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An uncountable number of II_1, II_∞-factors

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1. Introduction. Recently, great progress has been made in the investigation of the isomorphism classes of II_1-factors ([1], [2], [4], [6], [7], [8]).
In particular, McDuff [4] proved the existence of a countably infinite number of II_1-factors on a separable Hilbert space.

In this paper, by using the method of McDuff, we shall show the existence of an uncountable number of non-isomorphic II_1-factors on a separable Hilbert space.
Moreover, by using this result and tensor products, we shall show the existence of an uncountable number of II_∞ factors on a separable Hilbert space.

Concerning III-factors, Powers [11] has shown the existence of an uncountable number.

Added in proof. After writing this paper, the author received other two papers of McDuff [9], [10] in which she proves the existence of an uncountable number of II_1-factors. But, the construction is different from ours.

2. Construction of examples. Suppose G_1, G_2, ... ; H_1, H_2, ... are two sequences of groups. We denote by (G_1, G_2, ... ; H_1, H_2, ...) the group generated by the G_i's and the H_i's with additional relations that H_i, H_j

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commute elementwise for $i \neq j$ and G_i , H_j commute elementwise for $i \leqq j$. This situation was considered in [2].

Let $L_1 = (G_1, G_2, \ldots ; H_1, H_2, \ldots)$ with $G_i = Z$, and $H_i = Z$ for all i, where Z is the infinite cyclic group. Define L_k inductively by $L_k = (G_1, G_2, \ldots ; H_1, H_2, \ldots)$ with $G_i = Z$, $H_i = L_{k-1}$ for all i and $k \geqq 2$.

Let I be the set of all positive integers and let I_1 be a sequence of positive integers ~~an subset of I.~~ Let $M_n(I_1) = \sum_{i=1}^{n} \oplus L_{p_i}$, if $I_1 = (p_1, p_2, \ldots)$ is infinite, and $M_n(I_1) = \sum_{i=1}^{n} \oplus L_{p_i}$ for $n \leqq n_0$ and $M_n(I_1) = M_{n_0}$ for $n > n_0$, if $I_1 = (p_1, p_2, \ldots, p_{n_0})$ is finite.

Let $G[I_1] = (G_1, G_2, \ldots ; H_1, H_2, \ldots)$ with $G_i = Z$ and $H_i = M_i(I_1)$ for all i.

For a discrete group G, U(G) is the W^*-algebra generated by the left regular representation of G.

Then, we shall show the following theorem

Theorem 1. Let $I_1 = (p_i)$ and $I_2 = (q_j)$ be two ~~subsets~~ sequences of I such that I_2 contains a positive integer q such that ~~$q \geqq 2$~~, q, ~~$q \geqq 1$~~ $\notin I_1$. Then, $U(G[I_1])$ is not *-isomorphic to $U(G[I_2])$. In particular, $U(G[I_1])$ is not *-isomorphic to $U(G[I_2])$, if I_1 and I_2 are two subsets of ~~even~~ positive integers and $I_1 \neq I_2$ as a set.

As a corollary, we have

Corollary 1. There exists an uncountable number of non-isomorphic II_1-factors on a separable Hilbert space.

To prove Theorem 1, we shall provide some considerations.

Definition 1 ([4]). For a W^*-algebra $U(G)$ we shall write $(U(G))_1$ to denote the unit sphere of $U(G)$. If B and C are subalgebras of $U(G)$ and $\delta > 0$, then we shall write $B \overset{\delta}{\subset} C$ to mean that given any $T \in (B)_1$, there exists some $S \in (C)_1$ with $\|T - S\|_2 < \delta$ where $\|x\|_2$ is the $L^2(G)$-norm of x, when $U(G)$ is embedded into $L^2(G)$ canonically.

Let $A = U(G)$. A bounded sequence (T_n) of elements of A is called a central sequence, if for all $X \in A$, $\|[X, T_n]\|_2 \to 0$ ($n \to \infty$), where [,] is the Lie product. Central sequences (T_n) , (T_n') in A are called equivalent, if $\|T_n - T_n'\|_2 \to 0$ ($n \to \infty$).

Let H be a subgroup of a group G. H is called strongly residual in G, if it satisfies the following conditions : there exist a subset S of the complement $G \setminus H$ of H and elemwnts g_1, g_2 of G such that (i) $g_1 H g_1^{-1} = H$, (ii) $S \cup g_1 S g_1^{-1} = G \setminus H$, (iii) $\{ g_2^n S g_2^{-n} \}_{n = 0, \pm 1, \pm 2, \ldots}$ forms a family of disjoint subsets of $G \setminus H$.

By the above definition, we can easily see that only one strongly residual subgroup of a commutative group G is G itself - in this case, S is the empty set.

Lemma 1 ([4]). Let G_i ($i = 1, 2, \ldots\ldots, n$) be a finite family of groups and let H_i ($i = 1, 2, \ldots\ldots, n$) be a subgroup of G_i. Suppose that H_i is strongly residual in G_i for each i, then $\sum_{i=1}^{n} \oplus H_i$ is strongly residual in $\sum_{i=1}^{n} \oplus G_i$.

Let H be a strongly residual subgroup of G, then H must

contain the center of G.
Let $\{H_n\}$ be a sequence of subgroups of G. $\{H_n\}$ is called a residual sequence of G, if it satisfies the following conditions : (i) H_n is strongly residual in G ; (ii) $H_n = H_{n+1} \oplus K_n$, where K_n is a subgroup of G ; (iii) $\bigcup_{n=1}^{\infty} H_n' = G$, where H_n' is the commutant of H_n in G.

Let G_i $(i = 1, 2, \ldots, m)$ be a finite family of groups and let $\{H_{i,n}\}$ $(i = 1, 2, \ldots, m)$ be a residual sequence of G_i then $\{\sum_{i=1}^{m} \oplus H_{i,n}\}$ is a residual sequence of $\sum_{i=1}^{m} G_i$.
Any central sequence in U(G) is equivalent to a central sequenc whose elements lie in U(H), if H is canonically considered as a subalgebra of U(G), *and H is strongly residual in G.*

Definition 2 ([4]). A sequence (T_n) in the unite sphere $(A)_1$ of $A = \mathcal{U}(G)$ is an ϵ - approximate central sequence, if $\limsup \| [T_n, X] \|_2 < \epsilon$ for all $X \in (A)_1$.
The set of all ϵ-approximate sequences is denoted by $C_A(\epsilon)$.

If H is strongly residual in G, then for all $(T_n) \in C_{U(G)}(\epsilon)$, there exists a sequence (T_n') in the unit sphere of U(H) such that $\limsup \| T_n - T_n' \|_2 < 14\,\epsilon$ (cf. [3], [5], [6]).

Let $G = (G_1, G_2, \ldots ; H_1, H_2, \ldots)$ with $G_i = Z$ and let $Q(G,n) = \sum_{j=n}^{\infty} \oplus H_j$ and $Q(G, m, n) = \sum_{j=m}^{n} \oplus H_j$.
Then, $\{ Q(G, n) \}$ is a residual sequence in G. Let $(\Gamma_k \mid k = 1, 2, \ldots, r)$ be a finite family with the form $\Gamma_k = (G_1, G_2, \ldots : H_1, H_2, \ldots)$ with $G_i = Z$.
Let $Q(\sum_{k=1}^{r} \oplus \Gamma_k, n) = \sum_{k=1}^{r} \oplus Q(\Gamma_k, n)$ is a residual sequence in G. This residual sequence is called the

canonical residual sequence.

Denote $Q(\sum_{k=1}^{r} \oplus \Gamma_k, n, m) = \sum_{k=1}^{r} \oplus Q(\Gamma_k, n, m)$.

A group G is called of type 0, if it is commutative G is called of type i, if $G = \sum_{j=1}^{n} \oplus G_j$ with $G_j = L_i$; G is called of type i_∞, if $G = \sum_{j=1}^{\infty} \oplus G_j$ with $G_j = L_i$; G is called of type $(i_1, i_2, \ldots\ldots, i_n)$, if $G = \sum_{j=1}^{n} \oplus G_j$, where G_j is of type i_j ; G is called of type $(i_1, i_2, \ldots, i_n)_\infty$, if $G = \sum_{j=1}^{n} \oplus G_j$ and some of G_j are of type i_{j_∞} and others are of type i_j.

Now let $U(G[I_1]) = A$ and $U(G[I_2]) = B$. Suppose that A is *-isomorphic to B. Then, under the identification A = B, we have two expressions $U(G[I_1])$ and $U(G[I_2])$.

Henceforward, we shall assume that A = B and conclude a contradiction.

Lemma 2 ([4]). For $\delta > 0$ and a positive integer n_1 there exists a positive integer n_2 such that
$U(Q(G]I_2], n_2)) \overset{\delta}{\subset} U(Q(G[I_1], n_1))$

Moreover, since $U(Q(G[I_i], n, n+1))$ is a factor, we have

Lemma 3 ([4]). For a positive integer m_2 with $m_2 > n_2$, there exists a positive integer m_1 such that $m_1 > n_1$ and $U(Q(G[I_2], n_2, m_2)) \overset{9\delta}{\subset} U(Q(G[I_1], n_1, m_1))$.

Now let $I_1 = (p_i)$ and $I_2 = (q_j)$. Without the loss of generality, we can assume that $q = q_1$.

For $t = 10^{q_1}$, by applying Lemma 2 for I_1 and the symmetric form of Lemma 2 for I_2, we can choose positive integers n_1

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$,n_2,\ldots\ldots,n_t$ such that $n_2 < n_4 < n_6 < \ldots\ldots < n_t$, and $n_1 < n_3 < \ldots\ldots < n_{t-1}$ and

$$U(Q(G[I_2], n_t)) \overset{\delta}{\subset} U(Q(G[I_1], n_{t-1})) \overset{\delta}{\subset} \ldots\ldots \overset{\delta}{\subset} U(Q(G[I_2], n_2)) \overset{\delta}{\subset} U(Q(G[I_1], n_1)).$$

Then, by Lemma 3, we can choose positive integers $m_1, m_2, \ldots\ldots, m_t$ such that $m_2 > m_4 > \ldots\ldots > m_t$ and $m_1 > m_3 > \ldots\ldots > m_{t-1}$ with $m_t > n_t$ and

$$U(Q(G[I_2], n_t, m_t)) \overset{9\delta}{\subset} U(Q(G[I_1], n_{t-1}, m_{t-1})) \overset{9\delta}{\subset} \ldots\ldots \overset{9\delta}{\subset} U(Q(G[I_1], n_1, m_1)).$$

Since $Q(G[I_i], h, k)$ is a finite sum of the form $(G_1, G_2, \ldots H_1, H_2, \ldots\ldots)$ with $G_i = Z$, it has the canonical residual sequence $\{ Q(Q(G[I_i], h, k), n) \}$.

For simplicity, we shall denote $Q(G[I_i], h, k)$ (resp. $Q(Q(G[I_i], h, k), n)$) by $Q_i(h, k)$ (resp. $Q_i^2((h, k), n)$.

Lemma 4. Suppose $U(Q_1(h, k)) \overset{9\delta}{\subset} U(Q_2(i, j)) \overset{9\delta}{\subset} U(Q_1(p, q))$ with $h > p$ and $q > k$.

Then for arbitrary positive integers r and w, there exists a positive integer s such that $U(Q_1^2(h, k), s)) \overset{(10)^3\delta}{\subset} U(Q_2^2(i, j), r)$ and $s > w$.

Proof. Suppose this is not true, then there exists $T_n \in (U(Q_1^2((h, k), n)))_1$ for each n with $n > w$ such that $\| T_n - S \|_2 \geqq (10)^3 \delta$ for all $S \in (U(Q_2^2((i, j), r))_1$.

Since $\{ Q_1^2((h, k), n)) \}$ is a residual sequence in $U(Q_1(h, k))$, (T_n) is a central sequence in $U(Q_1(h, k))$.

On the other hand, $Q_1(p, q) = Q_1(h, k) \oplus C$, where C is a subgroup of $Q_1(p, q)$; hence (T_n) is a central sequence in $U(Q_1(p, q))$. Now, take $T_n' \in (U(Q_2(i, j))_1$ such that $\| T_n - T_n' \|_2 < 9\delta$ and for arbitrary $X' \in (U(Q_2(i, j))_1$, take $X \in (U(Q_1(p, q))_1$ such that $\| X - X' \| < 9\delta$.

Then,

$$\| [X', T_n'] \|_2 = \| [X', T_n' - T_n] \|_2 + \| [T_n, X] \|_2 + \| [T_n, X - X'] \|_2 \leqq 2 \| T_n' - T_n \|_2 + \| [T_n, X] \|_2 + 2 \| X - X' \|_2 .$$

Hence $\limsup \| [X', T_n'] \|_2 \leqq 18\delta + 18\delta < 37\delta$.

Therefore, there exists a sequence (T_n'') in $(U(Q_2{}^2(i,j), r)))_1$ such that $\limsup \| T_n' - T_n'' \|_2 < 14 \cdot 37\delta$.

Then, $\| T_n - T_n'' \|_2 \leqq \| T_n - T_n' \|_2 + \| T_n' - T_n'' \|_2 < 10^3 \delta$.

This is a contradiction and completes the proof.

Applying this lemma for I_1 and the symmetric one for I_2, there exist positive integers $r_2, r_3, \ldots\ldots, r_t$ such that $r_2 < r_4 < \ldots\ldots < r_t$ and $r_3 < r_4 < \ldots\ldots < r_{t-1}$ and $U(Q_2{}^2((n_t, m_t), r_t)) \overset{(10)^3\delta}{\subset} \ldots\ldots$

$$U(Q_1{}^2((n_3, m_3), r_3)) \overset{(10)^3\delta}{\subset} U(Q_2{}^2((n_2, m_2), r_2)).$$

$$Q(G[I_1], n, m) = \sum_{j=1}^{m} \oplus M_j(I_1) \text{ is of type } (p_1, p_2, \quad , p_m).$$

Then, $Q_1{}^2((n,m), r)$ is of type $(p_1 - 1, \ldots\ldots, p_m - 1)$.

Therefore, at this time, $Q_i{}^2((h, k), r)$ might contain a type 0 -group as a direct summand.

Now we shall define : for $r < s$, $RQ_i{}^2((h, k), (r, s))$ = the center of $Q_i{}^2((h, k), r) + (Q_i{}^2((h, k), r) \ominus Q_i{}^2((h, k), s + 1))$

Lemma 5. For arbitrary positive integer $s_t > r_t$, there exist positive integers $s_4, s_5, \ldots\ldots, s_t$ such that $s_4 > s_6 > \cdots\cdots > s_t$ and $s_5 > s_7 > \cdots\cdots > s_{t-1}$ and

$$U(RQ_2{}^2(n_t, m_t), (r_t, s_t))) \overset{(10)^5\delta}{\subset} \cdots\cdots\cdots\cdots\cdots \overset{(10)^5\delta}{\subset} U(RQ_1{}^2((n_5, m_5), (r_5, s_5)) \overset{(10)^5\delta}{\subset} U(RQ_2{}^2((n_4, m_4), (r_4, s_4))).$$

Proof. $Q_2((n_{t-2}, m_{t-2})) = Q_2((n_t, m_t) \oplus H$, where H is a subgroup of $Q_2((n_{t-2}, m_{t-2}))$.

Now, consider $Q_2{}^2((n_t, m_t), r) \oplus H$ in $Q_2((n_{t-2}, m_{t-2}))$, then $Q_2{}^2((n_t, m_t), r)$ is strongly residual in $Q_2((n_t, m_t))$ and so $Q_2{}^2((n_t, m_t), r) \oplus H$ is strongly residual in $Q_2((n_{t-2}, m_{t-2}))$ for each r.

On the other hand,

$$U(Q_1((n_t-1, m_{t-1}) \overset{9\delta}{\subset} U(Q((n_{t-2}, m_{t-2})) \overset{9\delta}{\subset} U(Q((n_{t-3}, m_{t-3}))).$$

Therefore, by the similar method with the proof of Lemma 4, for each r there exists k with $k-1 > r_{t-1}$ such that $U(Q_1^2((n_{t-1}, m_{t-1}), k) \overset{(10)^3\delta}{\subset} U(Q_2^2((n_t, m_t), r) \oplus H)$.

Take $r = s_t + 1$, then for $T \in (U(RQ_2{}^2((n_t, m_t), (r_t, s_t))))_1 \subset (U(Q_2{}^2((n_t, m_t), r_t)))_1$, there exists $T' \in (U(Q_1{}^2((n_{t-1}, m_{t-1}), r_{t-1})))_1$ such that $\|T - T'\|_2 < (10)^3\delta$.

For $X' \in (U(Q_1{}^2((n_{t-1}, m_{t-1}), k)))_1$, take $X \in (U(Q_2{}^2((n_t, m_t), r) \oplus H))_1$ such that $\|X - X'\|_2 < (10)^3\delta$, then

$$\|[T', X']\|_2 \leqq \|[T' - T, X']\|_2 + \|[X' - X, T]\|_2 + \|[T, X]\|_2 < 2(10)^3\delta + 2(10)^3\delta$$

, because $[T, X] = 0$.

Hence, there exists $T'' \in (U(Q_1{}^2((n_{t-1}, m_{t-1}), k))')_1 \cap U(Q_1{}^2((n_{t-1}, m_{t-1}), r_{t-1})$, where $(\quad)'$ is the commutant of the W^*-algebra ( ), such that $\|T' - T''\|_2 < 4\cdot 4(10)^3\delta$ (cf. Lemma 4 in [4]). Hence, $\|T - T''\|_2 \leq \|T - T'\|_2 + \|T' - T''\|_2 < (10)^5\delta$

Clearly, $U(Q_1{}^2((n_{t-1}, m_{t-1}), k)' \cap U(Q_1{}^2((n_{t-1}, m_{t-1}), r_{t-1})) = RQ_1{}^2((n_{t-1}, m_{t-1}), (r_{t-1}, k-1))$.

Take $k - 1$ as s_{t-1}. The remained part is quite similar.

This completes the proof.

Remark. The proof of Lemma 5 is due to B. Vowden.

$RQ_1{}^2((h, k), (i, j))$ is of type $(p_1-1, p_2-1, \ldots, p_k-1)$. and $RQ_2{}^2((h, k), (i, j))$ is of type $(q_1-1, q_2-1, \ldots, q_k-1)$. They might contain a type 0 group as a direct summand. $RQ_i{}^2((h, k), (i, j)) = D \oplus W$, where D is the center of $RQ_i{}^2((h, k), (i, j))$ and W is of type $(i_1, i_2, \ldots, r_n)$ with $i_u \geqq 1$ for $u = 1, 2, \ldots, n$.

Define the canonical residual sequence of $RQ_1{}^2((h, k), (i, j))$ as follows ; $Q_1RQ_1{}^2((h, k), (i, j), n) = D \oplus Q(W, n)$. Quite similarly, we define the canonical residual sequence of $RQ_2{}^2$.

Now we shall continue this process by q_1 times . Then we have the following situation.

$$U(\Omega_t) \overset{K\delta}{\subset} U(\Omega_{t-1}) \overset{K\delta}{\subset} U(\Omega_{t-2}) \overset{K\delta}{\subset} U(\Omega_{t-3}).$$

, where Ω_t contains a type 1-group as a direct summand and Ω_{t-1} does not contain a type 1-group as a direct summand : moreover $\Omega_{t-2} = \Omega_t \oplus R$, where R is a subgroup of Ω_{t-2} ; K is a constant , which does not depend on δ .

and by the $q_1 + 1$ th process, we kave

$U(\Delta_t) \overset{K_1\delta}{\subset} U(\Delta_{t-1}) \overset{K_1\delta}{\subset} U(\Delta_{t-2})$

, where K_1 does not depend on ϵ .

Moreover, let $\Omega_t = E \oplus H$, where E is the center of Ω_t , then $\Delta_t = E \oplus E_1 \oplus W$, where E_1 is contained in the center of Δ_t and $E_1 = Q(L_1, n)$ for some n.

On the other hand, the center of Δ_{t-1} is same with the center C of Ω_{t-1}, because Ω_{t-1} does not contain a type 1-group as a direct summand.

Lemma 6. For $X \in (U(E_1))_1$, there exists an element $X' \in (U(C))_1$ such that $\| X - X' \|_2 < 10^2 K_1 \delta$.

Proof. Put $X_n = X$, then (x_n) is a central sequence in $U(E_1)$; it is a central sequence in $U(\Delta_{t-2})$, because $\Delta_{t-2} = \Delta_t \oplus \Gamma$ for some subgroup Γ .

Let $Y' \in (U(\Delta_{t-1}))_1$ such that $\|X - Y'\|_2 < K_1 \delta$. then by the discussions in the proof of Lemma 4 , $\| [Y', W'] \|_2 < 5K_1\delta$ for all $W' \in (U(\Delta_{t-1}))_1$; hence there exists a central element X' of $(U(\Delta_{t-1}))_1$ such that $\|X' - Y'\|_2 \leqq 2\cdot 5 K_1$; hence $\|X - X'\|_2 \leqq \|X - Y'\|_2 + \|X' - Y'\|_2 < 10^2 K_1 \delta$. This completes the proof.

Now we shall prove Theorem 1.

Proof of Theorem 1. $(U(E_1))_1 \subset (U(L_1\cdots))_1 \subset (U(\Omega_t))_1 \overset{K\delta}{\subset} (U(\Omega_{t-1}))_1$. By lemma 6, for $X \in (U(E_1))_1$, there exists an $X' \in (U(C))_1$ such that $\| X - X' \|_2 < 10^2 K_1 \delta$

For arbitrary $Y \in (U(L_1))_1$, take $Y' \in (U(\Omega_{t-1}))_1$ such that $\| Y' - Y\|_2 < K \delta$. Then,

Then,

$$\|[Y, X]\|_2 \leq \|[Y - Y', X]\|_2 + \|[Y', X - X']\|_2$$
$$+ \|[Y', X']\|_2 \leq 2K\delta + 2\cdot 10^2 K_1 \delta .$$

Hence there exists an element $X'' \in U(L_1) \cap U(L_1)' = (\lambda 1)$, where λ are complex numbers, such that $\|X - X''\|_2 < 4(K + 10^2 K_1)\delta$.

We can choose δ as arbitrary small number ; hence $U(E_1)$ must be the center of $U(L_1)$.

On the other hand, $U(E_1)$ is not the center of $U(L_1)$, because $U(E_1) = U(\sum_{j=n}^{\infty} \oplus H_j)$ with $H_j = Z$.
This is a contradiction and completes the proof,

Next, we shall show the existence of an uncountable number of II_∞-factors.

Let F_2 be the free group of two generators g_1, g_2 .
Let S be the set of $g \in F_2$ which, when written as a power of g_1 , g_2 of minimum length, end with a $g_1{}^n$, $n = \pm 1, \pm 2, \ldots$, then it is clear that $S \cup g_1 S g_1^{-1} = F_2 \setminus \{e\}$, $g_1 e g_1^{-1} = e$, and $\{g_2{}^n S g_2{}^{-n}\}_{n = 0, \pm 1, \ldots}$ are disjoint subsets of $F_2 \setminus \{e\}$, where e is the unit of F_2 ; therefore $\{e\}$ is strongly residual .

Now let $R_j = F_2$ for $j = 1, 2, \ldots$ and $\Gamma' = \sum_{j=1}^{\infty} \oplus R_j$.
Put $\Gamma_n = \sum_{j=n}^{\infty} \oplus R_j$, then Γ_n is strongly residual in Γ , because Γ_n' is strongly residual in Γ_n and $\sum_{j=1}^{n-1} \oplus \{e_j\}$ is strongly residual in $\sum_{j=1}^{n-1} \oplus R_j$, where e_j is the unit of R_j ; hence $\sum_{j=n}^{\infty} \oplus R_j$ is strongly residual in Γ .

Moreover $\Gamma_n \ominus \Gamma_{n+1} = R_n$ and $\bigcup_{n=1}^{\infty} \Gamma_n' = \sum_{j=1}^{\infty} \oplus R_j = \Gamma$

; hence $\{ \Gamma'_n \}$ is a residual sequence in Γ' .
Put $\Phi_n(I_1) = \Gamma'_n \oplus Q(G.[I_1], n)$ for $n = 1, 2, \ldots$,
then $\{ \Phi_n(I_1) \}$ is a residual sequence in $\Gamma' \oplus G[I_1]$.

Now, we shall show

Theorem 2. Let I_1 and I_2 be two subsets of I satisfying the conditions of Theorem 1, then $U(\Gamma' \oplus G[I_1])$ is not *-isomorphic to $U(\Gamma' \oplus G[I_2])$.

Proof. Since $\Phi_n(I_i) \ominus \Phi_{n+1}(I_i) = (\Gamma'_n \ominus \Gamma'_{n+1}) \oplus (Q(G[I_i], n) \ominus Q(G[I_i], n+L))$, $U(\Phi_n(I_i) \ominus \Phi_{n+1}(I_i))$ is a factor for $n = 1, 2, \ldots$ and $i = 1, 2$..

Therefore, we can apply the lemmas of McDuff [4].
Now. suppose that $U(\Gamma' \oplus G[I_1]) = U(\Gamma' \oplus G[I_2])$, then we have the similar situations with Lemmas 2 and 3 for two residual sequences $\{ \Phi_n(I_i) \}$ ($i = 1, 2$).

On the other hand, $\Gamma'_{m,n} = \Gamma'_m \ominus \Gamma'_n$ $m < n$ has the strong residual subgroup $\{ e \}$; hence we have the same relation with the previous case such that $U(Q_2{}^2((n_t, m_t), r_t) \overset{(10)^3\delta}{\subset} \ldots \overset{(10)^3\delta}{\subset} U(Q_1{}^2((n_3, m_3), r_3) \overset{(10)^3\delta}{\subset} U(Q_2{}^2((n_2, m_2), r_2))$.
This is a contradiction and completes the proof.

Theorem 3, Suppose that I_1, I_2 satisfy the conditions of Theorem 1 and let B be a type I -factor, then $B \otimes U(\Gamma' \oplus G[I_1])$ is not *-isomorphic to $B \otimes U(\Gamma' \oplus G[I_2])$.

Proof. $B \otimes U(\Gamma' \oplus G[I_i]) = B \otimes U(\Gamma') \otimes U(G[I_i]) = B \otimes U(G[I_i]) \otimes \bigotimes_{n=1}^{\infty} U(R_n)$, where $\bigotimes_{n=1}^{\infty} U(R_n)$ is tha canonical infinite tensor product of $\{ U(R_n) \}$ (cf. [6]
; hence $B \otimes U(\Gamma' \oplus G[I_i]) \otimes A$ is *- isomorphic to

$B \otimes U(\Gamma \oplus G[I_i])$ ($i = 1, 2$), where A is the hyperfinite II_1 -factor (cf. [6]).

We shall denote $U(\Gamma \oplus G[I_1])$ by N. Let φ be a normal, faithful semi-finite trace on B, and let τ_1 (resp. τ_2) be the normalized trace on N (resp. A), then $\varphi \otimes \tau_1 \otimes \tau_2$ will define a normal, faithful semi-finite trace on $B \otimes N \otimes A$.

Now, let E be a minimal projection of B, then $E \otimes 1_N$ is a finite projection in $B \otimes N$, where 1_N is the unit of N ; moreover $(E \otimes 1_N) B \otimes N (E \otimes 1_N) = E \otimes N$; hence it is *-isomorphic to N.

For arbitrary positive α with $\alpha \leqq \varphi \otimes \tau_1 \otimes \tau_2 (E \otimes 1_N \otimes 1_A)$, we have a projection P in A such that $\varphi \otimes \tau_1 \otimes \tau_2 (E \otimes 1_N \otimes P) = \alpha$, where 1_A is the unit of A.

Now suppose that $B \otimes N \otimes A$ is *-isomorphic to $B \otimes U(\Gamma \oplus G[I_2]) \otimes A$. then there exists a finite projection E_1 in $B \otimes N \otimes A$ such that $E_1(B \otimes N \otimes A)E_1$ is *-isomorphic to $U(\Gamma \oplus G[I_2]) \otimes A$.

Take $P_0 \in A$ such that $n_0 \varphi \otimes \tau_1 \otimes \tau_2 (E \otimes 1_N \otimes P_0) = \varphi \otimes \tau_1 \otimes \tau_2 (E_1)$ for some positive integer n_0.

Then, there exists a family $(E_{1,i} \mid i = 1, 2, \ldots, n_0)$ of mutually orthogonal , equivalent projections in $B \otimes N \otimes A$ such that $E_{1,i} \sim E \otimes 1_N \otimes P_0$, $E_{1,i} \leqq E_1$ and $\sum_{i=1}^{n_0} E_{1,i} = E_1$.

Since $E_{1,i} \sim E \otimes 1_N \otimes P_0$, $E_{1,i}(B \otimes N \otimes A)E_{1,i}$ is *-isomorphic to $(E \otimes 1_N \otimes P_0)(B \otimes N \otimes A)(E \otimes 1_N \otimes P_0)$.

On the other hand, $(E \otimes 1_N \otimes P_0)(B \otimes N \otimes A)(E \otimes 1_N \otimes P_0) = E \otimes N \otimes P_0AP_0$; sinee P_0AP_0 is *-isomorphic to A,

$E \otimes N \otimes P_o A P_o$ is *-isomorphic to $N \otimes A$.

Since $E_1(B \otimes N \otimes A)E_1$ is *-isomorphic to $E_{1,i}(B \otimes N \otimes A)E_{1,i} \otimes B_{no}$ and so it is *-isomorphic to $N \otimes A$, where B_{no} is the type I_{no} -factor.

Hence, $U(\Gamma' \oplus G[I_2]) \otimes A$ is *-isomorphic to $N \otimes A$.

Since $U(\Gamma' \oplus G[I_i]) \otimes A$ is *-isomorphic to $U(\Gamma' \oplus G[I_i])$, we have a contradiction, where $i = 1, 2$.

This completes the proof.

As a corollary, we have

Corollary 2. There exists an uncountable number of II_∞-factors on a separable Hilbert space.

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