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Numerical Analysis of Subharmonic Solutions to Duffing's Equation

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1. Preliminaries

The present paper is concerned with subharmonic solutions of Duffing's equation

$$\frac{d^2q}{d\tau^2} + \kappa^2 q\,(1+\beta q^2) = P_0 \cos \gamma\tau . \tag{1.1}$$

By the transformation

$$\kappa\tau = t,\quad \frac{\kappa^2}{P_0} q = x,\quad \frac{\gamma}{\kappa} = \omega,\quad \frac{\beta P_0^2}{\kappa^4} = \varepsilon ,$$

equation (1.1) can be reduced to the equation

$$\frac{d^2x}{dt^2} + x(1+ \varepsilon x^2) = \cos\omega t, \tag{1.2}$$

which, replacing ωt by t, one can rewrite as follows:

$$\frac{d^2x}{dt^2} + \frac{1}{\Omega} x(1+\varepsilon x^2) = \frac{1}{\Omega} \cos t , \tag{1.3}$$

where

(1.4) $$\Omega = \omega^2 .$$

To a subharmonic solution of (1.1), corresponds a solution of (1.3) of the form

(1.5) $$x(t) = a_1 \cos \frac{t}{3} + a_2 \cos t + \dots .$$

Hence replacing t by $3t$ in (1.3) and (1.5), one can reduce the problem to the one to find a solution of the form

(1.6) $$x(t) = a_1 \cos t + a_2 \cos 3t + \dots$$

to the equation

(1.7) $$\frac{d^2x}{dt^2} + \frac{9}{\Omega} x(1+\varepsilon x^2) = \frac{9}{\Omega} \cos 3t .$$

In the present paper, approximations of solutions of the form (1.6) will be computed by Galerkin's method for various values of the parameters ε and ω. The existence of subharmonic solutions corresponding to these approximations will be assured from approximations themselves by the author's method developed in [1]. Error estimates of approximations computed and the stability of the corresponding subharmonic solutions will be given also.

2. Galerkin's method

Consider a real periodic differential system

$$\frac{dx}{dt} = X(x,t), \tag{2.1}$$

where x and $X(x,t)$ are vectors and $X(x,t)$ is periodic t of period 2π. To get an approximation of a periodic solution of (2.1), we consider a trigonometric polynomial

$$x_m(t) = a_0 + \sum_{n=1}^{m} (a_{2n-1} \sin nt + a_{2n} \cos nt) \tag{2.2}$$

with unknown coefficients $\alpha = (a_0, a_1, a_2, \ldots, a_{2m-1}, a_{2m})$, and we determine these unknown coefficients by the equation

$$\begin{aligned} \frac{dx_m(t)}{dt} &= \frac{1}{2\pi}\int_0^{2\pi} X[x_m(t), t]\, dt \\ &+ \frac{1}{\pi}\sum_{n=1}^{m} \left\{ \sin nt \cdot \int_0^{2\pi} X[x_m(s), s] \sin ns\, ds \right. \\ &\left. + \cos nt \cdot \int_0^{2\pi} X[x_m(s), s] \cos ns\, ds \right\}. \end{aligned} \tag{2.3}$$

Equation (2.3) is evidently equivalent to the equation

$$\left\{ \begin{aligned} &F_0(\alpha) \triangleq \frac{1}{2\pi}\int_0^{2\pi} X[x_m(t),t]\, dt = 0, \\ &F_{2n-1}(\alpha) \triangleq \frac{1}{\pi}\int_0^{2\pi} X[x_m(t),t] \sin nt\, dt + na_{2n} = 0, \end{aligned} \right. \tag{2.4}$$

$$\begin{cases} F_{2n}(\alpha) \triangleq \frac{1}{\pi}\int_0^{2\pi} X[x_m(t),t]\cos nt\, dt - na_{2n-1} = 0 \end{cases}$$

$(n=1,2,\ldots,m)$.

A trigonometric polynomial with coefficients satisfying (2.4) will be called a Galerkin approximation of a periodic solution of (2.1) and the equation (2.4) will be called the determining equation of Galerkin approximations. A method of getting an approximation of a periodic solution by computing a Galerkin approximation is called Galerkin's method.

Galerkin's method is based mathematically on the following theorem due to the author [1].

Theorem 1. Suppose that $X(x,t)$ and its Jacobian matrix $\Psi(x,t)$ with respect to x are continuously differentiable with respect to x and t in the region $D\times L$, where D is a closed bounded region of the x-space and L is the real line. If differential system (2.1) possesses an isolated periodic solution $x=\hat{x}(t)$ lying inside D, then for sufficiently large m_0, there is a Galerkin approximation $x=\bar{x}_m(t)$ to any order $m\geq m_0$ such that

$$\bar{x}_m(t) \longrightarrow \hat{x}(t), \qquad \dot{\bar{x}}_m(t) \longrightarrow \dot{\hat{x}}(t) \qquad (\cdot = d/dt)$$

uniformly as $m \longrightarrow \infty$.

By an isolated periodic solution, is meant a periodic solution such that the multipliers of the relative first variation equation are all different from one.

To compute a Galerkin approximation, it is necessary to solve numerically the determining equation, which is usually nonlinear. Newton's method is well adapted for this purpose, because as will be shown later, the starting approximations can be found usually without difficulty and the iterative process can be practised easily on a computer by means of well-known formulas of Fourier analysis (See[2]).

3. Assurance of the existence of an exact periodic solution and error estimation of approximate solutions

Our method is based on the following theorem due to the author [1].

Theorem 2. *Suppose that (2.1) possesses a periodic approximate solution $x=x(t)$ lying inside D and the multipliers of the relative first variation equation*

$$(3.1)\qquad \frac{dy}{dt} = \Psi[x(t),t]y$$

are all different from one.

Let $\Phi(t)$ be the fundamental matrix of (3.1) satisfying the initial condition $\Phi(0) = E$ (E is the unit matirix) and $H(t,s) = (H_{k\ell}(t,s))$ be a piecewise continuous matrix such that

$$(3.2)\quad H(t,s) = \begin{cases} \Phi(t)[E - \Phi(2\pi)]^{-1}\Phi^{-1}(s) & \text{for } 0\leqq s\leqq t\leqq 2\pi , \\ \Phi(t)[E - \Phi(2\pi)]^{-1}\Phi(2\pi)\Phi^{-1}(s) & \text{for } 0\leqq t< s\leqq 2\Pi , \end{cases}$$

Let M be a positive number such that

(3.3) $$[2\pi \cdot \max_{0\leq t\leq 2\pi} \int_0^{2\pi} \sum_{k,\ell} H_{k\ell}^2(t,s)\,ds]^{1/2} \leqq M,$$

and r be a non-negative number such that

(3.4) $$\left\| \frac{d\bar{x}(t)}{dt} - X[x(t),t] \right\| \leqq r,$$

where the symbol $\|\ldots\|$ denotes the Euclidean norm of vectors or the correponding norm of matrices.

If there are positive constants δ and $\kappa < 1$ such that

(3.5)
(i) $D_\delta = \left\{ x \;\middle|\; \|x-\bar{x}(t)\| \leqq \delta \text{ for some } t \right\} \subset D,$

(ii) $\|\Psi[x,t] - \Psi[\bar{x}(t),t]\| \leqq \kappa/M$ for all x, t such that $\|x-\bar{x}(t)\| \leqq \delta$,

(iii) $M\varepsilon/(1-\kappa) \leqq \delta$,

then the given system (2.1) possesses one and only one periodic solution $x=\hat{x}(t)$ in D_δ and this is an isolated periodic solution. Further, for $x=\hat{x}(t)$, it holds that

(3.6) $$\|\bar{x}(t) - \hat{x}(t)\| \leqq \frac{Mr}{1-\kappa} .$$

When a Galerkin approximation $\bar{x}_m(t)$ has been computed, one can find the corresponding value of the number M through the numerical integration of (3.1) for $\bar{x}(t)=\bar{x}_m(t)$ and one can find also the corresponding value of the number r by evaluating the Fourier coefficients of the function

$$\frac{d\bar{x}_m(t)}{dt} - X[\bar{x}_m(t),t] .$$

Then one can easily check the existence of the constants δ and κ satisfying the condition (3.5). If there exist such constants δ and κ, then by Theorem 2 one can assure the existence of a periodic solution of (2.1) and further by (3.6) one can get an error bound for the Galerkin approximation $\bar{x}_m(t)$.

Remark. In the computation of the number M, one computes the fundamental matrix $\Phi(t)$ of (3.1) satisfying the initial condition $\Phi(0) = E$. If $\bar{x}(t)$ is close to the exact solution $\hat{x}(t)$, then the eigenvalues of $\Phi(2\pi)$ is close to the multipliers of the first variation equation of (2.1) with respect to the exact solution. Hence one can decide the stability of an exact periodic solution by inspecting the module of eigenvalues of the matrix $\Phi(2\pi)$.

4. Numerical computation of subharmonic solutions

1° To get a rough approximation of a periodic solution of (1.7), we consider a trigonometric polynomial of the form

$$x = a_1 \cos t + a_2 \cos 3t. \tag{4.1}$$

Then the determining equation for a polynomial of the above form is

$$\begin{cases} a_1[(9-\Omega)+\frac{27}{4}\varepsilon(a_1^2 + a_1a_2 + 2a_2^2)] = 0, \\ (1-\Omega)a_2 + \frac{\varepsilon}{4}(a_1^3+6a_1^2a_2 + 3a_2^3) - 1 = 0 \end{cases} \tag{4.2}$$

Since $a_1=0$ does not correspond to the desired subharmonic solution, we assume that

(4.3) $$a_1 \neq 0.$$

Then from (4.2) follows

(4.4) $$\begin{cases} (9-\Omega) + \frac{27}{4}\varepsilon(a_1{}^2+a_1a_2+2a_2{}^2) = 0, \\ (1-\Omega)a_2 + \frac{\varepsilon}{4}(a_1{}^3+6a_1{}^2a_2+3a_2{}^3) - 1 = 0. \end{cases}$$

From the first of (4.4) we have

(4.5) $$\Omega = 9 + \frac{27}{4}\varepsilon(a_1{}^2+a_1a_2+2a_2{}^2).$$

Hence substituting (4.5) into the second of (4.4), we have

(4.6) $$51a_2{}^3+27a_1a_2{}^2+21a_1{}^2a_2-a_1{}^3+\frac{4}{\varepsilon}(8a_2+1) = 0.$$

Let us consider the case where

(4.7) $$\varepsilon > 0.$$

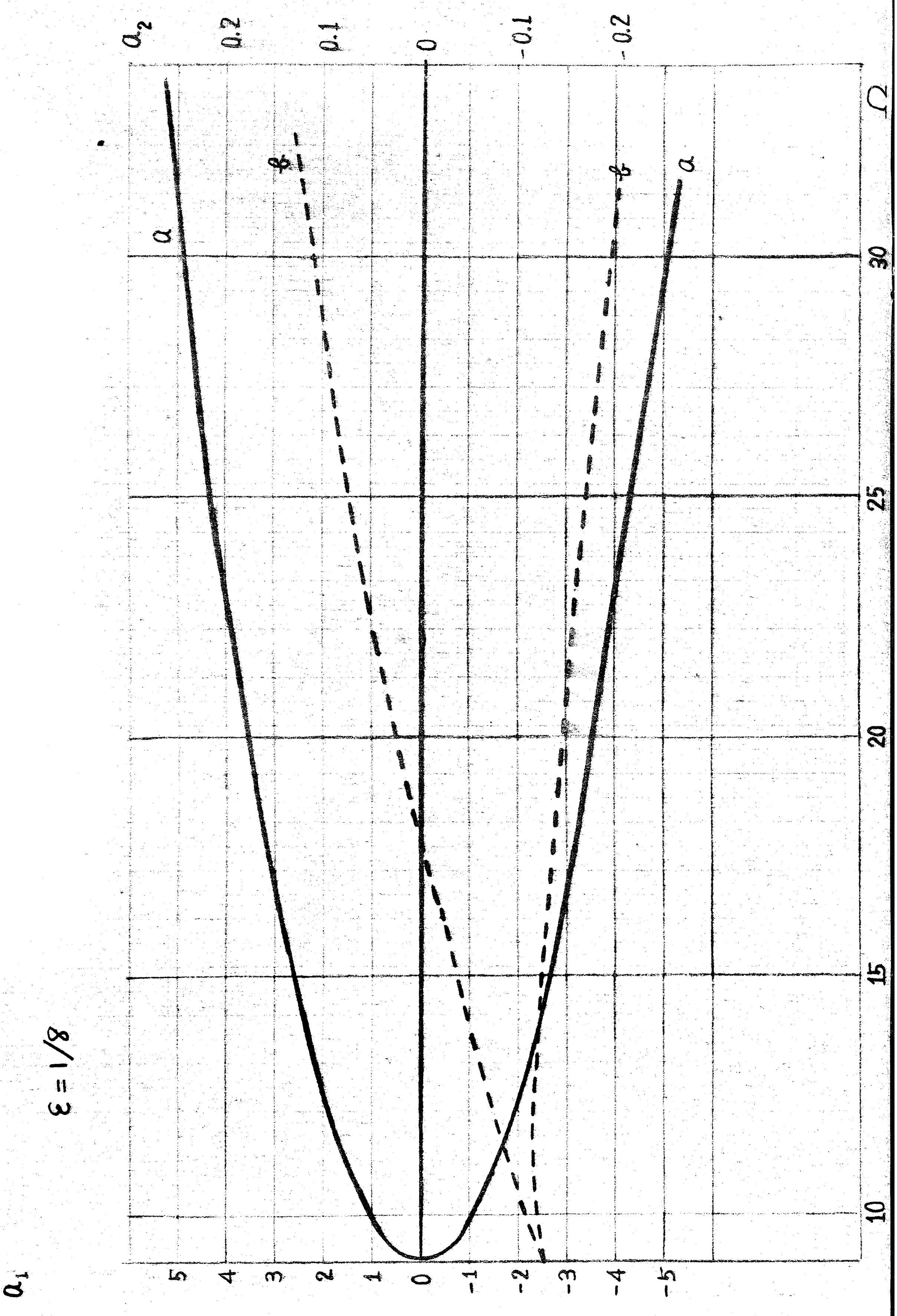
Then from (4.5), real solutions of (4.4) can exist only for

(4.8) $$\Omega > 9.$$

Now the derivative of the left member of (4.6) with respect to a_2 is always positive and hence (4.6) can have only one solution a_2 for any given value of a_1. Such being the case, for $\varepsilon=1/8$, 1/2, 1 and $a_1=-5(1)5$, we have computed a_2 satisfying (4.6) by Newton's method and then we have computed the corresponding values of Ω by (4.5). Drawing the graphs of (Ω, a_1) and (Ω, a_2) from the results obtained, we then have the approximate solutions of (4.4) for given values of Ω. Starting from these approximate solutions, we finally get the solutions of (4.4) by Newton's method for various values of Ω. Figure 1 shows the graphs of (Ω, a_1) and (Ω, a_2) for $\varepsilon=1/8$ and Table 1 shows the solutions of (4.4) obtained in the above way for $\varepsilon=1/8$.

TABLE 1

ω	Ω	a_1	a_2
4	16	2.88938 9313	-0.01832 4240
		-2.81281 9167	-0.12545 0320
3.2	10.24	1.25675 9007	-0.10488 5136
		-1.14426 3122	-0.11640 0863
3.1	9.61	0.89420 1639	-0.11557 8278
		-0.77575 5977	-0.11942 2055
3.05	9.3025	0.63741 8236	-0.12059 7225
		-0.51585 2082	-0.12171 4132


ε = 1/8
a_1
a_2
Ω
a
b
5
4
3
2
1
0
-1
-2
-3
-4
-5
0.2
0.1
0
-0.1
-0.2
10
15
20
25
30

2° To get a Galerkin approximation of a periodic solution of (1.7), we consider a trigonometric polynomial of the form

$$x(t) = \sum_{n=1}^{m} a_n \cos(2n-1)t. \tag{4.9}$$

If $x=x(t)$ is a solution of (1.7), then $x=x(-t)$ and $x=-x(t+\pi)$ are also solutions of (1.7). This suggests the form (4.9) of a periodic solution.

We have computed solutions of the determining equation by Newton's method starting from

$$a_1 = \bar{a}_1, \quad a_2 = \bar{a}_2, \quad a_3 = a_4 = \dots = a_m = 0,$$

where $(a_1, a_2)=(\bar{a}_1, \bar{a}_2)$ is a solution of equation (4.4) obtained in 1°. The Fourier coefficients appearing in the determining equation have been computed by the following formula [2]:

$$\frac{1}{\pi}\int_0^{2\pi} f(t)\cos pt\,dt = \frac{1}{N}\sum_{i=1}^{2N} f(t_i)\cos pt_i \qquad (p=1,2,\dots,\nu),$$

where

$$t_i = \frac{2i-1}{2N}\pi \qquad (i=1,2,\dots,2N)$$

and

$$N \geqq \nu+1.$$

Table 2 shows the values of coefficients a_n obtained in the above way with $N=64$.

TABLE 2

A_1 : $\varepsilon=1/8$, $\omega=4$

n	a_n	a_n
1	2.88939 64903	-2.81264 00834
2	-0.01834 34265	-0.12557 78220
3	-0.00034 06838	-0.00232 01890
4	-0.00000 20357	-0.00007 13067
5	0.00000 00117	-0.00000 19406
6	0.00000 00002	-0.00000 00528
7	0.0	-0.00000 00014
8	:	0.0
:	:	:
15	0.0	0.0

A_2 : $\varepsilon=1/8$, $\omega=3.2$

n	a_n	a_n
1	1.25671 76442	-1.14419 82013
2	-0.10489 29084	-0.11640 92077
3	-0.00052 42595	-0.00057 89532
4	0.00002 26175	-0.00002 82413
5	-0.00000 02233	-0.00000 07476
6	-0.00000 00080	-0.00000 00117
7	0.00000 00001	-0.00000 00003
8	0.0	0.0
:	:	:
15	0.0	0.0

A_3 : $\varepsilon = 1/8$, $\omega = 3.1$

n	a_n	a_n
1	0.89416 37367	-0.77570 72093
2	-0.11558 04311	-0.11942 42151
3	-0.00029 52928	-0.00030 39846
4	0.00002 15640	-0.00002 06942
5	-0.00000 04842	-0.00000 07033
6	-0.00000 00063	-0.00000 00064
7	0.00000 00002	-0.00000 00002
8	0.0	0.0
⋮	⋮	⋮
15	0.0	0.0

A_4 : $\varepsilon = 1/8$, $\omega = 3.05$

n	a_n	a_n
1	0.63738 94700	-0.51581 55306
2	-0.12059 77217	-0.12171 46120
3	-0.00015 03858	-0.00015 14040
4	0.00001 74607	-0.00001 45577
5	-0.00000 06320	-0.00000 07096
6	-0.00000 00038	-0.00000 00033
7	0.00000 00002	-0.00000 00002
8	0.0	0.0
⋮	⋮	⋮
15	0.0	0.0

1 1

B_1 : $\varepsilon=1/2$, $\omega=4$

n	a_n	a_n
1	1.46165 38180	-1.38496 30370
2	-0.04555 49212	-0.09836 10693
3	-0.00084 49916	-0.00180 57494
4	0.00000 59508	-0.00007 68957
5	0.00000 02492	-0.00000 26025
6	-0.00000 00018	-0.00000 00865
7	-0.00000 00001	-0.00000 00030
8	0.0	-0.00000 00001
.	.	0.0
.	.	.
.	.	.
15	0.0	0.0

B_2 : $\varepsilon=1/2$, $\omega=3.2$

n	a_n	a_n
1	0.64303 26531	-0.53088 90496
2	-0.10832 12797	-0.11289 93867
3	-0.00051 46997	-0.00053 17972
4	0.00005 10049	-0.00004 80110
5	-0.00000 13945	-0.00000 23212
6	-0.00000 00368	-0.00000 00375
7	0.00000 00014	-0.00000 00019
8	0.0	-0.00000 00001
9	.	0.0
.	.	.
.	.	.
15	0.0	0.0

B_3 : $\varepsilon=1/2$, $\omega=3.1$

n	a_n	a_n
1	0.45439 19264	-0.33641 48660
2	-0.11695 64970	-0.11793 31876
3	-0.00026 33429	-0.00026 41626
4	0.00004 53681	-0.00003 46282
5	-0.00000 22006	-0.00000 25224
6	-0.00000 00253	-0.00000 00203
7	0.00000 00017	-0.00000 00015
8	0.0	-0.00000 00001
9		0.0
15	0.0	0.0

B_4 : $\varepsilon=1/2$, $\omega=3.05$

n	a_n	a_n
1	0.31340 53558	-0.19237 42156
2	-0.12113 75427	-0.12103 80304
3	-0.00011 05583	-0.00011 03701
4	0.00003 47781	-0.00002 13758
5	-0.00000 26472	-0.00000 27122
6	-0.00000 00131	-0.00000 00086
7	0.00000 00015	-0.00000 00010
8	-0.00000 00001	-0.00000 00001
9	0.0	0.0
⋮	⋮	⋮
15	0.0	0.0

C_1 : $\varepsilon=1$, $\omega=4$

n	a_n	a_n
1	1.04265 32218	-0.96604 99115
2	-0.05366 54105	-0.09024 34248
3	-0.00099 23715	-0.00164 48577
4	0.00001 80707	-0.00008 58323
5	0.00000 04148	-0.00000 33351
6	-0.00000 00123	-0.00000 01232
7	-0.00000 00002	-0.00000 00049
8	0.0	-0.00000 00002
9	⋮	0.0
⋮		⋮
15	0.0	0.0

C_2 : $\varepsilon=1$, $\omega=3.2$

n	a_n	a_n
1	0.45801 73216	-0.34637 31608
2	-0.10946 46843	-0.11164 50565
3	-0.00048 31870	-0.00048 78993
4	0.00007 49046	-0.00006 06922
5	-0.00000 31416	-0.00000 42355
6	-0.00000 00747	-0.00000 00644
7	0.00000 00045	-0.00000 00045
8	-0.00000 00001	-0.00000 00002
9	0.0	0.0
⋮	⋮	⋮
15	0.0	0.0

C_3 : $\varepsilon=1$, $\omega=3.1$

n	a_n	a_n
1	0.31600 21647	-0.19866 15617
2	-0.11741 22723	-0.11732 07872
3	-0.00021 64496	-0.00021 57463
4	0.00006 37913	-0.00004 03006
5	-0.00000 45871	-0.00000 48456
6	-0.00000 00459	-0.00000 00309
7	0.00000 00050	-0.00000 00033
8	-0.00000 00002	-0.00000 00002
9	0.0	0.0
⋮	⋮	⋮
15	0.0	0.0

C_4 : $\varepsilon=1$, $\omega=3.05$

n	a_n	a_n
1	0.19874 95570	-0.07843 84541
2	-0.12123 55889	-0.12075 36311
3	-0.00005 65814	-0.00005 70524
4	0.00004 42107	-0.00001 73127
5	-0.00000 53693	-0.00000 53370
6	-0.00000 00176	-0.00000 00078
7	0.00000 00039	-0.00000 00015
8	-0.00000 00002	-0.00000 00003
9	0.0	0.0
⋮	⋮	⋮
15	0.0	0.0

5. Existence of subharmonic solutions, error estimation of Galerkin approximations and stability of subharmonic solutions

Equation (1.7) is equivalent to the system

$$(5.1)\quad \begin{cases} \dfrac{dx}{dt} = y\ , \\[2ex] \dfrac{dy}{dt} = -\dfrac{9}{\Omega}(x+\varepsilon x^3) + \dfrac{9}{\Omega}\cos 3t. \end{cases}$$

The Jacobian matrix $\Psi(x,y,t)$ of the right member of (5.1) with respect to x and y is

$$(5.2)\qquad \Psi(x,y,t) = \begin{bmatrix} 0 & 1 \\ -\frac{9}{\Omega}(1+3\varepsilon x^2) & 0 \end{bmatrix}.$$

For an approximate solution $(x,y) = (x(t), \bar{y}(t))$, we then have

$$\|\Psi[x,y,t] - \Psi[\bar{x}(t),\bar{y}(t),t]\| = \frac{27}{\Omega}\varepsilon|x^2-\bar{x}^2(t)| .$$

Hence, if

$$|x - \bar{x}(t)| \leqq \delta,$$

then we have

$$(5.3)\quad \|\Psi[x,y,t] - \Psi[\bar{x}(t),\bar{y}(t),t]\| \leqq \frac{27}{\Omega}\varepsilon\,\delta(\delta+2|\bar{x}(t)|).$$

Thus for sustem (5.1), condition (3.5) is satisfied if

$$(5.4)\quad \begin{cases} \dfrac{27}{\Omega}\varepsilon\delta\ (\delta+2\max\limits_{t}|x(t)|) \leqq \dfrac{\kappa}{M}, \\[2ex] \dfrac{Mr}{1-\kappa} \leqq \delta . \end{cases}$$

The method of verification of the above inequalities will be illustrated with an example.

Consider the case where $\varepsilon=1/8$, $\omega=3.05$ ($\Omega=9.3025$) and

(5.5) $\bar{x}(t) = 0.63738\,94700 \cos t - 0.12059\,77217 \cos 3t$
$-0.00015\,03858 \cos 5t + 0.00001\,74607 \cos 7t$
$-0.00000\,06320 \cos 9t - 0.00000\,00038 \cos 11t$
$+0.00000\,00002 \cos 13t.$

First, putting

$$\frac{d^2\bar{x}(t)}{dt^2} + \frac{9}{\Omega}[\bar{x}(t)+\varepsilon\bar{x}^3(t)] - \frac{9}{\Omega}\cos 3t = \sum_{k=1}^{\infty} d_k \cos(2k-1)t,$$

we compute

$$\sum_{k=1}^{25} |d_k| \quad ,$$

which is found to be $0.86207\,96094\times 10^{-8}$ for (5.5). Hence we may take r so that

(5.6) $$r = 8.63\times 10^{-9} .$$

Next, by the author's method developed in [3,4], we compute the fundamental matrix $\Phi(t)$ of the first variation equation of (5.1) with respect to $(x,y)=[\bar{x}(t), d\bar{x}(t)/dt]$ in the form

$$\Phi(t) = \frac{1}{2}A_0 + \sum_{n=1}^{30} A_n T_n(\frac{t}{\pi} - 1) \ ,$$

where $T_n(t)$ $(n=1,2,\ldots,30)$ are Chebyshev polynomials.

By (3.2), we then compute

$$H(t_p,\ t_q) \qquad (P,q=0,1,2,...,256),$$

where

$$t_p = p\pi/128 \qquad (p=0,1,2,...,256).$$

Making use of the values of $H(t_p,\ t_q)$ obtained, we compute the integrals

$$\int_0^{2\pi} \sum_{k,\ell} H_{k\ell}^2(t_p,s)\,ds \qquad (p=0,2,4,...,\ 256)$$

by Simpson's rule with mesh $h=1/128$. By (3.3), we then compute

$$[2\pi\cdot\max_p \int_0^{2\pi} \sum_{k\ell} H_{k\ell}^2\,(t_p,s)\,ds]^{1/2},$$

which is found to be $0.10143\ 69424\times10^3$ for (5.5). Hence for (5.5), we may take M so that

(5.7) $\qquad M = 101.5$

Now for (5.5),

(5.8) $\qquad \max_t |\bar{x}(t)| = 0.75815\ 56742.$

Hence by (5.6), (5.7) and (5.8), the inequalities of (5.4) can be written as follows:

$$(5.9)\quad \begin{cases} \dfrac{27}{9.3025} \times 0.125 \times (\ +1.51631\ 13484)\ \delta < \dfrac{\kappa}{101.5}\ , \\ \dfrac{101.5\times 8.63\times 10^{-9}}{1-\kappa} < \delta \end{cases}$$

Since

$$101.5\times 8.3\times 10^{-9} = 8.75945\times 10^{-7},$$

we assume tentatively that

$$(5.10)\qquad \delta < 9\times 10^{-7}.$$

Then the first inequality of (5.9) is satisfied if

$$\frac{27}{9.3025} \times 0.125\times 1.51631\ 22484\times \delta < \frac{\kappa}{101.5},$$

that is,

$$\frac{9.3025}{27\times 0.125\times 1.51631\ 22484\times 101.5}\ \kappa$$

$$= 0.01790\ 8374\ \ldots\times \kappa .$$

Then both inequalities of (5.9) are satisfied if

$$(5.11)\qquad \frac{8.75945\times 10^{-7}}{1-\kappa} < \delta < 0.01790 .$$

Consider the inequality

$$(5.12)\qquad 8.75945\times 10^{-7} < 0.01790\kappa(1-\kappa).$$

Since

$$\frac{8.75945\times 10^{-7}}{0.01790} = 4.893547\ \ldots\times 10^{-5},$$

we readily see that inequality (5.12) is satisfied by

(5.13) $\kappa = 5\times10^{-5}$.

For this value of κ ,

$$\frac{8.75945\times10^{-7}}{1-\kappa} = 8.75988\ 79943\ \ldots\times10^{-7},$$

$$0.01790\kappa = 8.950\ \ldots\times10^{-7}\ .$$

Hence taking into account the inequality (5.10), from (5.11) we have

(5.14) $8.75989\times10^{-7} < \delta < 8.950\times10^{-7}$,

which shows the existence of the positive numbers δ and $\kappa < 1$ satisfying (5.4). Thus by Theorem 2 we see the existence of an isolated periodic solution $x=\hat{x}(t)$ of (1.7) such that

(5.15) $|\hat{x}(t) - \bar{x}(t)| \leq 8.760\times10^{-7}$.

This proves that (1.7) <u>really possesses a subharmonic solution close to</u> $x=\bar{x}(t)$ <u>given by</u> (5.5) <u>and further that the approximation</u> (5.5) <u>of a subharmonic solution is within the error bound</u> 8.760×10^{-7}.

Now, if $x=\hat{x}(t)$ is a periodic solution of (1.7), then $x=\hat{x}(-t)$ and $x=-\hat{x}(t+\pi)$ are also periodic solutions of (1.7). However by (5.14) and (5.15),

$$\begin{cases} |\hat{x}(-t)-\bar{x}(t)| = |\hat{x}(-t)-\bar{x}(-t)| \leq \delta \\ |[-\hat{x}(t+\pi)]-\bar{x}(t)| = |[-\hat{x}(t+\pi)]-[-\bar{x}(t+\pi)]| \leq \delta\ . \end{cases}$$

Since there exists only one periodic solution in D_δ by Theorem 2, we then see that

$$\hat{x}(-t) = -\hat{x}(t+\pi) = \hat{x}(t),$$

which shows that $x=\hat{x}(t)$ is of the form

(5.16) $$\hat{x}(t) = \sum_{n=1}^{\infty} a_n \cos(2n-1)t.$$

From the value of $\Phi(2\pi)$, we readily see that the eigenvalues λ_1, λ_2 of $\Phi(2\pi)$ are

$$\lambda_1, \lambda_2 = 0.99321\ 02880 \pm 0.11633\ 28121i,$$

which implies

$$|\lambda_1| = |\lambda_2| = 0.99999\ 99996.$$

This shows that the stability of $x=\hat{x}(t)$ will be neutral.

For the case where $\varepsilon=1/8$, $\omega=3.05$ $(\Omega=9.3025)$ and

(5.17) $$\begin{aligned}\bar{x}(t) = &-0.51581\ 55306 \cos t - 0.12171\ 46120 \cos 3t \\ &-0.00015\ 14040 \cos 5t - 0.00001\ 45577 \cos 7t \\ &-0.00000\ 07096 \cos 9t - 0.00000\ 00033 \cos 11t \\ &-0.00000\ 00002 \cos 13t,\end{aligned}$$

we have

$$r = 1.31\times10^{-8}, \quad M=123.7.$$

Hence we again get the assurance for the existence of a subharmonic solution, and for (5.17), we get the error bound 1.6207×10^{-6}. However in this case, the eigenvalues of $\Phi(2\pi)$ are found to be

$$1.09964\ 4524 \quad \text{and} \quad 0.90938\ 47852,$$

which show that the subharmonic solution close to (5.17) is unstable.

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