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On the Cartier-Weil formula in the theory of quadratic forms

Shunichi Tanaka

Department of Mathematics, Osaka University

1. This note is an exposition of certain theorems (Theorem 1 and 2) connected with the Fourier transform of the character of the second degree obtained recently by P. Cartier [1] and A. Weil [3]. Weil has obtained these theorems as corollaries of his general theory of certain group of unitary operators. Cartier's proof is elementary and based only on the theory of Fourier transform and the Poisson summation formula.

We give here another (but not essentially different from that of Cartier or Weil) approach to these theorems.

2. Let G be a locally compact abelian group and G^* be its dual group. For $x \in G$ and $x^* \in G^*$, we write $\langle x, x^* \rangle$ instead of $x^*(x)$. Let dx be a Haar measure on G.

Let $S(G)$ denote the space of Schwartz-Bruhat functions on G with its usual topology. For $\Phi \in S(G)$, we denote the Fourier transform of Φ by Φ^*, that is, Φ^* is a function on G^* defined by

$$(1) \qquad \Phi^*(x) = \int \Phi(x) \langle x, x^* \rangle \, dx.$$

Then it is known that Φ^* is in $S(G^*)$ and

$$W : \Phi \longrightarrow \Phi^*$$

gives an (topological) isomorphism between $S(G)$ and $S(G^*)$. We choose the Haar measure dx^* on G^* so that the inversion formula of (1) is given by

$$\Phi(x) = \int \Phi^*(x) \overline{\langle x, x^* \rangle} \, dx^*.$$

Then W is extended to the unitary transformation between the Hilbert spaces $L^2(G, dx)$ and $L^2(G^*, dx^*)$.

We call a continuous function f on G a character of the second degree if $f(x) = 1$ for $x \varepsilon G$ and there exists a homomorphisms $\rho = \rho_f$ from G to G^* such that

$$(2) \qquad f(x + y) = f(x) f(y) \langle x, y\rho \rangle \quad \text{for } x, y \in G.$$

If f is a character of the second degree on G, $\Phi \longrightarrow \Phi f$ is a unitary operator on $L^2(G)$. We know moreover that $\Phi \longrightarrow \Phi f$ is an automorphism of $S(G)$ (see Section 12 in [3]).

Let S_h $(h \varepsilon G)$ be the unitary operator on $L^2(G)$ defined by

$$(S_h \Phi)(x) = \Phi(x + h)$$

and T_h be the unitary operator on $L^2(G^*)$ defined by

$$T_h = W S_h W^{-1} .$$

It is easy to see

$$(T_h\Psi)(x^*) = \Psi(x^*)\overline{\langle h, x^*\rangle}.$$

Lemma 1 (see Appendice in [2]). A bounded operator on $L^2(G^*)$ which commute with all T_h $(h \in G)$ is the multiplication operator by a (essentially) bounded function.

3. Let f be a character of the second degree such that $\rho = \rho_f$ is an isomorphism between G and G^*. For such ρ, we denote by $|\rho|$ the module of ρ, that is, the positive constant defined by the formula

$$\int_G \Psi(x\rho)\,dx = |\rho|^{-1} \int_{G^*} \Psi(x^*)\,dx^*, \quad \Psi \in L^1(G^*).$$

We consider the convolution of f and $\Phi \in S(G)$:

$$(f * \Phi)(x) = \int f(x - y)\Phi(y)\,dy.$$

By (2), it is also written as

$$(f * \Phi)(x) = f(x) \int \langle y, -x\rho\rangle f(-y)\Phi(y)\,dy.$$

So $\Phi \longrightarrow f * \Phi$ is the composite of the multiplication by the characters of the second degree, the Fourier transform and the change of the variable (by the isomorphism ρ). We can conclude that $\Phi \longrightarrow |\rho|^{\frac{1}{2}} f * \Phi$ is an isomorphism of $S(G)$ and a unitary operator on $L^2(G)$.

Theorem 1 (Cartier-Weil formula). There exists a constant

$\gamma(f)$ of the absolute value 1, such that

(3) $(f * \Phi)^*(x^*) = \gamma(f)|\rho|^{-\frac{1}{2}}\Phi^*(x^*)f(x^*\rho^{-1})$ for $\Phi \in S(G)$ and $x^* \in G^*$.

<u>Corollary</u>

(4) $\int(\int f(x-y)\Phi(y)dy)dx = \gamma(f)|\rho|^{-\frac{1}{2}}\int \Phi(x)dx$, for $\Phi \in S(G)$ (Put $x^* = 0$ in (3)).

Proof of Theorem 1.

Let U be the unitary operator on $L^2(G^*)$ (and the automorphism of $S(G^*)$) defined by

$$U : \Psi \longrightarrow W\left\{|\rho|^{\frac{1}{2}}f * (W^{-1}\Psi)\right\}.$$

By the fact that $f * (S_h\Phi) = S_h(f * \Phi)$, it is easy to check that U commute with T_h for all $h \in G$. By Lemma 1, U is defined by the multiplication by some bounded measurable function $M(x^*)$. Because U maps $S(G^*)$ into $S(G)$, U is defined by the multiplication by a continuous function $M(x^*)$ whose absolute value is 1 (by the unitarity of U). So we have

$$(|\rho|^{\frac{1}{2}}f * \Phi)^*(x^*) = M(x^*)\Phi^*(x^*)$$

for $\Phi \in S(G)$ and $x^* \in G^*$. Putting $x^* = 0$ in this formula, we have (4), where $\gamma(f) = M(0)$.

If we replace the function Φ by $\Phi(\cdot)\langle\cdot, x^*\rangle$ in (4), we have (3).

4. Let Γ be a closed subgroup of G and Γ_* be the closed subgroup of G^* associated with Γ by duality, i.e.

$$\Gamma_* = \left\{ \xi^* \varepsilon G^* ;\ \langle \xi, \xi^* \rangle = 1, \text{ for } \xi \varepsilon \Gamma \right\}.$$

Let $d\xi$ and $d\xi^*$ be Haar measures on Γ and Γ_* respectively. If we normalize $d\xi^*$ suitably, we have the following formula, which one may call the (generalized) Poisson summation formula (See 16, 17, 18 in [3]),

(5) $$\int_\Gamma \Phi(\xi)d\xi = \int_{\Gamma_*} \Phi^*(\xi^*)d\xi^*, \quad \text{for } \Phi \varepsilon S(G).$$

<u>Theorem 2</u>. Let f be a character of the second degree on G such that $\rho = \rho_f$ is an isomorphism between G and G^*. If, moreover, $f(\xi) = 1$ for $\xi \varepsilon \Gamma$ and ρ gives an isomorphism between Γ and Γ_*, we have $\gamma(f) = 1$.

Proof. We denote by c_1 and c_2 positive constans not depending on $\Phi \varepsilon S(G)$. (5) can be rewritten as

(6) $$\int_\Gamma \Phi(\xi)d\xi = c_i \int_\Gamma \Phi^*(-\xi\rho)d\xi.$$

We denote by $\check{f}$ the character of the second degree on G defined by $\check{f}(x) = f(-x)$. Then we have

$$\int_{\Gamma} \Phi(\xi)d\,\xi = \int_{\Gamma} (\check{f}\Phi)(\xi)d\xi$$

$$= c_1 \int_{\Gamma} (\check{f}\Phi)^*(-\,\xi\rho)d\xi, \qquad \text{by (6)},$$

$$= c_1 \int_{\Gamma} f(\xi)(\check{f}\Phi)^*(-\,\xi\rho)d\xi$$

$$= c_1 \int_{\Gamma} (f * \Phi)(\xi)d\xi$$

$$= c_1 \int_{\Gamma_*} (f * \Phi)^*(\xi^*)d\xi^*, \quad \text{by (5)},$$

$$= c_2\gamma(f) \int_{\Gamma_*} \Phi^*(\xi^*)\overline{f(\xi^*\rho^{-1})}d\xi^*, \quad \text{by Theorem 1},$$

$$= c_2\gamma(f) \int_{\Gamma_*} \Phi^*(\xi^*)d\xi^*$$

$$= c_2\gamma(f) \int_{\Gamma} \Phi(\xi)d\xi, \qquad \text{by (5)}.$$

So choosing $\Phi \in S(G)$ such that $\int_{\Gamma} \Phi(\xi)d\xi \neq 0$, we have $c_2\gamma(f) = 1$. As $\gamma(f)$ is of the absolute value 1, we have $\gamma(f) = 1$.

Bibliography

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