates the i-th component of a vector and $[\cdot]_{i,j}$ denotes the (i, j) element of the matrix $[\cdot]$.

Substituting (4.4) into the expression of the version of $dx(t|t)$ derived from (4.1), the filter dynamics is obtained as

$$d\hat{x}(t|t) = \hat{f}[t, x(t)]dt + (\widehat{hx}-\hat{h}\,\hat{x})(RR')^{-1}(dy-\hat{h}dt) \tag{4.7}$$

The result reveals that an exact realization of optimal nonlinear filters requires infinite dimensional filters which are practically impossible to realize except for a very special case. [11]

5. Quasi-linear stochastic differentials and an approximation to nonlinear filtering Equations

Up to the present time, several trials have been made on the physical realization of optimal nonlinear filters in an approximate form of finite dimensional filters. [12]∿[16] In this section, the author will introduce the reader to a method of stochastic linearization which is shown to play a useful role in the study of state estimation and which is hopefully of an extensive use to the version of optimal control problems.

With the hypotheses listed in section 3, a precise interpretation of Eq. (2.3) is given by Ito [17], who writes it as an integral equation,

$$x(t) = x(t_0) + \int_{t_0}^{t} f[s, x(s)]ds + \int_{t_0}^{t} Q(s)dw(s) \tag{5.1}$$

We expand the function $f[t, x(t)]$ in Eq. (5.1) into

$$f[t, x(t)] = a(t) + B(t)\{x(t) - \hat{x}(t|t)\} + e(t) \tag{5.2}$$

where $e(t)$ denotes the collection of n-dimensional vector error terms and $a(t)$, $B(t)$ are n-dimensional vector and n x n matrix respectively. We determine $a(t)$ and $B(t)$ in such a way that the conditional expectation of the squared norm of $e(t)$, $E\{\|e(t)\|^2|Y_t\}$, becomes minimal. The necessary and sufficient conditions to minimize $E\{\|e(t)\|^2|Y_t\}$ are given by

$$a(t) = E\{f[t, x(t)]|Y_t\} = \hat{f}[t, x(t)] \tag{5.3}$$

$$B(t) = E\{\{f[t,x(t)] - \hat{f}[t,x(t)]\}\{x(t) - \hat{x}(t|t)\}'|Y_t\}P(t|t)^{-1} \tag{5.4}$$

where

$$P(t|t) = \text{cov.}\ [x(t)|Y_t] \tag{5.5}$$

The scalar expressions of (5.3) and (5.4) are respectively as follows:

$$a_i(t) = E\{f_i[t, x(t)]|Y_t\} = \hat{f}_i[t, x(t)] \tag{5.6}$$

$$\sum_{\nu=1}^{n} b_{i\nu}(t)E\{\{x_\nu(t) - \hat{x}_\nu(t|t)\}\{x_j(t) - \hat{x}_j(t|t)\}|Y_t\}$$
$$= E\{[f_i[t, x(t)] - \hat{f}_i[t, x(t)]][x_j(t) - \hat{x}_j(t|t)]|Y_t\} \tag{5.7}$$

where $a_i(t)$ denotes the i-th component of $a(t)$ and $b_{ij}(t)$ the (i,j)-element of $B(t)$, $\hat{x}_j(t|t) = E\{x_j(t)|Y_t\}$ and where i, j=1, 2,..., n.

The same notion is applicable to the function $h[t, x(t)]$. We expand $h[t, x(t)]$ into

$$(5.8) \qquad h[t, x(t)] = h_1(t) + H_2(t)\{x(t) - \hat{x}(t|t)\} + e_h(t)$$

The vector function $h_1(t)$ and the matrix $H_2(t)$ should be determined so as to minimize $E\{\|e_h(t)\|^2|Y_t\}$ and these are given by

$$(5.9) \qquad h_1(t) = E\{h[t, x(t)]|Y_t\} = \hat{h}[t, x(t)]$$

$$(5.10) \qquad H_2(t) = E\{[h[t, x(t)] - \hat{h}[t, x(t)]][x(t) - x(t|t)]'|Y_t\}P(t|t)^{-1}$$

Based on the assumption that, for $t\varepsilon[t_0, T]$, the conditional probability density function $p\{x(t)|Y_t\}$ is gaussian with the mean value $\hat{x}(t|t)$ and covariance matrix $P(t|t)$, both $a(t)$ and $B(t)$ can be obtained in the form

$$(5.11) \qquad a(t) = a(t, \hat{x}(t|t), P(t|t))$$

and

$$(5.12a) \qquad B(t) = B(t, \hat{x}(t|t), P(t|t))$$

$$(5.12b) \qquad b_{ij}(t) = \frac{\partial a_i(t)}{\partial \hat{x}_j(t|t)}$$

Similarly, (5.9) and (5.10) become

$$(5.13) \qquad h_1(t) = h_1(t, \hat{x}(t|t), P(t|t))$$

and

$$(5.14a) \qquad H_2(t) = H_2(t, \hat{x}(t|t), P(t|t))$$

$$(5.14b) \qquad h_{ij}(t) = \frac{\partial h_i(t)}{\partial \hat{x}_j(t|t)}$$

A striking fact is that the random variables $a(t)$ and $B(t)$ are not independent but depend mutually on the state estimate $\hat{x}(t|t)$ and the error covariance matrix $P(t|t)$. From this point of view, in reality, more precise symbols, $a(t, \hat{x}(t|t), P(t|t))$ and $B(t, \hat{x}(t|t), P(t|t))$, should be introduced.

However, for the economy of description, we merely denote these by a(t) and B(t) without indicating the dependence on both $\hat{x}(t|t)$ and $P(t|t)$. Both $h_1(t)$ and $H_2(t)$ also follow this symbolic convention. We may thus define here the following n-dimensional quasi-linear stochastic differential of Ito type for Eq. (5.1),

(5.15) $$dx(t) = B(t)x(t)dt + \{a(t) - B(t)\hat{x}(t|t)\}dt + Q(t)dw(t)$$

and for the observation process

(5.16) $$dy(t) = H_2(t)x(t)dt + \{h_1(t) - H_2(t)\hat{x}(t|t)\}dt + R(t)dv(t)$$

However, respective diffusion terms in Eqs. (5.15) and (5.16) still remain unknown. We shall thus proceed to solve the problem including the computation of the state estimate $\hat{x}(t|t)$ and the error covariance matrix $P(t|t)$.

Let $\Phi(t, t_0)$ be the fundamental matrix associated with the homogeneous differential equation,

(5.17) $$\frac{dx(t)}{dt} = B(t)x(t)$$

The solution of Eq. (5.15) can formally be written as

(5.18) $$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, s)\{a(s) - B(s)\hat{x}(s|s)\}ds + \int_{t_0}^{t} \Phi(t, s)Q(s)dw(s)$$

From Eq. (5.18), it is a simple exercise to show that

(5.19) $$d\xi(t) = B(t)\xi(t)dt + Q(t)dw(t)$$

where

(5.20) $$\xi(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, s)Q(s)dw(s)$$

and

(5.21) $$\xi(t_0) = x(t_0)$$

On the other hand, it follows from Eq. (5.16) that

(5.22) $$y(t) = \int_{t_0}^{t} H_2(s)x(s)ds + \int_{t_0}^{t} \{h_1(s) - H_2(s)\hat{x}(s|s)\}ds + \int_{t_0}^{t} R(s)dw(s)$$

Starting from Eq. (5.22), a new stochastic process $\eta(t)$ can easily be defined by its stochastic differential

$$(5.23) \qquad d\eta(t) = H_2(t)\eta(t)dt + R(t)dv(t)$$

Let H_t be the σ-algebra of ω sets generated by the random variable $\eta(s)$ for $t_0 \leqq s \leqq t$. Then the y(t)-process is H_t-measurable and thus

$$(5.24) \qquad E\{\xi(t)|Y_t\} = E\{\xi(t)|H_t\} \triangleq \hat{\xi}(t|t)$$

Now we consider that the $\xi(t)$-process is the fictitious state variables determined by Eq. (5.19) and that Eq. (5.23) denotes the observations which are made on the $\eta(t)$-process. This situation implies that the current estimate $\hat{\xi}(t|t)$ is given by [18]

$$(5.25) \qquad d\hat{\xi}(t|t) = B(t)\hat{\xi}(t|t)dt + P_\xi(t|t)H_2(t)'\{B(t)B(t)'\}^{-1} \times \{d\eta - H_2(t)\hat{\xi}(t|t)dt\}$$

where

$$(5.26) \qquad P_\xi(t|t) = \text{cov. } [\xi(t)|H_t]$$

Substituting Eq. (5.23) into (5.25) and bearing Eq. (5.18) in mind, we have

$$(5.27) \qquad d\hat{x}(t|t) = \hat{f}[t, x(t)]dt + P(t|t)H_2(t)'\{R(t)R(t)'\}^{-1}(dy-\hat{h}dt)$$

and

$$(5.28) \qquad \hat{x}(t_0|t_0) = E\{x(t_0)\}$$

where (5.3) and (5.9) have been used. By combining (5.24) with (5.26), the version of $dP(t|t)/dt$ becomes

$$(5.29) \qquad \frac{dP}{dt} = B(t)P(t|t) + P(t|t)B(t)' + Q(t)Q(t)' - P(t|t)H_2(t)'\{R(t)R(t)'\}^{-1}H_2(t)P(t|t)$$

with

$$(5.30) \qquad P(t_0|t_0) = \text{cov. } [x(t_0)]$$

Eq. (5.27) with Eq. (5.29) describes the dynamic structure of a quasi-linear filter for generating a current estimate $\hat{x}(t|t)$.

6. Comparative discussions and their quantitative aspects

Although the most familiar technique is the introduction of Taylor series expansion on a nonlinear function, the basic notion of the approximation described here is the expansion of the nonlinear function and the determination of the coefficients by means of the minimal square error criterion, including the Gaussian assumption to the conditional probability density function. This implies that the infinite dimensional filter is approximated by the two dimensional filter consisting of the first and second moments.

Besides the approximated structure of filter dynamics developed here, a number of approximation has been proposed in the literature. The major differences in the derivations lie in the estimation criteria and in the approximation procedure applied. With the help of [9], various structures of filter dynamics are listed in Table 1, with the corresponding forms of error covariance equations in Table 2, where the one dimensional case is considered. To make comparative discussions more clear, two examples are shown in this section.

[Example-1] We shall first consider the one-dimensional case. The dynamical system considered here is schematically shown by the block diagram in Fig. 3.

From Fig. 3, the stochastic differential equation of the dynamical system is given by

$$dx = f(-x)dt + qdw \tag{6.1}$$

Table 1: Approximations to scalar nonlinear filtering equations

System dynamics: $dx = f(x)dt + q_0 dw$

Observation channel: $dy = h(x)dt + r_0 dv$

Filter dynamics: $d\hat{x} = m(\hat{x})dt + r_0^{-2} n(\hat{x})$

References	$m(\hat{x})$	$n(\hat{x})$
[9]	$f(\hat{x}) + \frac{1}{2}f''(\hat{x})p$	$ph'(\hat{x})[y - h(\hat{x}) - \frac{1}{2}ph''(\hat{x})]dt$
[12]	$f(\hat{x}) + \frac{1}{2}f''(\hat{x})p$	$ph'(\hat{x})[y - h(\hat{x}) - \frac{1}{2}ph''(\hat{x})]dt$
[13]	$f(\hat{x})$	$ph'(\hat{x})[y - h(\hat{x})]dt$
[14]	$f(\hat{x})$	$ph'(\hat{x})[y - h(\hat{x})]dt$
[16]	$\hat{f}(x)$	$ph_2(t)[dy - \hat{h}(x)dt]$

' denotes the derivative with respect to x

Table 2: A list of error covariance equations

Error covariance equation: $\frac{dp}{dt} = e_1(\hat{x}) + q_0^2 + r_0^{-2} e_2(\hat{x})$

References	$e_1(\hat{x})$	$e_2(\hat{x})$
[9]	$2pf'(\hat{x})$	$-p^2h'(\hat{x})^2 + p^2h''(\hat{x})[y-h(\hat{x})-\frac{1}{2}ph''(\hat{x})]$
[12]	$2pf'(\hat{x})$	$-p^2h'(\hat{x})^2$
[13]	$2pf'(\hat{x})$	$-p^2h'(\hat{x})^2 + \frac{1}{2}p^2h''(\hat{x})[y-h(\hat{x})]$
[14]	$2pf'(\hat{x})$	$-p^2h'(\hat{x})^2$
[16]	$2pb(t)$	$-p^2h_2(t)^2$

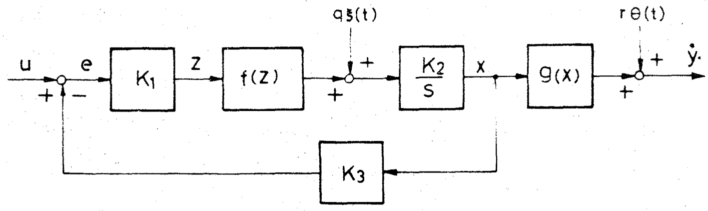

Fig. 3. Bolck diagram representation of the dynamical system in Example 1

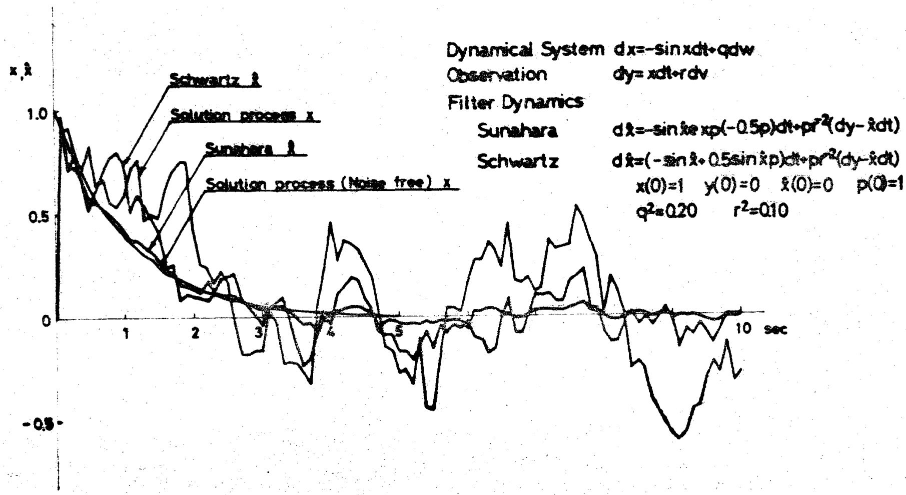

Fig. 4. State estimator dynamics

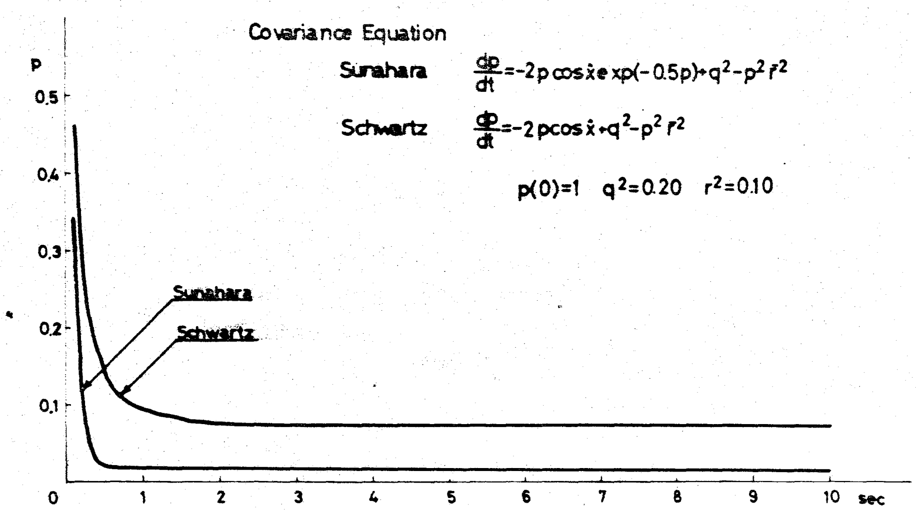

Fig. 5. Error covariance

with

(6.2) $$f(x) = \sin x$$

where $u = 0$, and $K_1 = K_2 = K_3 = 1$. The observation process is

(6.3) $$dy = xdt + rdv$$

Application of (5.11) and (5.12) to the present case brings

(6.4) $$a(t) = -\sin\hat{x}\, \exp(-0.5p)$$

(6.5) $$b(t) = -\cos\hat{x}\, \exp(-0.5p)$$

The approximated filter dynamics and related error covariance are respectively determined by

(6.6) $$d\hat{x} = -\sin\hat{x}\, \exp(-0.5p)dt + pr^{-2}(dy - \hat{x}dt)$$

and

(6.7) $$\frac{dp}{dt} = -2p\cos x\, \exp(-0.5p) + q^2 - p^2r^{-2}$$

As another possible method of approximation, we shall consider the method of Taylor expansion developed by [9]. We expand the nonlinear function into the following form,

(6.8) $$f_i(x) = f_i(\hat{x}) + f_{ij}^{(1)}(\hat{x})(\hat{x}_j - \hat{x}_j) + \frac{1}{2}f_{ijk}^{(2)}(\hat{x})(x_j-\hat{x}_j)(x_k-\hat{x}_k)$$

where f_i expresses the i-th component of the nonlinear vector function, f, $f_{ij}^{(1)} = \partial f_i/\partial x_j$ and $f_{ijk}^{(2)} = \partial^2 f_i/\partial x_j \partial x_k$, and where $i,j,k=1, 2,\ldots, n$. It follows from (6.8) that

(6.9) $$\hat{f}_i(x) = f_i(\hat{x}) + \frac{1}{2}f_{ijk}^{(2)}(\hat{x})p_{jk}$$

where p_{jk} expresses the (j, k) element of covariance matrix P. Bearing (6.9) in mind, a somewhat tedious calculation leads to the results,

(6.10) $$d\hat{x} = (-\sin\hat{x} + \frac{1}{2}\sin\hat{x}\, p)dt + pr^{-2}(dy - \hat{x}dt)$$

(6.11) $$\frac{dp}{dt} = -2p\cos\hat{x} - p^2r^{-2} + q^2$$

Comparison of numerical aspects of filter dynamics determined by Eqs. (6.6) and (6.7) with those given by Eqs. (6.10) and (6.11) is shown in Figs. 4 and 5.

[Example 2] Secondly, we shall consider the two-dimensional case. The dynamical system which is given by Fig. 6 is determined by

(6.12a) $$dx_1 = x_2 dt$$

(6.12b) $$dx_2 = [-x_2 + f(-x_1)]dt + q dw$$

with $x = x_1$, $dx/dt = x_2$ and with

(6.12c) $$f(x) = x - \frac{x^3}{8}$$

where $u = 0$ and $K_1 = K_2 = K_3 = 1$. The observation process is

(6.13a) $$dy_1 = x_1 dt + r_1 dv_1$$

(6.13b) $$dy_2 = x_2 dt + r_2 dv_2$$

where $r_1 = r_2 = r$. In this case, it is a simple exercise to obtain from (5.11) and (5.12) that

(6.14) $$a(t) = \begin{pmatrix} \hat{x}_2 \\ -\hat{x}_2 - \hat{x}_1 + \frac{1}{8}\hat{x}_1^3 + \frac{3}{8}p_{11}\hat{x}_1 \end{pmatrix}$$

and

$$\bar{B}(t) = \begin{pmatrix} 0 & 1 \\ -1 + \frac{3}{8}\hat{x}_1^2 + \frac{3}{8}p_{11} & -1 \end{pmatrix}$$

The approximated filter dynamics and the related error covariance are respectively given by the following form,

(6.16a) $$d\hat{x}_1 = \hat{x}_2 dt + p_{11}r^{-2}[(x_1-\hat{x}_1)dt + rdv_1] + p_{12}r^{-2}[(x_2-\hat{x}_2)dt + rdv_2]$$

(6.16b) $$d\hat{x}_2 = (-\hat{x}_2 - \hat{x}_1 + \frac{1}{8}\hat{x}_1^3 + \frac{3}{8}\hat{x}_1 p_{11})dt$$
$$+ p_{12}r^{-2}[(x_1 - \hat{x}_1)dt + rdv_1]$$
$$+ p_{22}r^{-2}[(x_2 - \hat{x}_2)dt + rdv_2]$$

(6.17a) $$\frac{dp_{11}}{dt} = 2p_{12} - p_{11}^2 r^{-2} - p_{12}^2 r^{-2}$$

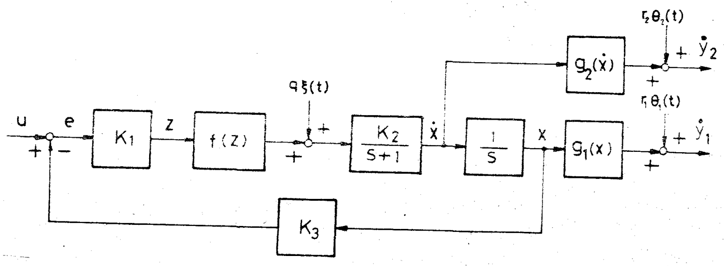

Fig. 6. Block diagram representation of the dynamical system considered in Example 2

Dynamical System $dx_1 = x_2 dt \quad dx_2 = (-x_2 - x_1 + \frac{1}{8}x_1^3)dt + q dw$

Observation $dy_1 = x_1 dt + r dv_1 \quad dy_2 = x_2 dt + r dv_2$

Filter Dynamics

Sunahara
Schwartz

$d\hat{x}_1 = \hat{x}_2 dt + p_{11} r^{-2}(dy_1 - \hat{x}_1 dt) + p_{12} r^{-2}(dy_2 - \hat{x}_2 dt)$

$d\hat{x}_2 = (-\hat{x}_2 - \hat{x}_1 + \frac{1}{8}\hat{x}_1^3)dt + \frac{3}{8}\hat{x}_1 p_{11} dt + p_{12} r^{-2}(dy_1 - \hat{x}_1 dt) + p_{22} r^{-2}(dy_2 - \hat{x}_2 dt)$

$x_1(0)=1.5 \quad x_2(0)=1.0 \quad y_1(0)=0 \quad y_2(0)=0 \quad \hat{x}_1(0)=0 \quad \hat{x}_2(0)=0$

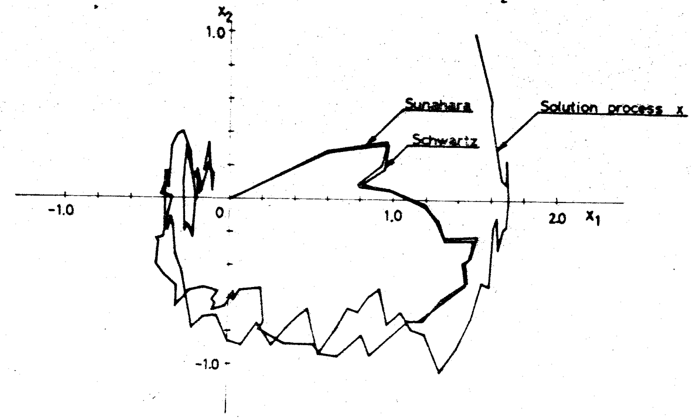

Fig. 7. State estimator dynamics

(6.17b) $$\frac{dp_{12}}{dt} = p_{11}(-1 + \frac{3}{8}\hat{x}_1^{\,2} + \frac{3}{8}p_{11}) + p_{22} - p_{12}$$
$$- p_{11}p_{12}r^{-2} - p_{12}p_{22}r^{-2}$$

(6.17c) $$\frac{dp_{22}}{dt} = 2[(-1 + \frac{3}{8}\hat{x}_1^{\,2} + \frac{3}{8}p_{11})p_{12} - p_{22}] + q^2$$
$$-p_{12}^{\,2}r^{-2} - p_{22}^{\,2}r^{-2}$$

The results obtained by using Taylor series expansion are listed below:

(6.18a) $$d\hat{x}_1 = \hat{x}_2 dt + p_{11}f^{-2}[(x_1 - \hat{x}_1)dt + rdv_1]$$
$$+ p_{12}r^{-2}[(x_2 - \hat{x}_2)dt + rdv_2]$$

(6.18b) $$d\hat{x}_2 = (-\hat{x}_2 - \hat{x}_1 + \frac{1}{8}\hat{x}_1^{\,3} + \frac{3}{8}\hat{x}_1 p_{11})dt$$
$$+ p_{12}r^{-2}[(x_1 - \hat{x}_1)dt + rdv_1]$$
$$+ p_{22}r^{-2}[(x_2 - \hat{x}_2)dt + rdv_2]$$

(6.19a) $$\frac{dp_{11}}{dt} = 2p_{12} - p_{11}^{\,2}r^{-2} - p_{12}^{\,2}r^{-2}$$

(6.19b) $$\frac{dp_{12}}{dt} = p_{11}(-1 + \frac{3}{8}\hat{x}_1^{\,2}) - p_{12} + p_{22} - p_{11}p_{12}r^{-2} - p_{12}p_{22}r^{-2}$$

(6.19c) $$\frac{dp_{22}}{dt} = 2[p_{12}(-1 + \frac{3}{8}\hat{x}_1^{\,2}) - p_{22}] + q^2 - p_{12}^{\,2}r^{-2} - p_{22}^{\,2}r^{-2}$$

Comparison of filter dynamics determined by Eqs. (6.16) and (6.17) with those given by Eqs. (6.18) and (6.19) is shown in Fig. 7.

7. Conclusions

In this short note, a special emphasis has been placed on the approximate method of state estimation for nonlinear dynamical systems with state-independent noise, including comparative discussions on quantitative aspects of approximated fashion of filter dynamics. Approximations to the optimal control for nonlinear dynamical systems under noisy observations are developing by the application of stochastic linearization technique in Markov processes mentioned in this short note. Furthermore, recently, various kinds of extensive studies on stochastic control processes are being performed, making the discussion more precise in Markovian frameworks. Particularly, the following subjects are problems at hand:

(1) Stochastic linearization method for nonlinear dynamical systems with state-dependent noise

(2) Accuracy of the stochastic linearization method

(3) Control performance deterioration due to the estimation error and its compensation

(4) Stochastic controllability

(5) Use of another type of stochastic differential equation [5]

Acknowledgment

The author should like to express his sincere thanks to Professor Yoshikazu Sawaragi for his enlightening discussions. Special thanks should be extended to Mr. Koji Ohminato, Mr. Koji Yamashita and Mr. Akira Osumi for performing numerical calculations by digital simulation studies. The statement should be added that all works of numerical version were carried out at Computing Center of Kyoto University.

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Special note

The descriptions in Sections 5 and 6 are entirely new. A full account including more detailed aspects of numerical results will be appeared later as a paper.