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Characteristic classes for spherical fiber spaces.

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1. Statement of results. Let $SF = SG = \lim_{n} SG(n)$, $SG(n) = \{ f : S^n \longrightarrow S^n, \text{ degree } 1 \}$, B_{SF} be the classifying space of SF. Our purpose is to determine $H_*(B_{SF}, Z_p)$ as a Hoph-algebra over Z_p, where p is an odd prime number. Coefficient is always Z_p, and we omit it later. Let $Q_0(S^0) = \lim_{\overrightarrow{n}} \Omega_0^n S^n$. Then $Q_0(S^0)$ has the same homotopy type of SF. Let $i : Q_0(S^0) \longrightarrow SF$ be the homotopy equivalence. Dyer-Lashof determined $H_*(Q_0(S^0))$ as a algebra over Z_p. $H_*(Q_0(S^0))$ is a free commutative algebra generated by x_J, $J \in H$, where $H = \{ J = (\varepsilon_1, J_1, \ldots, \varepsilon_r, J_r) \}$ satisfies the following properties : 1) $r \geq 1$ 2) $j_i \equiv 0, (p-1)$ 3) $j_r \equiv 0 \ (2(p-1))$ 4) $(p-1) \leq j_1 \leq j_2 \leq \cdots \leq j_r$ 5) $\varepsilon_i = 0$ or 1 6) if $\varepsilon_{i+1} = 0$ then $j_i/p-1$ and $j_{i+1}/p-1$ are even parity, if $\varepsilon_{i+1} = 1$ then $j_i/p-1$ and $j_{i+1}/p-1$ are odd parity. There is a continuous map $h_0 : L_p \longrightarrow Q_0(S^0)$, and $x_j \equiv h_{0*}(e_{2j(p-1)})$, where $e_i \in H_i(L_p)$ is a generator, and $x_I \equiv x_{(\varepsilon_1, j_1, \ldots, \varepsilon_r, j_r)} \equiv \beta_p^{\varepsilon_1} Q_{j_1} \ldots \beta_p^{\varepsilon_{r-1}} Q_{j_{r-1}} \beta_p^{\varepsilon_r} x_{j_r}$, where Q_j is the extended power operation defined by Dyer-Lashof. We identify $H_*(Q_0(S^0))$ and $H_*(SF)$ by i_* as a Z_p-module and we denote $\tilde{x} = i_*(x)$, if $x \in H_*(Q_0(S^0))$.

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Theorem I. $H_*(SF)$ is a free commutative algebra generated by $\widetilde{x}_J$; $J \in H$. Even though i_* is not a ring homomorphism.

Let H_1 be the subset of H consisting of $J = (\varepsilon_1, j_1, \ldots, \varepsilon_r, j_r)$, such that $j_1 \neq p-1$, and $r \geq 2$. Let $H_2 = \{ (\varepsilon, p-1, 1, j) \} \subseteq H$. And let $H_i^+ = \{ J \in H_i , \deg(x_J) = \text{even}, \}$ $H_i^- = \{ J \in H_i , \deg(x_J) = \text{odd} \}$ $i = 1, 2$. Let j ; $B_{SO} \longrightarrow B_{SF}$ be the inclusion map. Then by Peterson-Toda, $H_*(B_{SO})/\ker j^* \cong Z_p [z_1, z_2, \ldots]$, where $\deg(z_j) = 2j(p-1)$, and $\Delta(z_j) = \sum_{j_1+j_2=j} z_{j_1} \otimes z_{j_2}$, $z_0 = 1$. Let $\widetilde{z}_j = j_*(z_j) \in H(B_{SF})$.

Theorem II. $H_*(B_{SF}) = Z_p [\widetilde{z}_1, \widetilde{z}_2, \ldots] \otimes \Lambda(\sigma\widetilde{x}_1, \sigma\widetilde{x}_2, \ldots) \otimes C_*$.

C_* is a free commutative algebra generated by $\widetilde{x}_J$, $J \in H_1 \cup H_2$.

σ ; $H_*(SF) \longrightarrow H_*(B_{SF})$ is suspension. $\sigma\widetilde{x}_j$, $\sigma\widetilde{x}_J$ are primitive elements, and $\Delta(\widetilde{z}_j) = \sum_{j_1+j_2=j} \widetilde{z}_{j_1} \otimes \widetilde{z}_{j_2}$.

$H^*(B_{SF}) = Z_p [q_1, q_2, \ldots] \otimes \Lambda(\Delta q_1, \Delta q_2, \ldots) \otimes C$. $C = \bigotimes_{I \in H_1^+ \cup H_2^+}$

$\Lambda((\sigma\widetilde{x}_I)^*) \otimes \bigotimes_{J \in H_1^- \cup H_2^-} \Gamma_p [(\sigma\widetilde{x}_J)^*]$, where ( )* denote dual elements.

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2. H-structures on $Q_0(S^0)$. Let $SF(n) = \{ f : (S^n, *) \longrightarrow (S^n, *),$ degree $1 \}$. Then, $SG(n)$, and $SF(n)$ become H-spaces by composition of maps.

Let $SF(n) \wedge SF(n) \xrightarrow{\wedge} SF(2n)$, $SG(n) \times SG(n) \xrightarrow{*} SG(2n)$ be the map defined by reduced join and join respectively, then these three maps $\cdot$, $\wedge$, $*$, are homotopic in the stable range. Let $i_n : \Omega_0^n S^n \to \Omega_1^n S^n = SF(n)$ be the map defined by $i_n(\ell) = (i_n \vee \ell)$, and $i : Q_0S^0 \longrightarrow SF$ be the limit of i_n.

Proposition 2-1. The following deagram is homotopy commutative.

$$\begin{array}{ccccc}
Q_0S^0 \times Q_0S^0 & \xrightarrow[i \times i]{} SF \times SF \xrightarrow[\wedge]{} & SF & \xleftarrow[i]{} & Q_0S^0 \\
\downarrow{\scriptstyle \Delta \times \Delta} & & & & \uparrow{\scriptstyle \vee} \\
(Q_0S^0 \times Q_0S^0) \times (Q_0S^0 \times Q_0S^0) & \xrightarrow[id \times T \times id]{} & (Q_0S^0 \times Q_0S^0) \times (Q_0S^0 \times Q_0S^0) & \xrightarrow[\vee \times \wedge]{} & Q_0S^0 \times Q^0S^0
\end{array},$$

where $\vee : Q_0S^0 \times Q_0S^0$ be loop multiplication, and $\wedge : Q_0S^0 \times Q_0S^0 \longrightarrow Q_0S^0$ be the map defined by reduced join.

If K is a CW-complex, we put $Q(K) = \varinjlim_n \Omega^n S^n K$. $\theta : W \times_{\pi_p} Q(K)^p \longrightarrow Q(K)$ be the map defined by Dyer-Lashof. Let $Q(K) \times Q(L) \longrightarrow Q(K \wedge L)$ be the map defined by reduced join

Proposition 2-2. The following deagram is homotopy commutative.

$$\begin{array}{ccccc}
Q(K) \times (W \times_{\pi_p} Q(L)^p) & \xrightarrow[id \times \theta]{} & Q(K) \times Q(L) & \xrightarrow[\wedge]{} & Q(K \wedge L) \\
\downarrow & & & & \uparrow{\scriptstyle \theta} \\
W \times_{\pi_p} (Q(K) \times Q(L)^p) & \xrightarrow[id \times (\Delta_p \times id)]{} & W \times_{\pi_p} (Q(K) \times Q(L))^p & \xrightarrow[id \times_{\pi_p} (\wedge)^p]{} & W \times_{\pi_p} Q(K \wedge L)^p
\end{array}.$$

Let $h : L_p = W/\pi_p \longrightarrow Q(S^0) = \varinjlim_n \Omega^n S^n$ be the map defined by

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$h : L_p = W/\pi_p \longrightarrow W \times_{\pi_p} (w)^p \longrightarrow W \times_{\pi_p} Q(S^0)^p \xrightarrow{\theta} Q(S^0)$, $w \in Q_1(S^0)$,

$h_0 : L_p \xrightarrow{h} Q_p(S^0) \xrightarrow{(-p.id)} Q_0(S^0)$.

Proposition 2-3. The following diagram is homotopy commutative.

$$\begin{tikzcd} Q(K) \times L_p \arrow[r, "id \times h"] \arrow[d, "T"] & Q(K) \times Q(S^0) \arrow[r, "\wedge"] & Q(K \wedge S^0) = Q(K) \\ L_p \times Q(K) \arrow[r, "id \times_{\pi_p} \Delta_p"'] & W \times_{\pi_p} Q(K)^p \arrow[r, "\tilde{\theta}"] & Q(K) \arrow[u, "\cong"'] \end{tikzcd}$$

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3. Proof of Theorem I. We introduce a filtration into $H_*(Q_0(S^0))$. $H_*(Q_0(S^0)) = G_0 \geq G_1 \geq G_2 \ \ldots$ satisfy following properties. 1) $G_1 = \ker \varepsilon$, $\varepsilon : H_*(Q_0(S^0)) \longrightarrow Z_p$ is the augumentation. 2) $G_i \otimes G_j \longrightarrow G_{i+j}$

3) $x_J \in G_{p^{r-1}}$ where $J = (\varepsilon_1, j_1, \ldots, \varepsilon_r, j_r) \in H$, and $x_J \notin G_{p^{r-1}+1}$.

Proposition 3-1. There exsists unique filtration in $H_*(Q_0(S^0))$ satisfying the properties 1), 2), 3), and for $x \in H_*(Q_0(S^0))$, if $x \in G_j$ and $\Delta x = 1 \otimes x + x \otimes 1 + \sum x' \otimes x''$, then x', x'' belong G_j .

Proposition 3-2. Let $E_0H_*(Q_0(S^0))$ be the algebra associated to the above filtration. Then $H_*(Q_0(S^0))$ and $E_0H_*(Q_0(S^0))$ are isomorphuc as algebras.

Proposition 3-3. $\Lambda_*(x \otimes y) \in G_{pij}$, if $x \in G_i$ and $y \in G_j$.

Then Theorem I follows from Prop.2-1, Prop3-1, Prop3-2, and Prop3-3.

4. H_p - structure on B_{SF}. Let $\gamma_n \longrightarrow B_{SG(n)}$ be the universal oriented spherical fiber space with fiber S^{n-1} . Σ_p denotes the

permutation groop of p-element. $J^m\Sigma_p = \Sigma_p * \cdots * \Sigma_p$ denote m-th join of Σ_p.

Let $\gamma_n^{(p)} \longrightarrow B_{SG(n)}^{(p)}$ be exterior p-th Whiteney join of γ_n. Let $\pi_2^*(\gamma_n) \longrightarrow J^m\Sigma_p \times B_{SG(n)}^{(p)}$ denote the induced fibering of $\gamma_n^{(p)}$ by $\pi_2 \; J^m\pi_p \times B_{SG(n)}^{(p)} \longrightarrow B_{SG(n)}^{(p)}$.

Proposition 4-1. There exists a spherical fibring $P(\gamma_n) \longrightarrow J^\infty\Sigma_p \times_{\Sigma_p} B_{SG(n)}^{(p)}$

with fiber S^{Pn-1}, and bundle map

$$\begin{tikzcd} \pi_2^*(\gamma_n) \arrow[r, "q"] \arrow[d] & P(\gamma_n) \arrow[d] \\ J^\infty\Sigma_p \times B_{SG(n)}^{(p)} \arrow[r] & J^\infty\Sigma_p \times_{\Sigma_p} B_{SG(n)}^{(p)} \end{tikzcd}$$

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They satisfy following commutative diagram. $\forall \sigma \in \Sigma_p$

$$\begin{tikzcd} \pi_2^*(\gamma_n) \arrow[rr, "\sigma"] \arrow[dr, "q"] \arrow[dd] & & \pi_2^*(\gamma_n) \arrow[dl, "q"'] \arrow[dd] \\ & P(\gamma_n) \arrow[dd] & \\ J^\infty\Sigma_p \times B_{SG(n)}^{(p)} \arrow[rr] \arrow[dr] & & J^\infty\Sigma_p \times B_{SG(n)}^{(p)} \arrow[dl] \\ & J^\infty\Sigma_p \times_{\Sigma_p} B_{SG(n)}^{(p)} & \end{tikzcd}$$

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Let $E_{SG(n)} \longrightarrow B_{SG(n)}$ be the princepal fibering associated with γ_n,

i.e. $E_{SG(n)} = \left\{ f : \begin{array}{ccc} S^{n-1} & \longrightarrow & \gamma_n \\ \downarrow & & \downarrow \\ * & \longrightarrow & B_{SG(n)} \end{array} \; ; \text{ oriented fiber map} \right\}$.

$P_0(\gamma_n) \longrightarrow J^m\pi_p \times_{\pi_p} B_{SG(n)}^{(p)}$ denote restricted fibering of $P(\gamma_n)$, where π_p

denote cyclic group of order p.

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$\bar{\theta} : J\pi^m_p \underset{\pi_p}{\times} B_{SG(n)}^{(p)} \longrightarrow B_{SG(pn)}$ be the classifying map of $P(\gamma_n)$

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As the map $\bar{\theta} : J\pi^m_p/\pi_p \longrightarrow J\pi^m_p \underset{\pi_p}{\times} (e_0)^p \longrightarrow J\pi^m_p \underset{\pi_p}{\times} B_{SG(n)}^{(p)} \xrightarrow{\bar{\theta}} B_{SG(pn)}$,

$e_0 \in B_{SG(n)}$, is induced by the n-times of the reguler representation :

$\pi_p \longrightarrow SO(pn) \longrightarrow SG(pn)$, by the result of Kambe, we may suppose the above map is homotopic to constant map for suitable m, and n. And we may take m, sufficiently large for a suitablly sufficient large n. So we may assume $\bar{\theta}(J\pi^m_p/\pi_p) = e_0 \in B_{SG(pn)}$.

We define a map $\bar{\rho} : J\pi^m_p \longrightarrow SG(pn)$ in the following way. We identify $SG(n) = (E_{SG(n)})_{e_0} = \pi^{-1}(e_0)$, and $SG(pn) = (E_{SG(pn)}) = \pi^{-1}(e_0)$, respectively. We fix $i_n \in (E_{SG(n)})_{e_0}$ and for $w \in J\pi^m_p$, $\bar{\rho}(w)$ represents a following map.

$$\bar{\rho}(w) : \begin{tikzcd} S^{pn-1} \arrow[r, "{(w,\ id_n * \dots * id_n)}"] \arrow[d] & J\pi^m_p \times \gamma_n^{(p)} = \pi_2^*(\gamma_n) \arrow[r, "q"] \arrow[d] & P_0(\gamma_n) \arrow[r] \arrow[d] & \gamma_{pn} \arrow[d] \\ * \arrow[r] & J\pi^m_p \times B_{SG(n)}^{(p)} \arrow[r] & J\pi^m_p \underset{p\pi_p}{\times} B_{SG(n)}^{(p)} \arrow[r] & B_{SG(pn)} \end{tikzcd}$$

We define $\bar{\theta}' : J\pi^m_p \times E_{SG(n)}^{(p)} \longrightarrow E_{SG(pn)}$ be the following commutative diagram, for $(w, f_1, \dots, f_p) \in J\pi^m_p \times E_{SG(n)}^{(p)}$, $\bar{\theta}'(w, f_1, \dots, f_p)$:

$$
\begin{tikzcd}
S^{pn-1} \arrow[r, "\bar{\rho}^{-1}(w)"'] \arrow[d] & S^{pn-1} \arrow[r, "(w, f_1 * \cdots * f_p)"'] \arrow[d] & J^m_{\pi_p} \times \gamma_n^{(p)} \arrow[r, "q"'] \arrow[d] & P_0(\gamma_n) \arrow[r] \arrow[d] & \gamma_{pn} \arrow[d] \\
* \arrow[r] & * \arrow[r] & J^m_{\pi_p} \times B_{SG(n)}^{(p)} \arrow[r] & J^m_{\pi_p} \times_{\pi_p} B_{SG(n)}^{(p)} \arrow[r] & B_{SG(pn)}
\end{tikzcd}
$$

Proposition 4-2. $\bar{\theta}'$ is π_p-equivariant, we obtain following commutative diagram.

$$
\begin{tikzcd}
\bar{\theta} : J^m_{\pi_p} \times_{\pi_p} E_{SG(n)}^{(p)} \arrow[r] \arrow[d] & E_{SG(pn)} \arrow[d] \\
J^m_{\pi_p} \times_{\pi_p} B_{SG(n)}^{(p)} \arrow[r] & B_{SG(pn)}
\end{tikzcd}
$$

And $\bar{\theta}\,(J^m_{\pi_p} \times_{\pi_p} SG(n)^{(p)}) \subseteq SG(pn) \subseteq E_{SG(pn)}$, and $\bar{\theta}\,(w, f_1, \ldots, f_p)$ $= \bar{\rho}(w)(f_1 * \ldots * f_p)\ \bar{\rho}\,(w)^{-1}$, for any $(w, f_1, \ldots, f_p) \in J^m_{\pi_p} \times_{\pi_p} SG(n)^{(p)}$.

5. Decomposition of $\bar{\theta}$. Let $\Lambda = \{ J = (\varepsilon_1, \ldots, \varepsilon_p),\ \varepsilon_i = 0 \text{ or } 1 \}$, $|J|$ = number of $\{ \varepsilon_i = 1,\ J = (\varepsilon_1, \ldots, \varepsilon_p) \}$. π_p operates on Λ by permutation. We introduce in Λ an total ordering by the lexicographic order, for example, $(0, 1, \quad) \lneqq (1, \ldots)$. Let $\bar{\Lambda} = \Lambda / \pi_p$. We define the map $\bar{\Lambda} \xrightarrow{\pi} \Lambda$, by $\pi(\{J\})$ = the first element in $\{J\}$. Λ_0 denote the image of π. For each element $J_0 \in \Lambda_0$, we define η_{J_0} : 小文字 $(\Omega_0^{n-1} S^{n-1})^{p} \longrightarrow G(pn)$ as follows, where $G(pn) = \{ f : S^{pn-1} \longrightarrow S^{pn-1} \}$, $\varphi_2 : S^{n-1} \longrightarrow S_0^{n-1} \vee S_0^{n-1}$.

For $(\ell_1,\dots,\ell_p) \in (\Omega_0^{n-1}S^{n-1})^p$, $\eta_{J_0}(\ell_1,\dots,\ell_p)$ represents following map.

$$\begin{tikzcd}
\eta_{J_0}(\ell_1,\dots,\ell_p) : S^{n-1} * \dots * S^{n-1} \arrow[r, "\varphi_2 * \cdots * \varphi_2"'] \arrow[d, dashed] & (S_0^{n-1} \vee S_1^{n-1}) * \dots * (S_0^{n-1} \vee S_1^{n-1}) \arrow[d] \\
S^{pn-1} & \bigvee_{J \in \Lambda} S_J^{pn-1} \arrow[l, "\circledast"]
\end{tikzcd}$$

$\circledast$ is the map as follows, $\circledast|_{S_J} : S_J \to S$ represents, a) $0 * \cdots * 0$, if $J \neq \sigma J_0$ for any $\sigma \in \Pi_p$, $0 : S^{n-1} * \to S^{n-1}$. b) $\ell_1^{\varepsilon_1} * \dots * \ell_p^{\varepsilon_p}$, if $J = \sigma J_0 = (\varepsilon_1,\dots,\varepsilon_p)$ for some $\sigma \in \Pi$, where $\ell_i^0 = \mathrm{id}$, $\ell_i^1 = \ell_i$.

And $S_J^{pn-1} = S_{\varepsilon_1}^{n-1} * \dots * S_{\varepsilon_p}^{n-1}$.

We define $\bar{\theta}'_{J_0} : J\Pi_p^m \times (\Omega_0^{n-1} S^{n-1})^p \longrightarrow G(pn)$, for each $J_0 \in \Lambda_0$, as $\bar{\theta}'_{J_0}(w,\ell_1,\dots,\ell_p) = \bar{\rho}(\omega)\, \eta_{J_0}(\ell_1,\dots,\ell_p)\, \bar{\rho}(\omega)^{-1}$.

Proposition 5-1. $\bar{\theta}'_{J_0} : J\Pi_p^m \times (\Omega_0^{n-1} S^{n-1})^p \longrightarrow G(pn)$, is Π_p-equivariant, therefore it define a following map $\bar{\theta}_{J_0} : J\Pi_p^m \times_{\Pi_p} (\Omega_0^{n-1}S^{n-1})^p \longrightarrow G(pn)$.

Let $i : G(pn) \longrightarrow \Omega_0^{pn+1}S^{pn+1}$ be the inclusion

Proposition 5-2. $i\bar{\theta}$ and $\bigvee_{J_0 \in \Lambda_0} i\bar{\theta}_{J_0}$ are homotopic on (pn-5)-skelton as a map $J\Pi_p^m \times_{\Pi_p} (\Omega_0^{n-1}S^{n-1})^p \longrightarrow \Omega_0^{pn+1}S^{pn+1}$, where $\vee$ denote loop multiplication on $\Omega_0^{pn+1}S^{pn+1}$.

For $J_0 \in \Lambda_0$, $|J_0| \neq 0$, p, we define $h_{J_0} : J\Pi_p^m/\Pi_p \longrightarrow G(pn)$ as follows, fof $w \in J\Pi_p^m$,

$$\begin{tikzcd} h_{J_0}(w) : S^{pn-1} \arrow[r, "\bar{\varphi}(w)^{-1}"] \arrow[d] & S^{pn-1} \arrow[r, "\varphi_2 * \cdots * \varphi_2"] & \bigvee_{J \in \Lambda} S_J^{pn-1} \arrow[d, "\otimes"] \\ S^{pn-1} & & S^{pn-1} \arrow[ll, "\bar{\varphi}(w)"'] \end{tikzcd}$$

where $\otimes|_{S_J} : S_J^{pn-1} \longrightarrow S^{pn-1}$ represents a) $0* \quad *0$, if $J \neq \sigma J_0$, for any $\sigma \in \pi_p$. b) id_{pn-1}, if $J = \sigma J_0$, for some $\sigma \in \pi_p$. h_{J_0} is well defined.

Proposition 5-3. For $\Lambda_0 \in J_0 = (\varepsilon_1, \ldots, \varepsilon_p)$, $0 \neq |J| \neq p$, the following diagram is homotopy comutative.

$$\begin{tikzcd} J^m\pi_p \times_{\pi_p} \Omega_0^{n-1}S^{n-1} \arrow[r, "id \times_{\pi_p} \Delta_p"] \arrow[d, "h_{J_0} \times ( )^{\varepsilon_1} * \cdots * ( )^{\varepsilon_p}"'] & J^m\pi_p \times_{\pi_p} (\Omega_0^{n-1}S^{n-1})^{(p)} \arrow[r] & G_0(pn) \arrow[d, "*id_{pn-1}"] \\ G(pn) \times G_0(pn) \arrow[rr, "*"'] & & G_0(2pn) \end{tikzcd}$$

where $( )^{\varepsilon_1} * \cdots * ( )^{\varepsilon_p} : \Omega_0^{n-1}S^{n-1} \longrightarrow G(pn)$ is the map defined by $\ell \longmapsto (\ell)^{\varepsilon_1} * \ldots * (\ell)^{\varepsilon_p}$.

We define $\tilde{\theta}_p : J^m\pi_p \times_{\pi_p} (\Omega_0^{n-1}S^{n-1})^p \longrightarrow G(pn)$ by $\tilde{\theta}_p(w, \ell_1, \ldots, \ell_p) = \bar{\varphi}(w) \ (\ell_1 *, \ldots, * \ell_p) \ \bar{\varphi}(w)^{-1}$.

Proposition 5-4. $\tilde{\theta}_p \simeq \bar{\theta}(\ell, \ldots, \ell)$; homotopic.

6. Proof of Theorem II. $\bar{\theta} : J^\infty\pi_p \times_{\pi_p} SF^p \longrightarrow SF$, $\bar{\theta} : J^\infty\pi_p \times_{\pi_p} B_{SF}^p \longrightarrow B_{SF}$ are the maps corresponding to $\bar{\theta} : J^m\pi_p \times_{\pi_p} SG(n)^p \longrightarrow SG(pn)$,

$\overline{\theta} : J^m_{\pi_p} \underset{\pi_p}{\times} B^p_{SG(n)} \longrightarrow B_{SG(pn)}$ for large m and n. We define

$\overline{Q}_j : H_*(SF) \longrightarrow H_*(SF)$, $\overline{Q}_j : H_*(B_{SF}) \longrightarrow H_*(B_{SF})$, $j = 1,2,\ldots,$

by the $\overline{Q}_j(x) = \overline{\theta}_*(e_j \underset{\pi_p}{\otimes} x^p)$, for $x \in H_*(SF)$, or $\in H_*(B_{SF})$.

Proposition 6-1. In the homology spectral sequence associated with following fibering $SF \longrightarrow E_{SF} \longrightarrow B_{SF}$. $E^2_{**} = H_*(B_{SF}) \otimes H_*(SF)$.
If $x \in E^2_{2n,0}$ is transgresive. $y \in E^2_{0,2n-1}$, $\tau(x) = \{y\}$, then we obtain the following relations. $\{\tau \overline{Q}_0(x)\} = \{\tau(x^p)\} = \{\overline{Q}_{p-1}(y)\}$ in $E^{2np}_{0,2np-1}$, and $\{\tau(x^{p-1} \otimes y)\} = \{\overline{Q}_{p-2}(y)\}$ in $E^{2n(p-1)}_{0,2np-2}$.

Proposition 6-2. Let $\overline{h}_1 = h_{(1,0\ldots0)} : J^m_{\pi_p/\pi_p} \longrightarrow G(pn)$,

and $\overline{H}_1 : J^m_{\pi_p/\pi_p} \times S^{pn-1} \longrightarrow S^{pn-1}$ be the representative of $\overline{h}_1$. Then

in $H_*(C_{\overline{H}_1})$, $P^j(s)$ and $\Delta P^j(s)$ are non zero, where $s \in H^{pn-1}(C_{\overline{H}_1}) \cong Z_p$ is a generator, and $1 \leq j \leq m-1 / 2(p-1)$.

Proposition 6-3. If $\widetilde{x}_I \in H_*(SF)$ belongs to G_{p^j} , $j \geq 1$, where $I \in H$, then $\overline{Q}_{p-2}(x_I)$, $\overline{Q}_{p-1}(x_I)$ belong to $G_{p^{j+1}}$, and as elements of $G_{p^{j+1}}/G_{p^{j+1}+1}$+decomp . they coincide with $\widetilde{\beta_p Q_{p-1}(x_I)}$, $\widetilde{Q_p(x_I)}$ respectively.
This proposition is prooved by using the following lemmas.

Let $\overline{h}_1 : J^\infty_{\pi_p/\pi_p} \longrightarrow G(\infty) = \{f : S^\infty \longrightarrow S^\infty\}$, represent $\overline{h}_1 : J^m_{\pi_p/\pi_p} \longrightarrow G(pn)$ for large m, n. And $\overline{h}_{1,0} : J^\infty_{\pi_p/\pi_p} \xrightarrow{\overline{h}_1} G_p(\infty) \xrightarrow{((-p\cdot id))} Q_0 S^0$.

Lemma 6-1. $\overline{h}_{1,0_*}(e_{2i(p-1)}) = cx_i + x'$, $c \neq 0$, $x' \in G_2$,

$\overline{h}_{1,0_*}(e_{2i(p-1)-1}) = c'\beta_p x_i + x''$, $c' \neq 0$, $x'' \in G_2$. This proposition is proaved by using prop. 6-2.

Lemma 6-2. In $H_*(SF)$, we obtain the following relations.

$\langle \tilde{x}_j, \sigma(\Delta q_j) \rangle \neq 0$, $\langle \beta_p x_j, \sigma(q_j) \rangle \neq 0$, for $x \in G_2$, $\langle x, \sigma(\Delta q_j) \rangle = 0$,

$\langle x, \sigma(q_j) \rangle = 0$.

> lemma 6-3 ⊛ 下行.

Lemma 6-4. For any $x_I \in G_{pj}$, $I \in H$, $\overline{\theta}_{1*}(e_i \underset{\pi}{\otimes} x_I^p) \in G_{pj+1}$. And as an element of G_{pj+1}/G_{pj+1+1} + decop., it coincide with $Q_i(x_I)$.

Lemma 6-5. For any $x_I \in G_{pj}$, $I \in H$, $j \geq 1$. $\overline{\theta}_{(1,\ldots,1)*}(e_i \underset{\pi}{\otimes} x_I^p)$ belongs G_{pj+1+1}, if $i \leq p-1$.

We consider $j_* : H_*(SO) \longrightarrow H_*(SF)$, by Peterson-Toda, $H_*(SO)/\ker j_* \cong \Lambda(y_1, y_2, \ldots)$ $\deg(y_i) = 2i(p-1)-1$. Let $\tilde{y}_i \in H_*(SF)$, be $j_*(y_i)$.

Proposition 6-4. $H_*(SF)$ is a free commutative algebra generated by $\tilde{x}_j$, $\tilde{y}_j$, $j = 1,2,\ldots$, $\tilde{x}_I$, $I \in H_1^+ \cup H_2^+$, $\overline{Q}_{p-1} \ldots \overline{Q}_{p-1}(\tilde{x}_I)$, $I \in H_1^- \cup H_2^-$, $\overline{Q}_{p-1}$ operate on x_I k-times, $k \geq 0$. $\overline{Q}_{p-2}\overline{Q}_{p-1}, \ldots \overline{Q}_{p-1}(\tilde{x}_I)$, $I \in H_1^- \cup H_2^-$, $\overline{Q}_{p-2}$ operates on $\overline{Q}_{p-1} \ldots \overline{Q}_{p-1}(\tilde{x}_I)$ exactly one times, and $\overline{Q}_{p-1}$ operates on $\tilde{x}_I$, k-times, $k \geq 0$.

This proposition is proaved by using prop, 6-3, and structure of $H_*(SF)$ as an algebra. Then Theorem II follows from prop. 6-1, prop. 6-4 and the comparision theorem for spectral sequence.

lemma 6-3, If $J_0 \in \Lambda_0$, and $|J_0| \neq 0, 1, p$, then for any $x_I \in H_*(Q_0(S^0))$, $x_I \in G_{pj}$, $I \in H$, $\theta_{J_0 *}(e_i \underset{\pi_p}{\otimes} x_I^p) \in G_{pj+t+1}$.

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