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CAUCHY PROBLEMS FOR NON-LINEAR TRANSPORT MODELS

SUEO UENO and SONOYO MUKAI
Institute of Astrophysics
Faculty of Science
Kyoto University

1. Introduction : In recent years it has been shown that in the fields of radiative transfer (Ambartsumian (1), Bellman, Kagiwada, Kalaba, and Prestrud (2), Busbridge (3), Chandrasekher (4), Rybicki and Hummer (5), and Sobolev (6)), and of rarefied gas dynamics (Wing (7)), the transformation of the two-point boundary-value problems to the initial value problems plays an important role from the analytical and numerical aspects, because the latter problems contain computational advantages over the former ones. Whereas some recent trend in the theory of invariant imbedding was in expository manner presented in the linear transport processes by Ueno (8), however, in this note it will be shown how powerfully to convert the non-linear boundary-value problems into Cauchy problems with the aid of the invariant imbedding and quasilinearization (Bellman and Kalaba (9)). For the sake of simplicity the rod model with coherent scattering is used throughout the analytical treatment.

Recently, the mathematical theory of the invariant imbedding in the non-linear transport processes has been developed by the RAND-USC school (cf. Bellman and Kalaba (9), Kagiwada and Kalaba (10), and Wing (7)). On the other hand, Ambartsumian (11) has in recent years extended the invariance principle to the solution of non-linear radiative transfer problems. With the aid of this method some non-linear transport problems were dealt by Engibaryan (12) and the polychromatic diffuse reflection of light from an infinitely deep one-dimensional media with three-level atoms was discussed by Nikogosyan (13), whose theory is presented by the physical method based on the invariance principle. Furthermore, the Boltzman treatment for this problem was provided by Ueno (14), allowing for the photon emergence probability. Similarly, the dissipation function in the non-linear scattering processes of neutrons and plasma within the finite rod was found by Ueno and Mukai (15).

The characteristic feature of the non-linear radiative transfer is such that the radiation intensity depends not only upon the local optical properties of the medium, but also upon the impinging radiation field. In other words, the non-linearity comes from the fact that the parameters describing the optical properties of the medium with scattering are expressed in terms of the radiation field. The equation for the photon emergence probability plays an important role similar to the equation of transfer. The study of radiation-gas-dynamics is given by the simultaneous solution of the transfer equation and of the Boltzmann equation, allowing for the stochastic state of the media.

2. The Boltzmann formulation (see Ref. (7), (9), (10)) : Consider an one-dimensional model of optical length, x, illuminated by flux F of radiation incident on the right end z=x (see Fig. 1). At the left end z=0 the reflection effect is assumed.

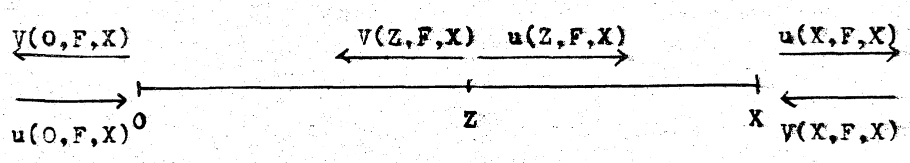

Fig. 1 The Physical rod model

Let the radiation fluxes at z, directed towards z=x and z=0, be denoted by u(z, F, x) and v(z, F, x), respectively.

The equation of transfer appropriate to this case is written in the form

$$(1) \qquad \frac{\partial u}{\partial Z} = f(u, V, Z) \quad ,$$

$$(2) \qquad -\frac{\partial V}{\partial Z} = g(u, V, Z) \qquad 0 \leqslant Z \leqslant X,$$

together with the two-point boundary conditions

(3) $$u(0,F,X) = \varphi(V(0,F,X)),$$

(4) $$V(X,F,X) = F.$$

On differentiating with respect to F, we get

(5) $$\frac{\partial^2 u}{\partial F \partial Z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial F} + \frac{\partial f}{\partial V}\frac{\partial V}{\partial F}, \quad \frac{\partial u}{\partial F}\bigg|_{Z=0} = \frac{\partial \varphi}{\partial V}\frac{\partial V}{\partial F}\bigg|_{Z=0},$$

(6) $$-\frac{\partial^2 V}{\partial F \partial Z} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial F} + \frac{\partial g}{\partial V}\frac{\partial V}{\partial F}, \quad \frac{\partial V}{\partial F}\bigg|_{Z=X} = 1.$$

Similarly, differentiating with respect to x, we have

(7) $$\frac{\partial^2 u}{\partial X \partial Z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial X} + \frac{\partial f}{\partial V}\frac{\partial V}{\partial X}, \quad \frac{\partial u}{\partial X}\bigg|_{Z=0} = \frac{\partial \varphi}{\partial V}\frac{\partial V}{\partial X}\bigg|_{Z=0},$$

(8) $$-\frac{\partial^2 V}{\partial X \partial Z} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial X} + \frac{\partial g}{\partial V}\frac{\partial V}{\partial X}, \quad \frac{\partial V}{\partial Z}\bigg|_{Z=X} + \frac{\partial V}{\partial X}\bigg|_{Z=X} = 0.$$

Furthermore, from eq. (2), we get

(9) $$\frac{\partial V}{\partial Z}\bigg|_{Z=X} = -g(u,V,X)\bigg|_{Z=X}.$$

Comparing eqs. (5) and (6) with eqs. (7) and (8), and assuming the uniqueness of the solution, we obtain

(10) $$\frac{\partial u}{\partial X}(Z,F,X) = g(R(F,X),F,X)\frac{\partial u}{\partial F}(Z,F,X),$$

(11) $$\frac{\partial V}{\partial X}(Z,F,X) = g(R(F,X),F,X)\frac{\partial V}{\partial F}(Z,F,X),$$

where

(12) $$R(F,X) = u(X,F,X), \quad V(Z,F,X)\bigg|_{Z=X} = F.$$

Now, the functional equation for R should be asked.

Differentiate eq. (12) with respect to x to obtain

$$(13) \qquad \frac{\partial R}{\partial X}(F,X) = \left.\frac{\partial u}{\partial Z}\right|_{Z=X} + \left.\frac{\partial u}{\partial X}\right|_{Z=X} .$$

On making use of eqs. (1) and (10), we have the required partial differential equation given by

$$(14) \qquad \frac{\partial R}{\partial X}(F,X) = f(R(F,X),F,X) + g(R(F,X),F,X)\,\frac{\partial R}{\partial F} ,$$

together with the initial condition

$$(15) \qquad R(F,0) = \varphi(F) .$$

Then, eqs. (10) and (11) are the desired partial differential equations governing u and v, together with the initial condition (12). In other words they give the solution for the Cauchy problem under consideration. It is noted that this requires only integration in the direction of increasing value of x.

3. Polychromatic diffuse reflection (see Ref. (13) and (14)) : Consider a semi-infinite rod consisting of three-level atoms, whose end z=x is illuminated by the constant flux of radiations at the frequencies i(i=1, 2 and 3)(see Fig. 2 and 3).

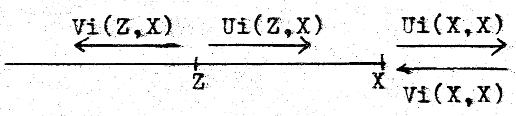

Fig. 2 The Physical rod model

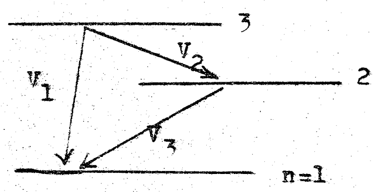

Fig. 3 Cyclical Transition

These three states are connected with each other by possible random transition, but connected with no other state. The redistribution of atoms with respect to levels under the influence of the incident fluxes results in the change in the transparency of the medium at some frequency.

For the sake of simplicity we assume the pure polychromatic scattering. Now we ask for the reflected fluxes $R_i(i=1,2,3)$ as functions of the fluxes F_i incident on the medium. For this purpose it is sufficient to find the R_i at one particular frequency, because of the law of the conservation of the photon number.

(16) $$\frac{F_i}{h\nu_i} + \frac{F_j}{h\nu_j} = \frac{R_i}{h\nu_i} + \frac{R_j}{h\nu_j} \qquad (i, j=1,2,3) .$$

In eq. (16) it is stated that the total number of photons in the incident fluxes at two frequencies is conserved in the reflected fluxes by the rod. In what follows, starting with the transfer equation, we shall find an equation for $R_1(F_1, F_2, F_3, x)$ at the first frequency. It is assumed that scattering of light in either direction is equally probable.

The equation of transfer takes the form

(17) $$\frac{\partial u_i(z,x)}{\partial z} = -\sigma_i u_i + \frac{1}{2}\lambda_i\sigma_i (u_i + v_i) ,$$

(18) $$-\frac{\partial v_i(z,x)}{\partial z} = -\sigma_i v_i + \frac{1}{2}\lambda_i\sigma_i (u_i + v_i) ,$$

where σ_i and $\lambda_i (i= 1,2,3)$ represent respectively the absorption coefficient and the albedo for single scattering, together with the boundary condition

(19) $$v_i(x,x) = F_i \qquad (i=1,2,3).$$

The albedo for single scattering denotes the survival probability of a photon after an elementary act of scattering.

Write

(20) $$R_i(z, v_j(z,x)) = u_i(z,x) \qquad (j=1,2,3) .$$

From eq. (19) we have

(21) $$\left.\frac{\partial v_i}{\partial z}\right|_{z=x} + \left.\frac{\partial v_i}{\partial x}\right|_{z=x} = 0 .$$

Assuming the linearity of the perturbation equation and uniqueness of solution, and using eqs. (17), (18) and (21), we get

(22) $$\frac{\partial R_1(F_i,x)}{\partial x_1} = \frac{\partial u_1}{\partial z}\Big|_{z=x} + \frac{\partial u_1}{\partial x}\Big|_{z=x} = \frac{\partial u_1}{\partial z}\Big|_{z=x} + \sum_i \frac{\partial u_1}{\partial v_i}\frac{\partial v_i}{\partial x}\Big|_{z=x}$$

$$= (-R_1+\frac{1}{2}\lambda_1(R_1+F_1)) + \sum_{i=1}^{3}(-F_i+\frac{1}{2}\lambda_i(R_i+F_i))\ \frac{\sigma_i}{\sigma_1}\frac{\partial R_1}{\partial F_i},$$

where $x_1 = \sigma_1 x$ and λi and σ_i/σ_1 depend upon the radiation field. On making use of eq. (16), we have

(23) $$(-F_i+\frac{1}{2}\lambda_i(R_i+F_i))\frac{\sigma_i}{\sigma_1} = (1-\lambda_1)\frac{R_i+F_i}{2}\frac{\nu_i}{\nu_1} + \frac{F_i-R_i}{2}\frac{\nu_i}{\nu_1}\frac{\sigma_i}{\sigma_1} \qquad (i=2,3).$$

Then, eq. (22) reduces to

(24) $$\frac{\partial R_1}{\partial x_1} = -(1-\lambda_1)\frac{F_1+R_1}{2} + \frac{1}{2}(F_1-R_1) + \sum_{i=1}^{3}\left((1-\lambda_1)\frac{R_i+F_i}{2} + \frac{F_i-R_i}{2}\frac{\sigma_i}{\sigma_1}\right)\frac{\nu_i}{\nu_1}\frac{\partial R_1}{\partial F_i}\delta_{1i},$$

where $\delta_{1i} = -1$ for $1=i$ and $\delta_{11}=1$. If $R_1(F_j, x) = \hat{R}_1(x)F_1$, then

(25) $$\frac{\partial \hat{R}_1}{\partial x_1} = \frac{\lambda}{2} + (\lambda-2)\hat{R}_1 + \frac{\lambda}{2}\hat{R}_1^2,$$

which is equivalent to the Riccati type equation. Eq.(24) reduces to that given by Nikogosyan (Ref.(13)).

Let the steady-state condition for the number density of atoms in the first and third states be given by

(26) $$n_1(B_{12}\rho_3+B_{13}\rho_1) = n_2\frac{g_1}{g_2}B_{12}(\sigma_3+\rho_3) + n_3\frac{g_1}{g_3}B_{13}(\sigma_1+\rho_1),$$

where n_i is the number density in the i-th state, ρ_i is the radiation density at $\nu=\nu_i$ given by $(F_i + R_i)/c$, B_{ij} is the Einstein coefficient of the transition probability of an atom from a lower state i to the higher state j, σ_i is equal to $8\pi h\nu_i^3/c^3$ and g_i is the statistical weight of the i-the state. Eq. (26) states that in non-local thermodynamical equilibrium the number of atoms leaving the 1-th state coinciedes with the number arriving at it.

Putting

(27) $$\lambda_1 = n_3B_{31}\sigma_1 / (n_1 - n_3 g_1/g_3)B_{13}\rho_1,$$

and

(28) $$\sigma_1 = \frac{h\nu_1}{\Delta\nu_1}(n_1-n_3\frac{g_1}{g_3})\frac{B_{13}}{c}, \quad \sigma_2 = \frac{h\nu_2}{\Delta\nu_2}(n_2-n_3\frac{g_2}{g_3})\frac{B_{23}}{c}, \quad \sigma_3 = \frac{h\nu_3}{\Delta\nu_3}(n_1-n_2\frac{g_1}{g_2})\frac{B_{12}}{c},$$

where $\Delta\nu_i$ is the effective width of the corresponding absorption line. In the atom λ_i and σ_i depend upon R_i.

On making use of eqs. (27) and (28), we have the terms for σ_2/σ_1, σ_3/σ_1 and $1-\lambda_1$, being functions of R_1. Substituting these derived expressions into eq. (24), after some transformations, we get the equations of the characteristics for the Cauchy problem. It is shown that, after some manipulations, the solution depends upon the atomic constants.

Finally, we obtain the functional equation for the photon emergence probability at the frequency ν_1.

$$(29)\qquad \frac{\partial p_1}{\partial x_1}(z_1,x_1)=-p_1(z_1,x_1)+p_1(z_1,x_1)\ \ p_1(x_1,x_1)+\sum_{i=1}^{3}\left(-F_i+\frac{1}{2}\lambda_i(F_i+R_i)\right)\frac{\sigma_i}{\sigma_1}\frac{\partial p_1}{\partial F_i},$$

where

$$(30)\qquad p_1(x_1,x_1)=\frac{\lambda_1}{2}\left(I+R_1(F_1,F_2,F_3,x)\right).$$

It represents such a probability that a photon absorbed at z_1 at a frequency ν_1 will appear from the medium at x_1 at the same frequency after one or more scattering processes, allowing for the cyclic transition between these three levels. If we take into account the collision process, the equation of the photon emergence probability makes the basis of the study of radiation-gas-dynamics in connection with the Boltzmann equation.

4. <u>Dissipation function of the non-linear transport process (see Ref(15))</u>:

Consider an inhomogeneous rod of the length x, where each second a single left moving particle is injected into the rod at z=x and no particle enters at the left end. We assume that the expected number density of particles lost per second in (z, z+dz) due to interaction between moving particles is

$$\phi(u(z,x),v(z,x))\Delta \quad + \ O(\Delta),$$

where the particle speed is considered to be unity. In the case of annihilation of particle ϕ -function is proportional to the product of the opposite stream fluxes (see Fig.1). Let the expected number density of particles at z and moving to the right be denoted by u(z,x) ; similarly, let the expected number density of particles at z and moving to the left be denoted by v(z,x).

The equation of transport takes the form

(31) $$\frac{du}{dz} = \alpha(z)u(z,x) + \beta(z)v(z,x) - \phi(u,v) ,$$

(32) $$-\frac{dv}{dz} = \beta(z)u(z,x) + \alpha(z)v(z,x) - \phi(u,v) ,$$

together with the boundary conditions

(33) $$u(0,x)=0 , \quad v(x,x)=F .$$

From eq. (33) we get

(34) $$\left.\frac{\partial v}{\partial z}\right|_{z=x} + \left.\frac{\partial v}{\partial x}\right|_{z=x} = 0 .$$

Putting

(35) $$R(v(z,x),x)=u(z,x) ,$$

where R is the expected number density of particles per second moving to the right at z due to an input of F particles per sencond at x, and differentiating u with respect to x in the limit z=x, we have

(36) $$\left.\frac{\partial u}{\partial x}\right|_{z=x} = \frac{\partial R}{\partial F}\left(-\frac{\partial v}{\partial z}\right)_{z=x} .$$

On making use of eq. (36), differentiating R with respect to x, we obtain

(37) $$\frac{\partial R(x,F)}{\partial X} = \left.\frac{\partial u}{\partial z}\right|_{z=x} + \left.\frac{\partial u}{\partial x}\right|_{z=x} = \alpha(x)R(F,x) + (\beta(x)R + \alpha(x)F - \phi(R,F))\frac{\partial R}{\partial F} + \beta(x)F - \phi(R,F).$$

When $\phi=0$ and u(x,x)=RF, eq. (37) reduces to the Riccati equation.

In a manner similar to that given in the derivation of the reflection coefficient R, we shall find the equation for the transmission coefficient T-function. Put

(38) $$T(v(z,x),z)\Big|_{z=x} = v(z,x)\Big|_{z=0} .$$

On differentiating eq. (38) with respect to z and passing to the limit z=0, we have the required partial differential equation

(39) $$\frac{\partial T(F,X)}{\partial x} = \frac{\partial v}{\partial z}\Big|_{z=0} + \frac{\partial v}{\partial x}\Big|_{z=0} = -\alpha(0)T(F,x) + (\alpha(x)F+\beta(x)R(F,X)) - -\phi(F,R))\frac{\partial T}{\partial F} + \phi(T,0).$$

The law of the conservation of the probability for the multiple scattering of particles takes the form

(40) $$v(z,x)-(u(z,x)+v(0,x)) = L(z,v(z,x))\, v(z,x),$$

where L-function is called the dissipation function. On differentiating with respect to z and passing to the limit at z=x, we get

(41) $$\frac{\partial v}{\partial z}\Big|_{z=x} - \frac{\partial u}{\partial z}\Big|_{z=x} = \left(\frac{\partial L}{\partial z} + \frac{\partial L}{\partial v}\frac{\partial v}{\partial z}\right)_{z=x} + L(x,v(x,x))\frac{\partial v}{\partial z}\Big|_{z=x}.$$

From eq. (41) we obtain the desired equation for the L-function

(42) $$F\cdot\frac{\partial L}{\partial x}(x,F) = (\alpha(x)F+\beta(x)R(F,x))L + (\alpha(x)F+\beta(x)R-\phi(R,F))\frac{\partial L}{\partial F} -(\alpha(x)+\beta(x))(R+F)-2\phi(R,F).$$

Such a equation will be useful for the study of the particle-particle interactions in the theory of neutron diffusion and plasma dynamics.

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